This book is dedicated to my family, for without their love and support, this work would have not been possible. Most importantly, let us not forget that our greatest gift are our children and our children’s children, who stand as a reminder, that curiosity and imagination is what drives our intellectual pursuits, detaching us from the mundane and confinement to Earthly values, and bringing us closer to the divine and the infinite.

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Escher detail above from “Belvedere” plate 230, [I].

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Preface

These notes were developed as part of a course on Differential Geometry at the advanced undergraduate, first year graduate level, which the author has taught for several years. There are many excellent texts in Differential Geometry but very few have an early introduction to differential forms and their applications to Physics. It is purpose of these notes to:

1. Provide a bridge between the very practical formulation of classical differential geometry created by early masters of the late 1800’s, and the more elegant but less intuitive modern formulation in terms of manifolds, bundles and differential forms. In particular, the central topic of curvature is presented in three different but equivalent formalisms.

2. Present the subject of differential geometry with an emphasis on making the material readable to physicists who may have encountered some of the concepts in the context of classical or quantum mechanics, but wish to strengthen the rigor of the mathematics. A source of inspiration for this goal is rooted in the shock to this author as a graduate student in the 70’s at Berkeley, at observing the gasping failure of communications between the particle physicists working on Gauge theories and differential geometers working on connection on fiber bundles. They seemed to be completely unaware at the time, that they were working on the same subject.

3. Provide the formal geometrical background which constitute the foundation of the Mathematical Theory of General Relativity.

4. Introduce examples of other applications of differential geometry to physics that might not appear in traditional texts used in courses for mathematics students.

This book should be accessible to students who have completed traditional training in Advanced Calculus, Linear Algebra, and Differential Equations. Students who master the entirety of this material will have gained insight in a very powerful tool in mathematical physics at the graduate level.

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Chapter 1

Vectors and Curves

1.1 Tangent Vectors

1.1.1 Definition Euclidean n-space $\mathbb{R}^n$ is defined as the set of ordered n-tuples $p = (p^1, \ldots, p^n)$, where $p^i \in \mathbb{R}$, for each $i = 1, \ldots, n$.

We may associate $p$ with the position vector of a point $p(p^1, \ldots, p^n)$ in n-space. Given any two n-tuples $p = (p^1, \ldots, p^n)$, $q = (q^1, \ldots, q^n)$ and any real number $c$, we define two operations:

\[
\begin{align*}
    p + q &= (p^1 + q^1, \ldots, p^n + q^n), \\
    c \cdot p &= (c p^1, \ldots, c p^n).
\end{align*}
\]

These two operations of vector sum and multiplication by a scalar satisfy all the 8 properties needed to give the set $V = \mathbb{R}^n$ a natural structure of a vector space.

1.1.2 Definition Let $x^i$ be the real valued functions in $\mathbb{R}^n$ such that

\[ x^i(p) = p^i \]

for any point $p = (p^1, \ldots, p^n)$. The functions $x^i$ are then called the natural coordinate functions. When convenient in $\mathbb{R}^3$ we revert to the usual names for the coordinates, $x^1 = x$, $x^2 = y$ and $x^3 = z$. A small awkwardness might occur in the transition to modern notation. In classical vector calculus a point in $\mathbb{R}^n$ is often denoted by $x$, in which case, we pick up the coordinates with the slot projection functions $u^i : \mathbb{R}^n \to \mathbb{R}$ defined by

\[ u^i(x) = x^i. \]

1.1.3 Definition A real valued function in $\mathbb{R}^n$ is of class $C^r$ if all the partial derivatives of the function up to order $r$ exist and are continuous. The space of infinitely differentiable (smooth) functions will be denoted by $C^\infty(\mathbb{R}^n)$ or $\mathcal{F}(\mathbb{R}^n)$.

---

1 In these notes we will use the following index conventions:
In $\mathbb{R}^n$, indices such as $i, j, k, l, m, n$, run from 1 to $n$.
In Space-Time, indices such as $\mu, \nu, \rho, \sigma$, run from 1 to 4.
On surfaces in $\mathbb{R}^3$, indices such as $\alpha, \beta, \gamma, \delta$, run from 1 to 2.
Spinor indices such as $A, B, \dot{A}, \dot{B}$ run from 1 to 2.
1.1.4 Definition Let $V$ and $V'$ be finite dimensional vector spaces such as $V = \mathbb{R}^k$ and $V' = \mathbb{R}^n$. The denote the space of linear transformations from $V$ to $V'$ by $L(V, V')$. The set of linear functionals $L(V, \mathbb{R})$ is called the dual vector space $V^*$. This space has the same dimension as $V$.

In calculus, vectors are usually regarded as arrows characterized by a direction and a length. Vectors as thus considered as independent of their location in space. Because of physical and mathematical reasons, it is advantageous to introduce a notion of vectors that does depend on location. For example, if the vector is to represent a force acting on a rigid body, then the resulting equations of motion will obviously depend on the point at which the force is applied.

In a later chapter we will consider vectors on curved spaces. In these cases the positions of the vectors are crucial. For instance, a unit vector pointing north at the earth’s equator is not at all the same as a unit vector pointing north at the tropic of Capricorn. This example should help motivate the following definition.

1.1.5 Definition A tangent vector $X_p$ in $\mathbb{R}^n$, is an ordered pair $\{x, p\}$. We may regard $x$ as an ordinary advanced calculus ”arrow-vector” and $p$ is the position vector of the foot of the arrow.

The collection of all tangent vectors at a point $p \in \mathbb{R}^n$ is called the tangent space at $p$ and will be denoted by $T_p(\mathbb{R}^n)$. Given two tangent vectors $X_p, Y_p$ and a constant $c$, we can define new tangent vectors at $p$ by $(X + Y)_p = X_p + Y_p$ and $(cX)_p = cX_p$. With this definition, it is clear that for each point $p$, the corresponding tangent space $T_p(\mathbb{R}^n)$ at that point has the structure of a vector space. On the other hand, there is no natural way to add two tangent vectors at different points.

The set $T(\mathbb{R}^n)$ (or simply $T\mathbb{R}^n$) consisting of the union of all tangent spaces at all points in $\mathbb{R}^n$ is called the tangent bundle. This object is not a vector space, but as we will see later it has much more structure than just a set.

![Fig. 1.1: Tangent Bundle](image)

1.1.6 Definition A vector field $X$ in $U \subset \mathbb{R}^n$ is a section of the tangent bundle, that is, a smooth function from $U$ to $T(U)$. The space of sections $\Gamma(T(U))$ is also denoted by $\mathcal{X}(U)$.

The difference between a tangent vector and a vector field is that in the latter case, the coefficients $v^i$ are smooth functions of $x^i$. Since in general there are not enough dimensions to depict a tangent bundle and vector fields as sections thereof, we use abstract diagrams such as shown Figure 1.1. In such a picture, the base space $M$ (in this case $M = \mathbb{R}^n$) is compressed into the continuum at the bottom of the picture in which several points $p_1, \ldots, p_k$ are shown. To each such point one attaches a tangent space. Here, the tangent spaces are just copies of $\mathbb{R}^n$ shown as vertical fibers in the diagram. The vector component $v_p$ of a tangent vector at the point $p$ is depicted as an arrow embedded in the fiber. The union of all such fibers constitutes the tangent bundle $T M = T\mathbb{R}^n$. A section of the bundle amounts to assigning a tangent vector to every point in the the base. It is required that such assignment of vectors is done in a smooth way so that there are no major ”changes” of the vector field between nearby points.

Given any two vector fields $X$ and $Y$ and any smooth function $f$, we can define new vector fields
X + Y and fX by

\[(X + Y)_p = X_p + Y_p \quad (1.2)\]
\[(fX)_p = fX_p,\]

so that \( \mathcal{X}(U) \) has the structure of a vector space over \( \mathbb{R} \). The subscript notation \( X_p \) to indicate the location of a tangent vector is sometimes cumbersome but necessary to distinguish them from vector fields.

Vector fields are essential objects in physical applications. If we consider the flow of a fluid in a region, the velocity vector field indicates the speed and direction of the flow of the fluid at that point. Other examples of vector fields in classical physics are the electric, magnetic and gravitational fields. The vector field in figure 1.2 represents a magnetic field around an electrical wire pointing out of the page.

1.1.7 Definition Let \( X_p = \{x, p\} \) be a tangent vector in an open neighborhood \( U \) of a point \( p \in \mathbb{R}^n \) and let \( f \) be a \( C^\infty \) function in \( U \). The directional derivative of \( f \) at the point \( p \), in the direction of \( x \), is defined by

\[X_p(f) = \nabla f(p) \cdot x, \quad (1.3)\]

where \( \nabla f(p) \) is the gradient of the function \( f \) at the point \( p \). The notation

\[X_p(f) \equiv \nabla_{X_p} f \]

is also used commonly. This notation emphasizes that, in differential geometry, we may think of a tangent vector at a point as an operator on the space of smooth functions in a neighborhood of the point. The operator assigns to a function the directional derivative of that function in the direction of the vector. Here we need not assume as in calculus that the direction vectors have unit length.

It is easy to generalize the notion of directional derivatives to vector fields by defining

\[X(f) \equiv \nabla_X f = \nabla f \cdot x, \quad (1.4)\]

where the function \( f \) and the components of \( x \) depend smoothly on the points of \( \mathbb{R}^n \).

The tangent space at a point \( p \) in \( \mathbb{R}^n \) can be envisioned as another copy of \( \mathbb{R}^n \) superimposed at the point \( p \). Thus, at a point \( p \) in \( \mathbb{R}^2 \), the tangent space consist of the point \( p \) and a copy of the vector space \( \mathbb{R}^2 \) attached as a "tangent plane" at the point \( p \). Since the base space is a flat 2-dimensional continuum, the tangent plane for each point appears indistinguishable from the base space as in figure 1.2.

Later we will define the tangent space for a curved continuum such as a surface in \( \mathbb{R}^3 \) as shown in figure 1.3. In this case, the tangent space at a point \( p \) consists of the vector space of all vectors actually tangent to the surface at the given point.

1.1.8 Proposition If \( f, g \in \mathcal{F}(\mathbb{R}^n) \), \( a, b \in \mathbb{R} \), and \( X \in \mathcal{X}(\mathbb{R}^n) \) is a vector field, then

\[X(af + bg) = aX(f) + bX(g) \quad (1.5)\]
\[X(fg) = fX(g) + gX(f)\]

1.1.9 Remark The space of smooth functions is a ring, ignoring a small technicality with domains. An operator such as a vector field with the properties above, is called a Linear Derivation on
Proof First, let us develop an mathematical expression for tangent vectors and vector fields that will facilitate computation.

Let \( p \in U \) be a point and let \( x^i \) be the coordinate functions in \( U \). Suppose that \( X_p = (x, p) \), where the components of the Euclidean vector \( x \) are \((v^1, \ldots, v^n)\). Then, for any function \( f \), the tangent vector \( X_p \) operates on \( f \) according to the formula

\[
X_p(f) = \sum_{i=1}^{n} v^i \left( \frac{\partial f}{\partial x^i} \right)(p). \tag{1.6}
\]

It is therefore natural to identify the tangent vector \( X_p \) with the differential operator

\[
X_p = \sum_{i=1}^{n} v^i \left( \frac{\partial}{\partial x^i} \right)_p \tag{1.7}
\]

\[
X_p = v^1 \left( \frac{\partial}{\partial x^1} \right)_p + \cdots + v^n \left( \frac{\partial}{\partial x^n} \right)_p.
\]

Notation: We will be using Einstein’s convention to suppress the summation symbol whenever an expression contains a repeated index. Thus, for example, the equation above could be simply written as

\[
X_p = v^i \left( \frac{\partial}{\partial x^i} \right)_p. \tag{1.8}
\]

This equation implies that the action of the vector \( X_p \) on the coordinate functions \( x^i \) yields the components \( v^i \) of the vector. In elementary treatments, vectors are often identified with the components of the vector and this may cause some confusion.

The quantities

\[
\{e_1, \ldots, e_k\}_p = \left\{ \left( \frac{\partial}{\partial x^1} \right)_p, \ldots, \left( \frac{\partial}{\partial x^n} \right)_p \right\}
\]

form a basis for the tangent space \( T_p(\mathbb{R}^n) \) at the point \( p \), and any tangent vector can be written as a linear combination of these basis vectors. The quantities \( v^i \) are called the \textbf{contravariant} components of the tangent vector. Thus, for example, the Euclidean vector in \( \mathbb{R}^3 \)

\[
x = 3i + 4j - 3k
\]

located at a point \( p \), would correspond to the tangent vector

\[
X_p = 3 \left( \frac{\partial}{\partial x} \right)_p + 4 \left( \frac{\partial}{\partial y} \right)_p - 3 \left( \frac{\partial}{\partial z} \right)_p.
\]
Let \( X = v^i \frac{\partial}{\partial x^i} \) be an arbitrary vector field and let \( f \) and \( g \) be real-valued functions. Then

\[
X(af + bg) = v^i \frac{\partial}{\partial x^i}(af + bg) = v^i \frac{\partial}{\partial x^i}(af) + v^i \frac{\partial}{\partial x^i}(bg) = av^i \frac{\partial f}{\partial x^i} + bv^i \frac{\partial g}{\partial x^i} = aX(f) + bX(g).
\]

Similarly,

\[
X(fg) = v^i \frac{\partial}{\partial x^i}(fg) = v^i f \frac{\partial}{\partial x^i}(g) + v^i g \frac{\partial}{\partial x^i}(f) = f v^i \frac{\partial g}{\partial x^i} + g v^i \frac{\partial f}{\partial x^i} = fX(g) + gX(f).
\]

To re-emphasize, any quantity in Euclidean space which satisfies relations 1.5 is called a linear derivation on the space of smooth functions. The word **linear** here is used in the usual sense of a linear operator in linear algebra, and the word derivation means that the operator satisfies Leibnitz’ rule.

The proof of the following proposition is slightly beyond the scope of this course, but the proposition is important because it characterizes vector fields in a coordinate-independent manner.

**1.1.10 Proposition** Any linear derivation on \( \mathcal{F}(\mathbb{R}^n) \) is a vector field.

This result allows us to identify vector fields with linear derivations. This step is a big departure from the usual concept of a “calculus” vector. To a differential geometer, a vector is a linear operator whose inputs are functions and the output are functions which at each point represent the directional derivative in the direction of the Euclidean vector.

**1.1.11 Example** Given the point \( p(1,1) \), the Euclidean vector \( \mathbf{x} = (3, 4) \) and the function \( f(x,y) = x^2 + y^2 \), we associate \( \mathbf{x} \) with the tangent vector

\[
X_p = 3 \frac{\partial}{\partial x} + 4 \frac{\partial}{\partial y}.
\]

Then,

\[
X_p(f) = 3 \left( \frac{\partial f}{\partial x} \right)_p + 4 \left( \frac{\partial f}{\partial y} \right)_p,
\]

\[
= 3(2x)|_p + 4(2y)|_p,
\]

\[
= 3(2) + 4(2) = 14.
\]

**1.1.12 Example** Let \( f(x,y,z) = xy^2z^3 \) and \( \mathbf{x} = (3x, 2y, z) \). Then
The linear transformation $DF$ is called an integral curve of $X$ if $\alpha'(0) = X_p$.

In local coordinates, the expression defining integral curves constitutes a system of first order differential equations, so the existence and uniqueness of solutions apply locally. We will treat this in more detail in section 7.1.3.

### 1.2 Differentiable Maps

#### 1.2.1 Definition

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a vector function defined by coordinate entries $F(p) = (f^1(p), f^2(p), \ldots, f^m(p))$. The vector function is called a mapping if the coordinate functions are all differentiable. If the coordinate functions are $C^\infty$, $F$ is called a smooth mapping. If $(x^1, x^2, \ldots, x^n)$ are local coordinates in $\mathbb{R}^n$ and $(y^1, y^2, \ldots, y^m)$ local coordinates in $\mathbb{R}^m$, a map $y = F(x)$ is represented in advanced calculus by $m$ functions of $n$ variables

$$y^i = f^i(x^i), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.$$  \hfill (1.9)

A map $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at a point $p \in \mathbb{R}^n$ if there exists a linear transformation $DF(p : \mathbb{R}^n) \rightarrow \mathbb{R}^m$ such that

$$\lim_{h \rightarrow 0} \frac{|F(p + h) - F(p) - DF(p)(h)|}{|h|} = 0.$$  \hfill (1.10)

The linear transformation $DF(p)$ is called the Jacobian.

#### Remarks

1. A differentiable mapping $F : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$ is what we called a curve. If $t \in I = [a, b]$, the mapping gives a parametrization $x(t)$, as we discussed in the previous section.

2. A differentiable mapping $F : R \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a coordinate transformation. Thus, for example, the mapping $F : (u, v) \in \mathbb{R}^2 \rightarrow (x, y) \in \mathbb{R}^2$, given by functions $x = x(u, v)$, $y = y(u, v)$, would constitute a change of coordinates from $(u, v)$ to $(x, y)$. The most familiar case is the polar coordinates transformation $x = r \cos \theta$, $y = r \sin \theta$.

3. A differentiable mapping $F : R \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is what in calculus we called a parametric surface. Typically one assumes that $R$ is a simple closed region, such as a rectangle. If one denotes the coordinates in $\mathbb{R}^2$ by $(u, v) \in R$, and $x \in \mathbb{R}^3$ the parametrization is written as $x(u, v) = (x(u, v), y(u, v), z(u, v))$. The treatment of surfaces in $\mathbb{R}^3$ is presented in chapter 4. If $\mathbb{R}^3$ is replaced by $\mathbb{R}^n$, the mapping locally represents a 2-dimensional surface in a space of $n$ dimensions.

For each point $p \in \mathbb{R}^n$, we say that the Jacobian induces a linear transformation $F_p$ from the tangent space $T_p\mathbb{R}^n$ to the tangent space $T_{F(p)}\mathbb{R}^m$. In differential geometry we this Jacobian map
1.2. DIFFERENTIABLE MAPS

is also called the push-forward. If we let $X$ be a tangent vector in $\mathbb{R}^n$, then the tangent vector $F_*X$ in $\mathbb{R}^m$ is defined by

$$F_*X(f) = X(f \circ F), \quad (1.11)$$

where $f \in \mathcal{C}(\mathbb{R}^m)$. (See figure 1.4)

![Fig. 1.4: Jacobian Map.](image)

As shown in the diagram, $F_*X(f)$ is evaluated at $F(p)$ whereas $X$ is evaluated at $p$. So, to be precise, equation (1.11) should really be written as

$$F_*X(f)(F(p)) = X(f \circ F)(p), \quad (1.12)$$

$$F_*X(f) \circ F = X(f \circ F), \quad (1.13)$$

As we we have learned from linear algebra, to find a matrix representation of a linear map in a particular basis, one applies the map to the basis vectors. If we denote by $\{\frac{\partial}{\partial x^i}\}$ the basis for the tangent space at a point $p \in \mathbb{R}^n$ and by $\{\frac{\partial}{\partial y^j}\}$ the basis for the tangent space at the corresponding point $F(p) \in \mathbb{R}^m$ with coordinates given by $y^j = f^j(x^i)$, the push-forward definition reads,

$$F_*\left(\frac{\partial}{\partial x^i}\right)(f) = \frac{\partial}{\partial y^j} \left( f \circ F \right),$$

$$= \frac{\partial f \partial y^j}{\partial y^j \partial x^i},$$

$$F_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}. \quad (1.14)$$

In other words, the matrix representation of $F_*$ in standard basis is in fact the Jacobian matrix. In classical notation, we simply write the Jacobian map in the familiar form,

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}. \quad (1.14)$$

1.2.2 Theorem If $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^p$ are mappings, then $(G \circ F)_* = G_* \circ F_*.$

Proof: Let $X \in T_p(\mathbb{R}^n)$, and $f$ be a smooth function $f : \mathbb{R}^p \to \mathbb{R}$. Then,

$$(G \circ F)_*(X)(f) = X(f \circ (G \circ F)),$$

$$= X((f \circ G) \circ F),$$

$$= F_*(X)(f \circ G),$$

$$= G_*(F_*(X)(f)),$$

$$= (G_* \circ F_*)(X)(f).$$

1.2.3 Inverse Function Theorem. When $m = n$, mappings are called change of coordinates. In the terminology of tangent spaces, the classical inverse function theorem states that if the Jacobian
map $F_*$ is a vector space isomorphism at a point, then there exists a neighborhood of the point in which $F$ is a diffeomorphism.

1.2.4 Remarks

1. The equation above shows that under change of coordinates, basis tangent vectors and by linearity all tangent vectors transform by multiplication by the matrix representation of the Jacobian. This is the source of the almost tautological definition in physics, that a contravariant tensor of rank one, is one that transforms like a contravariant tensor of rank one.

2. Many authors use the notation $dF$ to denote the push-forward map $F_*$, but we prefer to avoid this notation for now, since we have not yet introduced the notion of the differential as a linear map on tangent spaces.

3. If $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^p$ are mappings, we leave it as an exercise for the reader to verify that the formula $(G \circ F)_* = G_* \circ F_*$ for the composition of linear transformations corresponds to the classical chain rule.

4. As we will see later, the concept of the push-forward extends to manifold mappings $F : M \to N$. 
1.3 Curves in $\mathbb{R}^3$

1.3.1 Parametric Curves

1.3.1 Definition A curve $\alpha(t)$ in $\mathbb{R}^3$ is a $C^\infty$ map from an interval $I \subset \mathbb{R}$ into $\mathbb{R}^3$. The curve assigns to each value of a parameter $t \in \mathbb{R}$, a point $(\alpha^1(t), \alpha^2(t), \alpha^3(t)) \in \mathbb{R}^3$.

\[ I \subset \mathbb{R} \xrightarrow{\alpha} \mathbb{R}^3 \]
\[ t \mapsto \alpha(t) = (\alpha^1(t), \alpha^2(t), \alpha^3(t)) \]

One may think of the parameter $t$ as representing time, and the curve $\alpha$ as representing the trajectory of a moving point particle. In these notes we also use classical notation for the position vector

\[ x(t) = (x^1(t), x^2(t), x^3(t)), \quad (1.15) \]

which is more prevalent in vector calculus and elementary physics textbooks. Of course, what this notation really means is

\[ x^i(t) = (u^i \circ \alpha)(t), \quad (1.16) \]

where $u^i$ are the coordinate slot functions in an open set in $\mathbb{R}^3$.

1.3.2 Example Let

\[ \alpha(t) = (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3). \quad (1.17) \]

This equation represents a straight line passing through the point $p = (b_1, b_2, b_3)$, in the direction of the vector $v = (a_1, a_2, a_3)$.

1.3.3 Example Let

\[ \alpha(t) = (a \cos \omega t, a \sin \omega t, bt). \quad (1.18) \]

This curve is called a circular helix. Geometrically, we may view the curve as the path described by the hypotenuse of a triangle with slope $b$, which is wrapped around a circular cylinder of radius $a$. The projection of the helix onto the $xy$-plane is a circle and the curve rises at a constant rate in the $z$-direction (See Figure 1.5a). Similarly, The equation $\alpha(t) = (a \cosh \omega t, a \sinh \omega t, bt)$ is called a hyperbolic "helix." It represents the graph of curve that wraps around a hyperbolic cylinder rising at a constant rate.

Fig. 1.5: a) Circular Helix. b) Temple of Viviani

1.3.4 Example Let
\[\alpha(t) = (a(1 + \cos t), a \sin t, 2a \sin(t/2)). \quad (1.19)\]

This curve is called the Temple of Viviani. Geometrically, this is the curve of intersection of a sphere \(x^2 + y^2 + z^2 = 4a^2\) of radius 2a, and the cylinder \(x^2 + y^2 = 2ax\) of radius a with a generator tangent to the diameter of the sphere along the z-axis (See Figure 1.5(b)).

The Temple of Viviani is of historical interest in the development of calculus. The problem was posed anonymously by Viviani to Leibnitz, to determine on the surface of a semi-sphere, four identical windows, in such a way that the remaining surface be equivalent to a square. It appears as if Viviani was challenging the effectiveness of the new methods of calculus against the power of traditional geometry.

It is said that Leibnitz understood the nature of the challenge and solved the problem in one day. Not knowing the proposer of the enigma, he sent the solution to his Serenity Ferdinando, as he guessed that the challenge came from prominent Italian mathematicians. Upon receipt of the solution by Leibnitz, Viviani posted a mechanical solution solution without proof. He described it as using a boring device to remove from a semisphere, the surface area cut by two cylinders with half the radius, and which are tangential to a diameter of the base. Upon realizing this could not physically be rendered as a temple since the roof surface would rest on only four points, Viviani no longer spoke of a temple but referred to the shape as a ”sail.”

1.3.5 Definition Let \(\alpha : I \to \mathbb{R}^3\) be a curve in \(\mathbb{R}^3\) given in components as above \(\alpha = (\alpha^1, \alpha^2, \alpha^3)\). For each point \(t \in I\) we define the velocity of the curve to be the tangent vector

\[\alpha'(t) = \left( \frac{d\alpha^1}{dt}, \frac{d\alpha^2}{dt}, \frac{d\alpha^3}{dt} \right)_{\alpha(t)} \quad (1.20)\]

At each point of the curve, the velocity vector is tangent to the curve and thus the velocity constitutes a vector field representing the velocity flow along that curve. In a similar manner the second derivative \(\alpha''(t)\) is a vector field called the acceleration along the curve. The length \(v = \|\alpha'(t)\|\) of the velocity vector is called the speed of the curve. The classical components of the velocity vector are simply given by

\[\mathbf{v}(t) = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \left( \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right), \quad (1.21)\]

and the speed is

\[v = \sqrt{ \left( \frac{dx^1}{dt} \right)^2 + \left( \frac{dx^2}{dt} \right)^2 + \left( \frac{dx^3}{dt} \right)^2 }. \quad (1.22)\]

As is well known, the vector form of the equation of the line 1.17 can be written as \(\mathbf{x}(t) = \mathbf{p} + t\mathbf{v}\), which is consistent with the Euclidean axiom stating that given a point and a direction, there is only one line passing through that point in that direction. In this case, the velocity \(\dot{\mathbf{x}} = \mathbf{v}\) is constant and hence the acceleration \(\ddot{\mathbf{x}} = 0\). This is as one would expect from Newton’s law of inertia.

The differential \(d\mathbf{x}\) of the position vector given by

\[d\mathbf{x} = (dx^1, dx^2, dx^3) = \left( \frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right) dt \quad (1.23)\]
which appears in line integrals in advanced calculus is some sort of an \textbf{infinitesimal tangent vector}. The norm $\|d\mathbf{x}\|$ of this infinitesimal tangent vector is called the differential of arc length $ds$. Clearly, we have

$$ds = \|d\mathbf{x}\| = v\, dt.$$ \hfill (1.24)

If one identifies the parameter $t$ as time in some given units, what this says is that for a particle moving along a curve, the speed is the rate of change of the arc length with respect to time. This is intuitively exactly what one would expect.

As we will see later in this text, the notion of infinitesimal objects needs to be treated in a more rigorous mathematical setting. At the same time, we must not discard the great intuitive value of this notion as envisioned by the masters who invented Calculus, even at the risk of some possible confusion! Thus, whereas in the more strict sense of modern differential geometry, the velocity is a tangent vector and hence it is a differential operator on the space of functions, the quantity $d\mathbf{x}$ can be viewed as a traditional vector which, at the infinitesimal level is a linear approximation to the curve, and points tangentially in the direction of $v$.

\subsection*{1.3.2 Velocity}

For any smooth function $f: \mathbb{R}^3 \to \mathbb{R}$, we formally define the action of the velocity vector field $\alpha'(t)$ as a linear derivation by the formula

$$\alpha'(t)(f) |_{\alpha(t)} = \frac{d}{dt}(f \circ \alpha) |_{t}. \hfill (1.25)$$

The modern notation is more precise, since it takes into account that the velocity has a vector part as well as point of application. Given a point on the curve, the velocity of the curve acting on a function, yields the directional derivative of that function in the direction tangential to the curve at the point in question. The diagram in figure [1.6] below provides a more visual interpretation of the velocity vector formula 1.25 as a linear mapping between tangent spaces.

![Fig. 1.6: Velocity Vector Operator](image)

The map $\alpha(t)$ from $\mathbb{R}$ to $\mathbb{R}^3$ induces a push-forward map $\alpha_*$ from the tangent space of $\mathbb{R}$ to the tangent space of $\mathbb{R}^3$. The image $\alpha_*(\frac{d}{dt})$ in $T_{\alpha(t)}\mathbb{R}^3$ of the tangent vector $\frac{d}{dt}$ is what we call $\alpha'(t)$.

$$\alpha_*(\frac{d}{dt}) = \alpha'(t).$$

Since $\alpha'(t)$ is a tangent vector in $\mathbb{R}^3$, it acts on functions in $\mathbb{R}^3$. The action of $\alpha'(t)$ on a function $f$ on $\mathbb{R}^3$ is the same as the action of $\frac{d}{dt}$ on the composition $(f \circ \alpha)$. In particular, if we apply $\alpha'(t)$ to the coordinate functions $x^i$, we get the components of the the tangent vector

$$\alpha'(t)(x^i) |_{\alpha(t)} = \frac{d}{dt}(x^i \circ \alpha) |_{t}. \hfill (1.26)$$

To unpack the above discussion in the simplest possible terms, we associate with the classical velocity vector $\mathbf{v} = \dot{\mathbf{x}}$ a linear derivation $\alpha'(t)$ given by
\[ \alpha'(t) = \frac{d}{dt}(x^i \circ \alpha_t)(\partial / \partial x^i)_{\alpha(t)}, \]
\[ = \frac{dx^1}{dt} \frac{\partial}{\partial x^1} + \frac{dx^2}{dt} \frac{\partial}{\partial x^2} + \frac{dx^3}{dt} \frac{\partial}{\partial x^3}. \] (1.27)

So, given a real valued function on \( \mathbb{R}^3 \), the action of the velocity vector is given by the chain rule

\[ \alpha'(t)(f) = \frac{\partial f}{\partial x^1} \frac{dx^1}{dt} + \frac{\partial f}{\partial x^2} \frac{dx^2}{dt} + \frac{\partial f}{\partial x^3} \frac{dx^3}{dt} = \nabla f \cdot \mathbf{v}. \]

If \( \alpha(t) \) is a curve in \( \mathbb{R}^n \) with tangent vector \( X = \alpha'(t) \) and \( F : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable map, then \( F \circ X \) is a tangent vector to the curve \( F \circ \alpha \) in \( \mathbb{R}^m \). That is, \( F_\ast \) maps tangent vectors of \( \alpha \) to tangent vectors of \( F \circ \alpha \).

1.3.6 Definition

If \( t = t(s) \) is a smooth, real valued function and \( \alpha(t) \) is a curve in \( \mathbb{R}^3 \), we say that the curve \( \beta(s) = \alpha(t(s)) \) is a reparametrization of \( \alpha \).

A common reparametrization of curve is obtained by using the arc length as the parameter. Using this reparametrization is quite natural, since we know from basic physics that the rate of change of the arc length is what we call speed

\[ v = \frac{ds}{dt} = \| \alpha'(t) \|. \] (1.28)

The arc length is obtained by integrating the above formula

\[ s = \int \| \alpha'(t) \| \ dt = \int \sqrt{\left( \frac{dx^1}{dt} \right)^2 + \left( \frac{dx^2}{dt} \right)^2 + \left( \frac{dx^3}{dt} \right)^2} \ dt \] (1.29)

In practice it is typically difficult to actually find an explicit arc length parameterization of a curve since not only does one have calculate the integral, but also one needs to be able to find the inverse function \( t \) in terms of \( s \). On the other hand, from a theoretical point of view, arc length parameterizations are ideal since any curve so parametrized has unit speed. The proof of this fact is a simple application of the chain rule and the inverse function theorem.

\[ \beta'(s) = \left[ \alpha(t(s)) \right]' = \alpha'(t(s)) t'(s) = \alpha'(t(s)) \frac{1}{s'(t(s))} = \frac{\alpha'(t(s))}{\| \alpha'(t(s)) \|}, \]

and any vector divided by its length is a unit vector. Leibnitz notation makes this even more self evident

\[ \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{dx}{\frac{dt}{ds}} = \frac{\frac{dx}{dt}}{\| \frac{dx}{dt} \|} \]

1.3.7 Example Let \( \alpha(t) = (a \cos \omega t, a \sin \omega t, bt) \). Then

\[ \mathbf{v}(t) = (-a \omega \sin \omega t, a \omega \cos \omega t, b), \]
1.3. CURVES IN $\mathbb{R}^3$

$s(t) = \int_0^t \sqrt{(-a\omega \sin \omega u)^2 + (a\omega \cos \omega u)^2 + b^2} \, du$

$= \int_0^t \sqrt{a^2 \omega^2 + b^2} \, du$

$= ct$, where, $c = \sqrt{a^2 \omega^2 + b^2}$.

The helix of unit speed is then given by

$\beta(s) = (a \cos \frac{\omega s}{c}, a \sin \frac{\omega s}{c}, b \frac{\omega s}{c})$.

1.3.3 Frenet Frames

Let $\beta(s)$ be a curve parametrized by arc length and let $T(s)$ be the vector

$T(s) = \beta'(s)$. \hspace{1cm} (1.30)

The vector $T(s)$ is tangential to the curve and it has unit length. Hereafter, we will call $T$ the unit tangent vector. Differentiating the relation

$T \cdot T = 1$, \hspace{1cm} (1.31)

we get

$2T \cdot T' = 0$, \hspace{1cm} (1.32)

so we conclude that the vector $T'$ is orthogonal to $T$. Let $N$ be a unit vector orthogonal to $T$, and let $\kappa$ be the scalar such that

$T'(s) = \kappa N(s)$. \hspace{1cm} (1.33)

We call $N$ the unit normal to the curve, and $\kappa$ the curvature. Taking the length of both sides of last equation, and recalling that $N$ has unit length, we deduce that

$\kappa = \|T'(s)\|$ \hspace{1cm} (1.34)

It makes sense to call $\kappa$ the curvature since, if $T$ is a unit vector, then $T'(s)$ is not zero only if the direction of $T$ is changing. The rate of change of the direction of the tangent vector is precisely what one would expect to be a measure how much a curve is curving. In particular, if $T' = 0$ at a particular point, we expect that at that point, the curve is locally well approximated by a straight line.

We now introduce a third vector

$B = T \times N$, \hspace{1cm} (1.35)

which we will call the binormal vector. The triplet of vectors $(T, N, B)$ forms an orthonormal set; that is,

$T \cdot T = N \cdot N = B \cdot B = 1$

$T \cdot N = T \cdot B = N \cdot B = 0$. \hspace{1cm} (1.36)

If we differentiate the relation $B \cdot B = 1$, we find that $B \cdot B' = 0$, hence $B'$ is orthogonal to $B$. Furthermore, differentiating the equation $T \cdot B = 0$, we get

$B' \cdot T + B \cdot T' = 0$.

rewriting the last equation

$B' \cdot T = -T' \cdot B = -\kappa N \cdot B = 0$. 

we also conclude that $B'$ must also be orthogonal to $T$. This can only happen if $B'$ is orthogonal to the $TB$-plane, so $B'$ must be proportional to $N$. In other words, we must have

$$B'(s) = -\tau N(s)$$

for some quantity $\tau$, which we will call the torsion. The torsion is similar to the curvature in the sense that it measures the rate of change of the binormal. Since the binormal also has unit length, the only way one can have a non-zero derivative is if $B$ is changing directions. This means that if addition $B$ did not change directions, the vector would truly be a constant vector, so the curve would be a flat curve embedded into the $TN$-plane.

The quantity $B'$ then measures the rate of change in the up and down direction of an observer moving with the curve always facing forward in the direction of the tangent vector. The binormal $B$ is something like the flag in the back of sand dune buggy.

The set of basis vectors $\{T, N, B\}$ is called the Frenet frame or the repère mobile (moving frame). The advantage of this basis over the fixed $\{i, j, k\}$ basis is that the Frenet frame is naturally adapted to the curve. It propagates along the curve with the tangent vector always pointing in the direction of motion, and the normal and binormal vectors pointing in the directions in which the curve is tending to curve. In particular, a complete description of how the curve is curving can be obtained by calculating the rate of change of the frame in terms of the frame itself.

1.3.8 Theorem Let $\beta(s)$ be a unit speed curve with curvature $\kappa$ and torsion $\tau$. Then

$$\begin{align*}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= -\tau N
\end{align*}$$

Proof: We need only establish the equation for $N'$. Differentiating the equation $N \cdot N = 1$, we get $2N \cdot N' = 0$, so $N'$ is orthogonal to $N$. Hence, $N'$ must be a linear combination of $T$ and $B$.

$$N' = aT + bB.$$  

Taking the dot product of last equation with $T$ and $B$ respectively, we see that

$$a = N' \cdot T, \quad b = N' \cdot B.$$  

On the other hand, differentiating the equations $N \cdot T = 0$, and $N \cdot B = 0$, we find that

$$\begin{align*}
N' \cdot T &= -N \cdot T' = -N \cdot (\kappa N) = -\kappa \\
N' \cdot B &= -N \cdot B' = -N \cdot (-\tau N) = \tau.
\end{align*}$$

We conclude that $a = -\kappa$, $b = \tau$, and thus

$$N' = -\kappa T + \tau B.$$  

The Frenet frame equations (1.38) can also be written in matrix form as shown below.

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \quad (1.39)$$
The group-theoretic significance of this matrix formulation is quite important and we will come back to this later when we talk about general orthonormal frames. At this time, perhaps it suffices to point out that the appearance of an antisymmetric matrix in the Frenet equations is not at all coincidental.

The following theorem provides a computational method to calculate the curvature and torsion directly from the equation of a given unit speed curve.

1.3.9 Proposition  Let $\beta(s)$ be a unit speed curve with curvature $\kappa > 0$ and torsion $\tau$. Then

$$\kappa \equiv \|\beta''(s)\|$$

$$\tau = \frac{\beta' \cdot [\beta'' \times \beta''']}{\beta'' \cdot \beta''}$$

1.3.10 Example  Consider a circle of radius $r$ whose equation is given by

$$\alpha(t) = (r \cos t, r \sin t, 0).$$

Then,

$$\alpha'(t) = (-r \sin t, r \cos t, 0)$$

$$\|\alpha'(t)\| = \sqrt{(-r \sin t)^2 + (r \cos t)^2 + 0^2}$$

$$= \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}$$

$$= r.$$

Therefore, $ds/dt = r$ and $s = rt$, which we recognize as the formula for the length of an arc of circle of radius $t$, subtended by a central angle whose measure is $t$ radians. We conclude that

$$\beta(s) = (r \cos \frac{s}{r}, r \sin \frac{s}{r}, 0)$$
is a unit speed reparametrization. The curvature of the circle can now be easily computed

\[ T = \beta'(s) = (-\sin \frac{s}{r}, \cos \frac{s}{r}, 0) \]

\[ T' = (-\frac{1}{r} \cos \frac{s}{r}, -\frac{1}{r} \sin \frac{s}{r}, 0) \]

\[ \kappa = \| T'' \| = \| T' \| \]

\[ = \sqrt{\frac{1}{r^2} \cos^2 \frac{s}{r} + \frac{1}{r^2} \sin^2 \frac{s}{r}} + 0^2 \]

\[ = \sqrt{\frac{1}{r^2} \left( \cos^2 \frac{s}{r} + \sin^2 \frac{s}{r} \right)} \]

\[ = \frac{1}{r} \]

This is a very simple but important example. The fact that for a circle of radius \( r \) the curvature is \( \kappa = \frac{1}{r} \) could not be more intuitive. A small circle has large curvature and a large circle has small curvature. As the radius of the circle approaches infinity, the circle locally looks more and more like a straight line, and the curvature approaches 0. If one were walking along a great circle on a very large sphere (like the earth) one would be perceive the space to be locally flat.

1.3.11 Proposition Let \( \alpha(t) \) be a curve of velocity \( \mathbf{v} \), acceleration \( \mathbf{a} \), speed \( v \) and curvature \( \kappa \), then

\[ \mathbf{v} = vT, \]

\[ \mathbf{a} = \frac{dv}{dt}T + v^2\kappa N. \quad (1.41) \]

Proof: Let \( s(t) \) be the arc length and let \( \beta(s) \) be a unit speed reparametrization. Then \( \alpha(t) = \beta(s(t)) \) and by the chain rule

\[ \mathbf{v} = \alpha'(t) \]

\[ = \beta'(s(t))s'(t) \]

\[ = vT \]

\[ \mathbf{a} = \alpha''(t) \]

\[ = \frac{dv}{dt}T + vT'(s(t))s'(t) \]

\[ = \frac{dv}{dt}T + v(\kappa N)v \]

\[ = \frac{dv}{dt}T + v^2\kappa N \]

Equation [1.41] is important in physics. The equation states that a particle moving along a curve in space feels a component of acceleration along the direction of motion whenever there is a change of speed, and a centripetal acceleration in the direction of the normal whenever it changes direction. The centripetal acceleration and any point is

\[ a = v^2\kappa = \frac{v^2}{r} \]
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where $r$ is the radius of a circle called the **osculating circle**. The osculating circle has maximal tangential contact with the curve at the point in question. This is called contact of order 3, in the sense that the circle passes through three “consecutive points” in the curve. The osculating circle can be envisioned by a limiting process similar to that of the tangent to a curve in differential calculus. Let $p$ be point on the curve, and let $q_1$ and $q_2$ be two nearby points. The three points uniquely determine a circle. This circle is a “secant” approximation to the tangent circle. As the points $q_1$ and $q_2$ approach the point $p$, the “secant” circle approaches the osculating circle. The osculating circle, as shown in figure 1.8, always lies in the the $TN$-plane, which by analogy is called the **osculating plane**. The physics interpretation of equation 1.41 is that as a particle moves along a curve, in some sense at an infinitesimal level, it is moving tangential to a circle, and hence, the centripetal acceleration at each point coincides with the centripetal acceleration along the osculating circle. As the points move along, the osculating circles move along with them, changing their radii appropriately.

1.3.12 Example (Helix)

$$\beta(s) = (a \cos \frac{\omega s}{c}, a \sin \frac{\omega s}{c}, \frac{b s}{c}), \text{ where } c = \sqrt{a^2 \omega^2 + b^2}$$

$$\beta'(s) = \left(-\frac{a \omega}{c} \sin \frac{\omega s}{c}, \frac{a \omega}{c} \cos \frac{\omega s}{c}, \frac{b}{c}\right)$$

$$\beta''(s) = \left(-\frac{a \omega^2}{c^2} \cos \frac{\omega s}{c}, \frac{a \omega^2}{c^2} \sin \frac{\omega s}{c}, 0\right)$$

$$\beta'''(s) = \left(\frac{a \omega^3}{c^3} \sin \frac{\omega s}{c}, -\frac{a \omega^3}{c^3} \cos \frac{\omega s}{c}, 0\right)$$

$$\kappa^2 = \beta'' \cdot \beta''$$

$$= \frac{a^2 \omega^4}{c^4}$$

$$\kappa = \pm \frac{a \omega^2}{c^2}$$

$$\tau \equiv \frac{(\beta' \beta''' \beta''')}{\beta'' \cdot \beta''}$$

$$= \frac{b}{c} \left[ \frac{-a \omega^2}{c^3} \cos \frac{\omega s}{c} - \frac{a \omega^2}{c^3} \sin \frac{\omega s}{c} \right] \frac{c^4}{a^2 \omega^4}$$

$$= \frac{b a^2 \omega^5}{c} \frac{c^4}{a^2 \omega^5}$$

Simplifying the last expression and substituting the value of $c$, we get

$$\tau = \frac{b \omega}{a^2 \omega^2 + b^2}$$

$$\kappa = \pm \frac{a \omega^2}{a^2 \omega^2 + b^2}$$

Notice that if $b = 0$, the helix collapses to a circle in the $xy$-plane. In this case, the formulas above reduce to $\kappa = 1/a$ and $\tau = 0$. The ratio $\kappa/\tau = a \omega/b$ is particularly simple. Any curve where $\kappa/\tau = constant$ is called a helix, of which the circular helix is a special case.
1.3.13 Example (Plane curves) Let \( \alpha(t) = (x(t), y(t), 0) \). Then
\[
\alpha' = (x', y', 0) \\
\alpha'' = (x'', y'', 0) \\
\alpha''' = (x'''(t), y'''(t), 0) \\
\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \\
= \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} \\
\tau = 0
\]

1.3.14 Example Let \( \beta(s) = (x(s), y(s), 0) \), where
\[
x(s) = \int_0^s \cos \frac{t^2}{2c^2} \, dt \\
y(s) = \int_0^s \sin \frac{t^2}{2c^2} \, dt. \tag{1.42}
\]
Then, using the fundamental theorem of calculus, we have
\[
\beta'(s) = (\cos \frac{s^2}{2c^2}, \sin \frac{s^2}{2c^2}, 0),
\]
Since \( \|\beta'\| = v = 1 \), the curve is of unit speed, and \( s \) is indeed the arc length. The curvature is given by
\[
\kappa = \frac{\|x'y'' - y'x''\|}{(x'^2 + y'^2)^{3/2}} = (\beta' \cdot \beta')^{1/2} \\
= \frac{s}{c^2} \sin \frac{t^2}{2c^2}, \frac{s}{c^2} \cos \frac{t^2}{2c^2}, 0
\]
\[
= \frac{s}{c^2}.
\]
The integrals (1.42) are classical Fresnel integrals which we will discuss in more detail in the next section.

In cases where the given curve \( \alpha(t) \) is not of unit speed, the following proposition provides formulas to compute the curvature and torsion in terms of \( \alpha \).

1.3.15 Proposition If \( \alpha(t) \) is a regular curve in \( \mathbb{R}^3 \), then
\[
\kappa^2 = \frac{\|\alpha' \times \alpha''\|^2}{\|\alpha'\|^6} \tag{1.43} \\
\tau = \frac{(\alpha' \alpha'' \alpha''')}{\|\alpha' \times \alpha''\|^2}, \tag{1.44}
\]
where \( (\alpha' \alpha'' \alpha''') \) is the triple vector product \( [\alpha' \times \alpha''] \cdot \alpha''' \).
Proof:
\[
\alpha' = vT \\
\alpha'' = v'T + v^2 \kappa N \\
\alpha''' = (v^2 \kappa)N'(s(t))s'(t) + \ldots \\
= v^3 \kappa N' + \ldots \\
= v^3 \kappa \tau B + \ldots
\]
The other terms in $\alpha'''$ are unimportant here because $\alpha' \times \alpha''$ is proportional to $B$, so all we need is the $B$ component.

\[
\begin{align*}
\alpha' \times \alpha'' & = v^3 \kappa (T \times N) = v^3 \kappa B \\
\|\alpha' \times \alpha''\| & = v^3 \\
\kappa & = \frac{\|\alpha' \times \alpha''\|}{v^3} \\
(\alpha' \times \alpha'') \cdot \alpha''' & = v^6 \kappa^2 \tau \\
\tau & = \frac{(\alpha' \alpha'' \alpha''')}{v^6 \kappa^2} \\
& = \frac{||\alpha' \times \alpha''||^2}{v^6 \kappa^2}
\end{align*}
\]

### 1.4 Fundamental Theorem of Curves

The fundamental theorem of curves basically states that prescribing a curvature and torsion as functions of some parameter $s$, completely determines up to position and orientation, a curve $\beta(s)$ with that given curvature and torsion. Some geometrical insight into the significance of the curvature and torsion can be gained by considering the Taylor series expansion of an arbitrary unit speed curve $\beta(s)$ about $s = 0$.

\[
\begin{align*}
\beta(s) = \beta(0) + \beta'(0) s + \frac{\beta''(0)}{2!} s^2 + \frac{\beta'''(0)}{3!} s^3 + \ldots
\end{align*}
\]

(1.45)

Since we are assuming that $s$ is an arc length parameter,

\[
\begin{align*}
\beta'(0) & = T(0) = T_0 \\
\beta''(0) & = (\kappa N)(0) = \kappa_0 N_0 \\
\beta'''(0) & = (-\kappa^2 T + \kappa' N + \kappa \tau B)(0) = -\kappa_0^2 T_0 + \kappa_0' N_0 + \kappa_0 \tau_0 B_0
\end{align*}
\]

Keeping only the lowest terms in the components of $T$, $N$, and $B$, we get the first order Frenet approximation to the curve

\[
\beta(s) \approx \beta(0) + T_0 s + \frac{1}{2} \kappa_0 N_0 s^2 + \frac{1}{6} \kappa_0 \tau_0 B_0 s^3.
\]

(1.46)

The first two terms represent the linear approximation to the curve. The first three terms approximate the curve by a parabola which lies in the osculating plane ($TN$-plane). If $\kappa_0 = 0$, then locally the curve looks like a straight line. If $\tau_0 = 0$, then locally the curve is a plane curve contained on the osculating plane. In this sense, the curvature measures the deviation of the curve from a straight line and the torsion (also called the second curvature) measures the deviation of the curve from a plane curve. As shown in figure 1.3, a non-planar space curve locally looks like a wire that has first been bent in a parabolic shape in the $TN$ and twisted into a cubic along the $B$ axis.

So suppose that $p$ is an arbitrary point on a curve $\beta(s)$ parametrized by arc length. We position the curve so that $p$ is at the origin so that $\beta(0) = 0$ coincides with the point $p$. We chose the orthonormal basis vectors in $\mathbb{R}^3 \{e_1, e_2, e_3\}$ to coincide with the Frenet Frame $T_0, N_0, B_0$ at that point. then, the equation (1.46) provides a canonical representation of the curve near that point. This then constitutes a proof of the fundamental theorem of curves under the assumption the curve, curvature and torsion are analytic. (One could also treat the Frenet formulas as a system of differential equations and apply the conditions of existence and uniqueness of solutions for such systems.)
1.4.1 Proposition  A curve with $\kappa = 0$ is part of a straight line.
If $\kappa = 0$ then $\beta(s) = \beta(0) + sT_0$.

1.4.2 Proposition  A curve $\alpha(t)$ with $\tau = 0$ is a plane curve.
Proof:  If $\tau = 0$, then $(\alpha' \alpha'' \alpha''') = 0$. This means that the three vectors $\alpha'$, $\alpha''$, and $\alpha'''$ are linearly dependent and hence there exist functions $a_1(s), a_2(s)$ and $a_3(s)$ such that

$$a_3\alpha''' + a_2\alpha'' + a_1\alpha' = 0.$$  

This linear homogeneous equation will have a solution of the form

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3, \quad c_i = \text{constant vectors}.$$

This curve lies in the plane

$$(x - c_3) \cdot n = 0, \quad \text{where} \quad n = c_1 \times c_2$$

A consequence of the Frenet Equations is that given two curves in space $C$ and $C^*$ such that $\kappa(s) = \kappa^*(s)$ and $\tau(s) = \tau^*(s)$, the two curves are the same up to their position in space. To clarify what we mean by their "position" we need to review some basic concepts of linear algebra leading the notion the of an isometry.

1.4.1 Isometries

1.4.3 Definition  Let $x$ and $y$ be two column vectors in $\mathbb{R}^n$ and let $x^T$ represent the transpose row vector. To keep track on whether a vector is a row vector or a column vector, hereafter we write the components $\{x^i\}$ of a column vector with the indices up and the components $\{x_i\}$ of a row vector with the indices down. Similarly, if $A$ is an $n \times n$ matrix, we write its components as $A = (a^j_i)$.

The standard inner product is given by matrix multiplication of the row and column vectors

$$<x, y> = x^T y,$$ 

$$=<y, x>.$$ (1.48)
The inner product gives $\mathbb{R}^n$ the structure of a normed space by defining $\|x\| = \langle x, x \rangle^{1/2}$ and the structure of a metric space in which $d(x, y) = \|x - y\|$. The real inner product is bilinear (linear in each slot), from which it follows that

$$\|x \pm y\|^2 = \|x\|^2 \pm 2 \langle x, y \rangle + \|y\|^2,$$

and thus, we have the polarization identity

$$\langle x, y \rangle = \frac{1}{4} \|x + y\|^2 - \frac{1}{4} \|x - y\|^2.$$

The Euclidean inner product satisfies that relation

$$\langle x, y \rangle = \|x\| \cdot \|y\| \cos \theta,$$

where $\theta$ is the angle subtended by the two vectors. Two vectors are called orthogonal if $\langle x, y \rangle = 0$, and a set of basis vectors $B = \{e_1, \ldots, e_n\}$ is called an orthonormal basis if $\langle e_i, e_j \rangle = \delta_{ij}$. Given an orthonormal basis, the dual basis is the set of linear functionals $\{\alpha^i\}$ such that $\alpha^i(e_j) = \delta^i_j$. In terms of basis components, column vectors are given by $x = x^i e_i$, row vectors by $x^T = x_j \alpha^j$, and the inner product

$$\langle x, y \rangle = x^T y,$$

$$= (x_i \alpha^i)(y^j e_j),$$

$$= (x_i y^i) \alpha^i(e_j) = (x_i y^i) \delta^i_j,$$

$$= x_i y^i,$$

$$= \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$  (1.53)

Since $|\cos \theta| \leq 1$, there follows a special case of the Schwarz Inequality

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|. \quad (1.52)$$

Let $F$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $B = \{e_1, \ldots, e_n\}$ be an orthonormal basis. Then, there exists a matrix $A = [F]_B$ given by

$$A = (a^i_j) = \alpha^i(F(e_j))$$  (1.53)

or in terms of the inner product,

$$A = (a_{ij}) = \langle e_i, F(e_j) \rangle. \quad (1.54)$$

On the other hand, if $A$ is a fixed $n \times n$ matrix, the map $F$ defined by $F(x) = Ax$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose matrix representation in the standard basis is the matrix $A$ itself. It follows that given a linear transformation represented by a matrix $A$, we have

$$\langle x,Ay \rangle = x^T Ay,$$

$$= (A^T x)^T y,$$

$$= \langle A^T x, y \rangle. \quad (1.56)$$
1.4.4 Definition A real $n \times n$ matrix $A$ is called orthogonal if $A^T A = AA^T = I$. The linear transformation represented by $A$ is called an orthogonal transformation. Equivalently, the transformation represented by $A$ is orthogonal if

$$<x, Ay> = <A^{-1} x, y>.$$  

(1.57)

Thus, real orthogonal transformations are represented by symmetric matrices (Hermitian in the complex case) and the condition $A^T A = I$ implies that $\det(A) = \pm 1$.

1.4.5 Theorem If $A$ is an orthogonal matrix, then the transformation determined by $A$ preserves the inner product and the norm.

Proof: 

$$<Ax, Ay> = <A^T Ax, y>,$$

$$= <x, y>.$$ 

Furthermore, setting $y = x$:

$$<Ax, Ax> = <x, x>,$$

$$\|Ax\|^2 = \|x\|^2,$$

$$\|Ax\| = \|x\|.$$ 

As a corollary, if $\{e_i\}$ is an orthonormal basis, then so is $\{f_i = Ae_i\}$. That is, an orthogonal transformation represents a rotation if $\det A = 1$ and a rotation with a reflection if $\det A = -1$.

1.4.6 Definition A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ called an isometry if it preserves distances. That is, if for all $x, y$

$$d(F(x), F(y)) = d(x, y).$$  

(1.58)

1.4.7 Example (Translations) Let $q$ be fixed vector. The map $F(x) = x + q$ is called a translation. It is clearly an isometry since $\|F(x) - F(y)\| = \|x - y + p - (y + p)\| = \|x - y\|$.

1.4.8 Theorem An orthogonal transformation is an isometry.

Proof: Let $F$ be an isometry represented by an orthogonal matrix $A$. Then, since the transformation is linear and preserves norms, we have:

$$d(F(x), F(x)) = \|Ax - Ay\|,$$

$$= \|A(x - y)\|,$$

$$= \|x - y\|.$$ 

The composition of two isometries is also an isometry. The inverse of a translation by $q$ is a translation by $-q$. The inverse of an orthogonal transformation represented by $A$ is an orthogonal transformation represented by $A^{-1}$. Thus, the set of isometries consisting of translations and orthogonal transformations constitutes a group. Given an general isometry, we can use a translation to insure that $F(0) = 0$. We now prove the following theorem.

1.4.9 Theorem If $F$ is an isometry such that $F(0) = 0$, then $F$ is an orthogonal transformation.
1.4. FUNDAMENTAL THEOREM OF CURVES

**Proof:** We need to prove that $F$ preserves the inner product and that it is linear. We first show that $F$ preserves norms. In fact

$$
\|F(x)\| = d(F(x), 0),
= d(F(x), F(0)),
= d(x, 0),
= \|x - 0\|,
= \|x\|.
$$

Now, using 1.49 and the norm preserving property above, we have:

$$
d(F(x), F(y)) = d(x, y),
\|F(x), F(y)\| = \|x, y\|,
\|F(x)\|^2 - 2 < F(x), F(y) > + \|F(y)\|^2 = \|x\|^2 - 2 < x, y > + \|y\|^2.
< F(x), F(y) > = < x, y > .
$$

To show $F$ is linear, let $e_i$ be an orthonormal basis, and therefore $f_i = F(e_i)$ is also an orthonormal basis. Then

$$
F(ax + by) = \sum_{i=1}^{n} < F(ax + by), f_i > f_i ,
= \sum_{i=1}^{n} < F(ax + by), F(e_i) > f_i ,
= \sum_{i=1}^{n} < (ax + by), e_i > f_i ,
= a \sum_{i=1}^{n} < x, e_i > f_i + b \sum_{i=1}^{n} < y, e_i > f_i ,
= a \sum_{i=1}^{n} < F(x), f_i > f_i + b \sum_{i=1}^{n} < F(y), f_i > f_i ,
= aF(x) + bF(y).
$$

1.4.10 Theorem If $F : R^n \to R^n$ is an isometry then

$$
F(x) = Ax + q.
$$

where $A$ is orthogonal.

**Proof:** If $F(0) = q$, then $\bar{F} = F - q$ is an isometry with $\bar{F}(0) = 0$ and hence by the previous theorem $\bar{F}$ is an orthogonal transformation represented by an orthogonal matrix $\bar{F}x = Ax$. It follows that $F(x) = Ax + q$.

We have just shown that any isometry is the composition of translation and an orthogonal transformation. The latter is the linear part of the isometry. The orthogonal transformation preserves the inner product, lengths, and maps orthonormal bases to orthonormal bases.

1.4.11 Theorem If $\alpha$ is a curve in $R^n$ and $\beta$ is the image of $\alpha$ under a mapping $F$, then vectors tangent to $\alpha$ get mapped to tangent vectors to $\beta$. 
Proof: Let \( \beta = F \circ \alpha \). The proof follows trivially from the properties of the Jacobian map \( \beta_* = (F \circ \alpha)_* = F_* \alpha_* \) that takes tangent vectors to tangent vectors. If in addition \( F \) is an isometry, then \( F_* \) maps the Frenet frame of \( \alpha \) to the Frenet frame of \( \beta \).

We now have all the ingredients to prove the following:

1.4.12 Theorem (Fundamental theorem of Curves) If \( C \) and \( \tilde{C} \) are space curves such that \( \kappa(s) = \tilde{\kappa}(s) \), and \( \tau(s) = \tilde{\tau}(s) \) for all \( s \), the curves are isometric.

Proof: Given two such curves, we can perform a translation so that for some \( s = s_0 \) the corresponding points on \( C \) and \( \tilde{C} \) are made to coincide. Without lost of generality we can make this point be the origin. Now we perform an orthogonal transformation to make the Frenet frame \( \{ T_0, \tilde{N}_0, \tilde{B}_0 \} \) of \( C \) coincide with the Frenet frame \( \{ \tilde{T}_0, \tilde{N}_0, \tilde{B}_0 \} \) of \( \tilde{C} \). By Schwarz inequality, the inner product of two unit vectors is 1 if and only if the vectors are equal. With this in mind, let

\[
L = T \cdot \tilde{T} + N \cdot \tilde{N} + B \cdot \tilde{B}.
\]

A simple computation using the Frenet equations shows that \( L' = 0 \), so \( L = \) constant. But at \( s = 0 \) the Frenet frames of the two curves coincide, so the constant is 3 and this can only happen if for all \( s \), \( T = \tilde{T} \), \( N = \tilde{N} \), \( B = \tilde{B} \). Finally, since \( T = \tilde{T} \), we have \( \beta'(s) = \tilde{\beta}'(s) \), so \( \beta(s) = \tilde{\beta}(s) + \) constant. But since \( \beta(0) = \tilde{\beta}(0) \), the constant is 0 and \( \beta(s) = \tilde{\beta}(s) \) for all \( s \).

1.4.2 Natural Equations

The Fundamental Theorem of Curves states that up to an isometry, that is up to location and orientation, a curve is completely determined by the curvature and torsion. However The formulas for computing \( \kappa \) and \( \tau \) are sufficiently complicated that solving the Frenet system of differential equations could be a daunting task indeed. However with the invention of modern computers, obtaining and plotting numerical solutions is a routine matter. There is a plethora of differential equations solvers available nowadays, including the solvers built-in into Maple, Mathematica and Matlab.

![Fig. 1.10: Tangent](image)

For plane curves which as we know are characterized by \( \tau = 0 \), it is possible to find an integral formula for the curve coordinates in terms of the curvature. Given a curve parametrized by arc length, consider an arbitrary point with position vector \( \mathbf{x} = (x, y) \) on the curve and let \( \varphi \) be the angle that the tangent vector \( T \) makes with the horizontal, as shown in figure 1.10. Then, the Euclidean vector components of the unit tangent vector are given by

\[
\frac{dx}{ds} = T = (\cos \varphi, \sin \varphi).
\]

This means that

\[
\frac{dx}{ds} = \cos \varphi, \quad \text{and} \quad \frac{dy}{ds} = \sin \varphi
\]
From the first Frenet equation we also have
\[
\frac{dT}{ds} = (-\sin \varphi \frac{d\varphi}{ds}, \cos \varphi \frac{d\varphi}{ds}) = \kappa N
\]
so that,
\[
\| \frac{dT}{ds} \| = \frac{d\varphi}{ds} = \kappa.
\]
We conclude that
\[
x(s) = \int \cos \varphi \, ds, \quad y(s) = \int \sin \varphi \, ds, \text{ where, } \varphi = \int \kappa \, ds.
\]
Equations 1.60 are called the **Natural Equations** of a plane curve. Given the curvature \( \kappa \), the equation of the curve can be obtained by “quadratures,” the classical term for integrals.

**1.4.13 Example** Circle: \( \kappa = 1/R \)

The simplest natural equation is one where the curvature is constant. For obvious geometrical reasons we choose this constant to be \( 1/R \). Then, \( \varphi = s/R \) and
\[
x = (R \sin \frac{s}{R}, -R \cos \frac{s}{R}),
\]
which is the equation of a unit speed circle of radius \( R \).

**1.4.14 Example** Cornu Spiral: \( \kappa = \pi s \)

This is the most basic linear natural equation, except for the scaling factor of \( \pi \) which is inserted for historical conventions. Then \( \varphi = \frac{1}{2} \pi s^2 \), and
\[
x(s) = C(s) = \int \cos(\frac{1}{2} \pi s^2) \, ds; \quad y(s) = S(s) = \int \sin(\frac{1}{2} \pi s^2) \, ds.
\]

The functions \( C(s) \) and \( S(s) \) are called **Fresnel Integrals**. In the standard classical function libraries of Maple and Mathematica, they are listed as \( \text{FresnelC} \) and \( \text{FresnelS} \) respectively. The fast increasing frequency of oscillations of the integrands here make the computation prohibitive without the use of high speed computers. Graphing calculators are inadequate to render the rapid oscillations for \( s \) ranging from 0 to 15, for example, and simple computer programs for the trapezoidal rule as taught in typical calculus courses, completely falls apart in this range.

**Fig. 1.11:** Fresnel Diffraction

The **Cornu Spiral** is the curve \( x(s) = (x(s), y(s)) \) parametrized by Fresnel integrals (See figure 1.11a). It is a tribute to the mathematicians of the 1800’s that not only were they able to compute
the values of the Fresnel integrals to 4 or 5 decimal places, but they did it for the range of \( s \) from 0 to 15 as mentioned above, producing remarkably accurate renditions of the spiral.

Fresnel integrals appear in the study of diffraction. If a coherent beam of light such as a laser beam, hits a sharp straight edge and a screen is placed behind, there will appear on the screen a pattern of diffraction fringes. The amplitude and intensity of the diffraction pattern can obtained by a geometrical construction involving the Fresnel integrals. First consider the function \( \Psi(s) = ||x|| \) that measures the distance from the origin to the points in the Cornu spiral in the first quadrant. The square of this function is then proportional to the intensity of the diffraction pattern. The graph of \(|\Psi(s)|^2\) is shown in figure 1.11b. Translating this curve along an axis coinciding with that of the straight edge, generates a three dimensional surface as shown from "above" in figure 1.11c. A color scheme was used here to depict a model of the Fresnel diffraction by the straight edge.

1.4.15 Example Logarithmic Spiral \( \kappa = 1/(as + b) \)

A logarithmic spiral is a curve in which the position vector \( x \) makes a constant angle with the tangent vector, as shown in figure 1.12. A formula for the curve can be found easily if one uses the calculus formula in polar coordinates

\[
\tan \psi = \frac{r}{dr/d\theta}.
\]

Here, \( \psi \) is the angle between the polar direction and the tangent. If \( \psi \) is constant, then one can immediately integrate the equation to get the exponential function below, in which \( k \) is the constant of integration

\[
r(\theta) = ke^{\cot \psi \theta}
\]

Fn. 1.12: Logarithmic Spiral

Derivation of formula 1.62 has fallen through the cracks in standard fat calculus textbooks, at best relegated to an advanced exercise which most students will not do. Perhaps the reason is that the section on polar coordinates is typically covered in Calculus II, so students have not yet been exposed to the tools of vector calculus that facilitate the otherwise messy computation. To fill-in this gap, we present a short derivation of this neat formula. For a plane curve in parametric polar coordinates, we have

\[
x(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t)),
\]

\[
\dot{x} = (\dot{r} \cos \theta - r \sin \theta \dot{\theta}, \dot{r} \sin \theta + r \cos \theta \dot{\theta}).
\]

A direct computation of the dot product gives,

\[
| < x, \dot{x} > |^2 = (r \dot{r})^2.
\]
On the other hand,
\[ | \langle \mathbf{x}, \dot{\mathbf{x}} \rangle |^2 = \| \mathbf{x} \|^2 \| \dot{\mathbf{x}} \|^2 \cos^2 \psi, \]
\[ = r^2 (r^2 + r^2 \dot{\theta}^2) \cos^2 \psi. \]

Equating the two, we find,
\[ r^2 = (r^2 + r^2 \dot{\theta}^2) \cos^2 \psi, \]
\[ (\sin^2 \psi) r^2 = r^2 \dot{\theta}^2 \cos^2 \psi, \]
\[ (\sin \psi) d\theta = r \cos \psi d\theta, \]
\[ \tan \psi = \frac{r}{dr/d\theta}. \]

We leave it to the reader to do a direct computation of the curvature. Instead, we prove that if \( \kappa = 1/(as + b) \), where \( a \) and \( b \) are constant, then the curve is a logarithmic spiral. From the natural equations, we have,
\[ \frac{d\theta}{ds} = \kappa = \frac{1}{as + b}, \]
\[ \theta = \frac{1}{a} \ln(as + b) + C, \quad C = \text{const}, \]
\[ e^{a\theta} = A(as + b), \quad A = e^C, \]
\[ \frac{1}{\kappa} = Ae^{a\theta} = \frac{ds}{d\theta}, \]
\[ ds = Ae^{a\theta} \]

Back to the natural equations, the \( x \) and \( y \) coordinates are obtained by integrating,
\[ x = \int Ae^{a\theta} \cos \theta \, d\theta, \]
\[ y = \int Ae^{a\theta} \sin \theta \, d\theta. \]

We can avoid the integrations by parts by letting \( z = x + iy = re^{i\theta} \). We get
\[ z = A \int e^{a\theta} e^{i\theta} \, d\theta, \]
\[ = A \int e^{(a+i)\theta} \, d\theta, \]
\[ = \frac{A}{a+i} e^{(a+i)\theta}, \]
\[ = \frac{A}{a+i} e^{a\theta} e^{i\theta}. \]

Extracting the real part \( \| z \| = r \), we get
\[ r = \frac{A}{\sqrt{a^2 + 1}} e^{a\theta}, \quad (1.64) \]
which is the equation of a logarithmic spiral with $a = \cot \psi$. As shown in figure 1.12, families of concentric logarithmic spirals are ubiquitous in nature as in flowers and pine cones, in architectural designs. The projection of a conical helix as in figure 4.9 onto the plane through the origin, is a logarithmic spiral. The analog of a logarithmic spiral on a sphere is called a loxodrome as depicted in figure 4.2.

1.4.16 Example Meandering Curves: $\kappa = \sin s$

A whole family of meandering curves are obtained by letting $\kappa = A \sin ks$. The meandering graph shown in picture 1.13 was obtained by numerical integration for $A = 2$ and "wave number" $k = 1$. The larger the value of $A$ the larger the curvature of the "throats." If $A$ is large enough, the "throats" will overlap.

Fig. 1.13: Meandering Curve

Using superpositions of sine functions gives rise to a beautiful family of "multi-frequency" meanders with graphs that would challenge the most skillful calligraphists of the 1800’s. Figure 1.14 shows a rendition with two sine functions with equal amplitude $A = 1.8$, and with $k_1 = 1$, $k_2 = 1.2$.

Fig. 1.14: Bimodal Meander
Chapter 2

Differential Forms

2.1 One-Forms

One of the most puzzling ideas in elementary calculus is that of the differential. In the usual definition, the differential of a dependent variable \( y = f(x) \) is given in terms of the differential of the independent variable by \( dy = f'(x)dx \). The problem is with the quantity \( dx \). What does "\( dx \)" mean? What is the difference between \( \Delta x \) and \( dx \)? How much "smaller" than \( \Delta x \) does \( dx \) have to be? There is no trivial resolution to this question. Most introductory calculus texts evade the issue by treating \( dx \) as an arbitrarily small quantity (lacking mathematical rigor) or by simply referring to \( dx \) as an infinitesimal (a term introduced by Newton for an idea that could not otherwise be clearly defined at the time.)

In this section we introduce linear algebraic tools that will allow us to interpret the differential in terms of a linear operator.

2.1.1 Definition Let \( p \in \mathbb{R}^n \), and let \( T_p(\mathbb{R}^n) \) be the tangent space at \( p \). A 1-form at \( p \) is a linear map \( \phi \) from \( T_p(\mathbb{R}^n) \) into \( \mathbb{R} \), in other words, a linear functional. We recall that such a map must satisfy the following properties:
\[
\begin{align*}
\text{a) } & \quad \phi(X_p) \in \mathbb{R}, \quad \forall X_p \in \mathbb{R}^n \quad (2.1) \\
\text{b) } & \quad \phi(aX_p + bY_p) = a\phi(X_p) + b\phi(Y_p), \quad \forall a,b \in \mathbb{R}, \ X_p,Y_p \in T_p(\mathbb{R}^n)
\end{align*}
\]

A 1-form is a smooth assignment of a linear map \( \phi \) as above for each point in the space.

2.1.2 Definition Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a real-valued \( C^\infty \) function. We define the differential \( df \) of the function as the 1-form such that \( df(X) = X(f) \) (2.2) for every vector field in \( X \) in \( \mathbb{R}^n \).

In other words, at any point \( p \), the differential \( df \) of a function is an operator that assigns to a tangent vector \( X_p \) the directional derivative of the function in the direction of that vector.
\[
\text{df}(X)(p) = X_p(f) = \nabla f(p) \cdot X(p) \quad (2.3)
\]

In particular, if we apply the differential of the coordinate functions \( x^i \) to the basis vector fields, we get
\[
dx^i \left( \frac{\partial}{\partial x^j} \right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j \quad (2.4)
\]

The set of all linear functionals on a vector space is called the dual of the vector space. It is a standard theorem in linear algebra that the dual of a vector space is also a vector space of the
same dimension. Thus, the space \( T^*_p(\mathbb{R}^n) \) of all 1-forms at \( p \) is a vector space which is the dual of the tangent space \( T_p(\mathbb{R}^n) \). The space \( T^*_p(\mathbb{R}^n) \) is called the cotangent space of \( \mathbb{R}^n \) at the point \( p \). Equation (2.4) indicates that the set of differential forms \( \{(dx^1)_p, \ldots, (dx^n)_p\} \) constitutes the basis of the cotangent space which is dual to the standard basis \( \{\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^n}\right)_p\} \) of the tangent space. The union of all the cotangent spaces as \( p \) ranges over all points in \( \mathbb{R}^n \) is called the cotangent bundle \( T^*(\mathbb{R}^n) \).

2.1.3 Proposition Let \( f \) be any smooth function in \( \mathbb{R}^n \) and let \( \{x^1, \ldots, x^n\} \) be coordinate functions in a neighborhood \( U \) of a point \( p \). Then, the differential \( df \) is given locally by the expression

\[
df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \quad (2.5)
\]

Proof: The differential \( df \) is by definition a 1-form, so, at each point, it must be expressible as a linear combination of the basis elements \( \{(dx^1)_p, \ldots, (dx^n)_p\} \). Therefore, to prove the proposition, it suffices to show that the expression (2.5) applied to an arbitrary tangent vector coincides with definition 2.2. To see this, consider a tangent vector \( X_p = v^j \left( \frac{\partial}{\partial x^j} \right)_p \) and apply the expression above as follows:

\[
\left(\frac{\partial f}{\partial x^i} dx^i\right)_p(X_p) = \left(\frac{\partial f}{\partial x^i} dx^i\right)(v^j \frac{\partial}{\partial x^j})(p)
\]

\[
= v^j \left(\frac{\partial f}{\partial x^i} dx^i\right)\left(\frac{\partial}{\partial x^j}\right)(p)
\]

\[
= v^j \left(\frac{\partial f}{\partial x^i} \delta^i_j\right)(p)
\]

\[
= \left(\frac{\partial f}{\partial x^i} v^i\right)(p)
\]

\[
= \nabla f(p) \cdot \mathbf{x}(p)
\]

\[
= df(X)(p)
\]

The definition of differentials as linear functionals on the space of vector fields is much more satisfactory than the notion of infinitesimals, since the new definition is based on the rigorous machinery of linear algebra. If \( \alpha \) is an arbitrary 1-form, then locally

\[
\alpha = a_1(x)dx^1 + \ldots + a_n(x)dx^n, \quad (2.7)
\]

where the coefficients \( a_i \) are \( C^\infty \) functions. Thus, a 1-form is a smooth section of the cotangent bundle and we refer to it as a covariant tensor of rank 1, or simply a covector. The collection of all 1-forms is denoted by \( \Omega^1(\mathbb{R}^n) = \mathcal{A}^0(\mathbb{R}^n) \). The coefficients \( (a_1, \ldots, a_n) \) are called the covariant components of the covector. We will adopt the convention to always write the covariant components of a covector with the indices down. Physicists often refer to the covariant components of a 1-form as a covariant vector and this causes some confusion about the position of the indices. We emphasize that not all one forms are obtained by taking the differential of a function. If there exists a function \( f \), such that \( \alpha = df \), then the one form \( \alpha \) is called exact. In vector calculus and elementary physics, exact forms are important in understanding the path independence of line integrals of conservative vector fields.
2.2. TENSORS

As we have already noted, the cotangent space $T^*_p(\mathbb{R}^n)$ of 1-forms at a point $p$ has a natural vector space structure. We can easily extend the operations of addition and scalar multiplication to the space of all 1-forms by defining

$$ (\alpha + \beta)(X) = \alpha(X) + \beta(X) \quad (2.8) $$

$$ (f\alpha)(X) = f\alpha(X) $$

for all vector fields $X$ and all smooth functions $f$.

2.2 Tensors

As we mentioned at the beginning of this chapter, the notion of the differential $dx$ is not made precise in elementary treatments of calculus, so consequently, the differential of area $dxdy$ in $\mathbb{R}^2$, as well as the differential of surface area in $\mathbb{R}^3$, also need to be revisited in a more rigorous setting. For this purpose, we introduce a new type of multiplication between forms that not only captures the essence of differentials of area and volume, but also provides a rich algebraic and geometric structure generalizing cross products (which make sense only in $\mathbb{R}^3$) to Euclidean space of any dimension.

2.2.1 Definition A map $\phi : \mathcal{X}(\mathbb{R}^n) \times \mathcal{X}(\mathbb{R}^n) \rightarrow \mathbb{R}$ is called a bilinear map vector fields, if it is linear on each slot. That is,

$$ \phi(f^1X_1 + f^2X_2, Y_1) = f^1\phi(X_1, Y_1) + f^2\phi(X_2, Y_1) $$

$$ \phi(X_1, f^1Y_1 + f^2Y_2) = f^1\phi(X_1, Y_1) + f^2\phi(X_1, Y_2), \quad \forall X_i, Y_i \in \mathcal{X}(\mathbb{R}^n), \ f^i \in \mathcal{F}\mathbb{R}^n. $$

2.2.2 Definition Let $\alpha$ and $\beta$ be 1-forms. The tensor product of $\alpha$ and $\beta$ is defined as the bilinear map $\alpha \otimes \beta$ such that

$$ (\alpha \otimes \beta)(X,Y) = \alpha(X)\beta(Y) \quad (2.9) $$

for all vector fields $X$ and $Y$.

Thus, for example, if $\alpha = a_idx^i$ and $\beta = b_jdx^j$, then,

$$ (\alpha \otimes \beta)(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) = \alpha(\frac{\partial}{\partial x^k})\beta(\frac{\partial}{\partial x^l}) $$

$$ = (a_i\delta^i_k)(b_j\delta^j_l)(\frac{\partial}{\partial x^k})(\frac{\partial}{\partial x^l}) $$

$$ = a_k^ib_j^l $$

A quantity of the form $T = T_{ij}dx^i \otimes dx^j$ is called a covariant tensor of rank 2, and we may think of the set $\{dx^i \otimes dx^j\}$ as a basis for all such tensors. The space of covariant tensor fields of rank 2 is denoted $\mathcal{F}_2^0(\mathbb{R}^n)$. We must caution the reader again that there is possible confusion about the location of the indices, since physicists often refer to the components $T_{ij}$ as a covariant tensor of rank two, as long is it satisfies some transformation laws.

In a similar fashion, one can define the tensor product of vectors $X$ and $Y$ as the bilinear map $X \otimes Y$ such that

$$ (X \otimes Y)(f,g) = X(f)Y(g) \quad (2.10) $$

for any pair of arbitrary functions $f$ and $g$. 
If \( X = a^i \frac{\partial}{\partial x^i} \) and \( Y = b^j \frac{\partial}{\partial x^j} \), then the components of \( X \otimes Y \) in the basis \( \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \) are simply given by \( a^i b^j \). Any bilinear map of the form

\[
T = T^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}
\]  

is called a contravariant tensor of rank 2 in \( \mathbb{R}^n \).

The notion of tensor products can easily be generalized to higher rank, and in fact one can have tensors of mixed ranks. For example, a tensor of contravariant rank 2 and covariant rank 1 in \( \mathbb{R}^n \) is represented in local coordinates by an expression of the form

\[
T = T^{ij} \frac{\partial}{\partial x^i} \otimes dx^j.
\]  

This object is also called a tensor of type \( (2,1) \). Thus, we may think of a tensor of type \( (r,s) \) as map with three input slots. The map expects two functions in the first two slots and a vector in the third one. The action of the map is bilinear on the two functions and linear on the vector. The output is a real number. A tensor of type \( (r,s) \) is written in local coordinates as

\[
T = T^{i_1,...,i_r} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots dx^{j_s}.
\]

The tensor components are given by

\[
T^{i_1,...,i_r}_{j_1,...,j_s} = T(dx^{i_1},...,dx^{i_r}, \frac{\partial}{\partial x^{j_1}},...,\frac{\partial}{\partial x^{j_s}}).
\]

The set \( T^r_s(\mathbb{R}^n) \) of all tensors of type \( (r,s) \) at a point \( p \) has a vector space structure. The union of all such vector spaces is called the tensor bundle, and smooth sections of the bundle are called tensor fields \( \mathcal{T}^r_s(\mathbb{R}^n) \); that is, a tensor field is a smooth assignment of a tensor to each point in \( \mathbb{R}^n \).

### 2.2.2 Inner Products

Let \( X = a^i \frac{\partial}{\partial x^i} \) and \( Y = b^j \frac{\partial}{\partial x^j} \) be two vector fields and let

\[
g(X,Y) = \delta_{ij} a^i b^j.
\]

The quantity \( g(X,Y) \) is an example of a bilinear map that the reader will recognize as the usual dot product.

#### 2.2.3 Definition

A bilinear map \( g(X,Y) \equiv <X,Y> \) on vectors is called a real inner product if

1. \( g(X,Y) = g(Y,X) \),
2. \( g(X,X) \geq 0 \), \( \forall X \),
3. \( g(X,X) = 0 \) iff \( X = 0 \).

Since we assume \( g(X,Y) \) to be bilinear, an inner product is completely specified by its action on ordered pairs of basis vectors. The components \( g_{ij} \) of the inner product are thus given by

\[
g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g_{ij},
\]
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where $g_{ij}$ is a symmetric $n \times n$ matrix which we assume to be non-singular. By linearity, it is easy to see that if $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^j \frac{\partial}{\partial x^j}$ are two arbitrary vectors, then

$$< X, Y > = g(X, Y) = g_{ij}a^ib^j.$$  

In this sense, an inner product can be viewed as a generalization of the dot product. The standard Euclidean inner product is obtained if we take $g_{ij} = \delta_{ij}$. In this case, the quantity $g(X, X) = \|X\|^2$ gives the square of the length of the vector. For this reason $g_{ij}$ is called a **metric** and $g$ is called a **metric tensor**.

Another interpretation of the dot product can be seen if instead one considers a vector $X = a^i \frac{\partial}{\partial x^i}$ and a 1-form $\alpha = b^j dx^j$. The action of the 1-form on the vector gives $\alpha(X) = (b^j dx^j)(a^i \frac{\partial}{\partial x^i}) = b_j a^i \delta^i_j = a^i b_i$.

If we now define

$$b_i = g_{ij} b^j,$$  \hspace{1cm} (2.16)

we see that the equation above can be rewritten as

$$a^i b_j = g_{ij} a^ib^j,$$

and we recover the expression for the inner product.

Equation (2.16) shows that the metric can be used as a mechanism to lower indices, thus transforming the contravariant components of a vector to covariant ones. If we let $g^{ij}$ be the inverse of the matrix $g_{ij}$, that is

$$g^{ik} g_{kj} = \delta^i_j,$$  \hspace{1cm} (2.17)

we can also raise covariant indices by the equation

$$b^i = g^{ij} b_j.$$  \hspace{1cm} (2.18)

We have mentioned that the tangent and cotangent spaces of Euclidean space at a particular point $p$ are isomorphic. In view of the above discussion, we see that the metric $g$ can be interpreted on hand as a bilinear pairing of two vectors

$$g : T_p(\mathbb{R}^n) \times T_p(\mathbb{R}^n) \longrightarrow \mathbb{R},$$

and on the other as inducing a linear isomorphism

$$G_g : T_p(\mathbb{R}^n) \longrightarrow T^*_p(\mathbb{R}^n)$$

defined by

$$G_g X(Y) = g(X, Y),$$  \hspace{1cm} (2.19)

that maps vectors to covectors. To verify this definition is consistent with the action of lowering indices, let $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^j \frac{\partial}{\partial x^j}$. We show that that $G_g X = a_i dx^i$. In fact,

$$G_g X(Y) = (a^i dx^i)(b^j \frac{\partial}{\partial x^j}) = a_i b^j dx^i \frac{\partial}{\partial x^j} = a_i b^j \delta^i_j = g_{ij} a^j b^i = g(X, Y).$$
The inverse map $G^*: T^*_p(R^n) \to T_p(R^n)$ is defined by
\[
< G^*\alpha, X > = \alpha(X), \tag{2.20}
\]
for any 1-form $\alpha$ and tangent vector $X$. In Quantum Mechanics, it is common to use Dirac’s notation, in which a linear functional $\alpha$ on a vector space $V$ is called a \textbf{bra-vector} denoted by $\langle \alpha \rangle$, and a vector $X \in V$ is called a \textbf{ket-vector}, denoted by $|X\rangle$. The action of a bra-vector on a ket-vector is defined by the \textbf{bracket},
\[
\langle \alpha |X\rangle = \alpha(X). \tag{2.21}
\]
Thus, if the vector space has an inner product as above, we have
\[
\langle \alpha |X\rangle = < G^*\alpha, X > = \alpha(X) \tag{2.22}
\]
The mapping $C: T^*_p(R^n) \to R$ given by $(\alpha, X) \mapsto \langle \alpha |X\rangle = \alpha(X)$ is called a \textbf{Contraction}. In passing, we introduce a related concept called the \textbf{interior product}, or contraction of a vector and a form. If $\alpha$ is a $(k + 1)$-form and $X$ a vector, we define
\[
i_X\alpha(X_1, \ldots, X_k) = \alpha(X, X_1, \ldots, X_k) \tag{2.23}
\]
In particular, for a one form, we have
\[
i_X\alpha = \langle \alpha |X\rangle = \alpha(X).
\]
If $T$ is a type $(1_1)$ tensor, that is,
\[
T = T^i_j dx^j \otimes \frac{\partial}{\partial x^i},
\]
The contraction of the tensor is given by
\[
C(T) = T^i_j dx^j \frac{\partial}{\partial x^i},
\]
In other words, the contraction of the tensor is the trace of the $n \times n$ array that represents the tensor in the given basis. The notion of raising and lowering indices as well as contractions can be extended to tensors of all types. Thus, for example, we have
\[
g^{ij}T_{klm} = T^i_{klm}.
\]
A contraction between the indices $i$ and $l$ in the tensor above could be denoted by the notation
\[
C^d_1(T^i_{klm}) = T^i_{kim} = T_{km}.
\]
This is a very simple concept, but the notation for a general contraction is a bit awkward because one needs to keep track of the positions of the indices contracted. Let $T$ be a tensor of type $(1,1)$. A contraction $C^k_l$ yields a tensor of type $(1,0)$. Let $T$ be given in the form 2.12 Then
\[
C^k_l(T) = T^{i_1\ldots i_l-m_{i_1}i_{l+1}\ldots m_{i_l}j_1\ldots j_r} \frac{\partial}{\partial x^{m_{i_1}}} \otimes \ldots \otimes \frac{\partial}{\partial x^{m_{i_l}}} \otimes \ldots \otimes \frac{\partial}{\partial x^{m_{j_1}}} \otimes \ldots \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_r}, \tag{2.24}
\]
where the “hat” means that these are excluded. Here is a very neat and most useful result. If $S$ is a 2-tensor with symmetric components $T_{ij} = T_{ji}$ and $A$ is a 2-tensor with antisymmetric components $A^{ij} = -A^{ji}$ then the contraction
\[
S_{ij}A^{ij} = 0 \tag{2.25}
\]
The short proof uses the fact that summation indices are dummy indices and they can be relabelled at will by any other index that is not already used in an expression. We have

\[ S_{ij}A^{ij} = S_{ji}A^{ij} = -S_{ji}A^{ji} = -S_{kl}A^{kl} = -S_{ji}A^{ij} = 0, \]

since the quantity is the negative of itself.

In terms of the vector space isomorphism between the tangent and cotangent space induced by the metric, the gradient of a function \( f \), viewed as a differential geometry vector field, is given by

\[ \text{Grad} \, f = G^i df, \quad (2.26) \]

or in components

\[ (\nabla f)^i \equiv \nabla^i f = g^{ij} f_{,j}, \quad (2.27) \]

where \( f_{,j} \) is the commonly used abbreviation for the partial derivative with respect to \( x^j \).

In elementary treatments of calculus, authors often ignore the subtleties of differential 1-forms and tensor products and define the differential of arc length as

\[ ds^2 \equiv g_{ij} dx^i dx^j, \]

although what is really meant by such an expression is

\[ ds^2 \equiv g_{ij} dx^i \otimes dx^j. \quad (2.28) \]

### 2.2.4 Example

In cylindrical coordinates, the differential of arc length is

\[ ds^2 = dr^2 + r^2 d\theta^2 + dz^2. \quad (2.29) \]

In this case, the metric tensor has components

\[ g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.30) \]

### 2.2.5 Example

In spherical coordinates,

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta,
\end{align*}
\]

and the differential of arc length is given by

\[ ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (2.32) \]

In this case the metric tensor has components

\[ g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}. \quad (2.33) \]
2.2.3 Minkowski Space

An important object in mathematical physics is the so-called Minkowski space which is defined as the pair \((M_{1,3}, \eta)\), where

\[
M_{1,3} = \{(t, x^1, x^2, x^3) | t, x^i \in \mathbb{R}\}
\]  

(2.34)

and \(\eta\) is the bilinear map such that

\[
\eta(X, X) = t^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.
\]  

(2.35)

The matrix representing Minkowski’s metric \(\eta\) is given by

\[
\eta = \text{diag}(1, -1, -1, -1),
\]

in which case, the differential of arc length is given by

\[
ds^2 = \eta_{\mu\nu} dx^\mu \otimes dx^\nu = dt \otimes dt - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3 = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.
\]  

(2.36)

Note: Technically speaking, Minkowski’s metric is not really a metric since \(\eta(X, X) = 0\) does not imply that \(X = 0\). Non-zero vectors with zero length are called Light-like vectors and they are associated with particles that travel at the speed of light (which we have set equal to 1 in our system of units.)

The Minkowski metric \(g_{\mu\nu}\) and its matrix inverse \(g^{\mu\nu}\) are also used to raise and lower indices in the space in a manner completely analogous to \(\mathbb{R}^n\). Thus, for example, if \(A\) is a covariant vector with components

\[
A_\mu = (\rho, A_1, A_2, A_3),
\]

then the contravariant components of \(A\) are

\[
A^\mu = \eta^{\mu\nu} A_\nu = (-\rho, A_1, A_2, A_3)
\]

2.2.4 Wedge Products and 2-Forms

2.2.6 Definition A map \(\phi : T(\mathbb{R}^n) \times T(\mathbb{R}^n) \rightarrow \mathbb{R}\) is called alternating if

\[
\phi(X, Y) = -\phi(Y, X).
\]

The alternating property is reminiscent of determinants of square matrices that change sign if any two column vectors are switched. In fact, the determinant function is a perfect example of an alternating bilinear map on the space \(M_{2\times 2}\) of two by two matrices. Of course, for the definition above to apply, one has to view \(M_{2\times 2}\) as the space of column vectors.

2.2.7 Definition A 2-form \(\phi\) is a map \(\phi : T(\mathbb{R}^n) \times T(\mathbb{R}^n) \rightarrow \mathbb{R}\) which is alternating and bilinear.

2.2.8 Definition Let \(\alpha\) and \(\beta\) be 1-forms in \(\mathbb{R}^n\) and let \(X\) and \(Y\) be any two vector fields. The wedge product of the two 1-forms is the map \(\alpha \wedge \beta : T(\mathbb{R}^n) \times T(\mathbb{R}^n) \rightarrow \mathbb{R}\), given by the equation

\[
(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X).
\]  

(2.37)
2.2.9 Theorem If $\alpha$ and $\beta$ are 1-forms, then $\alpha \wedge \beta$ is a 2-form.

Proof: Let $\alpha$ and $\beta$ be 1-forms in $\mathbb{R}^n$ and let $X$ and $Y$ be any two vector fields. Then
\[
(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)
\]
Thus, the wedge product of two 1-forms is alternating.

To show that the wedge product of two 1-forms is bilinear, consider 1-forms, $\alpha, \beta$, vector fields $X_1, X_2, Y$ and functions $f^1, f^2$. Then, since the 1-forms are linear functionals, we get
\[
(\alpha \wedge \beta)(f^1 X_1 + f^2 X_2, Y) = \alpha(f^1 X_1 + f^2 X_2)\beta(Y) - \alpha(Y)\beta(f^1 X_1 + f^2 X_2)
\]
\[
= [f^1 \alpha(X_1) + f^2 \alpha(X_2)]\beta(Y) - \alpha(Y)[f^1 \beta(X_1) + f^2 \beta(X_2)]
\]
\[
= f^1 \alpha(X_1)\beta(Y) + f^2 \alpha(X_2)\beta(Y) + f^1 \alpha(Y)\beta(X_1) + f^2 \alpha(Y)\beta(X_2)
\]
\[
= f^1 [\alpha(X_1)\beta(Y) + \alpha(Y)\beta(X_1)] + f^2 [\alpha(X_2)\beta(Y) + \alpha(Y)\beta(X_2)]
\]
\[
= f^1 (\alpha \wedge \beta)(X_1, Y) + f^2 (\alpha \wedge \beta)(X_2, Y).
\]
The proof of linearity on the second slot is quite similar and is left to the reader.

The wedge product of two 1-forms has characteristics similar to cross products of vectors in the sense that both of these products anti-commute. This means that we need to be careful to introduce a minus sign every time we interchange the order of the operation. Thus, for example, we have
\[
dx^i \wedge dx^j = -dx^j \wedge dx^i
\]
if $i \neq j$, whereas
\[
dx^i \wedge dx^i = -dx^i \wedge dx^i = 0
\]
since any quantity that equals the negative of itself must vanish.

2.2.10 Example Consider the case of $\mathbb{R}^2$. Let
\[
\alpha = a \, dx + b \, dy,
\]
\[
\beta = c \, dx + d \, dy,
\]
since $dx \wedge dx = dy \wedge dy = 0$, and $dx \wedge dy = -dy \wedge dx$, we get,
\[
\alpha \wedge \beta = ad \, dx \wedge dy + bc \, dy \wedge dx,
\]
\[
= ad \, dx \wedge dy - bc \, dx \wedge dy,
\]
\[
= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \, dx \wedge dy.
\]

2.2.11 Example We can use the result above to make more evident the relation between wedge products in $\mathbb{R}^3$ and cross products. For combinatoric reasons, it is more convenient to label the coordinates as $\{x^1, x^2, x^3\}$. Let
\[
\alpha = a_1 \, dx^1 + a_2 \, dx^2 + a_3 \, dx^3,
\]
\[
\beta = b_1 \, dx^1 + b_2 \, dx^2 + b_3 \, dx^3,
\]
There are only three independent basis 2-forms, namely
\[
dy \wedge dz = \, dx^2 \wedge dx^3,
\]
\[
dx \wedge dz = -dx^1 \wedge dx^3,
\]
\[
dx \wedge dy = \, dx^1 \wedge dx^2.
\]
Computing the wedge products in pairs, we get
\[ \alpha \wedge \beta = \left| \begin{array}{cc} a_2 & a_3 \\ b_2 & b_3 \end{array} \right| dx^2 \wedge dx^3 + \left| \begin{array}{cc} a_1 & a_3 \\ b_1 & b_3 \end{array} \right| dx^1 \wedge dx^3 + \left| \begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array} \right| dx^1 \wedge dx^2. \]

If we consider vectors \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \), we see that the result above can be written as
\[ \alpha \wedge \beta = (\mathbf{a} \times \mathbf{b})_1 dx^2 \wedge dx^3 - (\mathbf{a} \times \mathbf{b})_2 dx^1 \wedge dx^3 + (\mathbf{a} \times \mathbf{b})_3 dx^1 \wedge dx^2 \quad (2.38) \]

It is worthwhile noticing that in one thinks of the indices in the formula above as permutations of the integers \( \{1, 2, 3\} \), the signs of the three terms correspond to the signature of the permutation. In particular, the middle term indices constitute an odd permutation, so the signature is minus one. One can get a good sense of the geometrical significance and the motivation for the creation of wedge products by considering a classical analogy in the language of vector calculus. As shown in figure 2.1, let us consider infinitesimal arc length vectors \( i\, dx, j\, dy \) and \( k\, dz \) pointing along the coordinate axes. Recall from the definition, that the cross product of two vectors is a new vector whose magnitude is the area of the parallelogram subtended by the two vectors and which points in the direction of a unit vector perpendicular to the plane containing the two vectors, oriented according to the right hand rule. Since \( i, j \) and \( k \) are mutually orthogonal vectors, the cross product of any pair is again a unit vector pointed in the direction of the third or the negative thereof. Thus, for example, in the \( xy \)-plane the differential of area is really an oriented quantity that can computed by the cross product \( (i\, dx \times j\, dy) = dx\, dy\, k \). A similar computation yields the differential of areas in the other two coordinate planes, except that in the \( xz \)-plane, the cross product needs to be taken in the reverse order. In terms of wedge products, the differential of area in the \( xy \)-plane is \( (dx \wedge dy) \), so that the oriented nature of the surface element is built-in. Technically, when reversing the order of variables in a double integral one should introduce a minus sign. This is typically ignored in basic calculus computations of double and triple integrals, but it cannot be ignored in vector calculus in the context of flux of a vector field through a surface.

The similarity between wedge products is even more striking in the next proposition but we emphasize again that wedge products are much more powerful than cross products, because wedge products can be computed in any dimension.

2.2.12 Example One could of course compute wedge products by just using the linearity properties. It would not be as efficient as grouping into pairs, but it would yield the same result. For example, Let
\[ \alpha = x^2 dx - y^2 dy \] and \( \beta = dx + dy - 2xydz. \) Then
\[ \alpha \wedge \beta = (x^2 dx - y^2 dy) \wedge (dx + dy - 2xydz) = x^2 dx \wedge dx + x^2 dx \wedge dy - 2x^3 y dx \wedge dz - y^2 dy \wedge dx - y^2 dy \wedge dy + 2xy^3 dy \wedge dz = x^2 dx \wedge dy - 2x^3 y dx \wedge dz - y^2 dy \wedge dx + 2xy^3 dy \wedge dz = (x^2 + y^2) dx \wedge dy - 2x^3 y dx \wedge dz + 2xy^3 dy \wedge dz. \]

In local coordinates, a 2-form can always be written in components as
\[ \phi = F_{ij} \, dx^i \wedge dx^j \quad (2.39) \]
If we think of $F$ as a matrix with components $F_{ij}$, we know from linear algebra that we can write $F$ uniquely as a sum of a symmetric and an antisymmetric matrix, namely,

$$F = S + A,$$

where,

$$F_{(ij)} = \frac{1}{2}(F_{ij} + F_{ji}),$$

$$F_{[ij]} = \frac{1}{2}(F_{ij} - F_{ji}),$$

are the completely symmetric and antisymmetric components. Since $dx^i \wedge dx^j$ is antisymmetric, and the contraction of a symmetric tensor with an antisymmetric tensor is zero, one may assume that the components of the 2-form in equation 2.39 are antisymmetric as well. With this mind, we can easily find a formula using wedges that generalizes the cross product that only makes sense in dimension 3, to any dimension.

Let $\alpha = a_i dx^i$ and $\beta = b_i dx^i$ be any two 1-forms in $\mathbb{R}^n$, and let $X$ and $Y$ be arbitrary vector fields. Then

$$\begin{align*}
(\alpha \wedge \beta)(X,Y) &= (a_i dx^i)(X)(b_j dx^j)(Y) - (a_i dx^i)(Y)(b_j dx^j)(X) \\
&= (a_i b_j)[dx^i(X)dx^j(Y) - dx^j(Y)dx^i(X)] \\
&= (a_i b_j)(dx^i \wedge dx^j)(X,Y).
\end{align*}$$

Because of the antisymmetry of the wedge product, the last of the above equations can be written as

$$\alpha \wedge \beta = \sum_{i=1}^{n} \sum_{j<i}^{n} (a_i b_j - a_j b_i)(dx^i \wedge dx^j),$$

$$= \frac{1}{2}(a_i b_j - a_j b_i)(dx^i \wedge dx^j).$$

In particular, if $n = 3$, the reader will recognize the coefficients of the wedge product as the components of the cross product of $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$, as shown earlier.

**Remark** Quantities such as $dx dy$ and $dy dz$ which often appear in calculus II, are not really well defined. What is meant by them are actually wedge products of 1-forms, but in reversing the order of integration, the antisymmetry of the wedge product is ignored. In performing surface integrals, however, the surfaces must be considered oriented surfaces and one has to insert a negative sign in the differential of surface area component in the $xz$-plane as shown later in 2.84.

### 2.2.5 Determinants

The properties of $n$-forms are closely related to determinants, so it might be helpful to digress a bit and review the fundamentals of determinants, as found in any standard linear algebra textbook such as [16]. Let $A \in M_n$ be an $n \times n$ matrix with column vectors

$$A = [v_1, v_2, \ldots, v_n]$$
2.2.13 Definition  A function $f : M_n \to \mathbb{R}$ is called multilinear if it is linear on each slot; that is,

$$f[v_1, \ldots, av_i, \ldots, bv_j, \ldots, v_n] = af[v_1, \ldots, v_i, \ldots, v_n] + bf[v_1, \ldots, v_j, \ldots, v_n].$$

2.2.14 Definition  A function $f : M_n \to \mathbb{R}$ is called alternating if it changes sign whenever any two columns are switched; that is,

$$f[v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n] = -f[v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n]$$

2.2.15 Definition  A determinant function is a map $D : M_n \to \mathbb{R}$ that is

a) Multilinear,

b) Alternating,

c) $D(I) = 1$.

One can then prove that this defines the function uniquely. In particular, if $A = (a_{ij})$, the determinant can be expressed as

$$\det(A) = \sum_\pi \text{sgn}(\pi) a_{\pi(1)}^1 a_{\pi(2)}^2 \cdots a_{\pi(n)}^n,$$  \hspace{1cm} (2.40)

where the sum is over all the permutations of $\{1, 2, \ldots, n\}$. The determinant can also be calculated by the cofactor expansion formula of Laplace. Thus, for example, the cofactor expansion along the entries on the first row $(a_{1j})$, is given by

$$\det(A) = \sum_k a_{1k} \Delta_{k1},$$ \hspace{1cm} (2.41)

where $\Delta$ is the cofactor matrix.

At this point it is convenient to introduce the totally antisymmetric Levi-Civita permutation symbol defined as follows:

$$\epsilon_{i_1i_2\ldots i_k} = \begin{cases} +1 & \text{if } (i_1, i_2, \ldots, i_k) \text{ is an even permutation of } (1, 2, \ldots, k) \\ -1 & \text{if } (i_1, i_2, \ldots, i_k) \text{ is an odd permutation of } (1, 2, \ldots, k) \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (2.42)

In dimension 3, there are only 6 $(3! = 6)$ non-vanishing components of $\epsilon_{ijk}$, namely,

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$

$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$$ \hspace{1cm} (2.43)

We set the Levi-Civita symbol with some or all the indices up, numerically equal to the permutation symbol will all the indices down. The permutation symbols are useful in the theory of determinants. In fact, if $A = (a_{ij})$ is an $n \times n$ matrix, then, equation (2.40) can be written as,

$$\det(A) = |A| = \epsilon_{i_1i_2\ldots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}.$$ \hspace{1cm} (2.44)

Thus, for example, for a $2 \times 2$ matrix,

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$\det(A) = \epsilon^{ij} a_{1i} a_{2j},$$

$$= \epsilon^{12} a_{11} a_{22} + \epsilon^{21} a_{12} a_{21},$$

$$= a_{11} a_{22} - a_{12} a_{21}.$$
2.2. TENSORS

We also introduce the generalized Kronecker delta symbol

$$\delta_{j_1j_2...j_k}^{i_1i_2...i_k} = \begin{cases} +1 & \text{if } (i_1, i_2, \ldots, i_k) \text{ is an even permutation of } (j_1, j_2 \ldots j_k) \\ -1 & \text{if } (i_1, i_2, \ldots, i_k) \text{ is an odd permutation of } (j_1, j_2 \ldots j_k) \\ 0 & \text{otherwise} \end{cases} \quad (2.45)$$

If one views the indices \(i_k\) as labelling rows and \(j_k\) as labelling columns of a matrix, we can represent the completely antisymmetric symbol by the determinant,

$$\delta_{j_1j_2...j_k}^{i_1i_2...i_k} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_k} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_k} \\ \cdots & \cdots & \cdots & \cdots \\ \delta^{i_k}_{j_1} & \delta^{i_k}_{j_2} & \cdots & \delta^{i_k}_{j_k} \end{vmatrix} \quad (2.46)$$

Not surprisingly, the generalized Kronecker delta is related to a product of Levi-Civita symbols by the equation

$$\epsilon_{i_1i_2...i_k}^{j_1j_2...j_k} = \delta_{j_1j_2...j_k}^{i_1i_2...i_k} \quad (2.47)$$

which is evident since both sides are completely antisymmetric. In dimension 3, the only non-zero components of \(\delta_{kl}^{ij}\) are,

\[
\begin{align*}
\delta_{12}^{12} &= \delta_{13}^{13} = \delta_{23}^{23} = 1 & \delta_{12}^{13} = \delta_{13}^{12} = \delta_{23}^{31} = -1 \\
\delta_{21}^{21} &= \delta_{31}^{31} = \delta_{32}^{32} = 1 & \delta_{12}^{21} = \delta_{13}^{31} = \delta_{23}^{32} = -1.
\end{align*}
\]

2.2.16 Proposition In dimension 3 the following identities hold

a) \(\epsilon_{i_1i_2...i_k}^{i_1i_2...i_k} = \delta_{i_1i_2...i_k}^{j_1j_2...j_k} = \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \cdots \delta_{i_k}^{j_k} - \delta_{i_1}^{j_k} \delta_{i_2}^{j_1} \cdots \delta_{i_k}^{j_{k-1}}\),

b) \(\epsilon_{i_1j_2...j_k} = 2\delta_{i_1}^{j_k}\),

c) \(\epsilon_{i_1j_2...j_k}^{i_1j_2...j_k} = 3!\)

Proof: For part (a), we compute the determinant by cofactor expansion on the first row

\[
\epsilon_{i_1j_2...j_k}^{i_1i_2...i_k} = \begin{vmatrix} \delta_{11}^{i_1} & \delta_{12}^{i_1} & \cdots & \delta_{1k}^{i_1} \\ \delta_{21}^{i_2} & \delta_{22}^{i_2} & \cdots & \delta_{2k}^{i_2} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{k1}^{i_k} & \delta_{k2}^{i_k} & \cdots & \delta_{kk}^{i_k} \end{vmatrix} = \delta_{11}^{i_1} \delta_{22}^{i_2} \cdots \delta_{kk}^{i_k} - \delta_{12}^{i_1} \delta_{21}^{i_2} \cdots \delta_{kk}^{i_k} + \cdots + (-1)^{i_1} \delta_{11}^{i_1} \delta_{22}^{i_2} \cdots \delta_{kk}^{i_k} = 3 \begin{vmatrix} \delta_{12}^{i_1} & \delta_{13}^{i_1} \delta_{22}^{i_2} & \cdots & \delta_{2k}^{i_2} \\ \delta_{k1}^{i_k} & \delta_{k2}^{i_k} \cdots & \cdots & \delta_{kk}^{i_k} \end{vmatrix} \]

Here we used the fact that the contraction \(\delta_{i_1}^{i_1}\) is just the trace of the identity matrix and the observation that we had to transpose columns in the last determinant in the next to last line. For part (b) follows easily for part (a), namely

\[
\epsilon_{i_1j_2...j_k}^{i_1j_2...j_k} = \delta_{i_1j_1}^{j_k} = \delta_{i_1k}^{j_k} - \delta_{i_1}^{j_k} \delta_{j_k}^{i_k} = 3\delta_{i_1}^{j_k} - \delta_{i_1}^{j_k} = 2\delta_{i_1}^{j_k}.
\]
From this, part (c) is obvious. With considerable more effort but inductively following the same scheme, one can establish the general formula,

\[ \epsilon_{i_1 \ldots i_k, i_{k+1}, \ldots i_n} \epsilon_{j_1 \ldots j_k, j_{k+1}, \ldots j_n} = k! \delta_{i_1 \ldots i_k}^{j_1 \ldots j_n}. \]  

(2.48)

**Vector Identities**

The permutation symbols are very useful in establishing and manipulating classical vector formulas. We present here a number of examples. For this purpose, let,

\[ \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}, \quad \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}, \quad \mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}, \quad \mathbf{d} = d_1 \mathbf{i} + d_2 \mathbf{j} + d_3 \mathbf{k}, \]

\[ \alpha = a_1 \, dx^1 + a_2 \, dx^2 + a_3 \, dx^3, \]

\[ \beta = b_1 \, dx^1 + b_2 \, dx^2 + b_3 \, dx^3, \]

\[ \gamma = c_1 \, dx^1 + c_2 \, dx^2 + c_3 \, dx^3, \]

\[ \delta = d_1 \, dx^1 + d_2 \, dx^2 + d_3 \, dx^3, \]

1. **Dot product and cross product**

\[ \mathbf{a} \cdot \mathbf{b} = \delta^{ij} a_i b_j = a_i b^i, \quad (\mathbf{a} \times \mathbf{b})_k = \epsilon_{kij} a_i b_j \]  

(2.49)

2. **Wedge product**

\[ \alpha \wedge \beta = \epsilon^{kij} (\mathbf{a} \times \mathbf{b})_k \, dx^i \wedge dx^j. \]  

(2.50)

3. **Triple product**

\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \delta_{ij} a^i (\mathbf{b} \times \mathbf{c})^j, \]

\[ = \delta_{ij} a^i \epsilon_{kl} b^k c^l, \]

\[ = \epsilon_{ikl} a^i b^k c^l, \]

\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \text{det}([\mathbf{a} \, \mathbf{b} \, \mathbf{c}]), \]  

(2.51)

\[ = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \]  

(2.52)

4. **Triple cross product: bac-cab identity**

\[ [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_l = \epsilon_{lmn} a_m (\mathbf{b} \times \mathbf{c})_n \]

\[ = \epsilon_{lmn} a_m (\epsilon_{njk} b_j c_k) \]

\[ = \epsilon_{lmn} \epsilon_{njk} a_m b_j c_k \]

\[ = \epsilon_{lmn} \epsilon_{njk} a_m b_j c_k \]

\[ = (\delta^k_m \delta^j_l - \delta^j_m \delta^k_l) a^m b_j c_k \]

\[ = b_l a^m c_m - c_l a^m b_m. \]

Rewriting in vector form

\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b}). \]  

(2.53)
5. Dot product of cross products

\((a \times b) \cdot (c \times d) = a \cdot (b \times c \times d),\)

\[= a \cdot [c(b \cdot d) - d(b \cdot c)]\]

\[= (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c),\]

\[= \begin{vmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{vmatrix} \quad (2.54)\]

6. Norm of cross-product

\[\|a \times b\|^2 = (a \times b) \cdot (a \times b),\]

\[= \begin{vmatrix} a \cdot a & a \cdot b \\ b \cdot a & b \cdot b \end{vmatrix},\]

\[= \|a\|^2 \|b\| - (a \cdot b)^2 \quad (2.55)\]

7. More wedge products. Let \(C = c^k \frac{\partial}{\partial x^k}, \quad D = d^m \frac{\partial}{\partial x^m}.\) Then,

\[\left(\alpha \wedge \beta\right)(C, D) = \begin{vmatrix} \alpha(C) & \alpha(D) \\ \beta(C) & \beta(D) \end{vmatrix},\]

\[= \begin{vmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{vmatrix} \quad (2.56)\]

8. Grad, Cur Div in \(\mathbb{R}^3\)

Let \(\nabla_i = \frac{\partial}{\partial x^i}, \ \nabla^i = \delta^{ij} \nabla_j, \ \mathbf{A} = \mathbf{a}\) and define

- \((\nabla f)_i = \nabla_i f\)
- \((\nabla \times \mathbf{A})_i = \epsilon^{ijk} \nabla_j a_k\)
- \(\nabla \cdot \mathbf{A} = \delta^{ij} \nabla_i a_j = \nabla_j a_i\)
- \(\nabla \cdot (\nabla f) \equiv \nabla^2 f = \nabla_i \nabla_i f\)

(a)

\[\nabla \times \nabla f = 0, \quad \nabla^2 f = 0 \quad (2.57)\]

(b)

\[\nabla \cdot (\nabla \times \mathbf{A}) = \delta^{ij} \nabla_i (\nabla \times \mathbf{a})_j, \]

\[= \delta^{ij} \nabla_i \epsilon^{kl} \nabla_k a_l, \]

\[= \epsilon^{kli} \nabla_i \nabla_j a_k, \]

\[\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (2.58)\]

where in the last step in the two items above we use the fact that we have a contraction of two symmetric with two antisymmetric indices is always 0.

(c) The same steps as in the bac-cab identity give

\[\nabla \times (\nabla \times \mathbf{A}) = \nabla_i (\nabla^m a_m) - \nabla^m \nabla_m a_i,\]

\[\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A},\]

where \(\nabla^2 \mathbf{A}\) means the the Laplacian of each component of \(\mathbf{A}\).

This last equation is crucial in the derivation of the wave equation for light from Maxwell’s equations for the the electromagnetic field.
2.2.6 n-Forms

2.2.17 Definition Let $\alpha^1, \alpha^2, \alpha^3$, be one forms, and $X_1, X_2, X_3 \in \mathcal{X}$. Let $\pi$ be the set of permutations of $\{1, 2, 3\}$. Then

\[
(\alpha^1 \wedge \alpha^2 \wedge \alpha^3)(X_1, X_2, X_3) = \sum_{\pi} \text{sign}(\pi) \alpha^1(X_{\pi(1)}) \alpha^2(X_{\pi(2)}) \alpha^3(X_{\pi(3)}),
\]

\[
= \epsilon^{ijk} \alpha^1(X_i) \alpha^2(X_j) \alpha^3(X_k).
\]

This trilinear map is an example of a alternating covariant 3-tensor.

2.2.18 Definition A 3-form $\phi$ in $\mathbb{R}^n$ is an alternating, covariant 3-tensor. In local coordinates, a 3-form can be written as an object of the following type

\[
\phi = A_{ijk} dx^i \wedge dx^j \wedge dx^k \tag{2.59}
\]

where we assume that the wedge product of three 1-forms is associative but alternating in the sense that if one switches any two differentials, then the entire expression changes by a minus sign. There is nothing really wrong with using definition $\tag{2.59}$. This definition however, is coordinate-dependent and differential geometers prefer coordinate-free definitions, theorems and proofs.

We can easily extend the concepts above to higher order forms.

2.2.19 Definition Let $T^0_k(\mathbb{R}^n)$ be the set multilinear maps

\[
t : T(\mathbb{R}) \times \ldots \times T(\mathbb{R}) \to \mathbb{R}
\]

from $k$ copies of $T(\mathbb{R})$ to $\mathbb{R}$. The map $t$ is called skew-symmetric if

\[
t(e_1, \ldots, e_k) = \text{sign}(\pi)t(e_{\pi(1)}, \ldots, e_{\pi(k)}),
\]

where $\pi$ is the set of permutations of $\{1, \ldots, k\}$. A skew-symmetry covariant tensor of rank $k$ at $p$, is called a k-form at $p$. denote by $\bigwedge^k_p(\mathbb{R}^n)$ the space of k-forms at $p \in \mathbb{R}^n$. This vector space has dimension

\[
dim \bigwedge^k_p(\mathbb{R}^n) = \binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

for $k \leq n$ and dimension 0 for $k > n$. We will identify $\bigwedge^0_p(\mathbb{R}^n)$ with the space of $C^\infty$ functions at $p$. The union of all $\bigwedge^k_p(\mathbb{R}^n)$ as $p$ ranges through all points in $\mathbb{R}^n$ is called the bundle of k-forms and will be denoted by

\[
\bigwedge^k(\mathbb{R}^n) = \bigcup_p \bigwedge^k_p(\mathbb{R}^n).
\]

Sections of the bundle are called k-forms and the space of all sections is denoted by

\[
\Omega^k(\mathbb{R}^n) = \Gamma(\bigwedge^k(\mathbb{R}^n)).
\]

A section $\alpha \in \Omega^k$ of the bundle technically should be called k-form field, but the consensus in the literature is to call such a section simply a k-form. In local coordinates, an k-form can be written as:

\[
\alpha = A_{i_1, \ldots, i_k}(x) dx^{i_1} \wedge \ldots dx^{i_k}. \tag{2.61}
\]
2.2. Definition The alternation map $A : T^0_k(\mathbb{R}^n) \to T^0_k(\mathbb{R}^n)$ is defined by

$$A(t_1, \ldots, t_k) = \frac{1}{k!} \sum_{\pi} (\text{sign } \pi) t(\pi(1), \ldots, \pi(k)).$$

2.2.21 Definition If $\alpha \in \Omega^k(\mathbb{R}^n)$ and $\beta \in \Omega^l(\mathbb{R}^n)$, then

$$\alpha \wedge \beta = \frac{(k + l)!}{k! l!} A(\alpha \otimes \beta) \quad (2.62)$$

If $\alpha$ is a $k$-form and $\beta$ an $l$-form, we have

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha. \quad (2.63)$$

Now for a little combinatorics. Factorials are unavoidable due to the permutation attributes of the wedge product. The convention here follows Marsden [19] and Spivak [31], which reduces proliferation of factorials later. Now, let us count the number of linearly independent differential forms in Euclidean space. More specifically, we want to find the vector space dimension of the space of $k$-forms in $\mathbb{R}^n$. As stated above, we will think of 0-forms as being ordinary functions. Since functions are the "scalars", the space of 0-forms as a vector space has dimension 1.

<table>
<thead>
<tr>
<th>$\mathbb{R}^2$</th>
<th>Forms</th>
<th>Dim</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-forms</td>
<td>$f$</td>
<td>1</td>
</tr>
<tr>
<td>1-forms</td>
<td>$f dx^1, g dx^2$</td>
<td>2</td>
</tr>
<tr>
<td>2-forms</td>
<td>$f dx^1 \wedge dx^2$</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mathbb{R}^3$</th>
<th>Forms</th>
<th>Dim</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-forms</td>
<td>$f$</td>
<td>1</td>
</tr>
<tr>
<td>1-forms</td>
<td>$f_1 dx^1, f_2 dx^2, f_3 dx^3$</td>
<td>3</td>
</tr>
<tr>
<td>2-forms</td>
<td>$f_1 dx^2 \wedge dx^3, f_2 dx^3 \wedge dx^1, f_3 dx^1 \wedge dx^2$</td>
<td>3</td>
</tr>
<tr>
<td>3-forms</td>
<td>$f_1 dx^1 \wedge dx^2 \wedge dx^3$</td>
<td>1</td>
</tr>
</tbody>
</table>

The binomial coefficient pattern should be evident to the reader.

It is possible define **tensor-valued differential forms.** Let $E = T^r_s(\mathbb{R}^n)$ be the tensor bundle. A tensor-valued $p$-form is defined as a section $T \in \Omega^p(\mathbb{R}^n, E) = \Gamma(E \otimes \Lambda^p(\mathbb{R}^n))$.

In local coordinates, a tensor-valued $k$-form is a $(r+s+p)$ tensor

$$T = T_{i_1 \ldots i_r j_1 \ldots j_s k_1 \ldots k_s} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \otimes dx^{k_1} \wedge \ldots \wedge dx^{k_s}. \quad (2.64)$$

Thus, for example, the quantity

$$\Omega^r_{j} = \frac{1}{2} R^r_{jkl} \ dx^k \wedge dx^l$$

would be called the components of a $(1)$-valued 2-form

$$\Omega = \Omega^r_{j} \frac{\partial}{\partial x^l} \otimes dx^j.$$ 

The notion of the wedge product can be extended to tensor-valued forms using tensor products on the tensorial indices and wedge products on the differential form indices.
CHAPTER 2. DIFFERENTIAL FORMS

2.3 Exterior Derivatives

In this section we introduce a differential operator that generalizes the classical gradient, curl and divergence operators.

2.3.1 Definition Let \( \alpha \) be a one form in \( \mathbb{R}^n \). The differential \( d\alpha \) is the two form defined by

\[
d\alpha (X, Y) = X(\alpha(Y)) - Y(\alpha(X)),
\]

for any pair of vector fields \( X \) and \( Y \).

To explore the meaning of this definition in local coordinates, let \( \alpha = f_i \, dx^i \) and let \( X = \frac{\partial}{\partial x^j}, \ Y = \frac{\partial}{\partial x^k} \), then

\[
d\alpha(X,Y) = \frac{\partial f_i}{\partial x^j} \left( \frac{\partial f_i}{\partial x^k} \right) \, dx^j \wedge dx^k,
\]

\[
d\left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) = \frac{\partial f_k}{\partial x^j} \, dx^j \wedge dx^k - \frac{\partial f_j}{\partial x^k} \, dx^k \wedge dx^j.
\]

Therefore, taking into account the antisymmetry of wedge products, we have.

\[
d\alpha = \frac{1}{2} \left( \frac{\partial f_k}{\partial x^j} - \frac{\partial f_j}{\partial x^k} \right) \, dx^j \wedge dx^k,
\]

\[
= \frac{\partial f_k}{\partial x^j} \, dx^j \wedge dx^k - \frac{\partial f_j}{\partial x^k} \, dx^k \wedge dx^j = df_k \wedge dx^k.
\]

The definition \(2.65\) of a differential of a 1-form can be refined to provide a coordinate-free definition in general manifolds (see \(6.27\)) and it can be extended to differentials of \( m \)-forms. For now, the computation immediately above suffices to motivate the following coordinate dependent definition (for a coordinate-free definition for general manifolds, see \(6.39\)).

2.3.2 Definition Let \( \alpha \) be an \( m \)-form, given in coordinates as in equation (2.61). The exterior derivative of \( \alpha \) is the \((m+1)\)-form \( d\alpha \) given by

\[
d\alpha = dA_{i_1...i_m} \wedge dx^{i_1} \ldots dx^{i_m} = \frac{\partial A_{i_1...i_m}}{\partial x^0}(x)\,dx^0 \wedge dx^{i_1} \ldots dx^{i_m}.
\]

In the special case where \( \alpha \) is a 0-form, that is, a function, we write

\[
df = \frac{\partial f}{\partial x^i} \, dx^i.
\]

2.3.3 Theorem

\begin{align*}
a) \quad d : \Lambda^m &\rightarrow \Lambda^{m+1} \\
b) \quad d^2 = d \circ d = 0 \\
c) \quad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta & \quad \forall \alpha \in \Lambda^p, \beta \in \Lambda^q
\end{align*}

(2.67)

Proof:
2.3. EXTERIOR DERIVATIVES

By definition,
\[
d(\alpha) = d(\frac{\partial f}{\partial x^i}) = \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i = \frac{1}{2} \left[ \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \right] dx^j \wedge dx^i = 0.
\]

Now, suppose that \( \alpha \) is represented locally as in equation (2.61). It follows from equation (2.66) that
\[
d(\alpha) = d(dA_{i_1 \ldots i_m}) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_m} = 0.
\]

2.3.4 Definition Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be differentiable mapping and let \( \alpha \) be a \( k \)-form in \( \mathbb{R}^m \). Then, at each point \( y \in \mathbb{R}^m \) with \( y = F(x) \), the mapping \( F \) induces a map called the pull-back \( F^* : \bigwedge^k_{(F(x))} \rightarrow \bigwedge^k_x \) defined by
\[
(F^* \alpha)_x(X_1, \ldots, X_k) = \alpha_{F(x)}(F_*(X_1), \ldots, F_*X_k),
\]
for any tangent vectors \( \{ X_1, \ldots, X_k \} \) in \( \mathbb{R}^n \).
If \( g \) is a 0-form, namely a function, \( F^*(g) = g \circ F \). we have the following theorem.

2.3.5 Theorem

a) \( F^*(g \alpha_1) = (g \circ F) F^* \alpha \),
b) \( F^*(\alpha_1 + \alpha_2) = F^* \alpha_1 + F^* \alpha_2 \),
c) \( F^*(\alpha \wedge \beta) = F^* \alpha \wedge F^* \beta \),
d) \( F^*(d \alpha) = d(F^* \alpha) \).
Part (d) is encapsulated in the commuting diagram in figure 2.2

**Proof:** Part (a) is basically the definition for the case of 0-forms and part (b) is clear from the linearity of the push-forward. We leave part (c) as an exercise and prove part (d). In the case of a 0-form, let \( g \) be a function and \( X \) a vector field in \( \mathbb{R}^m \). By a simple computation that amounts to recycling definitions, we have:

\[
\begin{align*}
\frac{d}{dF}(F^*g) &= (F^*dg)(X) \quad = X(g \circ F) = d(g \circ F)(X), \\
F^*dg &= d(g \circ F),
\end{align*}
\]

so, \( d(F^*g) = F^*(dg) \) is true by the composite mapping theorem. Let \( \alpha \) be a \( k \)-form

\[
\alpha = A_{i_1,...,i_k} dy^{i_1} \wedge ... dy^{i_k},
\]

so that

\[
d\alpha = (dA_{i_1,...,i_k}) \wedge dy^{i_1} ... dy^{i_k}.
\]

Then, by part (c),

\[
F^*\alpha = (F^*A_{i_1,...,i_k}) f^* dy^{i_1} \wedge ... f^* dy^{i_k},
\]

\[
d(F^*\alpha) = d(F^*A_{i_1,...,i_k}) \wedge f^* dy^{i_1} \wedge ... f^* dy^{i_k},
\]

\[
= F^*(dA_{i_1,...,i_k}) \wedge f^* dy^{i_1} \wedge ... f^* dy^{i_k},
\]

\[
= F^*(d\alpha).
\]

So again, the result rests on the chain rule.

To connect with advanced calculus, suppose that locally the mapping \( F \) is given by \( y^k = f^k(x^i) \). Then the pullback of the form \( dg \) given the formula above \( F^*dg = d(g \circ F) \) is given in local coordinates by the chain rule

\[
F^*dg = \frac{\partial g}{\partial x^j} dx^j.
\]

In particular, the pull-back of local coordinate functions is given by

\[
F^*(dy^i) = \frac{\partial y^i}{\partial x^j} dx^j.
\]  

(2.72)

Thus, pullback for the basis 1-forms \( dy^k \) is yet another manifestation of the differential as linear map represented by the Jacobian

\[
dy^k = \frac{\partial y^k}{\partial x^i} dx^i.
\]  

(2.73)
In particular, if $m = n$,

\[
\begin{align*}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \ldots \frac{\partial y_n}{\partial x_n} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n, \\
\epsilon_{i_1 i_2 \ldots i_n} & \frac{\partial y_1}{\partial x_{i_1}} \frac{\partial y_2}{\partial x_{i_2}} \ldots \frac{\partial y_n}{\partial x_{i_n}} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n, \\
& = |J| \wedge dx_2 \wedge \ldots \wedge dx_n.
\end{align*}
\] (2.74)

So, the pullback of the volume form,

\[
F^* d\Omega = |J| dx_1 \wedge \ldots \wedge dx_n,
\]
gives rise to the integrand that appears in the change of variables theorem for integration. More explicitly, let $R \subseteq \mathbb{R}^n$ be a simply connected region $F : R \rightarrow \mathbb{R}^m$, with $m \geq n$. If $\omega$ is a $k$-form in $\mathbb{R}^m$, then

\[
\int_{F(R)} \omega = \int_{R} F^* \omega
\]
(2.75)

We refer to this as integration by pull-back.

If $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism, one can push-forward forms with the inverse of the pull-back $F_* = (F^{-1})^*$.

2.3.6 Example Line Integrals

Let $\omega = f_i dx^i$ be a one form in $\mathbb{R}^3$ and let $C$ be the curve given by the mapping $\phi : I = t \in [a,b] \rightarrow \mathbf{x}(t) \in \mathbb{R}^3$. We can write $\omega = \mathbf{F} \cdot d\mathbf{x}$, where $\mathbf{F} = (f_1,f_2,f_3)$ is a vector field. Then the integration by pull-back equation 2.75 reads,

\[
\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C \omega, \\
= \int_{\phi(I)} \omega, \\
= \int_{\phi^*(I)} \omega, \\
= \int_I f^i(\mathbf{x}(t)) \frac{dx^i}{dt} dt, \\
= \int_I \mathbf{F}(\mathbf{x}(t)) \frac{d\mathbf{x}}{dt} dt
\]

This coincides with the definition of line integrals as introduced in calculus.

2.3.7 Example Polar Coordinates

Let $x = r \cos \theta$ and $y = r \sin \theta$ and $f = f(x,y)$. Then

\[
\begin{align*}
\int \int f(x,y) dx \wedge dy &= \int \int f(x(r,\theta), y(r,\theta)) r(dr \wedge d\theta)
\end{align*}
\] (2.76)
In this case, the element of arc length is diagonal
\[ ds^2 = dr^2 + r^2 d\theta^2, \]
as it should be for an orthogonal change of variables. The differential of area is
\[ dA = \sqrt{\det g} \, dr \wedge d\theta, \]
\[ = r(\,dr \wedge d\theta\,) \]
If the polar coordinates map is denoted by \( F : \mathbb{R}^2 \to \mathbb{R}^2 \), then equation \([2.76]\) is just the explicit expression for the pullback of \( F^*(f \, dA) \).

2.3.8 Example Polar coordinates are just a special example of the general transformation in \( \mathbb{R}^2 \) given by,
\[ x = x(u, v), \quad dx = \frac{\partial x}{\partial u} \, du + \frac{\partial x}{\partial v} \, dv, \]
\[ y = y(u, v), \quad dy = \frac{\partial y}{\partial u} \, du + \frac{\partial y}{\partial v} \, dv, \]
for which
\[ \phi^*(dx \wedge dy) = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| \, du \wedge dv \tag{2.77} \]

2.3.9 Example Surface Integrals
Let \( R \subseteq \mathbb{R}^2 \) be a simply connected region with boundary \( \partial R \) and let the mapping \( \phi : (u, v) \in R \to x(u)v)\mathbb{R}^2 \) describe a surface \( S \) with boundary \( C = \phi(\partial R) \). Here, \( \alpha = 1, 2 \), with \( u = u^1, \, v = u^2 \).
Given a vector field \( \mathbf{F} = (f_1, f_2, f_3) \), we assign to it the 2-form
\[ \omega = \mathbf{F} \cdot dS, \]
\[ = f_1 \, dx^2 \wedge dx^3 - f_2 \, dx^1 \wedge dx^3 + f_3 \, dx^1 \wedge dx^2, \]
\[ = \epsilon^{ij}_k f_i \, dx^j \wedge dx^k. \]
Then,
\[ \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_S \omega, \]
\[ = \int \int_R \phi^* \omega, \]
\[ = \int \int_R \epsilon^{ij}_k f_i \frac{\partial x^j}{\partial u^\alpha} \, du^\alpha \wedge \frac{\partial x^k}{\partial u^\beta} \, du^\beta, \]
\[ = \int \int_R \mathbf{F} \cdot \left( \frac{\partial \mathbf{x}}{\partial u^\alpha} \times \frac{\partial \mathbf{x}}{\partial u^\beta} \right) \, du^\alpha \wedge du^\beta \]
We expand this slick computation for the benefit of those readers who may have gotten got lost in the index manipulation.
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\[
\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int \omega,
\]

\[
= \int \int_R \phi^* \omega,
\]

\[
= \int \int_R [f_1 \phi^*(dx^2 \land dx^3) - f_2 \phi^*(dx^1 \land dx^3) + f_3 \phi^*(dx^1 \land dx^2)],
\]

\[
= \int \int_R \begin{vmatrix}
\frac{\partial x^2}{\partial u} & \frac{\partial x^2}{\partial v} \\
\frac{\partial x^1}{\partial u} & \frac{\partial x^1}{\partial v}
\end{vmatrix} - f_2 \begin{vmatrix}
\frac{\partial x^2}{\partial u} & \frac{\partial x^2}{\partial v} \\
\frac{\partial x^1}{\partial u} & \frac{\partial x^1}{\partial v}
\end{vmatrix} + f_3 \begin{vmatrix}
\frac{\partial x^2}{\partial u} & \frac{\partial x^2}{\partial v} \\
\frac{\partial x^1}{\partial u} & \frac{\partial x^1}{\partial v}
\end{vmatrix} du \land dv
\]

\[
= \int \int_R \mathbf{F} \cdot \left( \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial u} \right) du \land dv
\]

This pull-back formula for surface integrals is how most students are introduced to this subject in the third semester of calculus.

2.3.10 Remark

1. The differential of area in polar coordinates is of course a special example of the change of coordinate theorem for multiple integrals as indicated above.

2. As shown in equation 2.33 the metric in spherical coordinates is given by

\[
ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]

so the differential of volume is

\[
dV = \sqrt{\det g} \, dr \land d\theta \land d\phi,
\]

\[
= r^2 \sin \theta \, dr \land d\theta \land d\phi.
\]

2.3.2 Stokes’ Theorem in \( \mathbb{R}^n \)

Let \( \alpha = P(x, y) \, dx + Q(x, y) \, dy \). Then,

\[
da\alpha = \left( \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \land dx + \left( \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy \right) \land dy
\]

\[
= \frac{\partial P}{\partial y} dx \land dy + \frac{\partial Q}{\partial x} dy \land dx
\]

\[
= \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \land dy. \quad (2.78)
\]

This example is related to Green’s theorem in \( \mathbb{R}^2 \). For convenience, we include here a proof of Green’s Theorem in a special case.

We say that a region \( D \) in the plane is of **Type I** if it is enclosed between the graphs of two continuous functions of \( x \). The region inside the simple closed curve in figure 2.3 bounded by \( f_1(x) \) and \( f_2(x) \), between \( a \) and \( b \), is a region of type I. A region in the plane is of **Type II** if it lies between two continuous functions of \( y \). The region in 2.3 bounded between \( c \leq y \leq d \), would be a region of type II.

2.3.11 Green’s theorem

Let \( C \) be a simple closed curve in the \( xy \)-plane and let \( \partial P/\partial x \) and \( \partial Q/\partial y \) be continuous functions of \((x, y)\) inside and on \( C \). Let \( R \) be the region inside the closed curve so that the boundary \( \delta R = C \). Then
Fig. 2.3: Simple closed curve.

\[
\oint_{\partial R} P \, dx + Q \, dy = \iint_R \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \, dA. \quad (2.79)
\]

We first prove that for a type I region such as the one bounded between \(a\) and \(b\) shown in 2.3, we have

\[
\int_C P \, dx = -\iint_D \frac{\partial P}{\partial y} \, dA \quad (2.80)
\]

Where \(C\) is partitioned into curves \(C_1, C_2, C_3\) and \(C_4\). By the fundamental theorem of calculus, we have on the right,

\[
\iint_D \frac{\partial P}{\partial y} \, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} \, dy \, dx,
\]

\[
= \int_a^b [P(x, f_2(x)) - P(x, f_1(x))] \]

On the left, the integrals along \(C_2\) and \(C_4\) vanish, since there is no variation on \(x\). The integral along \(C_3\) is traversed in opposite direction of \(C_1\), so we have,

\[
\int_C P(x, y) \, dx = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} P(x, y) \, dx,
\]

\[
= \int_{C_1} P(x, y) \, dx - \int_{C_3} P(x, y) \, dx,
\]

\[
= \int_a^b P(x, f_1(x)) \, dx - \int_a^b P(x, f_2(x)) \, dx
\]

The result for type I regions is clear from just comparing the two sides. By a completely analogous process on type II regions we find that

\[
\oint_C Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA. \quad (2.81)
\]

The theorem follows by subdividing \(R\) into a grid of regions of both types, all oriented in the same direction as shown on the right in picture 2.3. Then one adds 2.80 or 2.81 as appropriate, for each of the subdomains. All contributions from internal boundaries cancel since each is traversed
twice in opposite directions, so all that remains of the line integrals is the contribution along the boundary $\delta R$.

Let $\alpha = P \, dx + Q \, dy$. Comparing with equation (2.78) we can write Green’s theorem in the form

$$\int_C \alpha = \int \int_D d\alpha. \quad (2.82)$$

It is possible to extend Green’s Theorem to more complicated regions that are not simple connected. Green’s theorem is a special case in dimension of two of,

### 2.3.12 Stoke’s Theorem

If $\omega$ is a $C^1$ one form in $\mathbb{R}^n$ and $S$ is $C^2$ surface with boundary $\delta S = C$, then

$$\int_{\delta S} \omega = \int \int_S d\omega. \quad (2.83)$$

**Proof:** The proof can be done by pulling back to the $uv$-plane and using the chain rule, thus allowing us to use Green’s theorem. Let $\omega = f_i \, dx^i$ and $S$ be parametrized by $x^i = x^i(u^\alpha)$, where $(u^1, u^2) \in R \subset \mathbb{R}^2$. We assume that the boundary of $R$ is a simple closed curve. Then

$$\int_C \omega = \int_{\delta S} f_i \, dx^i,$$

$$= \int_{\delta R} \frac{\partial x^i}{\partial u^\alpha} \, du^\alpha,$$

$$= \int \int_R \frac{\partial}{\partial u^\beta} (f_i \frac{\partial x^i}{\partial u^\alpha}) \, du^\beta \wedge du^\alpha,$$

$$= \int \int_R \left[ \frac{\partial f_i}{\partial x^k} \frac{\partial x^k}{\partial u^\beta} \frac{\partial x^i}{\partial u^\alpha} + f_i \frac{\partial^2 x^i}{\partial x^k \partial u^\alpha} \right] \, du^\beta \wedge du^\alpha,$$

$$= \int \int_R \left[ \frac{\partial f_i}{\partial x^k} \frac{\partial x^k}{\partial u^\beta} \frac{\partial x^i}{\partial u^\alpha} \right] \, du^\beta \wedge du^\alpha,$$

$$= \int \int_R \left[ \frac{\partial f_i}{\partial x^k} \frac{\partial x^k}{\partial u^\beta} \right] \, du^\beta \wedge \left[ \frac{\partial x^i}{\partial u^\alpha} \right] \, du^\alpha,$$

$$= \int \int_S f_i \frac{\partial x^i}{\partial x^k} \, dx^k \wedge dx^i = \int \int_S df_i \wedge dx^i.$$

We present a less intuitive but far more elegant proof. The idea is formally the same, namely, we pull-back to the plane by formula (2.75) apply Green’s theorem in the form given in equation (2.82) and then use the fact that the pull-back commutes with the differential as in theorem (2.71).

Let $\phi : R \subset \mathbb{R}^2 \to S$ denote the surface parametrization map. Assume that $\phi^{-1}(\delta S) = \delta(\phi^{-1} S)$, that is, the inverse of the boundary of $S$ is the boundary of the domain $R$. Then,
Closed and Exact forms

2.3.13 Example Let \( \alpha = M(x,y)dx + N(x,y)dy \), and suppose that \( d\alpha = 0 \). Then, by the previous example,

\[
d\alpha = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})
dx \wedge dy.
\]

Thus, \( d\alpha = 0 \) iff \( N_x = M_y \), which implies that \( N = f_y \) and \( M_x \) for some function \( f(x,y) \). Hence,

\[
\alpha = f_x \, dx + f_y \, dy = df.
\]

The reader should also be familiar with this example in the context of exact differential equations of first order and conservative force fields.

2.3.14 Definition A differential form \( \alpha \) is called **closed** if \( d\alpha = 0 \).

2.3.15 Definition A differential form \( \alpha \) is called **exact** if there exists a form \( \beta \) such that \( \alpha = d\beta \).

Since \( d \circ d = 0 \), it is clear that an exact form is also closed. For the converse to be true, one must require a topological condition that the space be contractible. A contractible space is one that can be deformed continuously to an interior point.

2.3.16 Poincaré Lemma. In a contractible space (such as \( \mathbb{R}^n \)), if a differential form is closed, then it is exact.

To prove this lemma we need much more machinery than we have available at this point. We present the proof in 7.1.2. The standard counterexample showing that the topological condition in the Poincaré Lemma is needed, is the form,

\[
\omega = \frac{-y \, dx + x \, dy}{x^2 + y^2}
\]

A short computation shows that \( d\omega = 0 \), so \( \omega \) is closed. Let \( \theta = \tan^{-1}(y/x) \) be the angle in polar coordinates. One can recognize that \( \omega = d\theta \), but this is only true in \( \mathbb{R}^2 - L \), where \( L \) is the non-negative \( x \)-axis, \( L = \{(x,0) \in \mathbb{R}^2 | x \geq 0\} \). If one computes the line integral from \((-1,0)\) to \((1,0)\) along the top half of the unit circle, the result is \( \pi \). But the line integral along the bottom half of the unit circle gives \(-\pi\). The integral is therefore not path independent, so \( \omega \neq d\theta \) on any region that contains the origin. If one tries to find another \( C^1 \) function \( f \) such that \( \omega = df \), one can easily show that \( f = \theta + \text{const} \), which is not possible along \( L \).
2.4 The Hodge $\ast$ Operator

2.4.1 Dual Forms

An important lesson students learn in linear algebra is that all vector spaces of finite dimension $n$ are isomorphic to each other. Thus, for instance, the space $P_3$ of all real polynomials in $x$ of degree 3, and the space $M_{2\times2}$ of real 2 by 2 matrices are, in terms of their vector space properties, basically no different from the Euclidean vector space $\mathbb{R}^4$. A good example of this is the tangent space $T_p\mathbb{R}^3$ which has dimension 3. The process of replacing $\frac{\partial}{\partial x}$ by $i$, $\frac{\partial}{\partial y}$ by $j$ and $\frac{\partial}{\partial z}$ by $k$ is a linear, 1-1 and onto map that sends the "vector" part of a tangent vector $a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$ to regular Euclidean vector $(a_1, a_2, a_3)$.

We have also observed that the tangent space $T_p\mathbb{R}^n$ is isomorphic to the cotangent space $T^*_p\mathbb{R}^n$. In this case, the vector space isomorphism maps the standard basis vectors $\{\frac{\partial}{\partial x}\}$ to their duals $\{dx\}$. This isomorphism then transforms a contravariant vector to a covariant vector. In terms of components, the isomorphism is provided by the Euclidean metric that maps the components of a contravariant vector with indices up to a covariant vector with indices down.

Another interesting example is provided by the spaces $\Lambda^1_p(\mathbb{R}^3)$ and $\Lambda^2_p(\mathbb{R}^3)$, both of which have dimension 3. It follows that these two spaces must be isomorphic. In this case the isomorphism is given as follows:

\[\begin{array}{ccc}
\text{dx} & \mapsto & dy \wedge dz \\
\text{dy} & \mapsto & -dx \wedge dz \\
\text{dz} & \mapsto & dx \wedge dy
\end{array}\]  

(2.84)

More generally, we have seen that the dimension of the space of k-forms in $\mathbb{R}^n$ is given by the binomial coefficient $\binom{n}{k}$. Since

\[\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!},\]

it must be true that

$\Lambda^k_p(\mathbb{R}^n) \cong \Lambda^{n-k}_p(\mathbb{R}^n)$.  

(2.85)

To describe the isomorphism between these two spaces, we introduce the following generalization of determinants,

2.4.1 Definition. Let $\phi: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. The unique constant $\det \phi$ such that,

$\phi^\ast: \Lambda^n(\mathbb{R}^n) \to \Lambda^n(\mathbb{R}^n)$

satisfies,

$\phi^\ast \omega = (\det \phi) \omega$,  

(2.86)

for all $n$-forms, is called the determinant of $\phi$. This is congruent with the standard linear algebra formula [2.44] since in a particular basis, the Jacobian of a linear map is the same as the matrix the represents the linear map in that basis. Let, $g(X, Y)$ be an inner product and $\{e_1, \ldots, e_n\}$ be an orthonormal basis with dual forms $\{\theta^1, \ldots, \theta^n\}$, then, the element of arc length is, the bilinear symmetric tensor

$ds^2 = g_{ij} \theta^i \otimes \theta^j$.

The metric then induces an $n$-form

$d\Omega = \theta^1 \wedge \theta^2 \ldots \wedge \theta^n$, 

CHAPTER 2. DIFFERENTIAL FORMS

called the Volume element. With this choice of form, the reader will recognize equation \ref{2.86} as the integrand in the change of variables theorem for multiple integration, as in example \ref{2.76}. More generally, if \( \{ f_1, \ldots, f_n \} \) is a positively oriented basis with dual basis \( \{ \phi^1, \ldots, \phi^n \} \), then,
\[
d\Omega = \sqrt{\det g} \phi^1 \wedge \ldots \wedge \phi^n.
\] (2.87)

### 2.4.2 Definition

Let \( g \) be the matrix representing the components of the metric in \( \mathbb{R}^n \). The Hodge \( \ast \) operator is the linear isomorphism \( \ast : \Lambda^p(\mathbb{R}^n) \rightarrow \Lambda^{n-k}(\mathbb{R}^n) \) defined in standard local coordinates by the equation,
\[
\ast (dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = \sqrt{\det g} \epsilon^{i_1, \ldots, i_k}_{i_{k+1}, \ldots, i_n} dx^{i_{k+1}} \wedge \ldots \wedge dx^{i_n},
\] (2.88)

For flat Euclidean space \( \sqrt{\det g} = 1 \), so the factor in the definition may appear superfluous. However, when we consider more general Riemannian manifolds, we will have to be more careful with raising and lowering indices with the metric, and take into account that the Levi-Civita symbol is not a tensor but something slightly more complicated called a tensor density. Including the \( \sqrt{\det g} \) is done in anticipation of this more general setting later. Since the forms \( dx^{i_1} \wedge \ldots \wedge dx^{i_k} \) constitute a basis of the vector space \( \Lambda^k(\mathbb{R}^n) \) and the \( \ast \) operator is assumed to be a linear map, equation \ref{2.88} completely specifies the map for all \( k \)-forms. In particular, if the components of a dual of a form are equal to the components of the form, the tensor is called self-dual. Of course, this can only happen if the tensor and its dual are of the same rank.

A metric \( g \) on \( \mathbb{R}^n \) induces an inner product on \( \Lambda^k(\mathbb{R}^n) \) as follows. Let \( \{ e_1, \ldots, e_n \} \) be an orthonormal basis with dual basis \( \theta^1, \ldots, \theta^n \). If \( \alpha, \beta \in \Lambda^k(\mathbb{R}^n) \), we can write
\[
\alpha = a_{i_1 \ldots i_k} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k},
\]
\[
\beta = b_{j_1 \ldots j_k} \theta^{j_1} \wedge \ldots \wedge \theta^{j_k}
\]
The induced inner product is defined by
\[
< \alpha, \beta >^{(k)} = \frac{1}{k!} a_{i_1 \ldots i_k} b^{i_1 \ldots i_k}.
\] (2.89)
If \( \alpha, \beta \in \Lambda^k(\mathbb{R}^n) \), then \( \ast \beta \in \Lambda^{n-k}(\mathbb{R}^n) \), so \( \alpha \wedge \ast \beta \) must be a multiple of the volume form. The Hodge \( \ast \) operator is the unique isomorphism such that
\[
\alpha \wedge \ast \beta = < \alpha, \beta >^{(k)} d\Omega.
\] (2.90)
Clearly,
\[
\alpha \wedge \ast \beta = \ast \star \alpha \wedge \beta
\]
When it is evident that the inner product is the induced inner product on \( \Lambda^k(\mathbb{R}^n) \) the indicator \( (k) \) is often suppressed. Some authors prefer to define the induced inner product of two \( k \)-forms by
\[
< \alpha, \beta > = \int (\alpha \wedge \ast \beta) \, d\Omega.
\] (2.91)
Then, if \( \alpha \) is a \( k \)-form and \( \beta \) is a \( (k-1) \)-form, one can define the adjoint or co-differential by
\[
< \delta \alpha, \beta > = < \alpha, d\beta >.
\] (2.92)
The adjoint is given by
\[
\delta = (-1)^{nk+n+1} \ast d \ast.
\] (2.93)
2.4. THE HODGE $\star$ OPERATOR

In particular,

$$\delta = \begin{cases} 
- \star d \star & \text{if } n \text{ is even} \\
(-1)^k \star d \star & \text{if } n \text{ is odd}
\end{cases} \quad (2.94)$$

The differential maps $(k-1)$-forms to $k$-forms, and the co-differential maps $k$-forms to $(k-1)$-forms. It is also the case that $\delta \circ \delta = 0$ The combination

$$\Delta = d \delta + \delta d \quad (2.95)$$

is called the flat metric Laplacian on forms. It maps $k$-forms to $k$-forms.

2.4.3 Example Hodge operator in $\mathbb{R}^2$

In $\mathbb{R}^2$,

$$\star dx = dy \quad \star dy = -dx,$$

or, if one thinks of a matrix representation of $\star : \Lambda(\mathbb{R}^2) \to \Lambda(\mathbb{R}^2)$ in standard basis, we can write the above as

$$\star \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

The reader might wish to peek at the symplectic matrix [5.50] in the discussion in chapter 5 on conformal mappings. Given functions $u = u(x,y)$ and $v = v(x,y)$, let $\omega = u \, dx - v \, dy$. Then,

$$d \omega = -(u_y + v_x) \, dx \wedge dy \quad \text{hence} \quad d \omega = 0 \Rightarrow u_y = -v_x$$

$$d \star \omega = (u_x - u_y) \, dx \wedge dy \quad \Rightarrow \quad d \star \omega = 0 \Rightarrow u_x = v_y. \quad (2.96)$$

Thus, the equations $d \omega = 0$ and $d \star \omega = 0$ are equivalent to the Cauchy-Riemann equations for a holomorphic function $f(z) = u(x,y) + iv(x,y)$. On the other hand,

$$du = u_x \, dx + u_y \, dy, \quad dv = v_x \, dx + v_y \, dy,$$

so the determinant of the Jacobian of the transformation $T : (x,y) \to (u,v)$, with the condition above on $\omega$, is given by,

$$|J| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x^2 + u_y^2 = v_x^2 + v_y^2.$$

If $|J| \neq 0$, we can set $u_x = R \cos \phi$, $u_y = R \sin \phi$, for some $R$ and some angle $\phi$. Then,

$$|J| = \begin{vmatrix} R & 0 \\ 0 & R \end{vmatrix} = \begin{vmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{vmatrix}.$$

Thus, the transformation is given by the composition of a dilation and a rotation. A more thorough discussion of this topic is found in the section on conformal maps in chapter 5.

2.4.4 Example Hodge operator in $\mathbb{R}^3$

$$\star dx^1 = \epsilon^1_{jk} dx^j \wedge dx^k$$

$$= \frac{1}{2!} \left[ \epsilon^1_{132} dx^2 \wedge dx^3 + \epsilon^1_{321} dx^3 \wedge dx^2 \right]$$

$$= \frac{1}{2!} \left[ dx^2 \wedge dx^3 - dx^3 \wedge dx^2 \right]$$

$$= \frac{1}{2!} \left[ dx^2 \wedge dx^3 + dx^2 \wedge dx^3 \right]$$

$$= dx^2 \wedge dx^3.$$
We leave it to the reader to complete the computation of the action of the \( \star \) operator on the other basis forms. The results are

\[
\begin{align*}
\star dx^1 &= +dx^2 \wedge dx^3 \\
\star dx^2 &= -dx^1 \wedge dx^3 \\
\star dx^3 &= +dx^1 \wedge dx^2, \\
\star(dx^2 \wedge dx^3) &= dx^1 \\
\star(-dx^3 \wedge dx^1) &= dx^2 \\
\star(dx^1 \wedge dx^2) &= dx^3,
\end{align*}
\]  

(2.97)

and

\[
\star(dx^1 \wedge dx^2 \wedge dx^3) = 1.
\]

(2.98)

In particular, if \( f : \mathbb{R}^3 \to \mathbb{R} \) is any 0-form (a function), then,

\[
\begin{align*}
\star f &= f(dx^1 \wedge dx^2 \wedge dx^3) \\
&= f dV,
\end{align*}
\]

(2.100)

where \( dV \) is the volume form.

2.4.5 Example Let \( \alpha = a_1 dx^1 a_2 dx^2 + a_3 dx^3 \), and \( \beta = b_1 dx^1 b_2 dx^2 + b_3 dx^3 \). Then,

\[
\begin{align*}
\star(\alpha \wedge \beta) &= (a_2 b_3 - a_3 b_2) \star(dx^2 \wedge dx^3) + (a_1 b_3 - a_3 b_1) \star(dx^1 \wedge dx^3) + \\
&\quad (a_1 b_2 - a_2 b_1) \star(dx^1 \wedge dx^2) \\
&= (a_2 b_3 - a_3 b_2) dx^1 + (a_1 b_3 - a_3 b_1) dx^2 + (a_1 b_2 - a_2 b_1) dx^3 \\
&= (\vec{a} \times \vec{b})_i dx^i,
\end{align*}
\]

(2.101)

The previous examples provide some insight on the action of the \( \land \) and \( \star \) operators. If one thinks of the quantities \( dx^1, dx^2 \) and \( dx^3 \) as analogous to \( \vec{i}, \vec{j} \) and \( \vec{k} \), then it should be apparent that equations (2.97) are the differential geometry versions of the well known relations

\[
\begin{align*}
i &= j \times k \\
j &= -i \times k \\
k &= i \times j.
\end{align*}
\]

2.4.6 Example In Minkowski space the collection of all 2-forms has dimension \( \binom{4}{2} = 6 \). The Hodge \( \star \) operator in this case splits \( \wedge_2(M_{1,3}) \) into two 3-dim subspaces \( \wedge_+^2 \), such that \( \star : \wedge_+^2 \to \wedge_+^2 \). More specifically, \( \wedge_+^2 \) is spanned by the forms \( \{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3\} \), and \( \wedge_-^2 \) is spanned by the forms \( \{dx^2 \wedge dx^3, -dx^1 \wedge dx^3, dx^1 \wedge dx^2\} \). The action of \( \star \) on \( \wedge_+^2 \) is

\[
\begin{align*}
\star(dx^0 \wedge dx^1) &= \frac{1}{2} \epsilon_{01}^{kl} dx^k \wedge dx^l = -dx^2 \wedge dx^3 \\
\star(dx^0 \wedge dx^2) &= \frac{1}{2} \epsilon_{02}^{kl} dx^k \wedge dx^l = +dx^1 \wedge dx^3 \\
\star(dx^0 \wedge dx^3) &= \frac{1}{2} \epsilon_{03}^{kl} dx^k \wedge dx^l = -dx^1 \wedge dx^2,
\end{align*}
\]

and on \( \wedge_-^2 \),

\[
\begin{align*}
\star(+dx^2 \wedge dx^3) &= \frac{1}{2} \epsilon_{23}^{kl} dx^k \wedge dx^l = dx^0 \wedge dx^1 \\
\star(-dx^1 \wedge dx^3) &= \frac{1}{2} \epsilon_{13}^{kl} dx^k \wedge dx^l = dx^0 \wedge dx^2 \\
\star(+dx^1 \wedge dx^2) &= \frac{1}{2} \epsilon_{12}^{kl} dx^k \wedge dx^l = dx^0 \wedge dx^3.
\end{align*}
\]
In verifying the equations above, we recall that the Levi-Civita symbols that contain an index with value 0 in the up position have an extra minus sign as a result of raising the index with \( \eta^{00} \). If \( F \in \Lambda^2(M) \), we will formally write \( F = F_+ + F_- \), where \( F_\pm \in \Lambda^2_\pm \). We would like to note that the action of the dual operator on \( \Lambda^2(M) \) is such that \( \star : \Lambda^2(M) \to \Lambda^2(M) \), and \( \star^2 = -1 \). In a vector space a map like \( \star \) with the property \( \star^2 = -1 \) is a linear involution of the space. In the case in question \( \Lambda^2_\pm \) are the eigenspaces corresponding to the +1 and -1 eigenvalues of this involution.

It is also worthwhile to calculate the duals of 1-forms in \( M_1, 3 \). The results are,

\[
\star dt = -dx^1 \wedge dx^2 \wedge dx^3 \\
\star dx^1 = +dx^2 \wedge dt \wedge dx^3 \\
\star dx^2 = +dt \wedge dx^1 \wedge dx^3 \\
\star dx^3 = +dx^1 \wedge dt \wedge dx^2.
\] (2.102)

### 2.4.2 Laplacian

Classical differential operators that enter in Green’s and Stokes’ Theorems are better understood as special manifestations of the exterior differential and the Hodge \( \star \) operators in \( \mathbb{R}^3 \). Here is precisely how this works:

1. Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be a \( C^\infty \) function. Then

\[
df = \frac{\partial f}{\partial x^i} dx^i = \nabla f \cdot dx
\] (2.103)

2. Let \( \alpha = A_i dx^i \) be a 1-form in \( \mathbb{R}^3 \). Then

\[
(*d)\alpha = \frac{1}{2} \left( \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right) \star (dx^i \wedge dx^j) \\
= (\nabla \times A) \cdot dx
\] (2.104)

3. Let \( \alpha = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \) be a 2-form in \( \mathbb{R}^3 \). Then

\[
d\alpha = \left( \frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3 \\
= (\nabla \cdot B) dV
\] (2.105)

4. Let \( \alpha = B_i dx^i \), then

\[
(*d*) \alpha = \nabla \cdot B
\] (2.106)

5. Let \( f \) be a real valued function. Then the Laplacian is given by:

\[
(*d*) df = \nabla \cdot \nabla f = \nabla^2 f
\] (2.107)

The Laplacian definition here is consistent with [2.95] because in the case of a function \( f \), that is, a 0-form, \( \delta f = 0 \) so \( \Delta f = \delta df \). The results above can be summarized in terms of short exact sequence called the De Rham Complex as shown in figure [2.4]. The sequence is called exact because successive application of the differential operator gives zero. That is, \( d \circ d = 0 \). Since there are no 4-forms in \( \mathbb{R}^3 \), the sequence terminates as shown. If one starts with a function in \( \Lambda^0(\mathbb{R}^3) \), then \( (d \circ d)f = 0 \) just says that \( \nabla \times \nabla f = 0 \) as in the case of conservative vector fields. If instead, one starts with a one form \( \alpha \) in \( \Lambda^1(\mathbb{R}^3) \), corresponding to a vector field \( \mathbf{A} \), then \( (d \circ d)f = 0 \) says that \( \nabla \cdot (\nabla \times \mathbf{A}) = 0 \) as in the case of incompressible vector fields. If one starts with a function,
but instead of applying the differential twice consecutively, one "hops" in between with the Hodge operator, the result is the Laplacian of the function.

If denote by $R$ a simply connected closed region in Euclidean space whose boundary is $\delta R$, then in terms of forms, the the fundamental theorem of Calculus, Stoke’s Theorem (See ref 2.83) and the Divergence theorem in $\mathbb{R}^3$ can be expressed by a single generalized Stokes theorem.

$$
\int_{\delta R} \omega = \int \int_R d\omega.
$$

(2.108)

We find it irresistible to point out that if one defines a complex one-form,

$$
\omega = f(z) \, dz,
$$

(2.109)

where $f(z) = u(x,y) + iv(x,y)$, and where one assumes that $u,v$ are differentiable with continuous derivatives, then the conditions introduced in equation 2.96 are equivalent to requiring that $d\omega = 0$. In other words, if the form is closed, then $u$ and $v$ satisfy the Cauchy-Riemann equations. Stokes theorem then tells us that in a contractible region with boundary $C$, the line integral

$$
\int_C \omega = \int_C f(z) \, dz = 0.
$$

The is Cauchy’s integral Theorem. We should also point point out the tantalizing appearance of equations 2.96 to Maxwell’s equations in the section that follows.

### 2.4.3 Maxwell Equations

The classical equations of Maxwell describing electromagnetic phenomena are

$$
\nabla \cdot E = 4\pi \rho \\
\nabla \times B = 4\pi J + \frac{\partial E}{\partial t} \\
\nabla \cdot B = 0 \\
\nabla \times E = -\frac{\partial B}{\partial t}
$$

(2.110)

We would like to formulate these equations in the language of differential forms. Let $x^\mu = (t, x^1, x^2, x^3)$ be local coordinates in Minkowski’s space $M_{1,3}$. Define the Maxwell 2-form $F$ by the equation

$$
F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3),
$$

(2.111)

where

$$
F_{\mu\nu} = \begin{bmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & B_z & -B_y \\
E_y & -B_z & 0 & B_x \\
E_z & B_y & -B_x & 0
\end{bmatrix}.
$$

(2.112)
Written in complete detail, Maxwell’s 2-form is given by
\[ F = -E_x dt \wedge dx^1 - E_y dt \wedge dx^2 - E_z dt \wedge dx^3 + 
B_z dx^1 \wedge dx^2 - B_y dx^1 \wedge dx^3 + B_x dx^2 \wedge dx^3. \] (2.113)

We also define the source current 1-form
\[ J = J_\mu dx^\mu = \rho dt + J_1 dx^1 + J_2 dx^2 + J_3 dx^3. \] (2.114)

2.4.7 Proposition Maxwell’s Equations (2.110) are equivalent to the equations
\[ dF = 0, \quad d \ast F = 4\pi \ast J. \] (2.115)

Proof: The proof is by direct computation using the definitions of the exterior derivative and the Hodge-\ast operator.

\[
\begin{align*}
dF &= \left( \frac{\partial E_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3 + \\
&\quad \left( \frac{\partial E_y}{\partial x^3} - \frac{\partial B_x}{\partial x^2} \right) dx^2 \wedge dt \wedge dx^3 + \\
&\quad \left( \frac{\partial E_z}{\partial x^1} - \frac{\partial B_y}{\partial x^2} \right) dt \wedge dx^1 \wedge dx^3 + \\
&\quad \left( \frac{\partial E_z}{\partial x^2} - \frac{\partial B_z}{\partial x^1} \right) dx^1 \wedge dt \wedge dx^2.
\end{align*}
\]

Collecting terms and using the antisymmetry of the wedge operator, we get
\[
\begin{align*}
dF &= \left( \frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3 + \\
&\quad \left( \frac{\partial E_y}{\partial x^3} - \frac{\partial B_x}{\partial x^2} \right) dx^2 \wedge dt \wedge dx^3 + \\
&\quad \left( \frac{\partial E_z}{\partial x^1} - \frac{\partial B_y}{\partial x^2} \right) dt \wedge dx^1 \wedge dx^3 + \\
&\quad \left( \frac{\partial E_z}{\partial x^2} - \frac{\partial B_z}{\partial x^1} \right) dx^1 \wedge dt \wedge dx^2.
\end{align*}
\]

Therefore, \( dF = 0 \) iff
\[
\begin{align*}
\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} &= 0, \\
\frac{\partial E_y}{\partial x^3} - \frac{\partial B_x}{\partial x^2} &= 0, \\
\frac{\partial E_z}{\partial x^1} - \frac{\partial B_y}{\partial x^2} &= 0, \\
\frac{\partial E_z}{\partial x^2} - \frac{\partial B_z}{\partial x^1} &= 0,
\end{align*}
\]
which is the same as
\[ \nabla \cdot \mathbf{B} = 0, \]
and
\[
\begin{align*}
\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} &= 0, \\
\frac{\partial E_y}{\partial x^3} - \frac{\partial B_x}{\partial x^2} &= 0, \\
\frac{\partial E_z}{\partial x^1} - \frac{\partial B_y}{\partial x^2} &= 0, \\
\frac{\partial E_z}{\partial x^2} - \frac{\partial B_z}{\partial x^1} &= 0,
\end{align*}
\]
which means that
\[-\nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0.\] (2.116)

To verify the second set of Maxwell equations, we first compute the dual of the current density 1-form (2.114) using the results from example 2.4.1. We get
\[\ast \mathbf{J} = \left[ -\rho dx^1 \wedge dx^2 \wedge dx^3 + J_1 dx^2 \wedge dt \wedge dx^3 + J_2 dt \wedge dx^1 \wedge dx^3 + J_3 dx^1 \wedge dt \wedge dx^2 \right].\] (2.117)

We could now proceed to compute \(d \ast \mathbf{F}\), but perhaps it is more elegant to notice that \(\mathbf{F} \in \wedge^2(M)\), and so, according to example (2.4.1), \(\mathbf{F}\) splits into \(\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_-\). In fact, we see from (2.112) that the components of \(\mathbf{F}_+\) are those of \(-\mathbf{E}\) and the components of \(\mathbf{F}_-\) constitute the magnetic field vector \(\mathbf{B}\). Using the results of example (2.4.1), we can immediately write the components of \(\ast \mathbf{F}\):
\[\ast \mathbf{F} = \left[ \begin{array}{cccc} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{array} \right].\] (2.118)

Effectively, the dual operator amounts to exchanging \(\mathbf{E} \mapsto -\mathbf{B}\) and \(\mathbf{B} \mapsto +\mathbf{E}\), in the left hand side of the first set of Maxwell equations. We infer from equations (2.116) and (2.117) that
\[\nabla \cdot \mathbf{E} = 4\pi \rho\]
and
\[\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{J}.\]

Most standard electrodynamic textbooks carry out the computation entirely in tensor components. To connect with this approach, we should mention that it \(\mathbf{F}^{\mu \nu}\) represents the electromagnetic tensor, then the dual tensor is
\[\mathbf{F}^{\ast \mu \nu} = \frac{\sqrt{\det g}}{2} \epsilon^{\mu \nu \sigma \tau} \mathbf{F}_{\sigma \tau}.\] (2.120)

Since \(d\mathbf{F} = 0\), in a contractible region there exists a one form \(\mathbf{A}\) such that \(\mathbf{F} = d\mathbf{A}\). The form \(\mathbf{A}\) is called the 4-vector potential. The components of \(\mathbf{A}\) are,
\[\mathbf{A} = A_\mu \, dx^\mu, \quad A_\mu = (\phi, \mathbf{A})\] (2.121)
where \(\phi\) is the electric potential and \(\mathbf{A}\) the magnetic vector potential. The components of the electromagnetic tensor \(\mathbf{F}\) are given by
\[F^{\mu \nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu},\] (2.122)
and the classical electromagnetic Lagrangian is
\[L_{EM} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + J^\mu A_\mu.\] (2.123)
The corresponding Euler-Lagrange equations are

\[ \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial A_\mu} \right) - \frac{\partial L}{\partial A_\mu} = 0 \]  

(2.124)

To carry out the computation we first use the Minkowski to write the Lagrangian with the indices down. The key is to keep in mind that \( A_{\mu,\nu} \) are treated as independent variables, so the derivatives of \( A_{\alpha,\beta} \) vanish unless \( \mu = \alpha \) and \( \nu = \beta \). We get,

\[ \frac{\partial L}{\partial (A_{\mu,\nu})} = -\frac{1}{4} \frac{\partial L}{\partial (A_{\mu,\nu})} (F_{\alpha\beta} F^{\alpha\beta}), \]

\[ = -\frac{1}{4} \frac{\partial L}{\partial (A_{\mu,\nu})} (F_{\alpha\beta} F_{\lambda\sigma} \eta^{\alpha\lambda} \eta^{\beta\sigma}), \]

\[ = \frac{1}{4} \eta^{\alpha\lambda} \eta^{\beta\sigma} [F_{\alpha\beta}(\delta_{\lambda}^{\mu} \delta_{\sigma}^{\nu} - \delta_{\sigma}^{\mu} \delta_{\lambda}^{\nu}) + F_{\lambda\sigma}(\delta_{\alpha}^{\mu} \delta_{\beta}^{\nu} - \delta_{\beta}^{\mu} \delta_{\alpha}^{\nu}) \]

\[ = -\frac{1}{4} [F_{\mu\nu} + F_{\mu\nu} - F_{\nu\mu} - F_{\nu\mu}], \]

\[ = -F_{\mu\nu}. \]

On the other hand,

\[ \frac{\partial L}{\partial A_\mu} = J_\nu. \]

Therefore, the field equations are

\[ \frac{\partial}{\partial x^\mu} F^{\mu\nu} = J_\mu. \]  

(2.125)

In the gauge theory formulation of classical electrodynamics the invariant expression for the Lagrangian is the square of the norm of the field \( F \) under the induced inner product

\[ < F, F > = -\int (F \wedge *F) \, d\Omega. \]  

(2.126)

This the starting point to generalize to non-Abelian gauge theories.
Chapter 3

Connections

3.1 Frames

This chapter is dedicated to professor Arthur Fischer. In my second year as an undergraduate at Berkeley, I took the undergraduate course in differential geometry which to this day is still called Math 140. The driving force in my career was trying to understand the general theory of relativity, which was only available at the graduate level. However, the graduate course (Math 280 at the time) read that the only prerequisite was Math 140. So I got emboldened and enrolled in the course taught that year by Dr. Fischer. The required book for the course was the classic by Adler, Bazin, Schiffer. I loved the book; it was clearly within my reach and I began to devour the pages with great satisfaction that I was getting a grasp of the mathematics and the physics. On the other hand, I was completely lost in the course. It seemed as if had nothing to do with the material I was learning on my own. Around the third week of classes, Dr. Fischer went through a computation with these mysterious operators, and upon finishing the computation he said if we were following, he had just derived the formula for the Christoffel symbols. Clearly I was not following, they looked nothing like the Christoffel symbols I had learned from the book. So, with great embarrassment I went to his office and explained my predicament. He smiled, apologized when he did not need to, and invited me to 1-1 sessions for the rest of the two-semester course. That is how I got through the book he was really using, namely Abraham-Marsden. I am forever grateful.

As we have already noted in Chapter 1, the theory of curves in $\mathbb{R}^3$ can be elegantly formulated by introducing orthonormal triplets of vectors which we called Frenet frames. The Frenet vectors are adapted to the curves in such a manner that the rate of change of the frame gives information about the curvature of the curve. In this chapter we will study the properties of arbitrary frames and their corresponding rates of change in the direction of the various vectors in the frame. These concepts will then be applied later to special frames adapted to surfaces.

3.1.1 Definition A coordinate frame in $\mathbb{R}^n$ is an n-tuple of vector fields $\{e_1, \ldots, e_n\}$ which are linearly independent at each point $p$ in the space.

In local coordinates $\{x^1, \ldots, x^n\}$, we can always express the frame vectors as linear combinations of the standard basis vectors

$$ e_i = \sum_{j=1}^{n} A^j_i \frac{\partial}{\partial x^j} = \partial_j A^j_i, \quad (3.1) $$

where $\partial_i = \frac{\partial}{\partial x^j}$. Placing the basis vectors $\partial_j$ on the left is done to be consistent with the summation convention, keeping in mind that the differential operators do not act on the matrix elements. We assume the matrix $A = (A^j_i)$ to be nonsingular at each point. In linear algebra, this concept is referred to as a change of basis, the difference being that in our case, the transformation matrix $A$
depends on the position. A frame field is called \textbf{orthonormal} if at each point,

\[
< e_i, e_j > = \delta_{ij}. \quad (3.2)
\]

Throughout this chapter, we will assume that all frame fields are orthonormal. Whereas this restriction is not necessary, it is convenient because it results in considerable simplification in computing the components of an arbitrary vector in the frame.

\textbf{3.1.2 Proposition} If \( \{e_1, \ldots, e_n\} \) is an orthonormal frame, then the transformation matrix is orthogonal (ie, \( AA^T = I \)).

\textbf{Proof:} The proof is by direct computation. Let \( e_i = \partial_j A^j_i \). Then

\[
\delta_{ij} = < e_i, e_j > = < \partial_j A^k_i, \partial_l A^l_j > = A^k_i A^l_j < \partial_j, \partial_l > = A^k_i A^l_j \delta_{kl} = A^k_i \delta_{kj} = A^k_i (A^T)_{jk}.
\]

Hence

\[
(A^T)_{jk} A^k_i = \delta_{ij}
\]

\[
(A^T)^i_k A^k_i = \delta^i_j
\]

\[
A^T A = I
\]

Given a frame vectors \( e_i \), we can also introduce the corresponding coframe forms \( \theta^i \) by requiring that

\[
\theta^i(e_j) = \delta^i_j. \quad (3.3)
\]

Since the dual coframe is a set of 1-forms, they can also be expressed in local coordinates as linear combinations

\[
\theta^i = B^i_k dx^k.
\]

It follows from equation (3.3), that

\[
\theta^i(e_j) = B^i_k dx^k(\partial_j A^l_j) = B^i_k A^l_j dx^k(\partial_l) = B^i_k A^l_j \delta^k_l = B^i_k A^k_j.
\]

\[
\delta^i_j = B^i_k A^k_j.
\]

Therefore, we conclude that \( BA = I \), so \( B = A^{-1} = A^T \). In other words, when the frames are orthonormal, we have

\[
e_i = \partial_k A^k_i \quad \theta^i = A^i_k dx^k. \quad (3.4)
\]
3.1. FRAMES

3.1.3 Example Consider the transformation from Cartesian to cylindrical coordinates:

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. \]  \hspace{1cm} (3.5)

Using the chain rule for partial derivatives, we have

\[ \frac{\partial}{\partial r} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \]
\[ \frac{\partial}{\partial \theta} = -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y} \]
\[ \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \]

The vectors \( \frac{\partial}{\partial r} \) and \( \frac{\partial}{\partial z} \) are clearly unit vectors. To make the vector \( \frac{\partial}{\partial \theta} \) a unit vector, it suffices to divide it by its length \( r \). We can then compute the dot products of each pair of vectors and easily verify that the quantities

\[ e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad e_3 = \frac{\partial}{\partial z}, \]  \hspace{1cm} (3.6)

are a triplet of mutually orthogonal unit vectors and thus constitute an orthonormal frame. The surfaces with constant value for the coordinates \( r, \theta \) and \( z \) respectively, represent a set of mutually orthogonal surfaces at each point. The frame vectors at a point are normal to these surfaces as shown in figure 3.1. Physicists often refer to these frame vectors as \( \{ \hat{r}, \hat{\theta}, \hat{z} \} \), or as \( \{ e_r, e_\theta, e_z \} \).

3.1.4 Example For spherical coordinates (2.31)

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta, \]

the chain rule leads to

\[ \frac{\partial}{\partial r} = \sin \theta \cos \phi \frac{\partial}{\partial x} + \sin \theta \sin \phi \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial z}, \]
\[ \frac{\partial}{\partial \theta} = r \cos \theta \cos \phi \frac{\partial}{\partial x} + r \cos \theta \sin \phi \frac{\partial}{\partial y} - r \sin \theta \frac{\partial}{\partial z}, \]
\[ \frac{\partial}{\partial \phi} = -r \sin \theta \sin \phi \frac{\partial}{\partial x} + r \sin \theta \cos \phi \frac{\partial}{\partial y}. \]

The vector \( \frac{\partial}{\partial r} \) is of unit length but the other two need to be normalised. As before, all we need to do is divide the vectors by their magnitude. For \( \frac{\partial}{\partial \theta} \) we divide by \( r \) and for \( \frac{\partial}{\partial \phi} \), we divide by \( r \sin \theta \). Taking the dot products of all pairs and using basic trigonometric identities, one can verify that we again obtain an orthonormal frame.

\[ e_1 = e_r = \frac{\partial}{\partial r}, \quad e_2 = e_\theta = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad e_3 = e_\phi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}, \]  \hspace{1cm} (3.7)

Furthermore, the frame vectors are normal to triply orthogonal surfaces, which in this case are spheres, cones and planes, as shown in figure 3.2. The fact that the chain rule in the two situations above leads to orthonormal frames is not coincidental. The results are related to the orthogonality of the level surfaces \( x^i = \text{constant} \). Since the level surfaces are orthogonal whenever they intersect, one expects the gradients of the surfaces to also be orthogonal. Transformations of this type are called triply orthogonal systems.
3.2 Curvilinear Coordinates

Orthogonal transformations such as spherical and cylindrical coordinates appear ubiquitously in mathematical physics because the geometry of a large number of problems in this discipline exhibit symmetry with respect to an axis or to the origin. In such situations, transformations to the appropriate coordinate system often result in considerable simplification of the field equations involved in the problem. It has been shown that the Laplace operator that appears in potential, heat, and wave equations, and in Schrödinger field equations is separable in twelve orthogonal coordinate systems. A simple and efficient method to calculate the Laplacian in orthogonal coordinates can be implemented using differential forms.

3.2.1 Example In spherical coordinates the differential of arc length is given by (see equation 2.32) the metric:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$  

Let

$$\begin{align*}
\theta^1 &= dr \\
\theta^2 &= r d\theta \\
\theta^3 &= r \sin \theta d\phi.
\end{align*}$$

Note that these three 1-forms constitute the dual coframe to the orthonormal frame derived in equation (3.7). Consider a scalar field $f = f(r, \theta, \phi)$. We now calculate the Laplacian of $f$ in spherical coordinates using the methods of section 2.4.2. To do this, we first compute the differential $df$ and express the result in terms of the coframe.

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi$$

The components $df$ in the coframe represent the gradient in spherical coordinates. Continuing with the scheme of section 2.4.2, we next apply the Hodge-$\star$ operator. Then, we rewrite the resulting 2-form in terms of wedge products of coordinate differentials so that we can apply the definition of the exterior derivative.

$$\star df = \frac{\partial f}{\partial r} \theta^2 \wedge \theta^3 - \frac{1}{r} \frac{\partial f}{\partial \theta} \theta^1 \wedge \theta^3 + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \theta^1 \wedge \theta^2$$

$$= r^2 \sin \theta \frac{\partial f}{\partial r} d\theta \wedge d\phi - r \sin \theta \frac{1}{r} \frac{\partial f}{\partial \theta} dr \wedge d\phi + r \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} dr \wedge d\theta$$

$$= r^2 \sin \theta \frac{\partial f}{\partial r} d\theta \wedge d\phi - \sin \theta \frac{\partial f}{\partial \theta} dr \wedge d\phi + \frac{1}{\sin \theta} \frac{\partial f}{\partial \phi} dr \wedge d\theta$$

$$d \star df = \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial f}{\partial r} \right) dr \wedge d\theta \wedge d\phi - \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) d\theta \wedge dr \wedge d\phi + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left( \frac{\partial f}{\partial \phi} \right) d\phi \wedge dr \wedge d\theta$$

$$= \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \phi^2} \right] dr \wedge d\theta \wedge d\phi.$$}

Finally, rewriting the differentials back in terms of the the coframe, we get

$$d \star df = \frac{1}{r^2 \sin \theta} \left[ \sin \theta \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \phi^2} \right] \theta^1 \wedge \theta^2 \wedge \theta^3.$$
3.2. CURVILINEAR COORDINATES

So, the Laplacian of $f$ is given by

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial f}{\partial r} \right] + \frac{1}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial f}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \right]$$  \hspace{1cm} (3.9)

The derivation of the expression for the spherical Laplacian through the use of differential forms is elegant and leads naturally to the operator in Sturm-Liouville form.

The process above can be carried out for general orthogonal transformations. A change of coordinates $x^i = x^i(u^k)$ leads to an orthogonal transformation if in the new coordinate system $u^k$, the line metric

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2$$  \hspace{1cm} (3.10)

only has diagonal entries. In this case, we choose the coframe

$$\theta^1 = \sqrt{g_{11}} du^1 = h_1 du^1$$
$$\theta^2 = \sqrt{g_{22}} du^2 = h_2 du^2$$
$$\theta^3 = \sqrt{g_{33}} du^3 = h_3 du^3$$

The quantities $\{h_1, h_2, h_3\}$ are classically called the weights. Please note that, in the interest of connecting to classical terminology, we have exchanged two indices for one and this will cause small discrepancies with the index summation convention. We will revert to using a summation symbol when these discrepancies occur. To satisfy the duality condition $\theta^i(e_j) = \delta^i_j$, we must choose the corresponding frame vectors $e_i$ as follows:

$$e_1 = \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial u^1} = \frac{1}{h_1} \frac{\partial}{\partial u^1}$$
$$e_2 = \frac{1}{\sqrt{g_{22}}} \frac{\partial}{\partial u^2} = \frac{1}{h_2} \frac{\partial}{\partial u^2}$$
$$e_3 = \frac{1}{\sqrt{g_{33}}} \frac{\partial}{\partial u^3} = \frac{1}{h_3} \frac{\partial}{\partial u^3}$$

**Gradient.** Let $f = f(x^i)$ and $x^i = x^i(u^k)$. Then

$$df = \frac{\partial f}{\partial x^k} dx^k$$
$$= \frac{\partial f}{\partial u^i} \frac{\partial u^i}{\partial x^k} dx^k$$
$$= \frac{\partial f}{\partial u^i} du^i$$
$$= \sum_i \frac{1}{h_i} \frac{\partial f}{\partial u^i} \theta^i$$
$$= e_i(f) \theta^i.$$  \hspace{1cm} (3.11)

As expected, the components of the gradient in the coframe $\theta^i$ are the just the frame vectors.

**Curl.** Let $F = (F_1, F_2, F_3)$ be a classical vector field. Construct the corresponding 1-form $F = F_i \theta^i$. 

\begin{align*}
\nabla &= \left( \frac{1}{h_1} \frac{\partial}{\partial u^1}, \frac{1}{h_2} \frac{\partial}{\partial u^2}, \frac{1}{h_3} \frac{\partial}{\partial u^3} \right) \quad (3.11)
\end{align*}
in the coframe. We calculate the curl using the dual of the exterior derivative.

\[
F = F_1 \theta^1 + F_2 \theta^2 + F_3 \theta^3 \\
= (h_1 F_1) du^1 + (h_2 F_2) du^2 + (h_3 F_3) du^3 \\
= (hF)_i du^i, \text{ where } (hF)_i = h_i F_i \\
dF = \frac{1}{2} \left[ \frac{\partial (hF}_i}{\partial u^j} - \frac{\partial (hF}_j}{\partial u^i} \right] du^i \wedge du^j \\
= \frac{1}{h_i h_j} \left[ \frac{\partial (hF}_i}{\partial u^j} - \frac{\partial (hF}_j}{\partial u^i} \right] \theta^i \wedge \theta^j \\
\ast dF = \epsilon^{ij} \left[ \frac{1}{h_i h_j} \left( \frac{\partial (hF}_i}{\partial u^j} - \frac{\partial (hF}_j}{\partial u^i} \right) \right] \theta^k = (\nabla \times F)_k \theta^k.
\]

Thus, the components of the curl are

\[
\left( \frac{1}{h_2 h_3} \left( \frac{\partial (h_1 F_3)}{\partial u^2} - \frac{\partial (h_2 F_3)}{\partial u^3} \right), \frac{1}{h_1 h_3} \left( \frac{\partial (h_1 F_2)}{\partial u^1} - \frac{\partial (h_2 F_1)}{\partial u^3} \right), \frac{1}{h_1 h_2} \left( \frac{\partial (h_1 F_1)}{\partial u^2} - \frac{\partial (h_2 F_2)}{\partial u^1} \right) \right) \quad (3.12)
\]

**Divergence.** As before, let \( F = F_i \theta^i \) and recall that \( \nabla \cdot F = \ast d \ast F \). The computation yields

\[
F = F_1 \theta^1 + F_2 \theta^2 + F_3 \theta^3 \\
\ast F = F_1 \theta^2 \wedge \theta^3 + F_2 \theta^3 \wedge \theta^1 + F_3 \theta^1 \wedge \theta^2 \\
= (h_2 h_3 F_1) du^2 \wedge du^3 + (h_1 h_3 F_2) du^3 \wedge du^1 + (h_1 h_2 F_3) du^1 \wedge du^2 \\
d \ast d F = \left[ \frac{\partial (h_2 h_3 F_1)}{\partial u^1} + \frac{\partial (h_1 h_3 F_2)}{\partial u^2} + \frac{\partial (h_1 h_2 F_3)}{\partial u^3} \right] du^1 \wedge du^2 \wedge du^3.
\]

Therefore,

\[
\nabla \cdot F = \ast d \ast F = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 h_3 F_1)}{\partial u^1} + \frac{\partial (h_1 h_3 F_2)}{\partial u^2} + \frac{\partial (h_1 h_2 F_3)}{\partial u^3} \right]. \quad (3.13)
\]

### 3.3 Covariant Derivative

In this section we introduce a generalization of directional derivatives. The directional derivative measures the rate of change of a function in the direction of a vector. We seek a quantity which measures the rate of change of a vector field in the direction of another.

**3.3.1 Definition** Given a pair \((X, Y)\) of arbitrary vector field in \( \mathbb{R}^n \), we associate a new vector field \( \nabla_X Y \), so that \( \nabla_X : \mathcal{X}(\mathbb{R}^n) \rightarrow \mathcal{X}(\mathbb{R}^n) \). The quantity \( \nabla \) called a **Koszul connection** if it satisfies the following properties:

1. \( \nabla_f X(Y) = f \nabla_X Y \),
2. \( \nabla_{(X_1 + X_2)} Y = \nabla_{X_1} Y + \nabla_{X_2} Y \),
3. \( \nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2 \),
4. \( \nabla_X f Y = X(f)Y + f \nabla_X Y \),

for all vector fields \( X, X_1, X_2, Y, Y_1, Y_2 \in \mathcal{X}(\mathbb{R}^n) \) and all smooth functions \( f \). Implicit in the properties, we set \( \nabla_X f = X(f) \). The definition states that the map \( \nabla_X \) is linear on \( X \) but behaves as a linear derivation on \( Y \). For this reason, the quantity \( \nabla_X Y \) is called the **covariant derivative** of \( Y \) in the direction of \( X \).
3.3.2 Proposition Let \( Y = f^i \frac{\partial}{\partial x^i} \) be a vector field in \( \mathbb{R}^n \), and let \( X \) another \( C^\infty \) vector field. Then the operator given by
\[
\nabla_X Y = X(f^i) \frac{\partial}{\partial x^i}
\] (3.14)
defines a Koszul connection.

Proof: The proof just requires verification that the four properties above are satisfied, and it is left as an exercise.

The operator defined in this proposition is called the standard connection compatible with the standard Euclidean metric. The action of this connection on a vector field \( Y \) yields a new vector field whose components are the directional derivatives of the components of \( Y \).

3.3.3 Example Let
\[
X = x \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y}, 
Y = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}.
\]
Then,
\[
\nabla_X Y = 2x^2 \frac{\partial}{\partial x} + (xy^2 + 2x^2yz) \frac{\partial}{\partial y}.
\]

3.3.4 Definition A Koszul connection \( \nabla_X \) is compatible with the metric \( g(Y, Z) = \langle Y, Z \rangle \) if
\[
\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.
\] (3.15)

if \( F: \mathbb{R}^n \to \mathbb{R}^n \) is an isometry so that \( \langle F_*X, F_*Y \rangle = \langle X, Y \rangle \), then it is connection preserving in the sense
\[
F_*(\nabla_X Y) = \nabla_{F_*X} F_*y.
\] (3.16)

In Euclidean space, the components of the standard frame vectors are constant, and thus their rates of change in any direction vanish. Let \( e_i \) be arbitrary frame field with dual forms \( \theta^i \). The covariant derivatives of the frame vectors in the directions of a vector \( X \) will in general yield new vectors. The new vectors must be linear combinations of the the basis vectors as follows:
\[
\nabla_X e_1 = \omega^1_1(X)e_1 + \omega^2_1(X)e_2 + \omega^3_1(X)e_3 \\
\nabla_X e_2 = \omega^1_2(X)e_1 + \omega^2_2(X)e_2 + \omega^3_2(X)e_3 \\
\nabla_X e_3 = \omega^1_3(X)e_1 + \omega^2_3(X)e_2 + \omega^3_3(X)e_3
\] (3.17)

The coefficients can be more succinctly expressed using the compact index notation,
\[
\nabla_X e_i = e_j \omega^i_j(X).
\] (3.18)

It follows immediately that
\[
\omega^i_j(X) = \theta^j(\nabla_X e_i).
\] (3.19)

Equivalently, one can take the inner product of both sides of equation (3.18) with \( e_k \) to get
\[
\langle \nabla_X e_i, e_k \rangle = \langle e_j \omega^i_j(X), e_k \rangle \\
= \omega^i_j(X) \langle e_j, e_k \rangle \\
= \omega^i_j(X)g_{jk}
\]
Hence,
\[ < \nabla_X e_i, e_k > = \omega_{ki}(X) \] (3.20)

The left hand side of the last equation is the inner product of two vectors, so the expression represents an array of functions. Consequently, the right hand side also represents an array of functions. In addition, both expressions are linear on \( X \), since by definition \( \nabla_X \) is linear on \( X \). We conclude that the right hand side can be interpreted as a matrix in which each entry is a 1-forms acting on the vector \( X \) to yield a function. The matrix valued quantity \( \omega_i^j \) is called the connection form. At the risk of being inconsistent with formalism of differential forms, but with the goal of connecting to classical notation we sometimes will write the above equation as
\[ < de_i, e_k > = \omega_{ki}, \] (3.21)

where \( \{ e_i \} \) are vector calculus vectors forming an orthonormal basis.

### 3.3.5 Definition

Let \( \nabla_X \) be a Koszul connection and let \( \{ e_i \} \) be a frame. The Christoffel symbols associated with the connection in the given frame are the functions \( \Gamma^k_{ij} \) given by
\[ \nabla_{e_i}e_j = \Gamma^k_{ij}e_k \] (3.22)

The Christoffel symbols are the coefficients that give the representation of the rate of change of the frame vectors in the direction of the frame vectors themselves. Many physicists therefore refer to the Christoffel symbols as the connection once again giving rise to possible confusion. The precise relation between the Christoffel symbols and the connection 1-forms is captured by the equations,
\[ \omega^k_i(e_j) = \Gamma^k_{ij}, \] (3.23)

or equivalently
\[ \omega^k_i = \Gamma^k_{ij} \theta^j. \] (3.24)

In a general frame in \( \mathbb{R}^n \) there are \( n^2 \) entries in the connection 1-form and \( n^3 \) Christoffel symbols. The number of independent components is reduced if one assumes that the frame is orthonormal. If \( X = T^i e_i \) is a general vector field, then
\[ \nabla_{e_j}X = \nabla_{e_j}(T^i e_i) \]
\[ = T^i_{j;i} e_i + T^i \Gamma^k_{ji} e_k \]
\[ = (T^i_{;j} + T^k \Gamma^i_{jk}) e_i \] (3.25)

Which is denoted classically as the covariant derivative
\[ T^i_{;j} = T^i_{;j} + \Gamma^i_{jk} T^k \] (3.26)

Where the comma in the subscript means regular derivative.

### 3.3.6 Proposition

Let and \( e_i \) be an orthonormal frame and \( \nabla_X \) be a Koszul connection compatible with the metric. Then
\[ \omega_{ji} = -\omega_{ij} \] (3.27)

**Proof:** Since it is given that \( < e_i, e_j > = \delta_{ij} \), we have
\[
0 = \nabla_X < e_i, e_j > \\
= < \nabla_X e_i, e_j > + < e_i, \nabla_X e_j > \\
= < \omega^k_i e_k, e_j > + < e_i, \omega^k_j e_k > \\
= \omega^k_i < e_k, e_j > + \omega^k_j < e_i, e_k > \\
= \omega^k_i g_{ij} + \omega^k_j g_{ik} \\
= \omega_{ji} + \omega_{ij}.
\]
3.3. COVARIANT DERIVATIVE

Thus proving that $\omega$ is indeed antisymmetric.

The covariant derivative can be extended to the full tensor field $T^r_s(\mathbb{R}^n)$ by requiring that

\begin{enumerate}[a)]
\item $\nabla_X: T^r_s(\mathbb{R}^n) \to T^r_s(\mathbb{R}^n)$,
\item $\nabla_X(T_1 \otimes T_2) = \nabla_X T_1 \otimes T_2 + T_1 \otimes \nabla_X T_2$
\end{enumerate}

Let $\nabla_X$ commute with all contractions, $\nabla_X(CT) = C(\nabla_X T)$.

Let us compute the covariant derivative of a one-form $\omega$ with respect to vector field $X$. The contraction of $\omega \otimes Y$ is the function $i_Y \omega = \omega(Y)$. Taking the covariant derivative, we have,

$$
\nabla_X (\omega(Y)) = (\nabla_X \omega)(Y) - \omega(\nabla_X Y),
$$

$$
X(\omega(Y)) = (\nabla_X \omega)(Y) - \omega(\nabla_X Y).
$$

Hence, the coordinate-free formula for the covariant derivative of one-form is,

$$
(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y).
$$

Let $\theta^i$ be the dual forms to $e_i$. We have

$$
\nabla_X (\theta^i \otimes e_j) = \nabla_X \theta^i \otimes e_j + \theta^i \otimes \nabla_X e_j.
$$

The contraction of $i_j \theta^i = \theta^i(e_j) = \delta^i_j$. Hence, taking the contraction of the equation above, we see that the left hand side becomes 0, and we conclude that,

$$
(\nabla_X \theta^i)(e_j) = -\theta^i(\nabla_X e_j).
$$

Let $\omega = T_i \theta^i$. Then,

$$
(\nabla_X \omega)(e_j) = (\nabla_X (T_i \theta^i))(e_j),
$$

$$
= X(T_i) \theta^i(e_j) + T_i (\nabla_X \theta^i)(e_j),
$$

$$
= X(T_j) - T_i \theta^i(\nabla_X e_j).
$$

If we now set $X = e_k$, we get,

$$
(\nabla (e_k) \omega)(e_i) = T_{j,k} - T_i \theta^i (\Gamma^l_{kj} e_l),
$$

$$
= T_{j,k} - T_i \delta^l_{ij} \Gamma^l_{kj},
$$

$$
= T_{j,k} - \Gamma^i_{jk} T_i.
$$

Classically, we write

$$
T_{j||k} = T_{j,k} - \Gamma^i_{jk} T_i.
$$

In general, let $T$ be a tensor of type $(\ell,\ell)$,

$$
T = T^i_{j_1,\ldots,j_{\ell}} e_{i_1} \otimes \cdots \otimes e_{i_{\ell}} \otimes \theta^{i_1} \otimes \cdots \theta^{i_{\ell}}.
$$

Then, since we know how to take the covariant derivative of a function, a vector, and a one form, we can use Leibnitz rule for tensor products and property of the covariant derivative commuting with contractions, to get by induction, a formula for the covariant derivative of an $(\ell,\ell)$-tensor,

$$
(\nabla_X T)(\theta^{i_1}, \ldots, \theta^{i_{\ell}}, e_{j_1}, \ldots, e_{j_{\ell}}) = X(T(\theta^{i_1}, \ldots, \theta^{i_{\ell}}, e_{i_1}, \ldots, e_{i_{\ell}}))
$$

$$
- T(\nabla_X \theta^{i_1}, \ldots, \theta^{i_{\ell}}, e_{j_1}, \ldots, e_{j_{\ell}}) - \cdots - T(\theta^{i_1}, \ldots, \nabla_X \theta^{i_{\ell}}, e_{j_1}, \ldots, e_{j_{\ell}}) - \cdots
$$

$$
- T(\theta^{i_1}, \ldots, \theta^{i_{\ell}}, \nabla_X e_{j_1}, \ldots, e_{j_{\ell}}) - \cdots - T(\theta^{i_1}, \ldots, \theta^{i_{\ell}}, e_{j_1}, \ldots, \nabla_X e_{j_{\ell}}).
$$
The covariant derivative picks up a term with a positive Christoffel symbol factor for each contravariant index and a term with a negative Christoffel symbol factor for each covariant index. Thus, for example, for a \( \frac{1}{2} \) tensor, components of the covariant derivative in classical notation are

\[
T^i_{jk,l} = T^i_{jk,l} + \Gamma^i_{lm}T^m_{jk} - \Gamma^m_{jl}T^i_{mk} - \Gamma^m_{kl}T^i_{mj}.
\]

(3.34)

In an orthonormal frame in \( \mathbb{R}^n \), the number of independent coefficients of the connection 1-form is \( \frac{1}{2}n(n-1) \) since by antisymmetry, the diagonal entries are zero, and one only needs to count the number of entries in the upper triangular part of the \( n \times n \) matrix \( \omega_{ij} \). Similarly, the number of independent Christoffel symbols gets reduced to \( \frac{1}{2}n^2(n-1) \). Raising one index with \( g_{ij} \), we find that \( \omega_{ij} \) is also antisymmetric, so in \( \mathbb{R}^3 \) the connection equations become

\[
\nabla_X [e_1, e_2, e_3] = [e_1, e_2, e_3] \begin{bmatrix}
0 & \omega_2^1(X) & \omega_3^1(X) \\
-\omega_3^1(X) & 0 & \omega_3^2(X) \\
-\omega_3^2(X) & -\omega_3^3(X) & 0
\end{bmatrix}
\]

(3.35)

Comparing the Frenet frame equation (1.39), we notice the obvious similarity to the general frame equations above. Clearly, the Frenet frame is a special case in which the basis vectors have been adapted to a curve resulting in a simpler connection in which some of the coefficients vanish. A further simplification occurs in the Frenet frame since here the equations represent the rate of change of the frame only along the direction of the curve rather than an arbitrary direction vector \( X \). To unpack this transition from classical to modern notation, consider a unit speed curve \( \beta(s) \). Then, as we discussed in section 1.15, we associate with the classical tangent vector \( T = \frac{dx}{ds} \) the vector field \( T = \beta'(s) = \frac{dx}{ds} \frac{\partial}{\partial x} \). Let \( W = W(\beta(s)) = w^j(s) \frac{\partial}{\partial x^j} \) be an arbitrary vector field constrained to the curve. The rate of change of \( W \) along the curve is given by

\[
\nabla_T W = \nabla \left( \frac{dx^i}{ds} \frac{\partial}{\partial x^i} \right) \left( w^j \frac{\partial}{\partial x^j} \right)
\]

\[
= \frac{dx^i}{ds} \nabla \left( \frac{\partial w^j}{\partial x^i} \frac{\partial}{\partial x^j} \right)
\]

\[
= \frac{dx^i}{ds} \frac{\partial}{\partial x^i} \frac{\partial w^j}{\partial x^j}
\]

\[
= \frac{dw^j}{ds} \frac{\partial}{\partial x^j}
\]

\[
= W'(s).
\]

3.4 Cartan Equations

Perhaps the most important contribution to the development of Differential Geometry is the work of Cartan culminating into the famous equations of structure discussed in this chapter.

First Structure Equation

3.4.1 Theorem Let \( \{e_i\} \) be a frame with connection \( \omega_j^i \) and dual coframe \( \theta^i \). Then

\[
\Theta^i \equiv \delta \theta^i + \omega_j^i \wedge \theta^j = 0
\]

(3.36)

Proof: Let

\[
e_i = \partial_j A_j^i.
\]
be a frame, and let $\theta^i$ be the corresponding coframe. Since $\theta^i(e_j)$, we have
$$\theta^i = (A^{-1})^i_j dx^j.$$ Let $X$ be an arbitrary vector field. Then
$$\nabla_X e_i = \nabla_X (\partial_j A^j_i)$$
$$e_j \omega^j_i(X) = \partial_j X(A^j_i)$$
$$= \partial_j d(A^j_i)(X)$$
$$= e_k (A^{-1})^k_j d(A^j_i)(X)$$
$$\omega^k_i(X) = (A^{-1})^k_j d(A^j_i)(X).$$
Hence,
$$\omega^k_i = (A^{-1})^k_j d(A^j_i),$$
or, in matrix notation,
$$\omega = A^{-1} dA. \quad (3.37)$$
On the other hand, taking the exterior derivative of $\theta^i$, we find that
$$d\theta^i = d(A^{-1})^i_j \wedge dx^j$$
$$= d(A^{-1})^i_j \wedge A^j_k \theta^k$$
$$d\theta = d(A^{-1})A \wedge \theta.$$
However, since $A^{-1}A = I$, we have $d(A^{-1})A = -A^{-1} dA = -\omega$, hence
$$d\theta = -\omega \wedge \theta. \quad (3.38)$$
In other words
$$d\theta^i + \omega^j_i \wedge \theta^j = 0.$$ \hfill (3.38)

3.4.2 Example $SO(2, \mathbb{R})$

Consider the polar coordinates part of the transformation in equation 3.5. Then the frame equations 3.6 in matrix form are given by:
$$[e_1, e_2] = \begin{bmatrix} \partial_x & \partial_y \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \quad (3.39)$$
Thus, the attitude matrix
$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad (3.40)$$
in this case, is a rotation matrix in $\mathbb{R}^2$. The set of all such matrices forms a continuous group (a Lie Group) called $SO(2, \mathbb{R})$. In such cases, the matrix
$$\omega = A^{-1} dA \quad (3.41)$$
in equation 3.37 is called the Maurer-Cartan form of the group. An easy computation shows that for the rotation group $SO(2)$, the connection form is
$$\omega = \begin{bmatrix} 0 & -d\theta \\ d\theta & 0 \end{bmatrix}. \quad (3.42)$$
Second Structure Equation

Let \( \theta^i \) be a coframe in \( \mathbb{R}^n \) with connection \( \omega^i_j \). Taking the exterior derivative of the first equation of structure and recalling the properties (2.67), we get

\[
\begin{align*}
    d(d\theta^i) + d(\omega^i_j \wedge \theta^j) &= 0 \\
    d\omega^i_j \wedge \theta^j - \omega^i_j \wedge d\theta^j &= 0.
\end{align*}
\]

Substituting recursively from the first equation of structure, we get

\[
\begin{align*}
    d\omega^i_j \wedge \theta^j - \omega^i_j \wedge (-\omega^k_j \wedge \theta^k) &= 0 \\
    d\omega^i_j \wedge \theta^j + \omega^k_j \wedge \omega^k_j \wedge \theta^j &= 0 \\
    (d\omega^i_j + \omega^k_j \wedge \omega^k_j) \wedge \theta^j &= 0 \\
    d\omega^i_j + \omega^k_j \wedge \omega^k_j &= 0.
\end{align*}
\]

3.4.3 Definition  The curvature \( \Omega \) of a connection \( \omega \) is the matrix valued 2-form,

\[
\Omega^i_j \equiv d\omega^i_j + \omega^k_j \wedge \omega^k_j.
\]

(3.43)

3.4.4 Theorem  Let \( \theta \) be a coframe with connection \( \omega \) in \( \mathbb{R}^n \). Then the curvature form vanishes:

\[
\Omega = d\omega + \omega \wedge \omega = 0.
\]

(3.44)

Proof:  Given that there is a non-singular matrix \( A \) such that \( \theta = A^{-1}dx \) and \( \omega = A^{-1}dA \), we have

\[
d\omega = d(A^{-1}) \wedge dA.
\]

On the other hand,

\[
\omega \wedge \omega = (A^{-1}dA) \wedge (A^{-1}dA) = -d(A^{-1})A \wedge A^{-1}dA = -d(A^{-1})(AA^{-1}) \wedge dA = -d(A^{-1}) \wedge dA.
\]

Therefore, \( d\omega = -\omega \wedge \omega \).

We should point out the slight abuse of the wedge notation. The connection \( \omega \) is matrix valued, so the symbol \( \omega \wedge \omega \) is really a composite of matrix and wedge multiplication.

3.4.5 Example  Sphere Frame

The frame for spherical coordinates [3.7] in matrix form is

\[
[e_r, e_{\theta}, e_{\phi}] = \begin{bmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{bmatrix} \begin{bmatrix}
\sin \theta \cos \phi & \cos \theta \cos \phi & -
\sin \phi \\
\sin \theta \sin \phi & \cos \theta \sin \phi & 
\cos \phi \\
\cos \theta & -\sin \theta & 0
\end{bmatrix}.
\]

Hence,

\[
A^{-1} = \begin{bmatrix}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{bmatrix}.
\]
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and

\[
\begin{bmatrix}
  \cos \theta \cos \phi \, d\theta - \sin \theta \sin \phi \, d\phi \\
  \cos \theta \sin \phi \, d\theta + \sin \theta \cos \phi \, d\phi \\
  - \sin \theta \, d\theta - \cos \theta \, d\phi
\end{bmatrix}.
\]

Since the \( \omega = A^{-1} dA \) is antisymmetric, it suffices to compute:

\[
\omega_1 = \left[ -\sin^2 \theta \cos^2 \phi - \sin^2 \theta \sin^2 \phi - \cos^2 \theta \right] d\theta \\
+ \left[ \sin \theta \cos \theta \cos \phi \, d\phi - \sin \theta \cos \phi \, d\phi \right].
\]

\[
\omega_2 = -d\theta.
\]

\[
\omega_3 = \left[ -\sin \theta \cos^2 \phi - \sin \theta \sin^2 \phi \right] d\phi = -\sin \theta \, d\phi.
\]

\[
\omega_2 = \left[ -\cos \theta \cos^2 \phi - \cos \theta \sin^2 \phi \right] d\phi = -\cos \theta \, d\phi.
\]

We conclude that the matrix-valued connection one form is

\[
\omega = \begin{bmatrix}
  0 & -d\theta & -\sin \theta \, d\phi \\
  d\theta & 0 & -\cos \theta \, d\phi \\
  \sin \theta \, d\phi & \cos \theta \, d\phi & 0
\end{bmatrix}.
\]

A slicker computation of the connection form can be obtained by a method of educated guessing working directly from the structure equations. We have that the dual one forms are:

\[
\theta^1 = dr \\
\theta^2 = r \, d\theta \\
\theta^3 = r \sin \theta \, d\phi.
\]

Then

\[
d\theta^2 = -d\theta \wedge dr \\
= -\omega_1^2 \wedge \theta^1 - \omega_3^2 \wedge \theta^3.
\]

So, on a first iteration we guess that \( \omega_1^1 = d\theta \). The component \( \omega_3^3 \) is not necessarily 0 because it might contain terms with \( d\phi \). Proceeding in this manner, we compute:

\[
d\theta^3 = \sin \theta \, dr \wedge d\phi + r \cos \theta \, d\theta \wedge d\phi \\
= -\sin \theta \, d\phi \wedge dr - \cos \theta \, d\phi \wedge r \, d\theta \\
= -\omega_1^2 \wedge \theta^1 - \omega_3^2 \wedge \theta^2.
\]

Now we guess that \( \omega_1^1 = \sin \theta \, d\phi \), and \( \omega_3^3 = \cos \theta \, d\phi \). Finally, we insert these into the full structure equations and check to see if any modifications need to be made. In this case, the forms we have found are completely compatible with the first equation of structure, so these must be the forms. The second equations of structure are much more straight-forward to verify. For example

\[
d\omega_3^3 = d(- \cos \theta \, d\phi) \\
= \sin \theta \, d\theta \wedge d\phi \\
= -d\theta \wedge (-\sin \theta \, d\phi) \\
= -\omega_1^1 \wedge \omega_3^3.
\]
Change of Basis

We briefly explore the behavior of the quantities $\Theta^i$ and $\Omega^i_j$ under a change of basis.

Let $e_i$ be frame in $M = \mathbb{R}^n$ with dual forms $\theta^i$, and let $\tau_i$ be another frame related to the first frame by an invertible transformation.

\[ e_i = e_j B^j_i, \]  

which we will write in matrix notation as $\tau = e B$. Referring back to the definition of connections (3.18), we introduce the covariant differential $\nabla$ which maps vectors into vector-valued forms,

\[ \nabla : \Omega^0(M,TM) \to \Omega^1(M,TM) \]

given by the formula

\[ \nabla e_i = e_j \otimes \omega^j_i \]
\[ \nabla e = e \omega \]  

(3.46)

where, once again, we have simplified the equation by using matrix notation. This definition is elegant because it does not explicitly show the dependence on $X$ in the connection (3.18). The idea of switching from derivatives to differentials is familiar from basic calculus. Consistent with equation 3.21, the vector calculus notation for equation 3.46 would be

\[ d e_i = e_j \omega^j_i. \]  

(3.47)

However, we point out that in the present context, the situation is much more subtle. The operator $\nabla$ here maps a vector field to a matrix-valued tensor of rank $(1,1)$. Another way to view the covariant differential is to think of $\nabla$ as an operator such that if $e$ is a frame, and $X$ a vector field, then $\nabla e(X) = \nabla X e$. If $f$ is a function, then $\nabla f(X) = \nabla_X f = df(X)$, so that $\nabla f = df$. In other words, $\nabla$ behaves like a covariant derivative on vectors, but like a differential on functions. The action of the covariant differential also extends to the entire tensor algebra, but we do not need that formalism for now, and we delay discussion to section 6.3 on connections on vector bundles. Taking the exterior differential of (3.45) and using (3.46) recursively, we get

\[ \nabla \tau = (\nabla e) B + e dB \]
\[ = e \omega B + e dB \]
\[ = \tau B^{-1} \omega B + e dB \]
\[ = \tau [B^{-1} \omega B + B^{-1} dB] \]
\[ = \tau \omega \]

(3.48)

provided that the connection $\omega$ in the new frame $\tau$ is related to the connection $\omega$ by the transformation law, (See 6.57)

\[ \omega = B^{-1} \omega B + B^{-1} dB. \]

(3.48)

It should be noted than if $e$ is the standard frame $e_i = \partial_i$ in $\mathbb{R}^n$, then $\nabla e = 0$, so that $\omega = 0$. In this case, the formula above reduces to $\omega = B^{-1} dB$, showing that the transformation rule is consistent with equation (3.37). The transformation law for the curvature forms is,

\[ \nabla \Omega = B^{-1} \Omega B. \]  

(3.49)

A quantity transforming as in (3.49) is said to be a Tensorial Form of Adjoint Type.

3.4.6 Example  Suppose that $B$ is a change of basis consisting of a rotation by an angle $\theta$ about $e_3$. The transformation is a an isometry that can be represented by the orthogonal rotation matrix
3.4. CARTAN EQUATIONS

\[ B = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]  
(3.50)

Carrying out the computation for the change of basis \(3.48\) we find:

\[ \omega^1_{22} = \omega^1_{22} - d\theta, \]
\[ \omega^1_{3} = \cos \theta \omega^1_{3} + \sin \theta \omega^2_{3}, \]
\[ \omega^2_{3} = -\sin \theta \omega^1_{3} + \cos \theta \omega^2_{3}. \]  
(3.51)

The \( B^{-1}dB \) part of the transformation only affects the \( \omega^1_{2} \) term, and the effect is just adding \( d\theta \) much like the case of the Maurer-Cartan form for \( SO_2 \) above.
Chapter 4

Theory of Surfaces

4.1 Manifolds

4.1.1 Definition A coordinate chart in $M \subset \mathbb{R}^3$ is a differentiable map $x$ from an open subset $V$ of $\mathbb{R}^2$ onto a set $U \subset M$.

$$x : V \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (u, v) \mapsto (x(u, v), y(u, v), z(u, v)) \quad (4.1)$$

Fig. 4.1: Surface

Each set $U = x(V)$ is called a Coordinate Neighborhood of $M$. We require that the Jacobian of the map has maximal rank. In local coordinates, a coordinate chart is represented by three equations in two variables:

$$x^i = f^i(u^\alpha), \text{ where } i = 1, 2, 3, \ \alpha = 1, 2. \quad (4.2)$$

The local coordinate representation allows us to use the tensor index formalism introduced in earlier chapters. The assumption that the Jacobian $J = (\partial x^i / \partial u^\alpha)$ be of maximal rank allows one to invoke the Implicit Function Theorem. Thus, in principle, one can locally solve for one of the coordinates, say $x^3$, in terms of the other two, like so:

$$x^3 = f(x^1, x^2). \quad (4.3)$$

The locus of points in $\mathbb{R}^3$ satisfying the equations $x^i = f^i(u^\alpha)$ can also be locally represented by an expression of the form

$$F(x^1, x^2, x^3) = 0 \quad (4.4)$$
4.1.2 Definition Let $U_i$ and $U_j$ be two coordinate neighborhoods of a point $p \in M$ with corresponding charts $x(u^1, u^2) : V_i \to U_i \subset \mathbb{R}^3$ and $y(v^1, v^2) : V_j \to U_j \subset \mathbb{R}^3$ with a non-empty intersection $U_i \cap U_j \neq \emptyset$. On the overlaps, the maps $\phi_{ij} = x^{-1}y$ are called transition functions or coordinate transformations. (see Figure 4.2)

Fig. 4.2: Coordinate Charts

4.1.3 Definition A differentiable manifold of dimension 2, is a space $M$ together with an indexed collection $\{U_\alpha\}_{\alpha \in I}$ of coordinate neighborhoods satisfying the following properties:

1. The neighborhoods $\{U_\alpha\}$ constitute an open cover $M$. That is, if $p \in M$, then $p$ belongs to some chart.

2. For any pair of coordinate neighborhoods $U_i$ and $U_j$ with $U_i \cap U_j \neq \emptyset$, the transition maps $\phi_{ij}$ and their inverses are differentiable.

3. An indexed collection satisfying the conditions above is called an Atlas. We require the atlas to be maximal in the sense that it contains all possible coordinate neighborhoods.

The overlapping coordinate patches represent different parametrizations for the same set of points in $\mathbb{R}^3$. Part (2) of the definition insures that on the overlap, the coordinate transformations are invertible. Part (3) is included for technical reasons, although in practice the condition is superfluous. A family of coordinate neighborhoods satisfying conditions (1) and (2) can always be extended to a maximal atlas. This can be shown from the fact that $M$ inherits a subspace topology consisting of open sets which are defined by the intersection of open sets in $\mathbb{R}^3$ with $M$.

If the coordinate patches in the definition map from $\mathbb{R}^n$ to $\mathbb{R}^m$ $n < m$ we say that $M$ is a $n$-dimensional submanifold embedded in $\mathbb{R}^m$. In fact, one could define an an abstract manifold without the reference to the embedding space by starting with a topological space $M$ that is locally Euclidean via homeomorphic coordinate patches and has a differentiable structure as in the definition above. However, it turns out that any differentiable manifold of dimension $n$ can be embedded in $\mathbb{R}^{2n}$, as proved by Whitney in a theorem that is beyond the scope of these notes.

A 2-dimensional manifold embedded in $\mathbb{R}^3$ in which the transition functions are $C^\infty$ is called a smooth surface. The first condition in the definition states that each coordinate neighborhood looks locally like a subset of $\mathbb{R}^2$. The second differentiability condition indicates that the patches are joined together smoothly as some sort of quilt. We summarize this notion by saying that a manifold is a space that is locally Euclidean and has a differentiable structure so that the notion of differentiation makes sense. Of course, $\mathbb{R}^n$ is itself an $n$ dimensional manifold.

The smoothness condition on the coordinate component functions $x^i(u^\alpha)$ implies that at any point $x^i(u^\alpha_0 + h^\alpha)$ near a point $x^i(u^\alpha_0) = x^i(u_0, v_0)$, the functions admit a Taylor expansion

$$x^i(u^\alpha_0 + h^\alpha) = x^i(u^\alpha_0) + h^\alpha \left( \frac{\partial x^i}{\partial u^\alpha} \right)_0 + \frac{1}{2!} h^\alpha h^\beta \left( \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} \right)_0 + \ldots$$

(4.5)
4.1. MANIFOLDS

Since the parameters $u^\alpha$ must enter independently, the Jacobian matrix

$$J \equiv \left[ \frac{\partial x^i}{\partial u^\alpha} \right] = \left[ \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} \frac{\partial z}{\partial u} \right]$$

must have maximal rank. At points where $J$ has rank 0 or 1, there is a singularity in the coordinate patch.

4.1.4 Example Consider the local coordinate chart for the unit sphere obtained by setting $r = 1$ in the equations for spherical coordinates 2.31

$$x(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

The vector equation is equivalent to three scalar functions in two variables:

$$x = \sin \theta \cos \phi,$$
$$y = \sin \theta \sin \phi,$$
$$z = \cos \phi.$$  \hspace{1cm} (4.6)

Clearly, the surface represented by this chart is part of the sphere $x^2 + y^2 + z^2 = 1$. The chart cannot possibly represent the whole sphere because, although a sphere is locally Euclidean, (the earth is locally flat) there is certainly a topological difference between a sphere and a plane. Indeed, if one analyzes the coordinate chart carefully, one will note that at the North pole ($\theta = 0, z = 1$, the coordinates become singular. This happens because $\theta = 0$ implies that $x = y = 0$ regardless of the value of $\phi$, so that the North pole has an infinite number of labels. In this coordinate patch, the Jacobian at the North Pole does not have maximal rank. To cover the entire sphere, one would need at least two coordinate patches. In fact, introducing an exactly analogous patch $y(u,v)$ based on South pole would suffice, as long as in overlap around the equator functions $x^{-1}y$, and $y^{-1}x$ are smooth. One could conceive more elaborate coordinate patches such as those used in baseball and tennis balls.

The fact that it is required to have two parameters to describe a patch on a surface in $\mathbb{R}^3$ is a manifestation of the 2-dimensional nature of of the surfaces. If one holds one of the parameters constant while varying the other, then the resulting 1-parameter equation describes a curve on the surface. Thus, for example, letting $\phi = constant$ in equation \hspace{1cm} (4.6), we get the equation of a meridian great circle.

4.1.5 Example Surface of Revolution

Given a function $f(r)$, the coordinate chart

$$x(r, \phi) = (r \cos \phi, r \sin \phi, f(r))$$  \hspace{1cm} (4.7)

represents a surface of revolution around the $z$-axis in which the cross section profile has the shape of the function. Horizontal cross-sections are circles of radius $r$. In figure \hspace{1cm} \hspace{1cm} 4.3 we have chosen the function $f(r) = e^{-r^2}$ to be a Gaussian, so the surface of revolution is bell-shaped. A lateral curve profile for $\phi = \pi/4$ is shown in black.

We should point out that this parametrization of surfaces of revolution is fairly constraining because of the requirement of $z = f(r)$ to be a function. Thus, for instance, the parametrization will not work for surfaces of revolution generated by closed curves. In the next example, we illustrate how one easily get around this constraint.

4.1.6 Example Torus
Consider the surface of revolution generated by rotating a circle $C$ of radius $r$ around a parallel axis located a distance $R$ from its center as shown in figure 4.4.

The resulting surface called a Torus can be parametrized by the coordinate patch

$$
\mathbf{x}(u, v) = ( (R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u ).
$$

(4.8)

Here the angle $u$ traces points around the $z$-axis, whereas the angle $v$ traces points around the circle $C$. (At the risk of some confusion in notation, (the parameters in the figure are bold-faced; this is done solely for the purpose of visibility.) The projection of a point in the surface of the Torus onto the $xy$-plane is located at a distance $(R + r \cos u)$ from the origin. Thus, the $x$ and $y$ coordinates of the point in the torus are just the polar coordinates of the projection of the point in the plane. The $z$-coordinate corresponds to the height of a right triangle with radius $r$ and opposite angle $u$.

### 4.1.7 Example Cartesian Equation

Surfaces in $\mathbb{R}^3$ are first introduced in vector calculus by a function of two variables $z = f(x, y)$. We will find it useful for consistency to use the obvious parametrization

$$
\mathbf{x}(u, v) = (u, v, f(u, v)).
$$

(4.9)

### 4.1.8 Notation

Given a parametrization of a surface in a local chart $\mathbf{x}(u, v) = \mathbf{x}(u^1, u^2) = \mathbf{x}(u^\alpha)$, we will denote the partial derivatives by any of the following notations:

$$
\begin{align*}
\mathbf{x}_u &= \mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u}, & \mathbf{x}_{uu} &= \mathbf{x}_{11} = \frac{\partial^2 \mathbf{x}}{\partial u^2} \\
\mathbf{x}_v &= \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial v}, & \mathbf{x}_{vv} &= \mathbf{x}_{22} = \frac{\partial^2 \mathbf{x}}{\partial v^2} \\
\mathbf{x}_\alpha &= \frac{\partial \mathbf{x}}{\partial u^\alpha}, & \mathbf{x}_{\alpha\beta} &= \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial v^\beta}
\end{align*}
$$

### 4.2 The First Fundamental Form

Let $\mathbf{x}^i(u^\alpha)$ be a local parametrization of a surface. Then, the Euclidean inner product in $\mathbb{R}^3$ induces an inner product in the space of tangent vectors at each point in the surface. This
4.2. THE FIRST FUNDAMENTAL FORM

The metric on the surface is obtained as follows:

\[ dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha, \]
\[ ds^2 = \delta_{ij} dx^i dx^j = \delta_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} du^\alpha du^\beta. \]

Thus,

\[ ds^2 = g_{\alpha\beta} du^\alpha du^\beta, \tag{4.10} \]

where

\[ g_{\alpha\beta} = \delta_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}. \tag{4.11} \]

We conclude that the surface, by virtue of being embedded in \( \mathbb{R}^3 \), inherits a natural metric (4.10) which we will call the induced metric. A pair \( \{ M, g \} \), where \( M \) is a manifold and \( g = g_{\alpha\beta} du^\alpha \otimes du^\beta \) is a metric is called a Riemannian manifold if considered as an entity in itself, and a Riemannian submanifold of \( \mathbb{R}^n \) if viewed as an object embedded in Euclidean space. An equivalent version of the metric (4.10) can be obtained by using a more traditional calculus notation:

\[ dx = x_u du + x_v dv \]
\[ ds^2 = dx \cdot dx = (x_u du + x_v dv) \cdot (x_u du + x_v dv) \]
\[ = (x_u \cdot x_u) du^2 + 2(x_u \cdot x_v) du dv + (x_v \cdot x_v) dv^2. \]

We can rewrite the last result as

\[ ds^2 = Edu^2 + 2Fdu dv + Gdv^2, \tag{4.12} \]

where

\[ E = g_{11} = x_u \cdot x_u, \]
\[ F = g_{12} = x_u \cdot x_v, \]
\[ = g_{21} = x_v \cdot x_u, \]
\[ G = g_{22} = x_v \cdot x_v. \]

That is

\[ g_{\alpha\beta} = x_\alpha \cdot x_\beta = <x_\alpha, x_\beta>. \]

4.2.1 Definition The element of arc length,

\[ ds^2 = g_{\alpha\beta} du^\alpha \otimes du^\beta, \tag{4.13} \]

is also called the first fundamental form. We must caution the reader that this quantity is not a form in the sense of differential geometry since \( ds^2 \) involves the symmetric tensor product rather than the wedge product.

The first fundamental form plays such a crucial role in the theory of surfaces that we will find it convenient to introduce a third, more modern version. Following the same development as in the theory of curves, consider a surface \( M \) defined locally by a function \( q = (u^1, u^2) \mapsto p = \alpha(u^1, u^2) \). We say that a quantity \( X_p \) is a tangent vector at a point \( p \in M \) if \( X_p \) is a linear derivation on the space of \( C^\infty \) real-valued functions \( F = \{ f : M \rightarrow \mathbb{R} \} \) on the surface. The set of all tangent vectors at a point \( p \in M \) is called the tangent space \( T_p M \). As before, a vector field \( X \) on the
surface is a smooth choice of a tangent vector at each point on the surface and the union of all tangent spaces is called the **tangent bundle** $TM$. Sections of the tangent bundle of $M$ are consistently denoted by $\mathcal{T}(M)$.

The coordinate chart map $\alpha : \mathbb{R}^2 \to M \subset \mathbb{R}^3$ induces a **Push-Forward** map $\alpha_* : T\mathbb{R}^2 \to TM$ which maps a vector $V$ at each point in $T_q(\mathbb{R}^2)$ into a vector $V_{\alpha(q)} = \alpha_*(V_q)$ in $T_{\alpha(q)}M$, as illustrated in the diagram 4.5.

![Fig. 4.5: Push-Forward](image)

The action of the Push-Forward is defined by

$\alpha_*(V) = V(\circ \alpha) \big|_q.$

(4.14)

Just as in the case of curves, when we revert back to classical notation to describe a surface as $x^i(u^\alpha)$, what we really mean is $(x^i \circ \alpha)(u^\alpha)$, where $x^i$ are the coordinate functions in $\mathbb{R}^3$. Particular examples of tangent vectors on $M$ are given by the push-forward of the standard basis of $T\mathbb{R}^2$. These tangent vectors which earlier we called $x$ are defined by

$\alpha_*(\partial_{u^\alpha})(f) = \partial_{u^\alpha}(f \circ \alpha) \big|_{u^\alpha}$

In this formalism, the first fundamental form $I$ is just the symmetric bilinear tensor defined by the induced metric,

$I(X,Y) = \langle X,Y \rangle = \langle x^\alpha, x^\beta \rangle$,

where $X$ and $Y$ are any pair of vector fields in $\mathcal{T}(M)$.

### Orthogonal Parametric Curves

Let $V$ and $W$ be vectors tangent to a surface $M$ defined locally by a chart $x(u^\alpha)$. Since the vectors $x_\alpha$ span the tangent space of $M$ at each point, the vectors $V$ and $W$ can be written as linear combinations,

$V = V^\alpha x_\alpha$

$W = W^\alpha x_\alpha$.

The functions $V^\alpha$ and $W^\alpha$ are called the **curvilinear coordinates** of the vectors. We can calculate the length and the inner product of the vectors using the induced Riemannian metric as follows:

$\|V\|^2 = \langle V,V \rangle = \langle V^\alpha x_\alpha, V^\beta x_\beta \rangle = V^\alpha V^\beta \langle x_\alpha, x_\beta \rangle$

$\|V\|^2 = g_{\alpha\beta} V^\alpha V^\beta$

$\|W\|^2 = g_{\alpha\beta} W^\alpha W^\beta,$

and

$\langle V,W \rangle = \langle V^\alpha x_\alpha, W^\beta x_\beta \rangle = V^\alpha W^\beta \langle x_\alpha, x_\beta \rangle = g_{\alpha\beta} V^\alpha W^\beta.$
4.2. THE FIRST FUNDAMENTAL FORM

The angle $\theta$ subtended by the the vectors $V$ and $W$ is given by the equation

$$\cos \theta = \frac{\langle V, W \rangle}{\|V\| \|W\|} = \frac{I(V, W)}{\sqrt[I]{I(V, V)} \sqrt[I]{I(W, W)}} = \frac{g_{\alpha_1 \beta_1} V^{\alpha_1} W^{\beta_1}}{g_{\alpha_2 \beta_2} V^{\alpha_2} W^{\beta_2} \cdot g_{\alpha_3 \beta_3} W^{\alpha_3} W^{\beta_3}},$$

(4.16)

where the numerical subscripts are needed for the $\alpha$ and $\beta$ indices to comply with Einstein’s summation convention.

Let $u^\alpha = \phi^\alpha(t)$ and $u^\alpha = \psi^\alpha(t)$ be two curves on the surface. Then the total differentials

$$du^\alpha = \frac{d\phi^\alpha}{dt} dt, \quad \delta u^\alpha = \frac{d\psi^\alpha}{dt} \delta t$$

represent infinitesimal tangent vectors to the curves. Thus, the angle between two infinitesimal vectors tangent to two intersecting curves on the surface satisfies the equation:

$$\cos \theta = \frac{g_{\alpha_1 \beta_1} du^{\alpha_1} \delta u^{\beta_1}}{\sqrt[g]{g_{\alpha_2 \beta_2} du^{\alpha_2} du^{\beta_2} \sqrt[g]{g_{\alpha_3 \beta_3} \delta u^{\alpha_3} \delta u^{\beta_3}}},$$

(4.17)

In particular, if the two curves happen to be the parametric curves, $u^1 = \text{const.}$ and $u^2 = \text{const.}$, then along one curve we have $du^1 = 0$, with $du^2$ arbitrary, and along the second $\delta u^1$ is arbitrary and $\delta u^2 = 0$. In this case, the cosine of the angle subtended by the infinitesimal tangent vectors reduces to:

$$\cos \theta = \frac{g_{12} \delta u^1 du^2}{\sqrt[g]{g_{11}(\delta u^1)^2} \sqrt[g]{g_{22}(du^2)^2} g_{11} g_{22} F} = \frac{F}{\sqrt[1]{EG}},$$

(4.18)

A simpler way to obtain this result is to recall that that parametric directions are given by $x_u$ and $x_v$, so

$$\cos \theta = \frac{\langle x_u, x_v \rangle}{\|x_u\| \|x_v\|} = \frac{F}{\sqrt[1]{EG}}.$$

(4.19)

It follows immediately from the equation above that:

4.2.2 Proposition The parametric curves are orthogonal if $F = 0$.

Orthogonal parametric curves are an important class because locally the coordinate grid on the surface is similar to coordinate grids in basic calculus, such as in polar coordinates for which $ds^2 = dr^2 + r^2 d\theta^2$.

4.2.3 Examples

a) Sphere

$$x = (a \sin \theta \cos \phi, a \sin \theta \sin \phi, a \cos \theta)$$

$$x_\theta = (a \cos \theta \cos \phi, a \cos \theta \sin \phi, -a \sin \theta)$$

$$x_\phi = (-a \sin \theta \sin \phi, a \sin \theta \cos \phi, 0)$$

$$E = x_\theta \cdot x_\theta = a^2$$

$$F = x_\theta \cdot x_\phi = 0$$

$$G = x_\phi \cdot x_\phi = a^2 \sin^2 \theta$$

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2$$

(4.20)
There are many interesting curves on a sphere, but amongst these the **Loxodromes** have a special role in history. A loxodrome is a curve that winds around a sphere making a constant angle with the meridians. In this sense, it is the spherical analog of a cylindrical helix and as such it is often called a spherical helix. The curves were significant in early navigation where they are referred as **Rhumb lines**. As people in the late 1400’s began to rediscover that earth was not flat, cartographers figured out methods to render maps on flat paper surfaces. One such technique is called the **Mercator** projection which is obtained by projecting the sphere onto a plane that wraps around the sphere as a cylinder tangential to the sphere along the equator.

As we will discuss in more detail later, a navigator travelling a constant bearing would be moving a on path that is a straight line on the Mercator projection map, but on the sphere it would be spiraling ever faster as one approached the poles. Thus it became important to understand the nature of such paths. It appears as if the first quantitative treatise of loxodromes was carried in the mid 1500’s by the Portuguese applied mathematician Pedro Nunes, who was chair of the department at the University of Coimbra. As an application, we will derive the equations of loxodromes and compute the arc length. A general spherical curve can be parametrized in the form \( \gamma(t) = (x(\theta(t), \phi(t)), x(\theta(t), \phi(t))) \). Let \( \sigma \) be the angle the curve makes with the meridians \( \phi = \text{constant} \). Then, recalling that \( \langle x_\theta, x_\phi \rangle = 0 \), we have:

\[
\begin{align*}
\gamma' &= x_\theta \frac{d\theta}{dt} + x_\phi \frac{d\phi}{dt}, \\
\cos \sigma &= \frac{\langle x_\theta, \gamma' \rangle}{\|x_\theta\| \|\gamma'\|} = \frac{E \frac{d\theta}{dt}}{\sqrt{E \frac{ds}{dt}}} = a \frac{d\theta}{ds}, \\
a^2 d\theta^2 &= \cos^2 \sigma \ ds^2, \\
a^2 \sin^2 \sigma \ d\theta^2 &= a^2 \cos^2 \sigma \sin^2 \theta \ d\phi^2, \\
\sin \sigma \ d\theta &= \pm \cos \sigma \sin \theta \ d\phi, \\
csc \theta \ d\phi &= \pm \cot \sigma \ d\phi.
\end{align*}
\]

The convention use throughout history by cartographers, is to measure the angle \( \theta \) from the equatorial plane. To better adhere to the history, but at the same time avoiding confusion, we replace \( \theta \) with \( \vartheta = \frac{\pi}{2} - \theta \), so that \( \vartheta = 0 \) corresponds to the equator. Integrating the last equation with this change, we get

\[
\sec \vartheta \ d\vartheta = \pm \cot \sigma \ d\phi
\]

\[
\ln \tan \left( \frac{\vartheta}{2} + \frac{\pi}{4} \right) = \pm \cot \sigma (\phi - \phi_0).
\]

Thus, we conclude that the equations of loxodromes and their arc lengths are given by
4.2. THE FIRST FUNDAMENTAL FORM

\[ \phi = \pm (\tan \sigma) \ln \tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) + \phi_0 \] \hspace{1cm} (4.21)

\[ s = a(\theta - \theta_0) \sec \sigma, \] \hspace{1cm} (4.22)

where \( \theta_0 \) and \( \phi_0 \) are the coordinates of the initial position. Figure 4.2 shows four loxodromes equally distributed around the sphere.

Loxodromes were the bases for a number of beautiful drawings and woodcuts by M. C. Escher. Figure 4.2 also shows one more manifestation of beautiful geometry in nature in a plant called Aloe Polyphylla. Not surprisingly, the plant has 5 loxodromes which is a Fibonacci number. We will show later under the discussion of conformal (angle preserving) maps in section 5.2.2 that loxodromes map into straight lines making a constant angle with meridians in the Mercator projection (See Figure 5.9).

b) Surface of Revolution

\[ x = (r \cos \theta, r \sin \theta, f(r)) \]
\[ x_r = (\cos \theta, \sin \theta, f'(r)) \]
\[ x_\theta = (-r \sin \theta, r \cos \theta, 0) \]
\[ E = x_r \cdot x_r = 1 + f'^2(r) \]
\[ F = x_r \cdot x_\theta = 0 \]
\[ G = x_\theta \cdot x_\theta = r^2 \]
\[ ds^2 = [1 + f'^2(r)] dr^2 + r^2 d\theta^2 \]

As in 4.7, we have chosen a Gaussian profile to illustrate a surface of revolution. Since \( F = 0 \) the parametric lines are orthogonal. The picture shows that this is indeed the case. At any point of the surface, the analogs of meridians and parallels intersect at right angles.

c) Pseudosphere

\[ x = (a \sin u \cos v, a \sin u \sin v, a(\cos u + \ln(\tan \frac{u}{2}))) \]
\[ E = a^2 \cot^2 u \]
\[ F = 0 \]
\[ G = a^2 \sin^2 u \]
\[ ds^2 = a^2 \cot^2 u \ du^2 + a^2 \sin^2 u \ dv^2 \]

The pseudosphere is a surface of revolution in which the profile curve is a tractrix. The tractrix curve was originated by a problem posed by Leibnitz to the effect of finding the path traced by a point initially placed on the horizontal axis at a distance \( a \) from the origin, as it was pulled along the vertical axis by a taught string of constant length \( a \), as shown in figure 4.7. The tractrix was later studied by Huygens in 1692. Colloquially this is the path of a reluctant dog at \((a, 0)\) dragged by a man walking up the \( z \)-axis. The tangent segment is the hypothenuse of a right triangle with base \( x \) and height \( \sqrt{a^2 - x^2} \), so the slope is \( dz/dx = -\sqrt{a^2 - x^2}/x \). Using the trigonometric substitution \( x = a \sin u \), we get \( z = a \int (\cos^2 u/\sin u) \ du \) which leads to the appropriate form for the profile of the surface of revolution. The pseudosphere was studied by Beltrami in 1868. He discovered that in spite of the surface extending asymptotically to infinity, the surface area is finite with \( S = 4\pi a^2 \) as in a sphere of the same radius, and the volume enclosed is half that sphere. We will have much more to say about this surface.

d) Torus
CHAPTER 4. THEORY OF SURFACES

Fig. 4.8: Examples of Surfaces

\[
x = ((b + a \cos u) \cos v, (b + a \cos u) \sin v, a \sin u)
\]

See [4.8]

\[
E = a^2
\]

\[
F = 0
\]

\[
G = (b + a \cos u)^2
\]

\[
ds^2 = a^2 du^2 + (b + a \cos u)^2 dv^2 \quad (4.23)
\]

e) Helicoid

\[
x = (u \cos v, u \sin v, av)
\]

Coordinate curves \( u = c \) are helices.

\[
E = 1,
\]

\[
F = 0,
\]

\[
G = u^2 + a^2,
\]

\[
ds^2 = du^2 + (u^2 + a^2) dv^2. \quad (4.24)
\]

f) Catenoid

\[
x = (u \cos v, u \sin v, c \cosh^{-1} \frac{u}{c})
\]

This is a catenary of revolution.

\[
E = \frac{u^2}{u^2 - c^2},
\]

\[
F = 0,
\]

\[
G = u^2,
\]

\[
ds^2 = \frac{u^2}{u^2 - c^2} du^2 + u^2 dv^2, \quad (4.25)
\]

g) Cone and Conical Helix

The equation \( z^2 = \cot^2 \alpha (x^2 + y^2) \), represents a circular cone whose generator makes an angle \( \alpha \) with the \( z \)-axis. In parametric form,

\[
x = (r \cos \phi, r \sin \phi, r \cot \alpha)
\]

\[
E = \csc^2 \alpha,
\]

\[
F = 0,
\]

\[
G = r^2,
\]

\[
ds^2 = \csc^2 \alpha \, dr^2 + r^2 d\phi^2. \quad (4.26)
\]
4.2. THE FIRST FUNDAMENTAL FORM

A conical helix a curve $\gamma(t) = x(r(t), \phi(t))$, that makes a constant angle $\sigma$ with the generators of the cone. Similar to the case of loxodromes, we have

$$\gamma' = x_r \frac{dr}{dt} + x_\phi \frac{d\phi}{dt}.$$

$$\cos \sigma = \frac{<x_r, \gamma'>}{\|x_r\| \|\gamma'\|} = \frac{E\frac{dr}{dt}}{\sqrt{E\frac{dr}{dt}}^2} = \sqrt{E}\frac{dr}{ds}.$$

$$E dr^2 = \cos^2 \sigma ds^2,$$

$$\csc^2 \alpha dr^2 = \cos^2 \sigma (\csc^2 \alpha dr^2 + r^2 d\phi^2),$$

$$\csc^2 \alpha \sin^2 \sigma dr^2 = r^2 \cos^2 \sigma d\phi^2,$$

$$\frac{1}{r} dr = \cot \sigma \sin \alpha d\phi.$$

Therefore, the equations of a conical helix are given by

$$r = ce^{\cot \sigma \sin \alpha \phi}. \quad (4.27)$$

As shown in figure 4.9, a conical helix projects into the plane as a logarithmic spiral. Many sea shells and other natural objects in nature exhibit neatly such conical spirals. The picture shown here is that of Lobatus gigas or Caracol Pala, previously known as Strombus gigas. This particular one is included here with certain degree of nostalgia for it has been a decorative item for decades in our family. The shell was probably found in Santa Cruz del Islote, Archipelago de San Bernando, located in the Gulf of Morrosquillo in the Caribbean coast of Colombia. In this densely populated island paradise, which then enjoyed the pulchritude of enchanting coral reefs, the shells are now virtually extinct as the coral has succumbed to bleaching with rising temperatures of the waters. The shell shows a cut in the spire which the island natives use to sever the columellar muscle and thus release the edible snail.
4.3 The Second Fundamental Form

Let $\mathbf{x} = \mathbf{x}(u^\alpha)$ be a coordinate patch on a surface $M$. Since $\mathbf{x}_u$ and $\mathbf{x}_v$ are tangential to the surface, we can construct a unit normal $\mathbf{n}$ to the surface by taking

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\| \mathbf{x}_u \times \mathbf{x}_v \|}$$  \hspace{1cm} (4.28)

Now, consider a curve on the surface given by $u^\beta = u^\beta(s)$. Without loss of generality, we assume that the curve is parametrized by arc length $s$ so that the curve has unit speed. Let $e = \{ \mathbf{T}, \mathbf{N}, \mathbf{B} \}$ be the Frenet Frame of the curve. Recall that the rate of change $\nabla_T W$ of a vector field $W$ along the curve correspond to the classical vector $w' = \frac{dw}{ds}$, so $\nabla W$ is associated with the vector $dw$. Thus the connection equation $\nabla e = e\omega$ is given by:

$$d[\mathbf{T}, \mathbf{N}, \mathbf{B}] = [\mathbf{T}, \mathbf{N}, \mathbf{B}] \begin{bmatrix} 0 & -\kappa ds & 0 \\ \kappa ds & 0 & -\tau ds \\ 0 & \tau ds & 0 \end{bmatrix}$$  \hspace{1cm} (4.29)

Following ideas first introduced by Darboux and subsequently perfected by Cartan, we introduce a new orthonormal frame $f = \{ \mathbf{T}, \mathbf{g}, \mathbf{n}, \mathbf{g} \}$ adapted to the surface, where at each point, $\mathbf{T}$ is the common tangent to the surface and to the curve on the surface, $\mathbf{n}$ is the unit normal to the surface and $\mathbf{g} = \mathbf{n} \times \mathbf{T}$. Since the two orthonormal frames must be related by a rotation that leaves the $\mathbf{T}$ vector fixed, we have $f = eB$, where $B$ is a matrix of the form

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}.$$  \hspace{1cm} (4.30)

We wish to find $\nabla f = f\varpi$. A short computation using the change of basis equations $\varpi = B^{-1}\omega B + B^{-1}dB$ (see equations 3.48 and 5.51) gives:

$$d[\mathbf{T}, \mathbf{g}, \mathbf{n}] = [\mathbf{T}, \mathbf{g}, \mathbf{n}] \begin{bmatrix} 0 & -\kappa \cos \theta ds & -\kappa \sin \theta ds \\ \kappa \cos \theta ds & 0 & -\tau ds + d\theta \\ \kappa \sin \theta ds & \tau ds - d\theta & 0 \end{bmatrix},$$  \hspace{1cm} (4.31)

and

$$d[\mathbf{T}, \mathbf{g}, \mathbf{n}] = [\mathbf{T}, \mathbf{g}, \mathbf{n}] \begin{bmatrix} 0 & -\kappa_\alpha ds & \kappa_\alpha ds \\ \kappa_\alpha ds & 0 & -\tau_\alpha ds \\ \kappa_\alpha ds & \tau_\alpha ds & 0 \end{bmatrix},$$  \hspace{1cm} (4.32)
where:
\[ \kappa_n = \kappa \sin \theta \] is called the **Normal Curvature**, \\
\[ \kappa_g = \kappa \cos \theta \] is called the **Geodesic Curvature**; \[ \mathbf{K}_g = \kappa_g \mathbf{g} \] the geodesic curvature vector, and \\
\[ \tau_g = \tau - d\theta/ds \] is called the **Geodesic Torsion**.

We conclude that we can decompose \( \mathbf{T}' \) and the curvature \( \kappa \) into their normal and surface tangent space components (see figure 4.11)

\[
\mathbf{T}' = \kappa_n \mathbf{n} + \kappa_g \mathbf{g}, \\
\kappa^2 = \kappa_n^2 + \kappa_g^2.
\]

The normal curvature \( \kappa_n \) measures the curvature of \( \mathbf{x}(u^\alpha(s)) \) resulting from the constraint of the curve to lie on a surface. The geodesic curvature \( \kappa_g \) measures the “sideward” component of the curvature in the tangent plane to the surface. Thus, if one draws a straight line on a flat piece of paper and then smoothly bends the paper into a surface, the straight line would now acquire some curvature. Since the line was originally straight, there is no sideward component of curvature so \( \kappa_g = 0 \) in this case. This means that the entire contribution to the curvature comes from the normal component, reflecting the fact that the only reason there is curvature here is due to the bend in the surface itself. In this sense, a curve on a surface for which the geodesic curvature vanishes at all points reflects locally the shortest path between two points. These curves are therefore called **geodesics** of the surface. The property of minimizing the path between two points is a local property. For example, on a sphere one would expect the geodesics to be great circles. However travelling from Los Angeles to San Francisco along one such great circle, there is a short path and a very long one that goes around the earth.

If one specifies a point \( \mathbf{p} \in M \) and a direction vector \( \mathbf{X}_p \in T_p M \), one can geometrically envision the normal curvature by considering the equivalence class of all unit speed curves in \( M \) that contain the point \( \mathbf{p} \) and whose tangent vectors line up with the direction of \( \mathbf{X} \). Of course, there are infinitely many such curves, but at an infinitesimal level, all these curves can be obtained by intersecting the surface with a “vertical” plane containing the vector \( \mathbf{X} \) and the normal to \( M \). All curves in this equivalence class have the same normal curvature and their geodesic curvatures vanish. In this sense, the normal curvature is more of a property pertaining to a direction on the surface at a point, whereas the geodesic curvature really depends on the curve itself. It might be impossible for a hiker walking on the undulating hills of the Ozarks to find a straight line trail, since the rolling hills of the terrain extend in all directions. It might be possible, however, for the hiker to walk on a path with zero geodesic curvature as long the same compass direction is maintained. We will come back to the Cartan Structure equations associated with the Darboux frame, but for computational purposes, the classical approach is very practical.

Using the chain rule, we see that the unit tangent vector \( \mathbf{T} \) to the curve is given by

\[
\mathbf{T} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{du^\alpha} \frac{du^\alpha}{ds} = \mathbf{x}_\alpha \frac{du^\alpha}{ds}.
\]
To find an explicit formula for the normal curvature we first differentiate equation \[4.35\]

\[
T' = \frac{dT}{ds} = \frac{d}{ds} (x_\alpha \frac{du^\alpha}{ds}) = \frac{d}{ds} (x_\alpha) \frac{du^\alpha}{ds} + x_\alpha \frac{d^2 u^\alpha}{ds^2} = \left( \frac{dx_\alpha}{du^\beta} \frac{du^\alpha}{ds} \right) \frac{du^\beta}{ds} + x_\alpha \frac{d^2 u^\alpha}{ds^2} = x_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} + x_\alpha \frac{d^2 u^\alpha}{ds^2}.
\]

Taking the inner product of the last equation with the normal and noticing that \(\langle x_\alpha, n \rangle = 0\), we get

\[
\kappa_n = \langle T', n \rangle = \langle x_{\alpha\beta}, n \rangle \frac{du^\alpha}{ds} \frac{du^\beta}{ds}
\]

\[
= b_{\alpha\beta} du^\alpha du^\beta
g_{\alpha\beta} du^\alpha du^\beta,
\]

(4.36)

where

\[
b_{\alpha\beta} = \langle x_{\alpha\beta}, n \rangle
\]

(4.37)

4.3.1 Definition The expression

\[
II = b_{\alpha\beta} du^\alpha \otimes du^\beta
\]

is called the second fundamental form .

4.3.2 Proposition The second fundamental form is symmetric.

Proof: In the classical formulation of the second fundamental form, the proof is trivial. We have \(b_{\alpha\beta} = b_{\beta\alpha}\), since for a \(C^\infty\) patch \(x(u^\alpha)\), we have \(x_{\alpha\beta} = x_{\beta\alpha}\), because the partial derivatives commute. We will denote the coefficients of the second fundamental form as follows:

\[
e = b_{11} = \langle x_{uu}, n \rangle
\]

\[
f = b_{12} = \langle x_{uv}, n \rangle
\]

\[
g = b_{22} = \langle x_{vv}, n \rangle,
\]

so equation (4.38) can be written as

\[
II = edu^2 + 2f du dv + g dv^2.
\]

(4.39)

It follows that the equation for the normal curvature (4.36) can be written explicitly as

\[
\kappa_n = \frac{II}{T} = \frac{edu^2 + 2f du dv + g dv^2}{E du^2 + 2F du dv + G dv^2}.
\]

(4.40)

It should pointed out that just as the first fundamental form can be represented as

\[
I = \langle dx, dx \rangle,
\]
we can represent the second fundamental form as

$$II = -\langle dx, dn \rangle$$

To see this, it suffices to note that differentiation of the identity, $$\langle x_\alpha, n \rangle = 0$$, implies that

$$\langle x_\alpha, n \rangle = -\langle x_\alpha, n_\beta \rangle.$$

Therefore,

$$\langle dx, dn \rangle = \langle x_\alpha du^\alpha, n_\beta du^\beta \rangle$$

$$= \langle x_\alpha du^\alpha, n_\beta du^\beta \rangle$$

$$= \langle x_\alpha, n_\beta \rangle du^\alpha du^\beta$$

$$= -\langle x_\alpha, n \rangle du^\alpha du^\beta$$

$$= -II$$

4.3.3 **Definition** Directions on a surface along which the second fundamental form

$$e du^2 + 2f du dv + g dv^2 = 0 \quad (4.41)$$

vanishes are called **asymptotic directions**, and curves having these directions are called **asymptotic curves**. This happens for example when there are straight lines on the surface as in the case of the intersection of the saddle $z = xy$ with the the plane $z = 0$.

We state without elaboration for now, that one can also define the third fundamental form by

$$III = \langle dn, dn \rangle = \langle n_\alpha, n_\beta \rangle du^\alpha du^\beta. \quad (4.42)$$

From a computational point a view, a more useful formula for the coefficients of the second fundamental formula can be derived by first applying the classical vector identity

$$(A \times B) \cdot (C \times D) = \begin{vmatrix} A \cdot C & A \cdot D \\ B \cdot C & B \cdot D \end{vmatrix} \quad (4.43)$$

to compute

$$\|x_u \times x_v\|^2 = (x_u \times x_v) \cdot (x_u \times x_v)$$

$$= det \begin{vmatrix} x_u \cdot x_u & x_u \cdot x_v \\ x_v \cdot x_u & x_v \cdot x_v \end{vmatrix}$$

$$= EG - F^2. \quad (4.44)$$

Consequently, the normal vector can be written as

$$n = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{x_u \times x_v}{\sqrt{EG - F^2}}$$

Thus, we can write the coefficients $b_{\alpha\beta}$ directly as triple products involving derivatives of ($x$). The expressions for these coefficients are

$$e = \frac{\langle x_\alpha x_\beta, x_{\alpha u} \rangle}{\sqrt{EG - F^2}}$$

$$f = \frac{\langle x_\alpha x_\beta, x_{\alpha v} \rangle}{\sqrt{EG - F^2}}$$

$$g = \frac{\langle x_\alpha x_\beta, x_{\beta v} \rangle}{\sqrt{EG - F^2}} \quad (4.45)$$
4.3.4 Example Sphere

Going back to example 4.20, we have:

\[ x_{\theta\theta} = (a \sin \theta \cos \phi, -a \sin \theta \sin \phi, -a \cos \theta) \]
\[ x_{\theta\phi} = (-a \cos \theta \sin \phi, a \cos \theta \cos \phi, 0) \]
\[ x_{\phi\phi} = (-a \sin \theta \cos \phi, -a \sin \theta \sin \phi, 0) \]
\[ n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]
\[ e = x_{\theta\theta} \cdot n = -a \]
\[ f = x_{\theta\phi} \cdot n = 0 \]
\[ g = x_{\phi\phi} \cdot n = -a \sin^2 \theta \]
\[ II = \frac{1}{a^2} I \]

The first fundamental form on a surface measures the square of the distance between two infinitesimally separated points. There is a similar interpretation of the second fundamental form as we show below. The second fundamental form measures the distance from a point on the surface to the tangent plane at a second infinitesimally separated point. To see this simple geometrical interpretation, consider a point \( x_0 = x(u_0^0) \in M \) and a nearby point \( x(u_0^0 + du^\alpha) \). Expanding on a Taylor series, we get

\[ x(u_0^0 + du^\alpha) = x_0 + (x_0)_\alpha du^\alpha + \frac{1}{2} (x_0)_{\alpha\beta} du^\alpha du^\beta + \ldots. \]

We recall that the distance formula from a point \( x \) to a plane which contains \( x_0 \) is just the scalar projection of \((x - x_0)\) onto the normal. Since the normal to the plane at \( x_0 \) is the same as the unit normal to the surface and \( < x_\alpha, n > = 0 \), we find that the distance \( D \) is

\[ D = < x - x_0, n > \]
\[ = \frac{1}{2} < (x_0)_{\alpha\beta}, n > du^\alpha du^\beta \]
\[ = \frac{1}{2} II_0 \]

The first fundamental form (or, rather, its determinant) also appears in calculus in the context of calculating the area of a parametrized surface. If one considers an infinitesimal parallelogram subtended by the vectors \( x_\alpha du \) and \( x_\beta dv \), then the differential of surface area is given by the length of the cross product of these two infinitesimal tangent vectors. That is,

\[ dS = ||x_\alpha \times x_\beta|| dudv \]
\[ S = \int \int \sqrt{E G - F^2} dudv \]

The second fundamental form contains information about the shape of the surface at a point. For example, the discussion above indicates that if \( b = |b_{\alpha\beta}| = eg - f^2 > 0 \) then all the neighboring points lie on the same side of the tangent plane, and hence, the surface is concave in one direction. If at a point on a surface \( b > 0 \), the point is called an elliptic point, if \( b < 0 \), the point is called hyperbolic or a saddle point, and if \( b = 0 \), the point is called parabolic.

4.4 Curvature

The concept of curvature and its relation to the fundamental forms, constitute the central object of study in differential geometry. One would like to be able to answer questions such as “what
quantities remain invariant as one surface is smoothly changed into another?" There is certainly something intrinsically different between a cone, which we can construct from a flat piece of paper, and a sphere, which we cannot. What is it that makes these two surfaces so different? How does one calculate the shortest path between two objects when the path is constrained to lie on a surface?

These and questions of similar type can be quantitatively answered through the study of curvature. We cannot overstate the importance of this subject; perhaps it suffices to say that, without a clear understanding of curvature, there would be no general theory of relativity, no concept of black holes, and even more disastrous, no Star Trek.

Studying the curvature of a hypersurface in $\mathbb{R}^n$ (a surface of dimension $n - 1$) begins by trying to understand the covariant derivative of the normal to the surface. If the normal to a surface is constant, then the surface is a flat hyperplane. Variations in the normal are what indicates the presence of curvature. For simplicity, we constrain our discussion to surfaces in $\mathbb{R}^3$, but the formalism we use is applicable to any dimension. We will also introduce in the modern version of the second fundamental form.

### 4.4.1 Classical Formulation of Curvature

The normal curvature $\kappa_n$ at any point on a surface measures the deviation from flatness as one moves along a direction tangential to the surface at that point. The direction can be taken as the unit tangent vector to a curve on the surface. We seek the directions in which the normal curvature attains the extrema. For this purpose, let the curve on the surface be given by $v = v(u)$ and let $\lambda = \frac{dv}{du}$. Then we can write the normal curvature in the form

$$
\kappa_n = \frac{II^*}{I^*} = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2},
$$

(4.46)

where $II'$ and $I'$ are the numerator and denominator respectively. To find the extrema, we take the derivative with respect to $\lambda$ and set it equal to zero. The resulting fraction is zero only when the numerator is zero, so from the quotient rule we get

$$
I^*(2f + 2g\lambda) - II^*(2F + 2G\lambda) = 0.
$$

It follows that,

$$
\kappa_n = \frac{II^*}{I^*} = \frac{f + g\lambda}{F + G\lambda}.
$$

(4.47)

On the other hand, combining with equation 4.46 we have,

$$
\kappa_n = \frac{(e + f\lambda) + \lambda(f + g\lambda)}{(E + F\lambda) + \lambda(F + G\lambda)} = \frac{f + g\lambda}{F + G\lambda}.
$$

This can only happen if

$$
\kappa_n = \frac{f + g\lambda}{F + G\lambda} = \frac{e + f\lambda}{E + F\lambda}.
$$

(4.48)

Equation 4.48 contains a wealth of information. On one hand, we can eliminate $\kappa_n$ which leads to the quadratic equation for $\lambda$

$$(Fg - gF)\lambda^2 + (Eg - Ge)\lambda + (Ef - Fe) = 0.$$

Recalling that $\lambda = \frac{dv}{du}$, and noticing that the coefficients resemble minors of a $3 \times 3$ matrix, we can elegantly rewrite the equation as
\[
\begin{vmatrix}
\frac{\partial^2}{\partial u^2} & -\frac{\partial}{\partial u} \frac{\partial}{\partial v} & \frac{\partial^2}{\partial v^2} \\
E & F & G \\
e & f & g
\end{vmatrix} = 0.
\] (4.49)

Equation (4.49) determines two directions \( \frac{\partial}{\partial u} \) along which the normal curvature attains the extrema, except for special cases when either \( b_{\alpha\beta} = 0 \), or \( b_{\alpha\beta} \) and \( g_{\alpha\beta} \) are proportional, which would cause the determinant to be identically zero. These two directions are called principal directions of curvature, each associated with an extremum of the normal curvature. We will have much more to say about these shortly.

On the other hand, we can write equations (4.48) in the form
\[
\begin{align*}
(e - E\kappa_n) + \lambda(f - F\kappa_n) &= 0 \\
(f - F\kappa_n) + \lambda(g - G\kappa_n) &= 0
\end{align*}
\]
Solving each equation for \( \lambda \) we can eliminate \( \lambda \) instead, and we are lead to a quadratic equation for \( \kappa_n \) which we can write as
\[
\begin{vmatrix}
e - E\kappa_n & f - F\kappa_n \\
f - F\kappa_n & g - G\kappa_n
\end{vmatrix} = 0.
\] (4.50)

It is interesting to note that equation (4.50) can be written as
\[
\left[ \begin{array}{cc}
e & f \\
f & g
\end{array} \right] - \kappa_n \left[ \begin{array}{cc}
E & F \\
F & G
\end{array} \right] = 0.
\]
In other words, the extrema for the values of the normal are the solutions of the equation
\[
\| b_{\alpha\beta} - \kappa_n g_{\alpha\beta} \| = 0.
\] (4.51)

Had it been the case that \( g_{\alpha\beta} = \delta_{\alpha\beta} \), the reader would recognize this as a eigenvalue equation for a symmetric matrix giving rise to two invariants, that is, the trace and the determinant of the matrix. We will treat this formally in the next section. The explicit quadratic expression for the extrema of \( \kappa_n \)
\[
(EG - F^2)\kappa_n^2 - (Eg - 2Ff + Ge)\kappa_n + (eg - f^2) = 0.
\]
We conclude there are two solutions \( \kappa_1 \) and \( \kappa_2 \) such that
\[
\kappa_1 \kappa_2 = \frac{eg - f^2}{EG - F^2},
\] (4.52)
and
\[
\frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} \frac{Eg - 2Ff + Ge}{EF - G^2}.
\] (4.53)

To understand better the deep significance of the last two equations, we introduce the modern formulation which will allow us to draw conclusions from the inextricable connection of these results with the center piece of the spectral theorem for symmetric operators in linear algebra.
4.4.2 Covariant Derivative Formulation of Curvature

4.4.1 Definition Let $X$ be a vector field on a surface $M$ in $\mathbb{R}^3$, and let $N$ be the normal vector. The map $L$, given by

$$LX = -\nabla_X N$$

(4.54)

is called the Weingarten map. The same definition applies if $M$ is an $n$-dimensional hypersurface in $\mathbb{R}^{n+1}$.

Here, we have adopted the convention to overline the operator $\nabla$ when it refers to the ambient space. The Weingarten map is natural to consider, since it represents the rate of change of the normal in an arbitrary direction tangential to the surface, which is what we wish to quantify.

4.4.2 Definition The Lie bracket $[X,Y]$ of two vector fields $X$ and $Y$ on a surface $M$ is defined as the commutator,

$$[X,Y] = XY - YX,$$

(4.55)

meaning that if $f$ is a function on $M$, then $[X,Y](f) = X(Y(f)) - Y(X(f))$.

4.4.3 Proposition The Lie bracket of two vectors $X, Y \in \mathcal{X}(M)$ is another vector in $\mathcal{X}(M)$.

Proof: If suffices to prove that the bracket is a linear derivation on the space of $C^\infty$ functions. Consider vectors $X, Y \in \mathcal{X}(M)$ and smooth functions $f, g$ in $M$. Then,

$$[X,Y](f + g) = X(Y(f + g)) - Y(X(f + g))$$

$$= X(Y(f) + Y(g)) - Y(X(f) + X(g))$$

$$= X(Y(f)) - Y(X(f)) + X(Y(g)) - Y(X(g))$$

$$= [X,Y](f) + [X,Y](g),$$

and

$$[X,Y](fg) = X(Y(fg)) - Y(X(fg))$$

$$= X[fgY(g) + gY(f)] - Y[fX(g) + gX(f)]$$

$$= X(f)Y(g) + fX(Y(g)) + X(g)Y(f) + gX(Y(f))$$

$$- Y(f)X(g) - f(Y(X(g)) - Y(g)X(f) - gY(X(f))$$

$$= f[X(Y(g)) - Y(X(g)) + g[X(Y(f)) - Y(X(f))]]$$

$$= f[X,Y](g) + g[X,Y](f).$$

4.4.4 Proposition The Weingarten map is a linear transformation on $\mathcal{X}(M)$.

Proof: Linearity follows from the linearity of $\nabla$, so it suffices to show that $L : X \rightarrow LX$ maps $X \in \mathcal{X}(M)$ to a vector $LX \in \mathcal{X}(M)$. Since $N$ is the unit normal to the surface, $< N, N >= 1$, so any derivative of $< N, N >$ is 0. Assuming that the connection is compatible with the metric,

$$\nabla_X < N, N > = < \nabla_X N, > + < N, \nabla_X N >$$

$$= 2 < \nabla_X N, N >$$

$$= 2 < -LX, N >= 0.$$ 

Therefore, $LX$ is orthogonal to $N$; hence, it lies in $\mathcal{X}(M)$.

In the previous section, we gave two equivalent definitions $< d\mathbf{x}, d\mathbf{x} >$, and $< X, Y >$ of the first fundamental form. We will now do the same for the second fundamental form.
4.4.5 Definition The **second fundamental form** is the bilinear map

\[ II(X,Y) = \langle LX, Y \rangle. \]  

(4.56)

4.4.6 Remark It should be noted that the two definitions of the second fundamental form are consistent. This is easy to see if one chooses \( X \) to have components \( x_\alpha \) and \( Y \) to have components \( x_\beta \). With these choices, \( LX \) has components \( -n_\alpha \) and \( II(X,Y) \) becomes \( b_{\alpha \beta} = -\langle x_\alpha, n_\beta \rangle \).

We also note that there is a third fundamental form defined by

\[ III(X,Y) = \langle LX, LY \rangle. \]  

(4.57)

In classical notation, the third fundamental form would be denoted by \( \langle d_n, d_n \rangle \). As one would expect, the third fundamental form contains third order Taylor series information about the surface.

4.4.7 Definition The **torsion** of a connection \( \nabla \) is the operator \( T \) such that \( \forall X, Y, \)

\[ T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \]  

(4.58)

A connection is called **torsion-free** if \( T(X,Y) = 0 \). In this case,

\[ \nabla_X Y - \nabla_Y X = [X,Y]. \]

We will say more later about the torsion and the importance of torsion-free connections. For the time being, it suffices to assume that for the rest of this section, all connections are torsion-free. Using this assumption, it is possible to prove the following important theorem.

4.4.8 Theorem The Weingarten map is a self-adjoint endomorphism on \( \mathcal{T} M \).

**Proof:** We have already shown that \( L: \mathcal{T} M \rightarrow \mathcal{T} M \) is a linear map. Recall that an operator \( L \) on a linear space is self adjoint if \( \langle LX, Y \rangle = \langle X, LY \rangle \), so the theorem is equivalent to proving that the second fundamental form is symmetric \( (II[X,Y] = II[Y,X]) \). Computing the difference of these two quantities, we get

\[ II[X,Y] - II[Y,X] = \langle LX, Y \rangle - \langle LY, X \rangle = \langle -\nabla_X N, Y \rangle - \langle -\nabla_Y N, X \rangle. \]

Since \( \langle X, N \rangle = \langle Y, N \rangle = 0 \) and the connection is compatible with the metric, we know that

\[ \langle -\nabla_X N, Y \rangle = \langle N, \nabla_X Y \rangle, \]

\[ \langle -\nabla_Y N, X \rangle = \langle N, \nabla_Y X \rangle, \]

hence,

\[ II[X,Y] - II[Y,X] = \langle N, \nabla_Y X \rangle - \langle N, \nabla_X Y \rangle, \]

\[ = \langle N, \nabla_Y X - \nabla_X Y \rangle = \langle N, [X,Y] \rangle = 0 \]  

(iff \( [X,Y] \in T(M) \)).

The central theorem of linear algebra is the Spectral Theorem. In the case of self-adjoint operators, the Spectral Theorem states that given the eigenvalue equation for a symmetric operator on a vector space with an inner product,

\[ LX = \kappa X, \]  

(4.59)
the eigenvalues are always real and eigenvectors corresponding to different eigenvalues are orthogonal. Here, the vector spaces in question are the tangent spaces at each point of a surface in $\mathbb{R}^3$, so the dimension is 2. Hence, we expect two eigenvalues and two eigenvectors:

$$LX_1 = \kappa_1 X_1$$  \hspace{1cm} (4.60)
$$LX_2 = \kappa_2 X_2.$$  \hspace{1cm} (4.61)

**4.4.9 Definition** The eigenvalues $\kappa_1$ and $\kappa_2$ of the Weingarten map $L$ are called the principal curvatures and the eigenvectors $X_1$ and $X_2$ are called the principal directions.

Several possible situations may occur, depending on the classification of the eigenvalues at each point $p$ on a given surface:

1. If $\kappa_1 \neq \kappa_2$ and both eigenvalues are positive, then $p$ is called an elliptic point.
2. If $\kappa_1\kappa_2 < 0$, then $p$ is called a hyperbolic point.
3. If $\kappa_1 = \kappa_2 \neq 0$, then $p$ is called an umbilic point.
4. If $\kappa_1\kappa_2 = 0$, then $p$ is called a parabolic point.

It is also known from linear algebra, that the determinant and the trace of a self-adjoint operator in a vector space of dimension two are the only invariants under an adjoint (similarity) transformation. Clearly, these invariants are important in the case of the operator $L$, and they deserve special names. In the case of a hypersurface of $n$-dimensions, there would $n$ eigenvalues, counting multiplicities, so the classification of the points would be more elaborate.

**4.4.10 Definition** The determinant $K = \det(L)$ is called the Gaussian curvature of $M$ and $H = \frac{1}{2} \Tr(L)$ is called the mean curvature.

Since any self-adjoint operator is diagonalizable and in a diagonal basis the matrix representing $L$ is $\text{diag}(\kappa_1, \kappa_2)$, if follows immediately that

$$K = \kappa_1\kappa_2$$
$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$  \hspace{1cm} (4.62)

An alternative definition of curvature is obtained by considering the unit normal as a map $N : M \to S^2$, which maps each point $p$ on the surface $M$, to the point on the sphere corresponding to the position vector $N_p$. The map is called the Gauss Map.

**4.4.11 Examples**

1. The Gauss map of a plane is constant. The image is a single point on $S^2$.
2. The image of the Gauss map of a circular cylinder is a great circle on $S^2$.
3. The Gauss map of the top half of a circular cone sends all points on the cone into a circle. We may envision this circle as the intersection of the cone and a unit sphere centered at the vertex.
4. The Gauss map of a circular hyperboloid of one sheet misses two antipodal spherical caps with boundaries corresponding to the circles of the asymptotic cone.
5. The Gauss map of a catenoid misses two antipodal points.
The Weingarten map is minus the derivative \( N_\ast = dN \) of the Gauss map. That is, \( LX = -N_\ast(X) \).

4.4.12 Proposition Let \( X \) and \( Y \) be any linearly independent vectors in \( \mathcal{X}(M) \). Then

\[
LX \times LY = K(X \times Y) \\
(LX \times Y) + (X \times LY) = 2H(X \times Y)
\]

(4.63)

**Proof:** Since \( LX, LY \in \mathcal{X}(M) \), they can be expressed as linear combinations of the basis vectors \( X \) and \( Y \).

\[
LX = a_1X + b_1Y \\
LY = a_2X + b_2Y.
\]

Computing the cross product, we get

\[
LX \times LY = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} X \times Y
\]

\[= \det(L)(X \times Y).\]

Similarly

\[
(LX \times Y) + (X \times LY) = (a_1 + b_2)(X \times Y)
\]

\[= \text{Tr}(L)(X \times Y)
\]

\[= (2H)(X \times Y).\]

4.4.13 Proposition

\[
K = \frac{eg - f^2}{EG - F^2}
\]

\[
H = \frac{1}{2} \frac{Eg - 2Ff + eG}{EG - F^2}
\]

(4.64)

**Proof:** Starting with equations (4.63), take the inner product of both sides with \( X \times Y \) and use the vector identity (4.43). We immediately get

\[
K = \begin{vmatrix} <LX,X> & <LX,Y> \\ <LY,X> & <LY,Y> \end{vmatrix}
\]

(4.65)

\[2H = \begin{vmatrix} <LX,X> & <LX,Y> \\ <Y,X> & <Y,Y> \end{vmatrix} + \begin{vmatrix} <X,X> & <X,Y> \\ <LY,X> & <LY,Y> \end{vmatrix}
\]

(4.66)

The result follows by taking \( X = x_u \) and \( Y = x_v \). Not surprisingly, this is in complete agreement with the classical formulas for the Gaussian curvature (equation 4.52) and for the Mean curvature (equation 4.53). The result is also consistent with the well known fact from Linear Algebra, that the only similarity transformation invariants of a symmetry quadratic form are the determinant and the trace.
4.5. FUNDAMENTAL EQUATIONS

If we denote by \( g \) and \( b \) the matrices of the fundamental forms whose components are \( g_{\alpha\beta} \) and \( b_{\alpha\beta} \) respectively, we can write the equations for the curvatures as:

\[
K = \det\left( \frac{b}{g} \right) = \det(g^{-1}b), \tag{4.67}
\]
\[
2H = \text{Tr}\left( \frac{b}{g} \right) = \text{Tr}(g^{-1}b) \tag{4.68}
\]

4.4.14 Example Sphere

From equations 4.20 and 4.3 we see that \( K = 1/a^2 \) and \( H = 1/a \). This is totally intuitive since one would expect \( \kappa_1 = \kappa_2 = 1/a \) since the normal curvature in any direction should equal the curvature of great circle. This means that a sphere is a surface of constant curvature and every point of a sphere is an umbilic point. This another way to think of the symmetry of the sphere in the sense that an observer at any point sees the same normal curvature in all directions.

The next theorem due to Euler gives a characterization of the normal curvature in the direction of an arbitrary unit vector \( X \) tangent to the surface surface \( M \) at a given point.

4.4.15 Theorem (Euler) Let \( X_1 \) and \( X_2 \) be unit eigenvectors of \( L \) and let \( X = (\cos \theta)X_1 + (\sin \theta)X_2 \). Then

\[
II(X,X) = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta. \tag{4.69}
\]

Proof: Simply compute \( II(X,X) = < LX,X > \), using the fact the \( LX_1 = \kappa_1 X_1 \), \( LX_2 = \kappa_2 X_2 \), and noting that the eigenvectors are orthogonal. We get

\[
< LX,X > = < \cos \theta \kappa_1 X_1 + \sin \theta \kappa_2 X_2, \cos \theta \kappa_1 X_1 + \sin \theta \kappa_2 X_2 > \\
= \kappa_1 \cos^2 \theta < X_1, X_1 > + \kappa_2 \sin^2 \theta < X_2, X_2 > \\
= \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta.
\]

4.4.16 Theorem The first, second and third fundamental forms satisfy the equation

\[
III - 2HII + KI = 0 \tag{4.70}
\]

Proof: The proof follows immediately from the fact that for a symmetric 2 by 2 matrix \( A \), the characteristic polynomial \( \kappa^2 - \text{tr}(A)\kappa + \det(A) = 0 \), and from the Cayley-Hamilton theorem stating that the matrix is annihilated by its characteristic polynomial.

4.5 Fundamental Equations

4.5.1 Gauss-Weingarten Equations

As we have just seen for example, the Gaussian curvature of sphere of radius \( a \) is \( 1/a^2 \). To compute this curvature we had to first compute the coefficients of the second fundamental form, and therefore, we first needed to compute the normal to the surface in \( \mathbb{R}^3 \). The computation therefore depended on the particular coordinate chart parametrizing the surface.

However, it would reasonable to conclude that the curvature of the sphere is an intrinsic quantity, independent of the embedding in \( \mathbb{R}^3 \). After all, a "two-dimensional" creature such as ant moving on the surface of the sphere would be constrained by the curvature of the sphere independent of the higher dimension on which the surface lives. This mode of thinking lead the brilliant mathematicians
Gauss and Riemann to question if the coefficients of the second fundamental form were functionally computable from the coefficients of the first fundamental form. To explore this idea, consider again the basis vectors at each point of a surface consisting of two tangent vectors and the normal, as shown in figure 4.12. Given a coordinate chart \( x(u^\alpha) \), the vectors \( x_\alpha \) live on the tangent space, but this is not necessarily true for the second derivative vectors \( x_{\alpha\beta} \). Here, \( x(u^\alpha) \) could refer to a coordinate patch in any number of dimensions, so all the tensor index formulas that follow, apply to surfaces of codimension 1 in \( \mathbb{R}^n \). The set of vectors \( \{x_\alpha, n\} \) constitutes a basis for \( \mathbb{R}^n \) at each point on the surface, we can express the vectors \( x_{\alpha\beta} \) as linear combinations of the bases vectors. Therefore, there exist coefficients \( \Gamma^\gamma_{\alpha\beta} \) and \( c_{\alpha\beta} \) such that

\[
x_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} x_\gamma + c_{\alpha\beta} n.
\] (4.71)

Taking the inner product of equation 4.71 with \( n \), noticing that the latter is a unit vector orthogonal to \( x_\gamma \), we find that \( c_{\alpha\beta} = <x_{\alpha\beta}, n> \), and hence these are just the coefficients of the second fundamental form. In other words, equation 4.71 can be written as

\[
x_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} x_\gamma + b_{\alpha\beta} n.
\] (4.72)

Equation 4.72 together with equation 4.75 below, are called the formulæ of Gauss. The covariant derivative formulation of the equation of Gauss follows in a similar fashion. Let \( X \) and \( Y \) be vector fields tangent to the surface. We can the decompose the co-variant derivative of \( Y \) in the direction of \( X \) into its tangential and normal components

\[
\nabla_X Y = \nabla_X Y + h(X,Y)N.
\]

But then,

\[
h(X,Y) = <\nabla_X Y, N>,
\]

\[
= - <Y, \nabla_X N>,
\]

\[
= - <L_X Y>,
\]

\[
= II(X,Y).
\]

Thus, the coordinate independent formulation of the equation of Gauss reads:

\[
\nabla_X Y = \nabla_X Y + II(X,Y)N.
\] (4.73)

The quantity \( \nabla_X Y \) represents a covariant derivative on the surface, so in that sense, it is intrinsic to the surface. If \( \alpha(s) \) is a curve on the surface with tangent \( T = \alpha'(s) \), we say that a vector field \( Y \) is parallel-transported along the curve if \( \nabla_T Y = 0 \). This notion of parallelism refers to parallelism on the surface, not the ambient space. To illustrate by example, Figure 4.13 shows a vector field \( Y \) tangent to a sphere along the circle with azimuthal angle \( \theta = \pi/3 \). The circle has unit tangent \( T = \alpha'(s) \), and at each point on the circle, the vector \( Y \) points North. To the inhabitants of the sphere, the vector \( Y \) appears parallel-transported on the surface along the curve, that is \( \nabla_T Y = 0 \). However, \( Y \) is clearly not parallel-transported in the ambient \( \mathbb{R}^3 \) space with respect to the connection \( \nabla \).
The torsion $T$ of the connection $\nabla$ is defined exactly as before (See equation 4.58).

\[ T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \]

Also, as in definition 3.15, the connection is compatible with the metric on the surface if

\[ \nabla_X <Y,Z> = <\nabla_X Y,Z> + <Y,\nabla_X Z>. \]

A torsion-free connection that is compatible with the metric is called a **Levi-Civita** connection.

### 4.5.1 Proposition
A Levi-Civita connection preserves length and angles under parallel transport.

**Proof:** Let $T = \alpha'(t)$ be tangent to curve $\alpha(T)$, and $X$ and $Y$ be parallel-transported along $\alpha$. By definition, $\nabla_T X = \nabla_T Y = 0$. Then

\[ \nabla_T <X,X> = 2 <\nabla_T X,X> = 0 \Rightarrow \|X\| = \text{constant}. \]
\[ \nabla_T <X,Y> = <\nabla_T X,Y> + <X,\nabla_T Y> \Rightarrow <X,Y> = \text{constant}, \]
\[ \cos \theta = \frac{<X,Y>}{\|X\| \cdot \|Y\|} = \text{constant}. \]

If one takes \( \{e_\alpha\} \) to be a basis of tangent space, the components of the connection in that basis are given by the familiar equation

\[ \nabla_{e_\alpha} e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma. \]

The $\Gamma$’s here are of course the same Christoffel symbols in the equation of Gauss 4.72. We have the following important result:

### 4.5.2 Theorem
In a manifold \( \{M,g\} \) with metric $g$ (the first fundamental form), there exists a unique Levi-Civita connection.

The proof is implicit in the computations that follow leading to equation 4.75, which express the components uniquely in terms of the metric. The entire equation (4.72) must be symmetric on the indices $\alpha\beta$, since $x^\alpha_{\alpha\beta} = x^\beta_{\alpha\beta}$, so $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$ is also symmetric on the lower indices. These quantities are called the **Christoffel symbols of the first kind**. Now we take the inner product with $x^\sigma$ to deduce that

\[ <x^\alpha_{\alpha\beta},x^\sigma> = \Gamma^\gamma_{\alpha\beta}\sigma, \]
\[ = \Gamma^\gamma_{\alpha\beta}g^\gamma\sigma; \]

where we have lowered the third index with the metric on the right hand side of the last equation. The quantities $\Gamma^\alpha_{\alpha\beta\sigma}$ are called **Christoffel symbols of the second kind**. Here we must note that not all indices are created equal. The Christoffel symbols of the second kind are only symmetric on the first two indices. The notation $\Gamma^\alpha_{\alpha\beta\sigma} = [\alpha\beta,\sigma]$ is also used in the literature.

The Christoffel symbols can be expressed in terms of the metric by first noticing that the derivative of the first fundamental form is given by

\[ g_{\alpha\gamma,\beta} = \frac{\partial}{\partial u^\beta} <x^\alpha, x^\gamma>, \]
\[ = <x^\alpha_{\alpha\beta}, x^\gamma> + <x^\alpha, x^\gamma_{\alpha\beta}> \]
\[ = \Gamma^\alpha_{\alpha\beta\gamma} + \Gamma^\gamma_{\gamma\beta\alpha}, \]
Taking other cyclic permutations of this equation, we get
\[
\begin{align*}
g_{\alpha \gamma, \beta} &= \Gamma_{\alpha \beta \gamma} + \Gamma_{\gamma \beta \alpha}, \\
g_{\beta \gamma, \alpha} &= \Gamma_{\alpha \beta \gamma} + \Gamma_{\gamma \alpha \beta}, \\
g_{\alpha \beta, \gamma} &= \Gamma_{\alpha \gamma \beta} + \Gamma_{\gamma \beta \alpha}.
\end{align*}
\]

Adding the first two and subtracting the third of the equations above, and recalling that the \( \Gamma \)'s are symmetric on the first two indices, we obtain the formula
\[
\Gamma_{\alpha \beta \gamma} = \frac{1}{2} \left( g_{\alpha \gamma, \beta} + g_{\beta \gamma, \alpha} - g_{\alpha \beta, \gamma} \right) \quad (4.74)
\]

Raising the third index with the inverse of the metric, we also have the following formula for the Christoffel symbols of the first kind (hereafter, Christoffel symbols refer to the symbols of the first kind, unless otherwise specified.)
\[
\Gamma^\sigma_{\alpha \beta} = \frac{1}{2} g^{\sigma \gamma} \left( g_{\alpha \gamma, \beta} + g_{\beta \gamma, \alpha} - g_{\alpha \beta, \gamma} \right). \quad (4.75)
\]

The Christoffel symbols are clearly symmetric in the lower indices
\[
\Gamma^\sigma_{\alpha \beta} = \Gamma^\sigma_{\beta \alpha}. \quad (4.76)
\]

Unless otherwise specified, a connection on \( \{M, g\} \) refers to the unique Levi-Civita connection.

We derive a well-known formula for the Christoffel symbols for the case \( \Gamma^\alpha_{\alpha \beta} \). From (4.76) we have:
\[
\Gamma^\alpha_{\alpha \beta} = \frac{1}{2} g^{\alpha \gamma} \left( g_{\alpha \gamma, \beta} + g_{\beta \gamma, \alpha} - g_{\alpha \beta, \gamma} \right).
\]

On the other hand,
\[
\begin{align*}
g^{\alpha \gamma} g_{\beta \gamma, \alpha} &= g^{\alpha \gamma} g_{\alpha \beta, \gamma},
\end{align*}
\]
as can be seen by switching the repeated indices of summation \( \alpha \) and \( \sigma \), and using the symmetry of the metric. The equation reduces to
\[
\Gamma^\alpha_{\alpha \beta} = \frac{1}{2} g^{\alpha \gamma} g_{\alpha \beta, \gamma}
\]

Let \( A \) be the cofactor transposed matrix of \( g \). From the linear algebra formula for the expansion of a determinant in terms of cofactors we can get an expression for the inverse of the metric as follows:
\[
\begin{align*}
det(g) &= g_{\alpha \gamma} A^{\alpha \gamma}, \\
\frac{\partial \det(g)}{\partial g_{\alpha \gamma}} &= A^{\alpha \gamma}, \\
g^{\alpha \gamma} &= \frac{A^{\alpha \gamma}}{\det(g)}, \\
&= \frac{1}{\det(g)} \frac{\partial \det(g)}{\partial g_{\alpha \gamma}}
\end{align*}
\]

so that
\[ \Gamma^\alpha_{\alpha \beta} = \frac{1}{2 \det(g)} \frac{\partial \det(g)}{\partial u^\alpha} \frac{\partial}{\partial u^\beta} g_{\alpha \gamma}, \]
\[ = \frac{1}{2 \det(g)} \frac{\partial}{\partial u^\beta} (\det(g)). \quad (4.77) \]

Using this result we can also get a tensorial version of the divergence of the vector field \( \mathbf{X} = v^\alpha e_\alpha \) on the manifold. Using the classical covariant derivative formula \[3.26\] for the components \( v^\alpha \), we define:

\[ \text{Div} \mathbf{X} = \nabla \cdot \mathbf{X} = v^\alpha_{\| \alpha} \]
\[ = \frac{\partial}{\partial u^\alpha} v^\alpha + \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial u^\alpha} (\sqrt{\det(g)} v^\alpha). \quad (4.78) \]

If \( f \) is a function on the manifold, \( df = f_{,\beta} \, du^\beta \) so the contravariant components of the gradient are

\[ (\nabla f)^\alpha = g^{\alpha \beta} f_{,\beta}. \quad (4.80) \]

Combining with equation above, we get a second order operator

\[ \Delta f = \text{Div}(\text{Grad} f), \]
\[ = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial u^\alpha} (\sqrt{\det(g)} g^{\alpha \beta} f_{,\beta}). \quad (4.81) \]

The quantity \( \Delta \) is called the Laplace-Beltrami operator on a function and it generalizes the Laplacian of functions in \( \mathbb{R}^n \) to functions on manifolds.

### 4.5.3 Example Laplacian in Spherical Coordinates

The metric in spherical coordinates is \( ds^2 = dr^2 + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \), so

\[ g_{\alpha \beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}, \quad g^{\alpha \beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{bmatrix}, \quad \sqrt{\det(g)} = r^2 \sin \theta. \]

The Laplace-Beltrami formula gives,

\[ \Delta f = \frac{1}{\sqrt{\det (g)}} \left[ \frac{\partial}{\partial u^1} (\sqrt{\det(g)} g^{11} \frac{\partial f}{\partial u^1}) + \frac{\partial}{\partial u^2} (\sqrt{\det(g)} g^{22} \frac{\partial f}{\partial u^2}) + \frac{\partial}{\partial u^3} (\sqrt{\det(g)} g^{33} \frac{\partial f}{\partial u^3}) \right], \]
\[ = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} (r^2 \sin \theta \frac{\partial f}{\partial r}) + \frac{\partial}{\partial \theta} (r^2 \sin \theta \frac{1}{r^2} \frac{\partial f}{\partial \theta}) + \frac{\partial}{\partial \phi} (r^2 \sin \theta \frac{1}{r^2 \sin^2 \theta} \frac{\partial f}{\partial \phi}) \right], \]
\[ = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \]

The result is the same as the formula for the Laplacian \[3.9\] found by differential form methods.
4.5.4 Example
As an example we unpack the formula for $\Gamma_{11}^1$. First, note that $\det(g) = \|g_{\alpha\beta}\| = EG - F^2$. From equation (4.75) we have

$$\Gamma_{11}^1 = \frac{1}{2} g^{\gamma\gamma} (g_{1\gamma,1} + g_{1\gamma,1} - g_{11,\gamma}),$$

$$= \frac{1}{2} g^{\gamma\gamma} (2g_{1\gamma,1} - g_{11,\gamma}),$$

$$= \frac{1}{2} [g^{11}(2g_{11,1} - g_{11,1}) + g^{12}(2g_{12,1} - g_{11,2})],$$

$$= \frac{1}{2 \det(g)} \{GE_u - F(2F_u - FE_v)\},$$

$$= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}.$$

Due to symmetry, there are five other similar equations for the other $\Gamma$’s. Proceeding as above, we can derive the entire set.

$$\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \quad \Gamma_{11}^2 = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}$$

$$\Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)} \quad \Gamma_{12}^2 = \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$\Gamma_{22}^1 = \frac{2GF_u - GG_u - FG_v}{2(EG - F^2)} \quad \Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$ (4.82)

They are a bit messy, but they all simplify considerably for orthogonal systems, in which case $F = 0$. Another reason why we like those coordinate systems.

4.5.5 Example Harmonic functions. A function $h$ on a surface in $\mathbb{R}^3$ is called Harmonic if it satisfies:

$$\Delta h = 0$$ (4.83)

Noticing that the matrix components of the inverse of the metric are given by

$$g^{\alpha\beta} = \frac{1}{\det(g)} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}$$ (4.84)

we get immediately from (4.81) the classical Laplace-Beltrami equation for surfaces,

$$\Delta h = \frac{1}{\sqrt{EG - F^2}} \left\{ \frac{\partial}{\partial u} \left[ \frac{\sqrt{G}}{\sqrt{EG - F^2}} \frac{\partial h}{\partial u} \right] + \frac{\partial}{\partial v} \left[ \frac{\sqrt{E}}{\sqrt{EG - F^2}} \frac{\partial h}{\partial v} \right] \right\} = 0.$$ (4.85)

If the coordinate patch is orthogonal so that $F = 0$, the equation reduces to:

$$\frac{\partial}{\partial u} \left[ \frac{\sqrt{G}}{\sqrt{E}} \frac{\partial h}{\partial u} \right] + \frac{\partial}{\partial v} \left[ \frac{\sqrt{E}}{\sqrt{G}} \frac{\partial h}{\partial v} \right] = 0$$ (4.86)

If in addition $E = G = \lambda^2$ so that the metric has the form,

$$ds^2 = \lambda^2 (du^2 + dv^2),$$ (4.87)

then,

$$\Delta h = \frac{1}{\lambda^2} \left[ \frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2} \right],$$ (4.88)
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so that \( \Delta^2 h = 0 \) is equivalent to \( \nabla^2 h = 0 \), where \( \nabla^2 \) is the Euclidean Laplacian. (Please compare to the discussion on isothermal coordinates \([4.5.2]\)). Two metrics that differ by a multiplicative factor are called conformally related. The result here means that the Laplacian is conformally invariant under this conformal transformation. This is true in general, and it is essential in applying the elegant methods of complex variables and conformal mappings to solve physical problems involving the Laplacian in the plane.

For a surface \( z = f(x, y) \), which we can write as a Monge patch \( x = \langle x, y, f(x, y) \rangle \), we have \( E = 1 + f_x^2, F = 2f_x f_y \) and \( G = 1 + f_y^2 = 0 \). A short computation shows that in this case, the Laplace-Beltrami equation can be written as, (compare to equation \([5.43]\))

\[
\Delta h = 1 \sqrt{1 + f_x^2 + f_y^2} \left\{ \frac{\partial}{\partial x} \left[ f_x \sqrt{1 + f_x^2 + f_y^2} \right] + \frac{\partial}{\partial y} \left[ f_y \sqrt{1 + f_x^2 + f_y^2} \right] \right\} = 0,
\]

or in terms of the Euclidean \( \mathbb{R}^2 \) del operator \( \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \),

\[
\nabla \cdot \left[ \nabla f \sqrt{1 + \|\nabla f\|^2} \right] = 0.
\]

\((4.89)\)

4.5.2 Curvature Tensor, Gauss’s Theorema Egregium

A fascinating set of relations can be obtained simply by equating \( x_{\beta\gamma\delta} = x_{\beta\delta\gamma} \). First notice that we can also write \( n_\alpha \) in terms of the frame vectors. This is by far easier since \( \langle n_\alpha, n_\alpha \rangle = 1 \) implies that \( \langle n_\alpha, n_\beta \rangle = 0 \), so \( n_\alpha \) lies on the tangent plane and it is therefore a linear combination the tangent vectors. As before, we easily verify that the coefficients are the second fundamental form with a raised index

\[
n_\alpha = -b^\gamma_\alpha x_\gamma.
\]

These are called the formulae of \textbf{Weingarten}.

Differentiating the equation of Gauss and recursively using the formulas of Gauss \([4.72]\) and Weingarten \([4.90]\) we write all the components in terms of the frame, we get

\[
\begin{align*}
x_{\beta\delta} &= \Gamma^\alpha_{\beta\delta} x_\alpha + b_{\beta\delta} n_\alpha, \\
x_{\beta\delta\gamma} &= \Gamma^\alpha_{\beta\delta}\gamma x_\alpha + \Gamma^\alpha_{\beta\delta} x_\alpha + b_{\beta\delta}\gamma n_\alpha + b_{\beta\delta} n_\gamma \\
&= \Gamma^\alpha_{\beta\delta}\gamma x_\alpha + \Gamma^\alpha_{\beta\delta} [\Gamma^\alpha_{\mu\gamma} x_\mu + b_{\alpha\gamma} n] + b_{\beta\delta}\gamma n - b_{\beta\delta} b^\gamma_\alpha x_\alpha, \\
x_{\beta\delta\gamma} &= \Gamma^\alpha_{\beta\delta}\gamma x_\alpha + \Gamma^\mu_{\beta\delta} x_\mu + b_{\beta\gamma} n_\alpha + b_{\beta\delta} n_\gamma, \\
x_{\beta\gamma\delta} &= \Gamma^\alpha_{\beta\gamma}\delta x_\alpha + \Gamma^\mu_{\beta\gamma} x_\mu + b_{\beta\gamma} b^\delta_\alpha x_\alpha + \Gamma^\alpha_{\beta\gamma} b_{\alpha\delta} n.
\end{align*}
\]

\((4.91)\)

The last equation above was obtained from the preceding one just by permuting \( \delta \) and \( \gamma \). Subtracting that last two equations and setting the tangential component to zero we get

\[
R^\alpha_{\beta\gamma\delta} = b_{\beta\delta} b^\alpha_{\gamma} - b_{\beta\gamma} b^\alpha_\delta,
\]

where the components of the \textbf{Riemann tensor} \( R \) are defined by
\[ R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\mu_{\beta\gamma} \Gamma^\alpha_{\delta\mu} - \Gamma^\mu_{\beta\delta} \Gamma^\alpha_{\gamma\mu}. \]  
(4.94)

Technically we are not justified at this point in calling \( R \) a tensor since we have not established yet the appropriate multi-linear features that a tensor must exhibit. We address this point at a later chapter. Lowering the index above we get

\[ R_{\alpha\beta\gamma\delta} = b_{\beta\delta} b_{\alpha\gamma} - b_{\beta\gamma} b_{\alpha\delta}. \]  
(4.95)

### 4.5.6 Theorema Egregium

Let \( M \) be a smooth surface in \( \mathbb{R}^3 \). Then,

\[ K = \frac{R_{1212}}{\det(g)}. \]  
(4.96)

**Proof:** Let \( \alpha = \gamma = 1 \) and \( \beta = \delta = 2 \) above. The equation then reads

\[ R_{1212} = b_{22} b_{11} - b_{21} b_{12}, \]
\[ = (e g - f^2), \]
\[ = K(E F - G^2), \]
\[ = K \det(g). \]

The remarkable result is that the Riemann tensor and hence the Gaussian curvature does not depend on the second fundamental form but only on the coefficients of the metric. Thus, the Gaussian curvature is an intrinsic quantity independent of the embedding, so that two surfaces that have the same first fundamental form have the same curvature. In this sense, the Gaussian curvature is a bending invariant!

Setting the normal components equal to zero gives

\[ \Gamma^\alpha_{\beta\delta} b_{\alpha\gamma} - \Gamma^\alpha_{\beta\gamma} b_{\alpha\delta} + b_{\beta\delta,\gamma} - b_{\beta\gamma,\delta} = 0 \]  
(4.97)

These are called the **Codazzi** (or Codazzi-Mainardi) equations.

Computing the Riemann tensor is labor intensive since one must first obtain all the non-zero Christoffel symbols as shown in the example above. Considerable gain in efficiency results from a form computation. For this purpose, let \( \{e_1, e_2, e_3\} \) be a Darboux frame adapted to the surface \( M \), with \( e_3 = n \). Let \( \{\theta^1, \theta^2, \theta^3\} \) be the corresponding orthonormal dual basis. Since at every point, a tangent vector \( X \in TM \) is a linear combination of \( \{e_1, e_2\} \), we see that \( \theta^3(X) = 0 \) for all such vectors. That is, \( \theta^3 = 0 \) on the surface. As a consequence, the entire set of the structure equations is

\[ d\theta^1 = -\omega^1_2 \wedge \theta^2, \]  
(4.98)
\[ d\theta^2 = -\omega^2_1 \wedge \theta^1, \]  
(4.99)
\[ d\theta^3 = -\omega^3_1 \wedge \theta^1 - \omega^3_2 \wedge \theta^2 = 0, \]  
(4.100)
\[ d\omega^1_2 = -\omega^1_3 \wedge \omega^2_3, \]  
Gauss Equation  
(4.101)
\[ d\omega^1_3 = -\omega^1_2 \wedge \omega^3_2, \]  
Codazzi Equations  
(4.102)
\[ d\omega^2_3 = -\omega^2_1 \wedge \omega^3_1, \]  
(4.103)

The key result is the following theorem

### 4.5.7 Curvature Form Equations
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\[ d\omega^1_2 = K \theta^1 \wedge \theta^2, \]
\[ \omega^3 \wedge \theta^2 + \omega^2 \wedge \theta^1 = -2H \theta^1 \wedge \theta^2. \]  

**Proof:** By applying the Weingarten map to the basis vector \( \{e_1, e_2\} \) of \( TM \), we find a matrix representation of the linear transformation:

\[ L e_1 = -\nabla e_1 e_3 = -\omega^1_3(e_1) e_1 - \omega^2_3(e_1) e_2, \]
\[ L e_2 = -\nabla e_2 e_3 = -\omega^1_3(e_2) e_1 - \omega^2_3(e_2) e_2. \]

Recalling that \( \omega \) is antisymmetric, we find:

\[ K = \det(L) = -[\omega^1_3(e_1) \omega^2_2(e_1) - \omega^1_3(e_2) \omega^2_2(e_1)] \]
\[ = -(\omega^1_3 \wedge \omega^2_2)(e_1, e_2) \]
\[ = d\omega^1_2(e_1, e_2). \]

Hence

\[ d\omega^1_2 = K \theta^1 \wedge \theta^2. \]

Similarly, recalling that \( \theta^1(e_j) = \delta^1_j \), we have

\[ (\omega^3 \wedge \theta^2 + \omega^2 \wedge \theta^1)(e_1, e_2) = \omega^1_3(e_1) - \omega^2_3(e_2) \]
\[ = \omega^1_3(e_1) + \omega^2_3(e_2) \]
\[ = \text{Tr}(L) = -2H \]

4.5.8 **Definition** A point of a surface at which \( K = 0 \) is called a **planar point**. A surface with \( K = 0 \) at all points is called a **flat** or **Gaussian flat** surface. A surface on which \( H = 0 \) at all points is called a **minimal** surface.

4.5.9 **Example** Sphere
Since the first fundamental form is \( I = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2 \), we have

\[ \theta^1 = a d\theta \]
\[ \theta^2 = a \sin \theta d\phi \]
\[ d\theta^2 = a \cos \theta d\theta \wedge d\phi \]
\[ = -\cos \theta d\phi \wedge \theta^1 = -\omega^1_2 \wedge \theta^1 \]
\[ \omega^1_2 = \cos \theta d\phi = -\omega^1_2 \]
\[ d\omega^1_2 = \sin \theta d\theta \wedge d\phi = \frac{1}{a^2} (a d\theta) \wedge (a \sin \theta d\phi) \]
\[ = \frac{1}{a^2} \theta^1 \wedge \theta^2 \]
\[ K = \frac{1}{a^2} \]

4.5.10 **Example** Torus
Using the the parametrization (See 4.23),
\[ \mathbf{x} = ((b + a \cos \theta) \cos \phi, (b + a \cos \theta) \sin \phi, a \sin u) \]

the first fundamental form is

\[ ds^2 = a^2 d\theta^2 + (b + a \cos \theta)^2 d\phi^2. \]

Thus, we have:

\[ \begin{align*}
\theta^1 &= a \ d\theta \\
\theta^2 &= (b + a \cos \theta) \ d\phi \\
d\theta^2 &= -a \sin \theta \ d\theta \wedge d\phi \\
&= \sin \theta \ d\phi \wedge \theta^1 = -\omega_1 \wedge \theta^1 \\
\omega_1 &= -\sin \theta \ d\phi = -\omega_2 \\
d\omega_1^2 &= \cos \theta \ d\theta \wedge d\phi = \frac{\cos \theta}{a(b + a \cos \theta)} (a \ d\theta) \wedge [(a + b \cos \theta) \ d\phi] \\
&= \frac{\cos \theta}{a(b + a \cos \theta)} \theta^1 \wedge \theta^2 \\
K &= \frac{\cos \theta}{a(b + a \cos \theta)}.
\end{align*} \]

This result makes intuitive sense.

When \( \theta = 0 \), the points lie on the outer equator, so \( K = \frac{1}{a(b + a)} > 0 \) is the product of the curvatures of the generating circle and the outer equator circle. The points are elliptic.

When \( \theta = \pi/2 \), the points lie on the top of the torus, so \( K = 0 \). The points are parabolic.

When \( \theta = \pi \), the points lie on the inner equator, so \( K = \frac{-1}{a(b - a)} < 0 \) is the product of the curvatures of the generating circle and the inner equator circle. The points are hyperbolic.

4.5.11 Example Orthogonal Parametric Curves

The examples above have the common feature that the parametric curves are orthogonal and hence \( F = 0 \). Using the same method, we can find a general formula for such cases. Since the first fundamental form is given by

\[ I = Edu^2 + Gdv^2 \]

We have:
\[ \theta^1 = \sqrt{E} \, du \]
\[ \theta^2 = \sqrt{G} \, dv \]
\[ d\theta^1 = (\sqrt{E})_v \, dv \wedge du = -(\sqrt{E})_u \, du \wedge dv \]
\[ = -\left( \frac{\sqrt{E}}{\sqrt{G}} \right) du \wedge \theta^2 = -\omega^1_2 \wedge \theta^2 \]
\[ d\theta^2 = (\sqrt{G})_u \, du \wedge dv = -(\sqrt{G})_v \, dv \wedge du \]
\[ = -\left( \frac{\sqrt{G}}{\sqrt{E}} \right) dv \wedge \theta^2 = -\omega^2_1 \wedge \theta^1 \]
\[ \omega^1_2 = \left( \frac{\sqrt{E}}{\sqrt{G}} \right) du - \left( \frac{\sqrt{G}}{\sqrt{E}} \right) dv \]
\[ d\omega^2_1 = -\frac{\partial}{\partial u} \left( \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) du \wedge dv + \frac{\partial}{\partial v} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) dv \wedge du \]
\[ = -\frac{1}{\sqrt{EG}} \left[ \frac{\partial}{\partial u} \left( \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right] \theta^1 \wedge \theta^2. \]

Therefore, the Gaussian Curvature of a surface mapped by a coordinate patch in which the parametric lines are orthogonal is given by:
\[ K = -\frac{1}{\sqrt{EG}} \left[ \frac{\partial}{\partial u} \left( \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right] \] (4.106)

Again, to connect with more classical notation, if a surface described by a coordinate patch \( x(u, v) \) has first fundamental for given by \( I = E \, du^2 + G \, dv^2 \), then
\[ dx = x_u \, du + x_v \, dv, \]
\[ = \frac{x_u}{\sqrt{E}} \, du + \frac{x_v}{\sqrt{G}} \, dv, \]
\[ = \frac{x_u}{\sqrt{E}} \theta^1 + \frac{x_v}{\sqrt{G}} \theta^2, \]
\[ dx = e_1 \theta^1 + e_2 \theta^2, \quad (4.107) \]
where
\[ e_1 = \frac{x_u}{\sqrt{E}}, \quad e_2 = \frac{x_v}{\sqrt{G}}. \]

Thus, when the parametric curves are orthogonal, the triplet \( \{ e_1, e_2, e_3 = n \} \) constitutes a moving orthonormal frame adapted to the surface. The awkwardness of combining calculus vectors and differential forms in the same equation is mitigated by the ease of jumping back and forth between the classical and the modern formalism. Thus, for example, covariant differential of the normal in \[4.103\] can be rewritten without the arbitrary vector in the operator \( LX \) as shown:
\[ \nabla_X e_3 = \omega^1_3(X) e_1 + \omega^2_3(X) e_2, \quad (4.108) \]
\[ de_3 = e_1 \omega^1_3 + e_2 \omega^2_3 = 0, \quad (4.109) \]
The equation just expresses the fact that the components of the Weingarten map, that is, the second fundamental form in this basis, can be written as some symmetric matrix given by:

\[
\begin{align*}
\omega^1_3 &= l\theta^1 + m\theta^2, \\
\omega^2_3 &= m\theta^1 + n\theta^2.
\end{align*}
\] (4.110)

In the case that \( E = 1 \), so that the metric is

\[
ds^2 = du^2 + G(u, v)dv^2,
\] (4.111)

the equation for curvature above reduces even further to:

\[
K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}.
\] (4.112)

The case is not as special as it appears at first. The change of parameters

\[
\hat{u}' = \int_0^u \sqrt{E} \, du
\]
results on \( d\hat{u}'^2 = E \, du^2 \), and thus it transforms an orthogonal system to one with \( E = 1 \). The parameters are reminiscent of polar coordinates \( ds^2 = dr^2 + r^2 d\phi^2 \). We will have more to say about this in the context of geodesics.

A slick proof of the Teorema Egregium can be obtained by differential forms. Let \( F : M \to \tilde{M} \) be an isometry between two surfaces with metrics \( g \) and \( \tilde{g} \) respectively. Let \( \{e_\alpha\} \) be an orthonormal basis for dual basis \( \{\theta^\alpha\} \). Define \( \tilde{e}_\alpha = F^* e_\alpha \). Recalling that isometries preserve inner products, we have

\[
<\tilde{e}_\alpha, \tilde{e}_\beta> = <F^* e_\alpha, F^* e_\beta> = <e_\alpha, e_\beta> = \delta_{\alpha\beta}.
\]

Thus, \( \{\tilde{e}_\alpha\} \) is also an orthonormal basis of the tangent space of \( \tilde{M} \). Let \( \tilde{\theta}^\alpha \) be the dual forms and denote with tilde's the connection forms and Gaussian curvature of \( \tilde{M} \).

**4.5.12 Theorem** (Teorema Egregium)

a) \( F^* \tilde{\theta}_\alpha = \theta_\alpha \),

b) \( F^* \tilde{\omega}^\alpha_\beta = \omega^\alpha_\beta \),

c) \( F^* \tilde{K} = K \).

**Proof:**

a) It suffices to show that the forms agree on basis vectors. We have

\[
F^* \tilde{\theta}_\alpha(e_\beta) = \tilde{\theta}_\alpha(F^* e_\beta),
\]

\[
= \tilde{\theta}_\alpha(\tilde{e}_\beta),
\]

\[
= \delta^{\alpha}_\beta,
\]

\[
= \theta_\alpha(e_\beta).
\]

b) We compute the pull-back of the first structure equation in \( \tilde{M} \):

\[
d\tilde{\theta}^\alpha + \tilde{\omega}^\alpha_\beta \wedge \tilde{\theta}^\beta = 0,
\]

\[
F^* d\tilde{\theta}^\alpha + F^* \tilde{\omega}^\alpha_\beta \wedge F^* \tilde{\theta}^\beta = 0,
\]

\[
d\theta^\alpha + F^* \tilde{\omega}^\alpha_\beta \wedge \theta^\beta = 0,
\]
The connection forms are defined uniquely by the first structure equation, so

\[ F^* \tilde{\omega}^\alpha_{\beta} = \omega^\alpha_{\beta} \]

c) In a similar manner, we compute the pull-back of the curvature equation:

\[
\begin{align*}
\mathcal{d}\tilde{\omega}^{12}_2 &= \tilde{\theta}^1 \wedge \tilde{\theta}^2, \\
F^* \mathcal{d}\tilde{\omega}^{12}_2 &= (F^* \tilde{\theta}) F^* \tilde{\theta}^1 \wedge F^* \tilde{\theta}^2, \\
\mathcal{d}F^* \tilde{\omega}^{12}_2 &= (F^* \tilde{\theta}) F^* \tilde{\theta}^1 \wedge F^* \tilde{\theta}^2, \\
\mathcal{d}\omega^{12}_2 &= (F^* K) \theta^1 \wedge \theta^2, \\
\end{align*}
\]

So again by uniqueness, \( F^* K = K \).

4.5.13 Example Catenoid - Helicoid

Perhaps the most celebrated classical is that of mapping between a helicoid \( M \) and a catenoid \( \tilde{M} \). Let \( a = c \), and label the coordinate patch for the former as \( x(u^\alpha) \) and \( y(\tilde{u}^\alpha) \) for the latter. The first fundamental forms are given as in 4.24 and 4.25.

\[
\begin{align*}
ds^2 &= du^2 + (u^2 + a^2) dv^2, & E &= 1, & G &= u^2 + a^2, \\
ds^2 &= \tilde{u}^2 \left( du^2 + \tilde{u}^2 dv^2 \right) \text{ with } \tilde{E} = \frac{\tilde{u}^2}{\tilde{u}^2 - a^2}, & \tilde{G} &= \tilde{u}^2.
\end{align*}
\]

Let \( F : M \to \tilde{M} \) be the mapping \( y = F(x) \), defined by \( \tilde{u}^2 = u^2 + a^2 \) and \( \tilde{v} = v \). Since \( \tilde{v} \, du = u \, dv \), we have \( \tilde{v} \, du^2 = u^2 \, dv^2 \) which shows that the mapping preserves the metric and hence it is an isometry. The Gaussian curvatures \( K \) and \( \tilde{K} \) follow from an easy computation using formula \( 4.106 \).

\[
\begin{align*}
K &= -\frac{1}{\sqrt{u^2 + a^2}} \left( \frac{\partial}{\partial u} \sqrt{u^2 + a^2} \right) = \frac{a^2}{(u^2 + a^2)^{3/2}}, \\
\tilde{K} &= -\frac{\sqrt{\tilde{u}^2 - a^2}}{\tilde{u}^2} \left( \frac{\partial}{\partial \tilde{u}} \sqrt{\tilde{u}^2 - a^2} \right) = -\frac{a^2}{\tilde{u}^4}.
\end{align*}
\]

It is immediately evident by substitution that as expected \( F^* \tilde{K} = K \). Figure 4.14 shows several stages of a one-parameter family \( M_t \) of isometries deforming a catenoid into a helicoid. The one-parameter family of coordinate patches chosen is

\[ z_t = (\cos t) \mathbf{x} + (\sin t) \mathbf{y} \]

Writing the equation of the coordinate patch \( z_t \) in complete detail, one can compute the coefficients

\[
\text{Fig. 4.14: Catenoid - Helicoid isometry}
\]
of the fundamental forms and thus establish the family of surfaces has mean curvature $H$ independent of the parameter $t$, and in fact $H = 0$ for each member of the family. We will discuss at a later chapter the geometry of surfaces of zero mean curvature.

4.5.14 Example Isothermal Coordinates.
Consider the case in which the metric has the form
\[ ds^2 = \lambda^2 (du^2 + dv^2), \] (4.116)
so that $E = G = \lambda^2$, $F = 0$. A metric in this form is said to be in isothermal coordinates. Inserting into equation \[4.106\] we get
\[
K = -\frac{1}{\lambda^2} \left[ \frac{\partial}{\partial u} \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial v} \right) \right],
\]
\[= -\frac{1}{\lambda^2} \left[ \frac{\partial}{\partial u} \frac{\partial}{\partial u} (\ln \lambda) + \frac{\partial}{\partial v} \frac{\partial}{\partial v} (\ln \lambda) \right].\]

Hence,
\[ K = -\frac{1}{\lambda^2} \nabla^2 (\ln \lambda). \] (4.117)

The tantalizing appearance of the Laplacian in this coordinate system gives an inkling that there is some complex analysis lurking in the neighborhood. Readers acquainted with complex variables will recall that the real and imaginary parts of holomorphic functions satisfy Laplace’s equations and that any holomorphic function in the complex plane describes a conformal map. In anticipation of further discussion on this matter, we prove the following:

4.5.15 Theorem Define the mean curvature vector $H = Hn$. If $x(u, v)$ is an isothermal parametrization of a surface, then
\[ x_{uu} + x_{vv} = 2\lambda^2 H. \] (4.118)

Proof: Since the coordinate patch is isothermal, $E = G = \lambda^2$ and $F = 0$. Specifically, we have $\langle x_u, x_u \rangle = \langle x_v, x_v \rangle$, and $\langle x_u, x_v \rangle = 0$. Differentiation then gives:
\[
\langle x_{uu}, x_u \rangle = \langle x_{vu}, x_u \rangle, \]
\[= -\langle x_{uv}, x_u \rangle, \]
\[\langle x_{uu} + x_{vv}, x_u \rangle = 0. \]

In the same manner,
\[
\langle x_{uv}, x_v \rangle = \langle x_{vu}, x_v \rangle, \]
\[= -\langle x_{uu}, x_v \rangle, \]
\[\langle x_{uu} + x_{vv}, x_v \rangle = 0. \]

If follows that $x_{uu} + x_{vv}$ is orthogonal to the surface and points in the direction of the normal $n$. On the other hand,
\[ \frac{E_g + G_e}{2EG} = H, \]
\[ \frac{g + e}{2\lambda^2} = H, \]
\[ e + g = 2\lambda^2 H, \]
\[ \langle x_{uu} + x_{uv}, n \rangle = 2\lambda^2 H, \]
\[ x_{uu} + x_{uv} = 2\lambda^2 H. \]
Chapter 5

Geometry of Surfaces

5.1 Surfaces of Constant Curvature

5.1.1 Ruled and Developable Surfaces

We present a brief discussion of surfaces of constant curvature $K = 0$, since $K = k_1 k_2$, a surface with zero Gaussian curvature at each point must have a principal direction with zero normal curvature, that is, either $k_1 = 0$ or $k_2 = 0$. It is therefore a necessary condition for a surface to have $K = 0$, that at each point there be a principal direction which is a straight line. A surface having this property of containing a straight line or segment of a straight line at each point is called a ruled surface. We may think of a ruled surface as a surface generated by the motion of a straight line. Given a point $p$ on a ruled surface, let $\alpha(t)$ be a curve with $\alpha(0) = p$, and let $X(t)$ be a unit vector field on the curve and pointing along the lines at their intersection points with the curve. On can then parametrize the surface near $p$ by a coordinate patch $y(t,v) = \alpha(t) + v X(t)$ as shown in the figure.

Having a straight line passing through each point in the surface is a necessary but not sufficient condition to insure that $K = 0$, as illustrated by the following examples

1) Saddle. Consider the saddle $z = xy$ which is trivially parametrized by the coordinate patch $y(u,v) = (u, v, uv)$. The patch can be written as $y(u,v) = (u, 0, v) + v(0, 1, u)$ or as $y(u,v) = (v, 0, u) + u(1, 0, v)$, so that the surface is doubly-ruled as shown in figure 5.1(a). The rulings are the coordinate curves $u =$ constant, and $v =$ constant. This neat fact is reflected in some architectural designs of simple structures with roofs made of straight slabs arranged in the shape of a hyperbolic paraboloid. A short computation gives $K = -(1 + u^2 + v^2)^{-2} = -(1 + x^2 + y^2)^{-2}$.

2) Hyperboloid. A common calculus example of a doubly ruled surface is given by circular hyperboloids of one sheet. Consider the circle $\alpha(u) = (\cos u, \sin u, 0)$, and the vector field $X(u) = \dot{\alpha} + k$ which points at a constant skew angle $\pi/4$. Then the $(x, y, z)$ coordinates in the parametrization $y(u,v) = \alpha(u) + v X(u)$, 

$$y(u,v) = (\cos u, \sin u, 0) + v(-\sin u, \cos u, 1),$$

$$y(u,v) = (\cos u - v \sin u, \sin u + v \sin u, v),$$

satisfy the equation $x^2 + y^2 - z^2 = 1$; that is, the surface is a circular hyperboloid of one sheet. The coordinate curves $s = $constant are straight line generators. If instead, we choose $X(u) = -\dot{\alpha} + k$, we get the same surface, but with an orthogonal set of line-generators as shown in figure 5.1(b).
5.1(b). This is an example of a surface in which the asymptotic curves are orthogonal at each point. Tangent planes to the surface at any point in the circle \( x^2 + y^2 = 1 \) at the throat, intersect the hyperboloid in a pair of line generators. The Gaussian curvature is also negative and is given by

\[
K = -(1 + 2v^2)^{-2} = -(1 + 2z^2)^{-2}.
\]

The double-ruled nature of the circular hyperboloid has been exploited by civil engineers in the design of heavy-duty gears with long teeth engaging along the lines. The double-ruling is also advantageous for the construction of the metal frame to support the cooling towers of nuclear reactors.

3) Möbius Band. The formal definition of an orientable surface \( M \) is that there exists a two-form that is non-zero at every point of \( M \). The idea is that the 2-form represents the oriented differential of surface area \( dS = \sqrt{\det g} \, du \wedge dv \). For the present purpose, an intuitive characterization is that there exists a unit normal vector field on \( M \). The Möbius Band is the most famous example of a non-orientable surface. It can be parametrized by the coordinate patch

\[
y(u, v) = \alpha(u) + vX(u),
\]

\[
\alpha(u) = (\cos 2u, \sin 2u, 0),
\]

\[
X(u) = (\cos u \cos 2u, \cos u \sin 2u, \sin u),
\]

\[
x(u, v) = (\cos 2u + v \cos u \cos 2u, \sin 2u + v \cos u \sin 2u, v \sin u).
\]  

(5.1)

The curve \( \alpha(u) \) is a circle, and the vector \( X(u) \) on the circle points in a direction that winds around by an angle \( \pi \) in one revolution. The parameter \( v \) here is restricted to \([-0.2, 0.2]\), so the generating line segment flips after one turn as shown in figure 5.1. Indeed, at \( u = 0 \) the generating line segment is \( vX(0) = v(1, 0, 0) \) but after one revolution at \( u = \pi \), the generating line segment given by \( vX(\pi) = v(-1, 0, 0) \), points in the opposite direction. We can interpret the topology of the surface as a rectangle with a pair of opposite sides identified (see figure 9.2). Clearly, it is impossible to choose a well defined normal vector field. The Gaussian curvature is somewhat messy, but the computation shows that \( K \) is negative everywhere.

![Example of Ruled Surfaces](image)

Fig. 5.1: Examples of Ruled Surfaces

Let \( M \) be a ruled surface with unit normal \( N \) and let \( X \) be a unit vector field tangent along the straight lines that generate the surface. The straight lines are geodesic in \( \mathbb{R}^3 \) so \( \nabla_X X = 0 \). By Gauss’s equation 4.73, \( \nabla_X X = 0 \), so the line is also a geodesic on the surface and (\( < LX, X > = 0 \), that is, the generator lines are asymptotic. Let \( Y \) be a unit tangent vector field orthogonal to \( X \) so that the pair constitutes an orthogonal basis of the tangent space at each point. Then

\[
K = < LX, X > < LY, Y > - < LX, Y >^2,
\]

\[
= - < LX, Y >^2 = - f^2,
\]

so we conclude that \( K \leq 0 \). If the vectors \( X \) and \( Y \) are not an orthogonal basis, the result must be
5.1. SURFACES OF CONSTANT CURVATURE

modified as in equation 4.65 which gives,

\[ K = -\frac{f^2}{EG - F^2}. \] (5.2)

The general formula for the Gaussian curvature of a ruled surface is obtained by a straightforward computation. We have:

\[ y_u = \alpha' + vX', \]
\[ y_v = X, \]
\[ y_u \times y_v = (\alpha' + vX') \times X, \]
\[ EG - F^2 = \|y_u \times y_v\| = \|X \times (\alpha' + vX')\|. \]
\[ y_{uv} = X', \quad y_{vv} = 0. \]

Hence, \( g = \langle y_{vv}, N \rangle = 0 \), and \( f = \langle y_{uv}, N \rangle = (\alpha'XX')/\sqrt{EG - F^2} \), where we are using the notation for the triple product. The resulting curvature is:

\[ K = -\frac{(\alpha'XX')}{\|X \times (\alpha' + vX')\|^4}. \] (5.3)

We may choose the orthogonal trajectories \( \alpha(u) \) to be integral curves of \( Y \) parametrized by arc length, so that \( Y = \alpha'(u) = T \). Since by choice, \( T \) and \( X \) are orthogonal unit vectors tangent to the surface, \( N = T \times X \) is normal to the surface and we have an orthonormal frame at each point. The covariant derivatives of the frame with respect to \( T \) along the one-parameter curves \( \alpha \) is just derivative with respect to the parameter, so we we get Frenet-like frame

\[ T' = c_1X + c_2N, \]
\[ X' = -c_1T + c_3N, \]
\[ N' = -c_2T - c_3X. \] (5.4)

A one-line computation gives:

\[ c_3 = -\langle X, N' \rangle = \langle N, X' \rangle = (XX'). \]

The function

\[ p(u) = \frac{(XX')}{\|X'\|^2} = \frac{c_3}{c_1^2 + c_3^2}, \]

is called the distribution parameter. Substituting 5.4 into 5.3 we rewrite the Gaussian curvature as

\[ K = -\frac{1}{\|X \times (T + vX')\|^4}, \]
\[ = -\frac{c_3^2}{[1 - 2c_1v + c_1^2v^2 + v^2c_3^2]^2}. \]

The special curve along which \( c_1 = 0 \) that is, \( (T', X) = 0 \) is called the stricture curve. Using the parametrization with the base curve being the stricture curve, we have \( p(u) = 1/c_3 \) and

\[ K = -\frac{c_3^2}{(1 + v^2c_3^2)} = \frac{p^2(u)}{(p^2(u) + v^2)}. \] (5.5)
A beautiful example is the hyperboloid of revolution in figure 5.1(b). The circle \( \alpha(t) \) at the throat used to generate the surface is the stricture curve. It turns out that \( X \) does not need to be orthogonal to \( T \) as it is the case here, as long as \( \|x\| = 1 \) and \( (T', X) = 0 \).

A ruled surface is called a **developable surface** if in addition, \( LX = \nabla_X N = 0 \), that is, the normal vector is parallel along the generating lines. Then equality holds in \( K \leq 0 \) and we have the following theorem

**5.1.1 Theorem**  A necessary and sufficient condition for surface to be developable is to have Gaussian curvature \( K = 0 \).

It is surprising that the general case of a closed and connected surface with \( K = 0 \) to be developable was not proved until 1961 in a short paper by Massey.

A particularly interesting type of developable surfaces are those in which the vector \( X \) is taken to be the tangent vector \( T \) of the curve \( \alpha \) itself. A surface with this property is called a **tangential developable**.

**5.1.2 Example** Developable Helicoid. A helicoid \( x(u,v) = (0,0,v) \) can be written in the form \( x = \alpha(v) + uX(v) \), where \( \alpha(v) = (\cos v, \sin v, 0) \) and \( X(v) = (\cos v, \sin v, 0) \), so it is a ruled surface. The surface has negative curvature as computed in 4.113 and the stricture curve is the \( z \)-axis. A neat related surface is obtained by the tangential developable of a helix. We choose \( \alpha(u) = (\cos u, \sin u, u) \) and \( X = T = (−\sin u, \cos u, 1) \) so that

\[
x(u,v) = (\cos u - v \sin u, \sin u + v \cos u, u + v).
\]

(5.6)

Since this is flat surface having \( K = 0 \) it is isometric to a plane. Indeed, if one takes a thin cardboard annulus with a slit in the \( xy \)-plane with the appropriate radius, one can bend the annulus around a cylinder by lifting one edge of the slit, thus creating a ribbon that wraps around the cylinder as shown in figure 5.2. A magnificent architectural example is exhibited by base of the spiral staircase near the pyramid of the Louvre museum. For the maple-generated image, a small numerical computation was carried to figure out the vertical shift and radius of the helicoid so that the staircase and the supporting developable match at the helix of intersection.

![Fig. 5.2: Developable Helicoid](image)

**5.1.2 Surfaces of Constant Positive Curvature**

In this section we prove a few global theorems. We assume the reader is acquainted with the notion of a compact space. In particular, in \( \mathbb{R}^n \) a compact set is closed and bounded so it is contained in a ball of sufficiently large radius, centered at the origin. We are concerned with compact manifolds,
which by definition are locally Euclidean and have a differentiable structure. Thus a compact surface in $\mathbb{R}^3$ can not have any edges or creases, and the tangent space is well defined at all points.

![Compact Surface](image)

**Fig. 5.3: Compact Surface**

5.1.3 **Theorem** In any compact surface in $\mathbb{R}^3$ there exists at least one point $p$ at which $K(p) > 0$.

**Proof:** Let $M$ be compact. Consider the function $f : M \to \mathbb{R}$ defined by $f = \|x\|$, where $x$ are local coordinates of a point on the surface. This is a continuous function on compact space, so it attains a maximum at a least one point $p$. The geometric interpretation of $p$ is that it is farthest away from the origin. The intuition about this theorem is simply that near the point $p$, the surface is entirely on one one side of the tangent plane as shown in figure [5.3](image), so the principal curvatures have the same sign. We make this formal. Let $R$ be the distance from $p$ to the origin, and construct a sphere of radius $R$ centered at the origin. The sphere will be tangential to the surface at $p$. Given a unit tangent $T$, let $\alpha(t)$ be unit speed integral curve near $p$, that is, $\alpha(0) = p$ and $T = \alpha'(0)$. The composite function $f(\alpha(t))$ also has a maximum at $p$, so by the second derivative test, we have $[f(\alpha)]''(0) = 0$ and $[f(\alpha)]'''(0) < 0$. By the definition of $f$, $f(\alpha) = \|\alpha^2\| = (\alpha, \alpha)$, so $f(\alpha)''(0) = 2(\alpha, \alpha')(0) = 0$. We conclude that the position vector $x_o = \alpha(0)$ of the point $p$ is orthogonal to $T$. Since this is true for any such $T$, the vector $x_o$ is also normal to the surface, so that the unit normal is $n = x_o/R$. Computing the second derivative we get

$$\frac{1}{2} [f(\alpha)]'''(0) = <\alpha, \alpha''>'(0),$$

$$= <T, T> + <x_o, \alpha''>(0),$$

$$= 1 + R <n, \alpha''(0) > < 0,$$

But clearly $<n, \alpha''(0) >$ is the normal curvature along $T$ so

$$k_n(p) < -\frac{1}{R},$$

Again, since $T$ was arbitrary, the normal curvature is less than $-1/R$ in any direction, a geometric indication that the surface is bending more than the sphere as intuitively shown by the picture. Therefore

$$K(p) = \kappa_1\kappa_2 > \frac{1}{R^2} > 0.$$

5.1.4 **Theorem** In a surface in which the coordinate directions are chosen to be the principal
directions of curvature, the Codazzi equations are

\[
\begin{align*}
\frac{\partial \kappa_1}{\partial u} &= \frac{1}{2} \frac{E}{E} (\kappa_2 - \kappa_1), \\
\frac{\partial \kappa_2}{\partial u} &= \frac{1}{2} \frac{G}{G} (\kappa_1 - \kappa_2)
\end{align*}
\]

\tag{5.7}

**Proof:** Let \(X\) and \(Y\) be eigenvectors of \(L\), so that

\[
LX = \kappa_1 X, \quad \kappa_1 = \frac{<LX,X>}{<X,X>} = \frac{e}{E},
\]

\[
LY = \kappa_2 Y, \quad \kappa_1 = \frac{<LY,Y>}{<Y,Y>} = \frac{g}{G}.
\]

If \(\kappa_1 \neq \kappa_2\), the eigenvectors are orthogonal, so taken them as the parametric directions means that \(F = f = 0\). The Codazzi equations \tag{4.97} are obtained by setting to zero the normal component of \(x_{\alpha\beta\gamma} - x_{\beta\alpha\gamma} = 0\). In terms of the covariant derivative formulation of the Gauss \tag{4.73} with \(X = e_1 = x_u, Y = e_2 = x_v, Z = e_\gamma\), the normal component of \(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = 0\) result in the equations of Codazzi in the form: (See \tag{6.21})

\[
\begin{align*}
<\nabla_X L Y - \nabla_Y L X, Z> &= 0, \\
\nabla_X L Y - \nabla_Y L X &= 0
\end{align*}
\]

We proceed to expand this equation:

\[
\nabla_{e_1} Le_2 - \nabla_{e_2} Le_1 = 0,
\]

\[
\nabla_{e_1} (\kappa_2 e_2) - \nabla_{e_2} (\kappa_1 e_1) = 0,
\]

\[
\frac{\partial \kappa_2}{\partial u} e_2 + \kappa_2 \Gamma^{\alpha}_{12} e_\alpha - \frac{\partial \kappa_1}{\partial v} e_1 - \kappa_1 \Gamma^{\alpha}_{21} e_\alpha = 0.
\]

Setting the \(e_1\) and \(e_2\) components to zero, we get:

\[
\begin{align*}
\frac{\partial \kappa_2}{\partial u} &= \kappa_1 \Gamma^{2}_{21} - \kappa_2 \Gamma^{2}_{12} = (\kappa_1 - \kappa_2) \Gamma^{2}_{12}, \\
\frac{\partial \kappa_1}{\partial v} &= \kappa_2 \Gamma^{1}_{12} - \kappa_1 \Gamma^{1}_{21} = (\kappa_2 - \kappa_1) \Gamma^{1}_{21}.
\end{align*}
\]

The result follows immediately from the expressions for the Christoffel symbols \tag{4.82} after setting \(F = 0\).

5.1.5 Proposition (Hilbert) Let \(p\) be a non-umbilic point and \(\kappa_1(p) > \kappa_2(p)\). If \(\kappa_1\) has a local maximum at \(p\) and \(\kappa_2\) has a local minimum at \(p\), the \(K(p) < 0\).

**Proof:** Take the asymptotic curves as parametric curves as in the preceding proposition. Suppose \(\kappa_1(p) > \kappa_2(p)\) and that the principal curvatures are local extrema. Then \((\kappa_1)_u = (\kappa_2)_v = 0\) so by equation \tag{5.7} we have \(E_v = G_u = 0\). Applying the second derivative test by differentiating \tag{5.7} at \(p\) we get

\[
\begin{align*}
(\kappa_1)_{vv} &= \frac{1}{2} \frac{E}{E} (\kappa_2 - \kappa_1) \leq 0, \\
(\kappa_2)_{uu} &= \frac{1}{2} \frac{G}{G} (\kappa_1 - \kappa_2) \geq 0,
\end{align*}
\]
which implies that \( E_{v v} \geq 0 \) and \( G_{u u} \geq 0 \). On the other hand, noting as above that \( E_v = G_u = 0 \), the Gaussian curvature formula 4.106 gives

\[
K = -\frac{1}{2EG}(E_{v v} + G_{u u}) \leq 0.
\]

5.1.6 Theorem (Liebmann) A compact manifold \( M \) in \( \mathbb{R}^3 \) of constant Gaussian curvature \( K \) is a sphere of radius \( R \) with \( K = 1/R^2 \).

Proof: Since \( M \) is compact, there is at least one point at which \( K > 0 \), and since \( K \) is constant, \( K > 0 \) everywhere. We prove by contradiction that all points are umbilic. Suppose there exists an non-umbilic point. Without loss of generality we assume that the larger principal curvature is \( \kappa_1 \). The principal curvatures are continuous functions in a compact space, so there is a point \( p \) at which \( \kappa_1 \) is maximum. Since \( K = \kappa_1 \kappa_2 = \text{constant} \), then at \( p \), \( \kappa_2 \) is a minimum. By Hilbert’s theorem above, \( K(p) < 0 \) which is a contradiction. So \( M \) is a sphere so some radius \( R \) and \( K = 1/R^2 \).

5.1.3 Surfaces of Constant Negative Curvature

The geometry of surfaces of constant negative curvature is very rich and is has a number of neat applications to physics. If \( K < 0 \), then it must be the case that the principal curvatures \( \kappa_1 \) and \( \kappa_2 \) have different signs. All points on the surface are hyperbolic, and by Hilbert’s theorem there are no compact surfaces of constant negative curvature. In addition, since \( \kappa_1 \neq \kappa_2 \), there always exist orthogonal asymptotic curves with asymptotic directions along the eigenvectors of the the second fundamental form. The prototype of a surface of constant negative curvature is the pseudosphere introduced in equation 4.23, which we repeat here for convenience.

\[
\mathbf{x}(u, v) = (a \sin u \cos v, a \sin u \sin v, a (\cos u + \ln(\tan \frac{u}{2}))).
\]

To compute the Gaussian curvature we first verify that the first fundamental form is as stated in 4.23. We have:

\[
\begin{align*}
\mathbf{x}_u &= (a \cos u \cos v, a \cos u \sin v, a \cos^2 u), \\
\mathbf{x}_v &= (-a \sin u \sin v, a \sin u \cos v, 0), \\
E &= a^2 \cos^2 u + a^2 \cos^4 u, \\
&= a^2 \cos^2 u(1 + \frac{\cos^2 u}{\sin^2 u}), \\
&= a^2 \cot^2 u, \\
F &= 0, \\
G &= a^2 \sin^2 u.
\end{align*}
\]

So the parametric curves are orthogonal, and

\[
I = a^2 \cot^2 u \, du^2 + a^2 \sin^2 u \, dv^2.
\]
Inserting into formula 4.106, we get
\[ K = -\frac{1}{a^2 \cos u} \left[ \frac{\partial}{\partial u} \left( \frac{\sin u}{a \cos u} \frac{\partial}{\partial u} (a \sin u) \right) \right], \]
\[ = -\frac{1}{a^2 \cos u} \frac{\partial}{\partial u} (\sin u), \]
\[ = -\frac{1}{a^2}. \] (5.8)

Another common parametrization of the pseudosphere is obtained by the substitution
\[ \mu = a \ln \tan \left( \frac{u}{2} \right). \] (5.9)

Without real loss of generality, we set \( a = 1 \), so \( e^\mu = \tan(u/2) \). The substitution is somewhat related to the classical Gudermannian. We have:
\[ \text{sech} \ \mu = \frac{2}{e^{u/2} + e^{-u/2}}, \]
\[ = \frac{2}{\tan(u/2) + \cot(u/2)}, \]
\[ = 2 \sin(u/2) \cos(u/2), \]
\[ = \sin u, \]
\[ \text{tanh} \ \mu = \frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}}, \]
\[ = \frac{\tan(u/2) - \cot(u/2)}{\tan(u/2) + \cot(u/2)}, \]
\[ = \sin^2(u/2) - \cos^2(u/2), \]
\[ = -\cos u. \]

In simplifying the equations above we multiplied top and bottom of the fractions by \( \sin(u/2) \cos(u/2) \). In terms of the parameter \( \mu \), the coordinate patch for the pseudosphere becomes
\[ x(\mu, v) = (a \text{sech} \ \mu \ \cos v, a \text{sech} \ \mu \ \sin v, a(\mu - \text{tanh} \ \mu)), \] (5.10)
and the fundamental forms are:
\[ I = \text{tanh}^2 \mu \ d\mu^2 + \text{sech}^2 \mu \ dv^2, \]
\[ II = -\text{sech} \mu \ d\mu^2 + \text{sech} \mu \ \text{tanh} \mu \ dv^2. \]

Using this latter parametrization we compute the surface area and volume of the top half of the pseudosphere.
\[ S = \int \int \sqrt{EG - F^2}, \]
\[ = \int_0^\pi \int_0^\infty (a \text{sech} \ \mu)(a \text{tanh} \ \mu) \ d\mu \ dv, \]
\[ = 4\pi a^2, \] (5.11)
\[ V = \pi a^3 \int_{-\infty}^{\infty} \text{sech}^2 \mu \ \text{tanh} \mu \ d\mu, \] (5.12)
\[ = \frac{2}{3}\pi a^3. \] (5.13)

It is interesting to note that the surface area is exactly the same as that of a sphere of radius \( a \), whereas the volume of revolution is half the volume of the sphere. Without loss of understanding of the geometry, for the rest of this section we set \( a = 1 \), so that \( K = -1 \).

We have the following theorem:
5.1. SURFACES OF CONSTANT CURVATURE

5.1.7 Theorem Let $M$ be a surface with constant negative curvature $K = -1$. If the parametric curves are chosen to be the asymptotic directions, there exists some quantity $\omega$ so that the first fundamental form can be written as:

$$ I = \cos^2 \omega \, du^2 + \sin^2 \omega \, dv^2, \quad (5.14) $$

Proof: The proof amounts to analyzing the integrability conditions represented by the Codazzi equations. In the brilliant book by Eisenhart, the author writes down the Codazzi equations and notes that the choice of $E$ and $G$ above are solutions of the equations. We take a more humble approach and show the steps on how to find a solution using the Cartan formalism. We have seen that using the asymptotic curves as parametric curves means that $F = f = 0$, $\kappa_1 = E/e$ and $\kappa_1 = G/e$. Let $E = \alpha^2$ and $G = \beta^2$ so that the first fundamental form is:

$$ I = \alpha^2 \, du^2 + \beta^2 \, dv^2. $$

We choose $\theta^1 = \alpha \, du$, and $\theta^2 = \beta \, dv$, as basis for the cotangent space dual to $\{e_1, e_1\}$. Let $e_3$ be the unit normal to the surface. We have:

$$ Le_1 = \kappa_1 e_1 = \nabla_{e_1} e_3 = \omega^3 (e_1) e_1, $$
$$ Le_2 = \kappa_2 e_2 = \nabla_{e_2} e_3 = \omega^3 (e_2) e_1, $$

so,

$$ \omega^1_3 = \kappa_1 \theta^1 = \kappa_1 \alpha \, du, $$
$$ \omega^2_3 = \kappa_2 \theta^2 = \kappa_2 \beta \, dv. $$

On the other hand,

$$ d\theta^1 = \alpha_v \, dv \wedge du = -\left( \frac{\alpha_v}{\beta} \, du \right) \wedge \theta^2 = -\omega^1_2 \wedge \theta^2, $$
$$ d\theta^2 = \beta_u \, du \wedge dv = -\left( \frac{\beta_u}{\alpha} \, dv \right) \wedge \theta^1 = -\omega^2_1 \wedge \theta^1, $$
$$ \omega^1_2 = \frac{\alpha_v}{\beta} \, du - \frac{\beta_u}{\alpha} \, dv. \quad (5.15) $$

Recall the Codazzi equations 4.103

$$ dw^1_3 = -\omega^1_2 \wedge \omega^2_3, $$
$$ dw^2_3 = -\omega^2_1 \wedge \omega^1_3. $$

Inserting the connection forms into the first Codazzi equation gives:

$$ (\kappa_1 \alpha)_v \, dv \wedge du + \left[ \frac{\alpha_v}{\beta} \, du - \frac{\beta_u}{\alpha} \, dv \right] \wedge \kappa_2 \beta \, dv = 0. $$

Since $\kappa_1 \kappa_2 = -1$, we can eliminate $\kappa_2$ and solve for $\alpha$. 

\[(\kappa_1 \alpha)_v - \alpha \kappa_2 = 0,\]
\[(\kappa_1 - \kappa_2) \alpha_v + (\kappa_1)_{v\alpha} = 0,\]
\[
\frac{\alpha_v}{\alpha} = - \frac{(\kappa_1)_{v\alpha}}{\kappa_1 - \kappa_2},
\]
\[
= - \frac{(\kappa_1)_{v\alpha}}{\kappa_1 + (1/\kappa_1)^{1/2}}
\]
\[
= - \frac{\kappa_1 (\kappa_1)_{v\alpha}}{\kappa_1^{2} + 1},
\]
\[
\frac{\partial}{\partial v} (\ln \alpha) = - \frac{\partial}{\partial v} \ln \left[ (\kappa_1^{2} + 1)^{1/2} \right].
\]

We may set
\[
\kappa_1 = \tan \omega, \quad \kappa_2 = - \cot \omega,
\]
for some \(\omega\). Then \((\kappa_1^{2} + 1)^{1/2} = \sec \omega\), so
\[
\frac{\partial}{\partial v} (\ln \alpha) = - \frac{\partial}{\partial v} \ln (\sec \omega) = \frac{\partial}{\partial v} \ln (\cos \omega).
\]

We choose the simplest solution \(\alpha = \cos \omega\). By a completely analogous computation using the second Codazzi equation, we get \(\beta = \sin \omega\) and that proves the theorem.

### 5.1.8 Theorem

If a surface with \(K = -1\) has first fundamental form written as \(I = \cos^2 \omega \, du^2 + \sin^2 \omega \, dv^2\), then \(\omega\) satisfies the so-called sine-Gordon equation:
\[
\omega_{uu} - \omega_{vv} = \sin \omega \cos \omega.
\]

#### Proof

Here \(E = \cos^2 \omega\) and \(G = \sin^2 \omega\). The theorem follows immediately from inserting these into the Gauss curvature equation in orthogonal coordinates [4.106] and setting \(K = -1\)
\[
K = - \frac{\omega_{uu} - \omega_{vv}}{\sin \omega \cos \omega} = -1
\]

The following transformation is often made:
\[
u = \hat{u} + \hat{v} \quad u = \hat{u} - \hat{v}.
\]
A quick computation yields a transformed fundamental form
\[
\hat{I} = d\hat{u}^2 + 2 \cos \hat{\omega} \, d\hat{u}d\hat{v} + d\hat{v}^2,
\]
where \(\omega = \hat{\omega}/2\). A coordinate system in which the first fundamental form is of this type is called a **Tchebychev Patch** (eventually one has to make a choice on how to transliterate from the Cyrillic alphabet). The corresponding curvature equation is
\[
\hat{\omega}_{uv} = \sin \hat{\omega}
\]

The sine-Gordon equation is one of class of very special type of nonlinear partial differential equations which admit **soliton** solutions. This is an incredibly rich area of research that would take us
into whole new branch of mathematics. We constrain our discussion to certain transformations that allow one to obtain new solutions from known solutions, and associate these with pseudospherical surfaces, that is, surfaces in $\mathbb{R}^3$ with constant negative curvature. We note that if in the Sine-Gordon equation \[5.17\] one sets $v = t$ where $t$ is a time parameter, what we have is a non-linear wave equation with speed $v = 1$. The reader will then recognize the transformation $u = \hat{u} + \hat{t}$ as the equations of characteristics. It is thus not surprising that the equation has solutions of the form $f(u - t)$.

### 5.1.4 Bäcklund Transforms

**5.1.9 Definition** Let $M$ be a surface with $K = -1$ and let $F : M \to \hat{M}$ be a map to another surface $\hat{M}$. Let $\hat{p} = F(p)$ and $N(p)$ and $\hat{N}(\hat{p})$ be the unit normals at $p$ and $\hat{p}$ respectively. $\hat{M}$ is called a **Bäcklund Transform** (BT) of $M$ with constant angle of inclination $\sigma$, if for all $p$:

a) The angle between $N$ and $\hat{N}$ is $\sigma$,

b) The distance $\lambda$ between $p$ and $\hat{p}$ is $\sin \sigma$,

c) the segment $p\hat{p}$ is tangent to $M$ at $p$.

![Fig. 5.4: Bäcklund Transform](image)

Bäcklund proved in 1883, that $F$ maps pseudospherical surfaces to pseudospherical surfaces and asymptotic lines to asymptotic lines. He also showed that given any unit tangent vector at $p$ which is not an asymptotic direction, a BT exists with $p\hat{p}$ in the direction of that tangent. The idea behind the proof of the BT theorem is basically to quantify the transformation from orthonormal frame at $p$ to the orthonormal frame at $\hat{p}$, find the conditions required for $\hat{K} = -1$, and write down the integrability conditions for the Cartan structure equations. The transformation consists of a rotation by an angle $\sigma$, a translation from $p$ to $\hat{p}$ and a rotation by an angle $\theta$ to align the frame with segment $p\hat{p}$. This could be done all at once, but we prefer to carry out the process in two stages. In the first stage, we apply the translation of the frame assuming that the segment joining $p$ and $\hat{p}$ is parallel to the basis vector $e_1$ at $p$, followed by a rotation by an angle $\sigma$ around the $e_1$ direction. We use this to seek conditions to guarantee that $\hat{K} = -1$. In stage two, we apply a rotation by an angle $\theta$ in the tangent plane.

### 5.1.10 Theorem** Let $M$ have Gaussian curvature $K = -1$, and let $\mathbf{x}(u, v)$ a coordinate patch for $M$ so that $I = E \, du^2 + G \, dv^2$. Let $\{e_1, e_2, e_3 = n\}$ be an orthonormal frame aligned with the asymptotic directions. Denote the frame and Cartan forms at $\hat{p}$ with hats. Consider the transformation $\hat{\mathbf{x}} = \mathbf{x} + \lambda e_1$ along with a rotation by an angle $\sigma$ around the $e_1$ axis. Then $\hat{K} = -1$ if and only if $\lambda = \sin \sigma$.

**Proof:** A rotation by an angle $\sigma$ around $e_1$ leaves the tangent vector $e_1$ and its dual form $\theta^1$ fixed.
The rotation has a matrix representation as shown below.

\[
\begin{bmatrix}
\hat{e}_1 \\
\hat{e}_2 \\
\hat{e}_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \sigma & -\sin \sigma \\
0 & \sin \sigma & \cos \sigma
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}.
\]

(5.20)

We compute the Cartan frame equations. We have: The coframe forms are \(\theta^1 = \sqrt{E} \, du\), \(\theta^2 = \sqrt{G} \, dv\) and \(\theta^3 = 0\) on \(M\). Thus

\[
d\hat{x} = x_u \, du + x_v \, dv,
\]

\[
= \frac{x_u}{\sqrt{E}} \, \sqrt{\epsilon} \, du + \frac{x_v}{\sqrt{G}} \, \sqrt{\epsilon} \, dv,
\]

\[
= \frac{x_u}{\sqrt{E}} \theta^1 + \frac{x_v}{\sqrt{G}} \theta^2,
\]

\[
= e_1 \theta^1 + e_2 \theta^2.
\]

We now compute \(d\hat{x}\) taking into account the rotation and then the translation:

\[
d\hat{x} = \hat{e}_1 \hat{\theta}^1 + \hat{e}_2 \hat{\theta}^2,
\]

\[
= e_1 \theta^1 + [\cos \sigma \, e_2 - \sin \sigma e_3] \hat{\theta}^2,
\]

\[
d\hat{x} = d\hat{x} + \lambda d\hat{e}_1,
\]

\[
= e_1 \theta^1 + e_2 \theta^2 + \lambda (e_2 \omega^1_2 + e_3 \omega^3_1)
\]

Equating the coefficients of \(e_2\) and \(e_3\) in two equations above, we get

\[
\cos \sigma \, \hat{\theta}^2 = \theta^2 + \lambda \omega^2_1,
\]

\[
-\sin \sigma \, \hat{\theta}^2 = \lambda \omega^3_1.
\]

(5.21)

Recall from 4.110 that \(\omega^1_3 = l \theta^1 + m \theta^2\) and \(\omega^2_3 = m \theta^1 + n \theta^2\) yields symmetric matrix components of the second fundamental form in the given basis. Using this fact and wedging with \(\hat{\theta}^1\) the second equation above, we get

\[
-\sin \sigma \, \hat{\theta}^1 \wedge \hat{\theta}^2 = \lambda \, \theta^1 \wedge \omega^3_1,
\]

\[
= \lambda \, \theta^1 \wedge (l \theta^1 + m \theta^2),
\]

\[
\hat{\theta}^1 \wedge \hat{\theta}^2 = -\frac{\lambda m}{\sin \sigma} \theta^1 \wedge \theta^2.
\]

(5.22)

Multiplying the first equation in [5.21] by \(\sin \sigma\), the second by \(\cos \sigma\), and adding, we get

\[
\sin \sigma [\theta^2 + \lambda \omega^2_1] = -\lambda \cos \sigma \omega^3_1,
\]

\[
\theta^2 = -\frac{\lambda}{\sin \sigma} [\sin \sigma \omega^2_1 + \cos \sigma \omega^3_1].
\]

(5.23)

(5.24)

Next, we compute \(\hat{\omega}_{32}\):
\[
\hat{\omega}_{32} (X) = \langle \nabla_X \hat{e}_2, \hat{e}_3 \rangle,
\]
\[
= \langle \cos \sigma \nabla_X e_3 - \sin \sigma \nabla_X e_3, \sin \sigma e_2 + \cos \sigma e_3 \rangle,
\]
\[
= \cos^2 \sigma \langle \nabla_X e_2, e_3 \rangle - \sin^2 \sigma \langle \nabla_X e_3, e_2 \rangle,
\]
\[
= (\cos^2 \sigma + \sin^2 \sigma) \langle \nabla_X e_2, e_3 \rangle,
\]
\[
= \omega_{32} (X),
\]
\[
\hat{\omega}_{32}^3 = \omega_{32}^3 = -m \theta^1 - n \theta^2.
\]

By the same process, we calculate \( \hat{\omega}_{31} \):
\[
\hat{\omega}_{31} (X) = \langle \nabla_X \hat{e}_1, \hat{e}_3 \rangle,
\]
\[
= \langle \nabla_X e_3 + \nabla_X e_3, \sin \sigma e^2 + \cos \sigma e^3 \rangle,
\]
\[
= \langle \omega_{31}^2 (X) e_2 + \omega_{31}^3 (X) e_3, \sin \sigma e_2 + \cos \sigma e_3 \rangle,
\]
\[
= \sin \sigma \omega_{32} (X) + \sin \sigma \omega_{31} (X),
\]
\[
\hat{\omega}_{31}^3 = \frac{\sin \sigma}{\lambda} \theta^2.
\]

Finally, putting these results together, we get
\[
d\hat{\omega}_{2} = \hat{\omega}_{3}^1 \wedge \hat{\omega}_{2}^3,
\]
\[
= - \left[ \frac{m \sin \sigma}{\lambda} \right] \theta^1 \wedge \theta^2,
\]
\[
= \left[ \frac{m \sin \sigma}{\lambda} \right] \left[ \frac{\sin \sigma}{m \lambda} \right] \hat{\theta}^1 \wedge \hat{\theta}^2,
\]
\[
= \left[ \frac{\sin \sigma}{\lambda} \right]^2 \hat{\theta}^1 \wedge \hat{\theta}^2.
\]

Hence
\[
K = - \left[ \frac{\sin \sigma}{\lambda} \right]^2 = -1
\]
if and only if \( \lambda = \pm \sin \sigma \). We choose \( \lambda = \sin \sigma \).

The conclusion of the theorem explains the condition in the definition of a BT that requires this equation to hold. With this condition, equation \(5.23\) takes the form:
\[
\omega_{12} = \cot \sigma \omega_{31} + \csc \sigma \theta^2.
\]

5.1.11 Theorem Let \( M \) be a pseudospherical surface with first fundamental form as in \[5.14\] and let \( F : M \rightarrow \hat{M} \) be a BT with angle of inclination \( \sigma \). If the segment \( \overline{pp} \) makes a constant angle \( \alpha \) with the basis vector \( e_1 \) at each point \( p \in M \), then
\[
\sin \sigma (\theta_u + \omega_u) = \sin \theta \cos \omega - \cos \sigma \cos \theta \sin \omega,
\]
\[
\sin \sigma (\theta_v + \omega_v) = -\cos \theta \sin \omega + \cos \sigma \sin \theta \cos \omega.
\]

Proof: Suppose \( \overline{pp} \) makes an angle \( \theta \) with the tangent vector \( e_1 \). In this case we first perform a rotation of axis around the normal vector \( e_3 \) to align the frame with \( e_1 \). The rotation can be
represented by a matrix \( e_i = e_j B^j_i \)

\[
B = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\] (5.27)

The effect on the Cartan frame is much easier establish since all we have to do is apply the change of basis formula 3.48 as shown in 3.51.

\[
\omega_1^2 = \omega_1^2 - d\theta,
\omega_1^3 = \cos \theta \omega_1^3 + \sin \theta \omega_2^3,
\omega_2^3 = -\sin \theta \omega_1^3 + \cos \theta \omega_2^3.
\] (5.28)

The dual forms transform with \( A = B^{-1} = B^T \), that is \( \tilde{\theta}^i = A^i_j \theta^j \). In particular,

\[
\tilde{\theta}^2 = -\sin \theta \theta^1 + \cos \theta \theta^2.
\] (5.29)

If we start with a pseudospherical surface with \( I = \cos^2 \omega \ du^2 + \sin^2 \omega \ dv^2 \), the Cartan forms are:

\[
\theta^1 = \cos \omega \ du, \quad \theta^2 = \sin \omega \ dv,
\omega_1^3 = \sin \omega \ du, \quad \omega_2^3 = -\cos \omega \ dv
\omega_1^2 = -\omega_v \ du - \omega_u \ dv.
\]

The BT transformation is the composition of the stage one and stage two. This means that we must subject the Cartan forms to the change of basis by a rotation by \( \theta \), as described by equations 5.28 and 5.29 followed by substitution into equation 5.25. We get:

\[
(\omega_1^2 - d\theta) = \cot \sigma (\cos \theta \omega_1^3 + \sin \theta \omega_2^3) + \csc \sigma (-\sin \theta \theta^1 + \cos \theta \theta^2).
\]

If we start with a pseudospherical surface with \( I = \cos^2 \omega \ du^2 + \sin^2 \omega \ dv^2 \), the Cartan forms are:

\[
\theta^1 = \cos \omega \ du, \quad \theta^2 = \sin \omega \ dv,
\omega_1^3 = \sin \omega \ du, \quad \omega_2^3 = -\cos \omega \ dv
\omega_1^2 = -\omega_v \ du - \omega_u \ dv.
\]

We extract two formulas obtained by equating the coefficients of \( du \) and \( dv \) respectively.

\[
-\omega_v - \theta_u = \cot \sigma \cos \theta \sin \omega - \csc \sigma \sin \theta \cos \omega,
-\omega_u - \theta_v = -\cot \sigma \sin \theta \cos \omega + \csc \sigma \cos \theta \sin \omega.
\]

The theorem follows by multiplying these equations by \( \sin \sigma \), and rearranging terms. The system of equations 5.26 is the classical B"{a}cklund Transform. In the special case in which \( \sigma = \pi/2 \), the angle between the normals \( e_3 \) and \( \hat{e}_3 \) is a right angle, so \( \hat{e}_3 \) is parallel to a tangent vector of \( M \). This is called a Bianchi Transform. Equations 5.26 then reduce to the much simpler system:

\[
\theta_u + \omega_v = \sin \theta \cos \omega,
\theta_v + \omega_u = -\cos \theta \sin \omega.
\] (5.30)

We can rewrite the BT-equations in the so called asymptotic coordinates. Let

\[
u = x + t, \quad \text{so that} \quad u = \frac{1}{2}(u + v),
v = x - t, \quad \text{so that} \quad v = \frac{1}{2}(u - v).
\]

By the chain rule, we have:

\[
\theta_u = \frac{1}{2}(\theta_x + \theta_t), \quad \omega_u = \frac{1}{2}(\omega_x + \omega_t),
\theta_v = \frac{1}{2}(\theta_x - \theta_t), \quad \omega_v = \frac{1}{2}(\theta_x - \theta_t).
\]
Adding and subtraction equations \(5.26\), the system reduces to
\[
\begin{align*}
\theta_x + \omega_x &= \frac{1 + \cos \sigma}{\sin \sigma} \sin(\theta - \omega), \\
\theta_t - \omega_t &= \frac{1 - \cos \sigma}{\sin \sigma} \sin(\theta + \omega),
\end{align*}
\]
which we can rewrite as
\[
\begin{align*}
\theta_t &= \omega_t + s \sin(\theta + \omega), \\
\theta_x &= -\omega_x + \frac{1}{s} \sin(\theta - \omega),
\end{align*}
\] (5.31)
where \(s = \tan(\sigma/2).\) We denote the BT \(5.31\) by the notation \(F = F(\omega, \theta, s).\) Given a pseudospherical Surface \(S\) associated with a solution \(\omega\) of the Sine Gordon equation, the transform \(F = F(\omega, \theta, \sigma)\) produces a new solution \(\theta\) associated with a new pseudospherical surface \(S'.\) Of course, the process can be iterated to produce new surfaces and new solutions. The neat thing is that further iterations can be carried out algebraically without the need to solve more differential equations. This remarkable result is encapsulated in the following theorem (see [18]).

5.1.12 Theorem (Bianchi Permutability) Let \(\{S, \omega\}\) be pair consisting of a pseudospherical surface corresponding to SGE solution \(\omega.\) Suppose that \(\sin^2 \theta_1 \neq \sin^2 \theta_2,\) and that \(\{S_1, \theta_1\}\) and \(\{S_2, \theta_2\}\) are pseudospherical surfaces generated respectively from surface \(S\) by BT’s
\[
\begin{align*}
S &\xrightarrow{F(\omega, \theta_1, s_1)} S_1, \\
S &\xrightarrow{F(\omega, \theta_2, s_2)} S_2.
\end{align*}
\]
Then, pair \(\{S', \Omega\}\) consisting of a pseudospherical surface \(S'\) with SGE solution \(\Omega\) can be found algebraically by requiring the compatibility of BT’s
\[
\begin{align*}
S_1 &\xrightarrow{F(\theta_1, \Omega, s_1)} S', \\
S_2 &\xrightarrow{F(\theta_2, \Omega, s_2)} S'.
\end{align*}
\]
Proof: It suffices to use only one of each pair of BT’s. By assumption,
\[
\begin{align*}
(\theta_1)_t &= \omega_t + s_1 \sin(\theta_1 + \omega) \quad \text{and} \quad \Omega_t = (\theta_1)_t + s_2 \sin(\Omega + \theta_1) \\
(\theta_2)_t &= \omega_t + s_2 \sin(\theta_2 + \omega) \quad \text{and} \quad \Omega_t = (\theta_2)_t + s_1 \sin(\Omega + \theta_2)
\end{align*}
\]
Adding the difference of the two equations on the left with the difference of the two equations on the right, we see that all derivatives cancel, and we get:
\[
s_1 [\sin(\theta_1 + \omega) - \sin(\Omega + \theta_2)] + s_2 [\sin(\Omega + \theta_1) - \sin(\theta_2 + \omega)] = 0.
\]
If we have quantities \(A\) and \(B\) such that \(As_1 + Bs_2 = 0,\) then
\[
\begin{align*}
\frac{s_1}{s_2} &= \frac{B}{A}, \\
1 + \frac{s_1}{s_2} &= \frac{s_2 + s_1}{s_2} = \frac{A - B}{A}, \\
1 - \frac{s_1}{s_2} &= \frac{s_2 - s_1}{s_2} = \frac{A + B}{A}, \\
\frac{s_2 + s_1}{s_2 - s_1} &= \frac{A - B}{A + B}.
\end{align*}
\]
Applying this to the equation above, we have

\[
\frac{s_2 + s_1}{s_2 - s_1} = \left[ \sin(\theta_1 + \theta) - \sin(\Omega + \theta_2) \right] - \left[ \sin(\Omega + \theta_1) - \sin(\theta_2 + \theta) \right],
\]

\[
= \frac{\sin(\theta_1 + \theta) - \sin(\Omega + \theta_1)}{\sin(\theta_1 + \theta) - \sin(\Omega + \theta_2)} + \frac{\sin(\Omega + \theta_1) - \sin(\theta_2 + \theta)}{\sin(\Omega + \theta_1) - \sin(\theta_2 + \theta)}. \tag{5.33}
\]

Using the sum-product formulas for sine functions we rewrite the equation as,

\[
\frac{s_2 + s_1}{s_2 - s_1} = \frac{2\sin[\frac{1}{2}(\omega - \Omega)]\cos[\frac{1}{2}(\Omega + \omega + 2\theta_1)] + 2\sin[\frac{1}{2}(\omega - \Omega)]\cos[\frac{1}{2}(\Omega + \omega + 2\theta_2)]}{2\cos[\frac{1}{2}(\omega - \Omega)]\sin[\frac{1}{2}(\Omega + \omega + 2\theta_1)] - 2\cos[\frac{1}{2}(\omega - \Omega)]\sin[\frac{1}{2}(\Omega + \omega + 2\theta_2)]},
\]

\[
= \frac{2\sin[\frac{1}{2}(\Omega - \omega)]\{\cos[\frac{1}{2}(\Omega + \omega + 2\theta_1)] + \cos[\frac{1}{2}(\Omega + \omega + 2\theta_2)]\}}{2\cos[\frac{1}{2}(\omega - \Omega)]\{\sin[\frac{1}{2}(\Omega + \omega + 2\theta_1)] - \sin[\frac{1}{2}(\Omega + \omega + 2\theta_2)]\}},
\]

Now, using the sum-product formulas again, we get,

\[
\frac{s_2 + s_1}{s_2 - s_1} = \frac{4\sin[\frac{1}{2}(\Omega - \omega)]\{\cos[\frac{1}{2}(2\theta_1 - 2\theta_2)] + \sin[\frac{1}{2}(2\Omega + 2\omega + 2\theta_1 + 2\theta_2)]\}}{4\cos[\frac{1}{2}(\Omega - \omega)]\{\sin[\frac{1}{2}(2\theta_1 - 2\theta_2)] + \cos[\frac{1}{2}(2\Omega + 2\omega + 2\theta_1 + 2\theta_2)]\}},
\]

\[
= \frac{4\sin[\frac{1}{2}(\Omega - \omega)]\cos[\frac{1}{2}(\theta_1 - \theta_2)]}{4\cos[\frac{1}{2}(\omega - \Omega)]\sin[\frac{1}{2}(\theta_2 - \theta_1)]},
\]

We conclude that,

\[
\tan\left(\frac{\Omega - \omega}{2}\right) = \frac{s_2 + s_1}{s_2 - s_1} \tan\left(\frac{\theta_2 - \theta_1}{2}\right). \tag{5.32}
\]

It is easy to write coordinate patch equations for a BT. The vector \( \hat{x} - x \) must be a vector of length \( \sin \sigma \) which is tangent to \( M \) at each point \( p \) and makes an angle \( \theta \) with \( e_1 \). Therefore, we must have

\[
\hat{x} - x = \sin \sigma [\cos \theta \, e_1 + \sin \theta \, e_2].
\]

But, \( e_1 = x_u/\sqrt{E} \) and \( e_2 = x_v/\sqrt{G} \), so we have:

\[
\hat{x} = x + \sin \sigma \left[ \frac{\cos \theta}{\cos \omega} x_u + \frac{\sin \theta}{\sin \omega} x_v \right]. \tag{5.33}
\]

### 5.1.13 Example Pseudosphere and One-Soliton Solution

If we are willing to sacrifice a bit of rigor for the sake of intuition, we can motivate the derivation of the standard parametric representation of the pseudosphere directly from the Bianchi Transform. Recall that in the steps leading to the SG equation, we chose the principal curvatures (see equation \[5.16\]) such that \( \kappa_1 = \tan \omega \) and \( \kappa_2 = -\cot \omega \). Then, as \( \omega \) approaches zero, \( \kappa_1 \) also approaches zero, while \( \kappa_2 \) becomes arbitrarily large, so as to maintain \( K = -1 \). The result is a degenerate surface that collapses onto a straight line. We may think of it as an infinitely long trumpet of infinitesimal diameter. As such, we pick a degenerate patch of the form

\[
x(u, v) = (0, 0, u)
\]

We set \( e_1 = (0, 0, 1) \) and, anticipating a surface of revolution, we pick \( e_2 = (\cos \phi, \sin \phi, 0) \). Recalling the BT coordinate patch in equation \[5.33\]

\[
\hat{x} = x + \sin \sigma [\cos \theta \, e_1 + \sin \theta \, e_2], \tag{5.34}
\]
5.1. SURFACES OF CONSTANT CURVATURE

we consider the case with \( \sin \sigma = 1 \). With \( \omega \) arbitrarily close to zero, the Bianchi Transform equations \[5.30\] become,
\[
\theta_u = \sin \theta, \quad \theta_v = 0.
\]
We set \( v = \phi \) and without loss of generality, we pick the constant of integration in the first equation above to be 0. The elementary integral gives immediately the stationary one-soliton solution
\[
u = \ln(\tan \frac{\theta}{2}), \quad \text{or} \quad \theta = 2 \tan^{-1}(e^\nu).
\]
The coordinate patch for the corresponding surface then gives,
\[
\hat{x} = (0, 0, \ln(\tan \frac{\theta}{2})) + \cos \theta(0, 0, 1) + \sin \theta(\cos \phi, \sin \phi, 0),
\]
which agrees with the parametrization of the pseudosphere given in equation 4.23. In other words, the angle \( \theta \) is precisely as shown in figure 4.7, consistent with the geometry of the Bianchi Transform stating that the segment joining the corresponding points is tangential to the surface generated.

5.1.14 Example Dini's Surface

Another well known surface that can be obtained by BT results in 'twisting' the pseudosphere in a helicoidal manner. Dini's surface is obtained by removing the special condition \( \sin \sigma = 1 \). Again, if one begins with the trivial solution \( \omega = 0 \) of the Sine-Gordon equation \[5.19\] the BT equations \[5.31\] reduce to,
\[
\theta_t = s \sin \theta,
\]
\[
\theta_x = \frac{1}{s} \sin \theta.
\]
Integrating, we get
\[
\ln(\tan \frac{\theta}{2}) = sx + \frac{1}{s} t, \quad (5.35)
\]
where we have set the constant of integration to 0. Solving for \( \theta \), we are lead to the moving one-soliton solution
\[
\theta(x, t) = 2 \tan^{-1}(e^{sx+\frac{1}{s}t}),
\]
We carry out the parametrization using the BT equations in \[5.20\] Again, assuming \( \omega \) is arbitrarily close to zero, the equations reduce to
\[
(sin \sigma)\theta_u = \sin \theta, \quad \text{or} \quad (csc \theta)\theta_u = 1/\sin \sigma
\]
\[
(sin \sigma)\theta_v = \cos \sigma \sin \theta. \quad \text{or} \quad (csc \theta)\theta_v = \cos \sigma / \sin \sigma
\]
Rewriting the equations in differential form
\[
d [\ln(\tan \frac{\theta}{2})] = \frac{1}{\sin \sigma} du + \frac{\cos \sigma}{\sin \sigma} dv,
\]
we can integrate immediately,
\[
\ln(\tan \frac{\theta}{2}) = \frac{u + v \cos \sigma}{\sin \sigma} = \chi, \quad (5.36)
\]
Here we have used \( \chi \) to denote the expression on the right hand side and we set the integration constant to 0. In terms of these coordinates, the moving soliton solution is
\[
\theta(u, v) = 2 \tan^{-1}(e^\chi)
\]
Using the same degenerate patch 5.34, the parametrization of the surface becomes

\[ \hat{x} = (0, 0, u) + \sin \sigma [\cos \theta (0, 0, 1) + \sin \theta (\cos \phi, \sin \phi, 0)], \]
\[ = (\sin \sigma \sin \theta \cos \phi, \sin \sigma \sin \theta \sin \phi, u - \sin \sigma \cos \theta) \]

Finally, using the results in equations 5.10 we rewrite the parametrization of Dini’s surface as

\[ \hat{x} = (\sin \sigma \cos \phi \sech \chi, \sin \sigma \sin \theta \sin \phi \sech \chi, u - \sin \sigma \tanh \chi) \]  \hspace{1cm} (5.37)

Notice that as expected, when \( \sigma = \pi/2 \), that is, when \( \sin \sigma = 1 \), we get \( \chi = u \), and the equation reduces to a pseudosphere (see equation 5.10). Another common parametrization of Dini’s surface in which the geometry is more intuitive is given by:

\[ \hat{x}(u, v) = (a \cos u \cos v, a \cos u \sin v, a(\cos u + \ln \tan(\frac{u}{2})) + bv). \]  \hspace{1cm} (5.38)

This surface has curvature \( K = -1 \) when \( a^2 + b^2 = 1 \), and it has an unfolding infundibular shape, as shown in figure 5.5 with parameters \( u \in [0, 2] \), \( v \in [0, 4\pi] \) and \( a = 1, b = 0.5, 0.2 \). The surface is essentially generated by revolving the tractrix profile curve of the pseudosphere about the central axis, while at the same time translating the curve at a constant rate parallel to the axis. The meridians traced by the parametric curves \( u = \text{constant} \) are helices. When \( b = 0 \), the equation gives a pseudosphere.

5.1.15 Example Kuen Surface

Applying the permutability theorem 5.32 to solution 5.1.4 we obtain immediately the two-soliton solution

\[ \Omega = 2 \tan^{-1} \left[ \frac{s_2 + s_1}{s_2 - s_1} \left( e^{s_1 + \frac{1}{2}t} - e^{s_2 + \frac{1}{2}t} \right) \right]. \]  \hspace{1cm} (5.39)

In this example, we perform a Bianchi transformation of a one-soliton pseudosphere to obtain a Kuen surface which is associated with a two-soliton solution. We begin with the parametrization of a pseudosphere given by 5.10 with \( a = 1 \),

\[ \hat{x}(u, v) = (\sech u \cos v, \sech u \sin v, (\mu - \tanh \mu)). \]

Let \( \omega = 2 \tan^{-1}(e^{\mu}) \), so that

\[ \mu = \ln \tan(\omega/2), \]
Then,
\[ \omega = 2 \tan^{-1}(e^\mu), \]
\[ \sin \omega = \sech \mu, \]
\[ \cos \omega = -\tanh \mu. \]

We will find \( \theta \) by solving the Bianchi equations 5.30. We compute:
\[ \omega_\mu = \frac{2e^\mu}{1 + e^{2\mu}} = \frac{2}{e^\mu + e^\mu} = \sech \mu, \]
\[ \omega_\nu = 0. \]

Substituting into the Bianchi equations, we get:
\[ \left\{ \begin{array}{l}
\theta_\mu = -\sin \theta \tanh \mu, \\
\theta_\nu = -\cos \theta \sech \mu - \sech \mu = -\sech \mu(1 + \cos \theta) = -2\cos^2\left(\frac{\theta}{2}\right) \sech \mu.
\end{array} \right. \]

Separate variables
\[ \left\{ \begin{array}{l}
\csc \theta \theta_\mu = -\tanh \mu, \\
\frac{1}{2} \sec^2\left(\frac{\theta}{2}\right) \theta_\nu = -\sech \mu,
\end{array} \right. \]
and integrate. The result is:
\[ \left\{ \begin{array}{l}
\tan\left(\frac{\theta}{2}\right) = -h_1(\nu) \sech \mu, \\
\tan\left(\frac{\theta}{2}\right) = -\nu \sech \mu + h_2(\mu),
\end{array} \right. \]
where \( h_1 \) and \( h_2 \) are the arbitrary functions of integration. Consistency of the equations requires \( h_1 = 1 \) and \( h_2 = 0 \). The solution is therefore
\[ \tan\left(\frac{\theta}{2}\right) = -\nu \sech \mu = -\frac{\nu}{\cosh \mu}, \]
\[ \theta = 2 \tan^{-1}(-\nu \sech \mu). \]

Only the cosine and the sine of the angle \( \theta \) enter into the Bianchi coordinate patch. Thinking of \( \tan\left(\frac{\theta}{2}\right) \) as the ratio of the opposite over the adjacent side of the right triangle with hypotenuse \( \sqrt{\cosh^2 \mu + \nu^2} \), we can compute the sine and cosine from the double angle formulas:
\[ \cos \theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \frac{\cosh^2 \mu - \nu^2}{\cosh^2 \mu + \nu^2}, \]
\[ \sin \theta = 2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) = -\frac{2\nu \cosh \mu}{\cosh^2 \mu + \nu^2}. \]

It remains to go through the algebraic gymnastics of computing the coordinate patch:
\[ \dot{x} = x + \frac{\cos \theta}{\cos \omega} x_u + \frac{\sin \theta}{\sin \omega} x_v, \]
\[ = ( \sech \mu \cos \nu, \sech \mu \sin \nu, (\mu - \tanh \mu) ) - \frac{\cos \theta}{\tanh \mu} ( -\sech \mu \tanh \mu \cos \nu, -\sech \mu \sin \nu, \tanh^2 \mu ) + \frac{\sin \theta}{\sech \mu} ( -\sech \mu \sin \nu, \sech \mu \cos \nu, 0 ), \]
\[ = ( \sech \mu \cos \nu, \sech \mu \sin \nu, (\mu - \tanh \mu) ) + \cos \theta( \sech \mu \cos \nu, \sech \mu \sin \nu, -\tanh \mu ) + \sin \theta( -\sin \nu, \cos \nu, 0 ), \]
CHAPTER 5. GEOMETRY OF SURFACES

Fig. 5.6: Surfaces with $K = -1$

The $x$-component of $x$ is:

$$x_{(x)} = (1 + \cos \theta) \operatorname{sech} \mu \cos v - \sin \theta \sin v$$

$$= \left[ 1 + \frac{\cosh^2 \mu - v^2}{\cosh^2 \mu + v^2} \right] \cos v + \left[ -\frac{2v \cosh \mu}{\cosh^2 \mu + v^2} \right] \sin v,$$

$$= \left[ \frac{2 \cosh^2 \mu}{\cosh^2 \mu + v^2} \right] \cos v + \left[ \frac{2v \cosh \mu}{\cosh^2 \mu + v^2} \right] \sin v,$$

$$= \frac{2 \cosh \mu (\cos v + v \sin v)}{\cosh^2 \mu + v^2}.$$  

The computation of the other two components is left as an exercise. The result is

$$x(u,v) = \left( \frac{2 \cosh \mu (\cos v + v \sin v)}{\cosh^2 \mu + v^2}, \frac{2 \cosh \mu (\sin v - v \cos v)}{\cosh^2 \mu + v^2}, \mu - \frac{2 \sinh 2 \mu}{\cosh^2 \mu + v^2} \right) \quad (5.40)$$

The Kuen surface in figure 5.6a is plotted with parameters $u \in [-1.4, 1.4]$ and $v \in [-4, 4]$.

As noted by Terng and Uhlenbeck [34], if in the 2-soliton equation 5.39 one sets $s_1 = e^{i\theta}$ and $s_2 = -e^{-i\theta}$, we get a real-valued solution

$$\phi = 2 \tan^{-1} \left[ \frac{\sin \theta \sin(\eta \cos \theta)}{\cos \theta \cosh(\xi \sin \theta)} \right], \quad (5.41)$$

where $\xi = x - t$ and $\eta = x + t$. This is a periodic solution called a breather. A rendition of the surface associated with this solution is shown in figure 5.6, using the parametrization derived by Rogers and Schief [29].

5.2 Minimal Surfaces

5.2.1 Minimal Area Property

In an earlier chapter we defined a minimal surface to be a surface of mean curvature $H = 0$. From the formula for the mean curvature 4.64, a surface in $\mathbb{R}^3$ is minimal if

$$2H = \operatorname{Tr}(g^{-1}b) = 0, \quad \text{which implies} \quad Eg - 2Ff + Ge = 0.$$ 

For historical reasons we first consider this condition for a surface with equation $z = f(x,y)$. We rewrite in parametric form using a Monge coordinate patch

$$x(x,y) = (x, y, f(x,y))$$
A quick computation yields the coefficients of the first and second fundamental forms.

\[
E = 1 + f_x^2, \quad e = f_{xx}/D, \quad F = f_x f_y, \quad f = f_{xy}/D
\]

\[
G = 1 + f_y^2, \quad g = f_{yy}/D
\]

where \( D = \sqrt{EG - F^2} = \sqrt{1 + f_x^2 + f_y^2} \). The surface has Gaussian curvature \( K = 0 \) if \( f(x,y) \) satisfies the Monge-Ampere equation

\[
f_{xx}f_{yy} - f_{xy}^2 = 0.
\]

We have already determined that the solutions are developable surfaces. The condition \( H = 0 \) for having a minimal surface is that \( f(x, y) \) satisfy the quasilinear differential equation:

\[
(1 + f_x^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_y^2)f_{yy} = 0.
\] (5.42)

Using the notation \( p = f_x, q = f_y \) the condition that the surface area over a region be minimal, follows from the variational equation:

\[
\delta \int \int \sqrt{EG - F^2} \, dy \, dx = \delta \int \int \sqrt{1 + p^2 + q^2} \, dy \, dx, \quad = \delta \int \int F(z, x, y, p, q) \, dy \, dx = 0.
\]

The Euler-Lagrange equation for this functional is

\[
\nabla \cdot \left( \frac{\nabla f}{1 + |\nabla f|^2} \right) = \frac{d}{dx} \frac{p}{\sqrt{1 + p^2 + q^2}} + \frac{d}{dy} \frac{q}{\sqrt{1 + p^2 + q^2}} = 0.
\] (5.43)

It was proved by Lagrange in 1762, that these are equivalent to the condition \( H = 0 \) exemplified in equation 5.42, but he was unable to find non-trivial solutions. In 1776 Meusnier showed that the catenoid and the helicoid satisfied Euler-Lagrange equations 5.43 and thus had zero mean curvature.

### 5.2.1 Example Scherk’s Surface

The catenoid and the helicoid remained the only known minimal surfaces until 1830, when Scherk obtained a new solution under the assumption that \( f(x, y) \) has the special form.

\[
f(x, y) = U(x) + V(y)
\]

In this case, the minimal surface equation 5.42 can be separated

\[
U''(1 - V'^2) + V''(1 + U'^2) = 0,
\]

\[
\frac{U''}{1 + U'^2} = -\frac{V''}{1 + V'^2} = C.
\]

The ordinary differential equations only involve the first and second derivatives of the variables, so they can be easily integrated. First, we let \( R = U' \), and solve for \( U \), setting the integration constants to zero.

\[
\frac{R'}{1 + R^2} = C,
\]

\[
\int \frac{R}{1 + R^2} \, dR = \int C \, dx,
\]

\[
\tan^{-1} R = Cx,
\]

\[
U' = \tan(Cx)
\]

\[
U = \ln[\sec Cx].
\]
The integral for \( V \) is done the same way, and the result is \( V = -\tan[\ln \sec y] \), so
\[
f(x, y) = \ln[\sec Cx] - \ln[\sec Cy].
\]
Setting \( C = 1 \) and rewriting in terms of cosines, we get
\[
f(x, y) = \ln \frac{\cos y}{\cos x}.
\]
(5.44)
The is the classical, doubly periodic \textbf{Scherk Surface} which we render in figure 5.7.

![Fig. 5.7: Scherk’s Surface.](image)

Next, we prove the area-minimizing property of for more general surfaces with zero mean curvature. The integrand in the formula for surface area is the square root of the determinant of the first fundamental form, so to perform a variation, we will need to take derivatives of determinants. The main idea in obtaining a formula for the derivative of a determinant rests on a neat result from linear algebra which at the risk of digressing a bit, it is worth proving now. This theorem due to Jacobi, will be most important later when we discuss the exponential map in the context of Lie groups and Lie Algebras (See section 7.2.1).

5.2.2 \textbf{Theorem} \ Let \( A \) be a square matrix. Then
\[
\det e^A = e^{\text{Tr}A} \quad (5.45)
\]
\textbf{Proof:}

First consider the case where \( A \) is a diagonal \( n \times n \) matrix \( A = \text{diag}[\kappa_1, \kappa_1, \ldots, \kappa_n] \) by an . Defining \( e^A \) by the exponential power series (See equation 7.44), it is immediately verified that:
\[
e^A = \text{diag}[e^{\kappa_1}, e^{\kappa_2}, \ldots, e^{\kappa_n}],
\]
\[
\det e^A = e^{\kappa_1}e^{\kappa_2} \cdots e^{\kappa_n},
\]
\[
= e^{\text{Tr}A}.
\]
Next, consider the case where \( A \) is diagonalizable. Then, there exists a similarity transformation \( Q \) such that \( A = Q^{-1}DQ \), where \( D \) is the diagonal matrix with the eigenvalues along the diagonal, \( D = \text{diag}[\kappa_1, \kappa_1, \ldots, \kappa_n] \). We recall that determinant and trace are invariant under similarity transformations. We have
\[
A^2 = (Q^{-1}DQ)(Q^{-1}DQ),
\]
\[
= Q^{-1}D(QQ^{-1})DQ,
\]
\[
= Q^{-1}DIDQ,
\]
\[
= Q^{-1}D^2Q.
\]
By induction, we can easily prove that \( A^n = Q^{-1}D^nQ \) and hence,

\[
e^A = \sum_{n=0}^{\infty} \frac{1}{n!}A^n,
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!}Q^{-1}D^nQ,
\]

\[
= Q^{-1}(\sum_{n=0}^{\infty} \frac{1}{n!}D^n)Q,
\]

\[
= Q^{-1}e^DQ,
\]

\[
\det A = \det Q^{-1}(\det e^D)\det Q,
\]

\[
= \det e^D = e^{\text{Tr}D} = e^{\text{Tr}A}.
\]

Finally, \( A \) might not diagonalizable, but it can be reduced to canonical Jordan form \( J \) by a similarity transformation \( A = Q^{-1}JQ \). The Jordan matrix \( J \) has the eigenvalues along the diagonal. with a block structure of the form \( J_\lambda = \lambda I + N \) where \( N \) is nilpotent, it remains true that \( \det e^J = e^{\text{Tr}J} \). Thus the argument above holds in this case as well.

Back to our topic. If in equation 5.45 we replace \( A \) by \( \ln A \), we get,

\[
\det A = e^{\text{Tr}((\ln A))}.
\]

(5.46)

Suppose that instead of single matrix \( A \), we have a \( C^\infty \), one-parameter family of matrices \( A_t \). Then, we can use the equation above to differentiate with respect to \( t \).

\[
\frac{d}{dt} \det A_t = e^{\text{Tr}((\ln A_t))} \frac{d}{dt} (\text{Tr}(\ln A_t)), \quad \text{hence,}
\]

\[
= (\det A_t) \text{Tr} \frac{d}{dt} \ln A_t,
\]

\[
= (\det A_t)\text{Tr}(A_t^{-1}\frac{dA_t}{dt})
\]

(5.47)

We unpack this formula for a special kind of variation defined as follows:

\[\text{5.2.3 Definition}\]

Let \( x(u,v) : U \rightarrow \mathbb{R}^3 \) be a coordinate patch for a surface \( \{M,g\} \) defined over a set \( U \subset \mathbb{R}^2 \), and let \( \phi : U \rightarrow \mathbb{R} \) be a \( C^\infty \) function. A normal deformation of the surface is a one-parameter family of surfaces \( M_t \) with coordinate patches of the form \( x_t(u,v) = x(u,v) + t\phi(u,v)n \), where \( t \) is a small parameter, \( t \in [-\epsilon, \epsilon] \), and \( n \) is the unit normal.

Let \( g_t \) be the matrix of the first fundamental form induced on the surface \( M_t \) and let \( \det(g_t) \) denote the determinant. The elements of surface area are \( dS_t = \sqrt{\det(g_t)} \, du \wedge dv \), and the areas are given by:

\[
A_t = \int_U \int dS_t = \int_U \int \sqrt{\det(g_t)} \, du \wedge dv.
\]

(5.48)

At \( t = 0 \), \( g_0 = g \) and \( dS_0 = dS \) represent the metric and the differential of surface area element of \( M \). We have the following theorem:

\[\text{5.2.4 Theorem}\]

The variation of surface area satisfies

\[
A_t'(0) = -2 \int_U \phi H dS,
\]

(5.49)
so that $A_t'(0) = 0$ if, and only if, $H = 0$.

**Proof:** The proof is by computation, using the formula \[5.47\] for derivatives of determinants.

\[
\frac{d}{dt} A_t = \frac{d}{dt} \int_U dS_t = \int_U \frac{1}{2\sqrt{\det(g_t)}} \frac{d}{dt} \det(g_t) \, du \wedge dv,
\]

\[
= \int_U \frac{1}{2\sqrt{\det(g_t)}} \det(g_t) \text{Tr}(g_t^{-1} \frac{dg_t}{dt}) \, du \wedge dv,
\]

\[
= \frac{1}{2} \int_U \text{Tr}(g_t^{-1} \frac{dg_t}{dt}) \, dS_t.
\]

It remains to compute the derivative of the one-parameter family of first fundamental forms along the normal deformation.

\[
dx_t = dx + t(\phi \, dn + d\phi \, n),
\]

\[
<dx_t, dx_t> = <dx, dx> + 2t\phi <dx, dn> + O(t^2),
\]

\[
I_t = I - 2t\phi t I + O(t^2),
\]

\[
(g_t)_{\alpha\beta} = g_{\alpha\beta} - 2\phi t b_{\alpha\beta} + O(t^2).
\]

Therefore,

\[
\frac{d}{dt} g_t|_{t=0} = -2\phi b
\]

We deduce that

\[
A_t'(0) = -\int \int_U \phi \text{Tr}(g^{-1} b) \, dS = -2 \int \int_U \phi H dS,
\]

for any function $\phi$, so this concludes the proof. Actually, a more complete proof would include analysis of the second variation, but this will not be treated in these notes.

### 5.2.5 Example: Surface of revolution

Let $M$ be a surface of revolution that is also a minimal surface. The standard coordinate patch is given by 4.7

\[
x(r, \phi) = (r \cos \phi, r \sin \phi, f(r))
\]

with fundamental form coefficients

\[
E = 1 + f'^2
\]

\[
e = f''/\sqrt{1 + f'^2},
\]

\[
F = 0,
\]

\[
f = 0,
\]

\[
G = r^2
\]

\[
g = rf'/\sqrt{1 + f'^2}.
\]

For this to be a minimal surface we must have:

\[
H = Eg + Ge = 0,
\]

\[
rf'(1 + f'^2) + r^2 f'' = 0
\]

The equation is easily integrated. Let $p = f'$, separate variables and integrate by partial fractions:

\[
rp' = p(1 + p^2),
\]

\[
\frac{1}{p(1 + p^2)} dp = -\frac{1}{r} dr,
\]

\[
\frac{p}{\sqrt{1 + p^2}} = \frac{A}{r},
\]

\[
A = r f'(1 + f'^2).
\]
where $A$ is a constant of integration. Squaring both sides and solving for $p = f'$ we get:

$$p^2r^2 = A^2(1 + p^2),$$

$$p = f' = \frac{A}{\sqrt{r^2 - 1}},$$

$$f = A \cosh^{-1} r.$$

The conclusion is that a catenoid is the only minimal surface of revolution. The mean curvature is not a bending invariant because it depends specifically on the second fundamental form. However, it is notable by a short computation, that the deformation described in equation 4.115 that bends a catenoid into a helicoid, results on a one-parameter family of minimal surfaces $z_t$, independent of $t$.

A surface of type $x = (r \cos \phi, r \sin \phi, h(\phi))$ is called a conoid. The helicoid is the only minimal surface that is also a conoid.

### 5.2.2 Conformal Mappings

In this section we explore the connection between minimal surfaces and conformal maps. For this purpose, it will be useful to insert a pedestrian review of some basic concepts in complex variables.

We denote by $\mathbb{C}$ the usual vector space of complex numbers of the form $z = x + iy$. Complex numbers can also be represented by antisymmetric matrices

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

in which the binary algebraic operations are mapped to matrix operations. Thus, for example, multiplying two complex numbers $z_1, z_2$, is equivalent to multiplying the corresponding matrices. By Euler’s formula, any complex number of unit length can be written in the form $z = e^{i\theta} = \cos \theta + i \sin \theta$. The matrix version of a unit vector is a rotation matrix

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

The set all such matrices forms a group called $SO(2)$. There are two special elements in this set that comprise a basis for the vector space, the identity matrix $I$ and the symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ (5.50)

The former corresponds to a rotation matrix with $\theta = 0$ and the latter to a rotation by $\theta = \pi/2$. Clearly $J^2 = -I$ showing that $J$ plays the role of the imaginary number $i$ in the matrix representation.

A fundamental result from complex variables is that if a function $w = f(z) = u + iv$ is differentiable in the complex sense (i.e. holomorphic), then the following properties hold:

1. The real and imaginary parts satisfy the Cauchy-Riemann equations $u_x = v_y; \ u_y = -v_x$.
2. The functions $u$ and $v$ are (conjugate) harmonic: $\nabla^2 u = \nabla^2 v = 0$.
3. The families of curves $u(x,y) =$constant and $v(x,y) =$constant, are mutually orthogonal. That is, $(\text{Grad} \ u, \text{Grad} \ v) = 0$. In the context of heat flow these curves are called isothermal lines.
4. The map is conformal, that is, it preserves angles.
5. The conformal factor is given by the Jacobian $|f'(z)|^2 = |\nabla u|^2 = |\nabla v|^2$. 
Let \( z = x + iy \) and \( \tau = x - iy \). The chain rule gives:

\[
\partial_z \equiv \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \partial_\tau \equiv \frac{\partial}{\partial \tau} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

7. \( w = f(z) \) is holomorphic if and only if \( \partial_\tau f = \frac{\partial f}{\partial \tau} = 0 \). This is equivalent to the Cauchy-Riemann equations.

8. \( I = dx^2 + dy^2 = dzd\tau \), and \( \Delta f \equiv f_{xx} + f_{yy} = 4 \frac{\partial^2 f}{\partial x \partial \tau} \).

Since \( \mathbb{R}^2 \) as a vector space is isomorphic to \( \mathbb{C} \), we can extend to a surface \( M \) in \( \mathbb{R}^3 \) the complex structure of \( \mathbb{C} \) by requiring that the locally Euclidean property be replaced by complex coordinate patches that are holomorphic. This can actually be done intrinsically for an oriented surface by introducing a \((1,1)\) tensor \( J : TM \to TM \) so that for an orthogonal basis \( \{e_1, e_2\} \) of the tangent space, \( J(e_1) = e_2 \) and \( J(e_2) = -e_1 \). This results on a matrix representation of \( J \) at each point identical to the symplectic matrix \((5.50)\) and it represents a rotation by \( \pi/2 \). The tensor \( J \) can always be introduced in any coordinate patch by starting with \( x_u \) or \( x_v \) and using the Gram-Schmidt process to find an orthogonal vector. An easy computation gives:

\[
\begin{align*}
v_1 &= x_u, \quad v_2 = x_v - F \frac{E}{F} x_u = E x_v - F x_u, \\
w_1 &= x_v, \quad w_2 = x_u - F \frac{G}{F} x_v = G x_u - F x_v.
\end{align*}
\]

Orientation is preserved if the differential of surface satisfies \( dS(X, J(X)) > 0 \) for all \( X \neq 0 \). Since \( J \) represents a rotation by \( \pi/2 \), there must be constants \( c_1 \) and \( c_2 \) such that \( Jx_u = c_1 (Ex_v - Fx_u) \) and \( Jx_v = c_2 (Fx_v - Gx_u) \). Setting \( \|x_u\|^2 = \|Jx_u\|^2 \) and \( \|x_v\|^2 = \|Jx_v\|^2 \) we find that \( c_1 = c_2 = 1/\sqrt{\det g} \), hence, the components of \( J \) in the parametric coordinate frame are

\[
\begin{align*}
Jx_u &= \frac{Ex_v - Fx_u}{\sqrt{EG - F^2}}, \\
Jx_v &= \frac{Fx_v - Gx_u}{\sqrt{EG - F^2}}.
\end{align*}
\]

5.2.6 Definition Let \( \{M, g\} \) and \( \{M', g'\} \) be Riemannian manifolds. A map \( F : M \to M' \) is called \textit{conformal} if there exists a function \( \lambda \), such that

\[
\langle F_*X, F_*Y \rangle' = \lambda^2 \langle X, Y \rangle,
\]

for all tangent vectors. If \( \lambda = 1 \), the map is an isometry.

Conformal maps on manifolds as defined above have the following properties:

1. \( \|F_*X\|^2 = |\lambda|^2 \|X\|^2 \). The reason is that \( \|X\|^2 = \langle X, X \rangle \).

2. \( F \) preserves angles, since at each point, the \( \lambda \)'s cancel in the formula for the cosine of the angle as in \(1.51\) and \(1.116\).

3. \( F_*(J(X)) = J(F_*(X)) \). This is easily shown by applying the maps to a basis and using the fact that \( F \) preserves angles and \( J \) represents a rotation.

5.2.7 Example An isothermal system as in \(4.116\) in which the metric takes the form \( ds^2 = \lambda^2 \left( dt^2 + dv^2 \right) = \lambda^2 \left( du^2 + dv^2 \right) \) can be considered as a map from \( \mathcal{U} \subset \mathbb{R}^2 \) as a plane surface, to \( M \). This is an
example of the most basic conformal map. As previously stated in theorem 4.5.2, the coordinate patch in this case satisfies the equation,

$$x_{uu} + x_{uv} = 2\lambda^2 H.$$  

The conclusion of this theorem can be restated by saying that that surface in isothermal coordinates is a minimal surface if and only if its coordinate patch functions are harmonic in the usual Euclidean sense.

5.2.8 Definition Given an isothermal, conformal patch $x(u, v)$, we call a map $y(u, v)$ a **conjugate patch** if it satisfies the Cauchy-Riemann equations

$$x_u = y_v, \quad x_v = y_u. \quad (5.54)$$

In complex variables, the real and imaginary parts of a holomorphic function are conjugate harmonic functions. Given one of these, one can determine the other by integrating the Cauchy-Riemann equations. In a similar way, given an isothermal conformal patch, we may determine a conjugate patch and this conjugate patch is also isothermal and conformal. The conjugate patches can be rendered as the real and imaginary parts of a complex holomorphic patch.

5.2.9 Definition Given conjugate harmonic patches $x$ and $y$, we define the **associated family** to be the one-parameter family

$$z_t = Re[e^{it}(x + iy)] = (\cos t)x + (\sin t)y. \quad (5.55)$$

5.2.10 Existence of Isothermal Coordinates The existence of isothermal coordinates is more subtle and requires some harmonic analysis. Given $\{M, g\}$ with a coordinate patch $x(u, v)$, we seek a map $F(p) = (h(p), k(p))$, in which the metric is isothermal. The condition that the parametric curves be orthogonal means we must have $(\nabla h, \nabla k) = 0$. Using the definition of the gradient 2.27 and recalling the components of the inverse of the metric 4.84, we get

$$(\nabla h, \nabla k) = 0 \text{ holds if there exists a function } \lambda \text{ so that}$$

$$k_u = \lambda (Fh_u - Eh_v), \quad k_v = \lambda (Gh_u - Fh_v).$$

To get $|\nabla k|^2 = |\nabla h|^2$ we set $\lambda = 1/\sqrt{\det(g)}$. The integrability condition $k_{uv} = k_{vu}$ yields the classical Laplace-Beltrami equation 4.55

$$\frac{\partial}{\partial u} \left[ \frac{Gh_u - Fh_v}{\sqrt{EG - F^2}} \right] + \frac{\partial}{\partial v} \left[ \frac{Eh_v - Fh_u}{\sqrt{EG - F^2}} \right] = 0.$$ 

Hence, the existence of non-trivial solutions is tied to the harmonic analysis on the existence of solutions of the elliptic operator.
An equivalent approach for the analytic case was obtained by Gauss by the simple but ingenious method of factoring the first fundamental form \( ds^2 = E du^2 + 2F dudv + G dv^2 \),

\[
ds^2 = \left[ \sqrt{E} du + \frac{F + i\sqrt{EG - F^2}}{\sqrt{E}} dv \right] \left[ \sqrt{E} du + \frac{F - i\sqrt{EG - F^2}}{\sqrt{E}} dv \right].
\]

The idea is to find coordinates \( h \) and \( k \) and a conformal factor \( \lambda \), such that

\[
ds^2 = \lambda^2 dhdk.
\]

This would suffice, since having found \( h \) and \( k \), one could set \( h = \phi + i\psi \) and \( k = \phi - i\psi \). Following Eisenhart [18], let \( t_1 \) and \( t_2 \) be integrating factors of the system

\[
t_1 \left( \sqrt{E} du + \frac{F + i\sqrt{EG - F^2}}{\sqrt{E}} dv \right) = dh = h_u du + h_v dv,
\]

\[
t_2 \left( \sqrt{E} du + \frac{F - i\sqrt{EG - F^2}}{\sqrt{E}} dv \right) = dk = k_u du + k_v dv,
\]

where \( \lambda^2 = 1/(t_1 t_2) \).

The first equation above is equivalent to,

\[
t_1 \sqrt{E} = h_u, \quad t_1 \left( \frac{F + iH}{\sqrt{E}} \right) = h_v,
\]

where \( H = \sqrt{\det(g)} = \sqrt{EG - F^2} \). Solving for \( t_1 \) in the first equation and substituting on the second, we get

\[
h_u(F + iH) = Eh_v, \quad (5.56)
\]

\[
ih_u = \frac{Eh_v - Fh_u}{H}. \quad (5.57)
\]

On the other hand, multiplying the first equation above by \( F - iH \) we get

\[
h_u(F^2 + H^2) = E(F - iH)h_v,
\]

\[
h_u(F^2 + EG - F^2) = E(F - iH)h_v,
\]

\[
EGh_u = E(F - iH)h_v.
\]

Hence

\[
ih_v = \frac{Fh_v - Gh_u}{H}. \quad (5.58)
\]

The integrability condition \( h_{uv} = h_{vu} \) for equations [5.57] and [5.58] gives again the Laplace Beltrami equation

\[
\frac{\partial}{\partial u} \left[ \frac{Gh_u - Fh_v}{\sqrt{EG - F^2}} \right] + \frac{\partial}{\partial v} \left[ \frac{Eh_u - Fh_v}{\sqrt{EG - F^2}} \right] = 0.
\]

By a completely analogous computation, one can verify that \( k \) satisfies the same integrability condition.

The existence of isothermal coordinates is closely related to the the existence of complex structures. Even for the case of 2-dimensional manifolds, if the topological conditions on the metric are weakened, the problem is not simple (See: [19]).

**5.2.11 Exercise** Let \( ds^2 = E du^2 + 2F dudv + F dv^2 = \nu|d\zeta|^2 \), where \( \zeta = u + iv \).
5.2. MINIMAL SURFACES

1) Substitute $du$ and $dv$ in terms of $z$, and equate coefficients to show that,

$$\mu = \frac{1}{4\nu}(E - G + 2Fi), \quad \nu(1 + \mu\overline{\nu}) = \frac{1}{2}(E + G).$$

2) Solve for $\nu$ and thus show that,

$$\nu = \frac{1}{4}(E + G + 2\sqrt{EG - F^2}).$$

3) We want $ds^2 = \lambda^2 dh d\overline{h} = \lambda^2 |dh|^2$, where $h = h(z, \overline{z})$. Show that $h$ exists if, and only if, it satisfies the Beltrami equation

$$\frac{\partial h}{\partial \overline{z}} = \mu \frac{\partial h}{\partial z}.$$

5.2.12 Example Stereographic Projection. Consider the unit sphere $S^2$: $x^2 + y^2 + z^2 = 1$. For each point $P(x, y, z)$ on the sphere, we draw a line segment from the north pole to $P$ and extend the segment until it intersects the $xy$-plane, viewed as a copy of the Complex plane, at point $\zeta = X + iY$. By simple ratio and proportions of corresponding sides of similar triangles, as partially shown in 5.8, we have,

$$X = \frac{x}{1 - z},$$

$$Y = \frac{y}{1 - z}, \quad \text{so that}$$

$$\zeta = \frac{x + iy}{1 - z} \quad (5.59)$$

The map $\pi : S^2 \to \mathbb{C}$ projects each point $P$ except for the North pole to the unique complex number $\zeta$ given by the equation above. The closer the point $P$ is to the North pole, the larger the norm of $\zeta$. This gives rise to the geometric interpretation that under the stereographic projection, the North pole correspond to the point at infinity in the complex plane. To find the inverse $\pi^{-1}$ of the
projection, first notice that:

\[ \zeta \bar{\zeta} = \frac{x^2 + y^2}{(1 - z)^2}, \]

\[ \zeta \bar{\zeta} + 1 = \frac{x^2 + y^2 + 1}{(1 - z)^2}, \]

\[ = \frac{2 - 2z}{(1 - z)^2}, \]

\[ = \frac{2}{1 - z}. \]

Combining that last equation with (5.59), we find the x, y and z coordinates on the sphere,

\[ \pi^{-1}(\zeta) = (x, y, z), \]

\[ = \left( \frac{\zeta + \bar{\zeta}}{\zeta \bar{\zeta} + 1}, \frac{\zeta - \bar{\zeta}}{i(\zeta \bar{\zeta} + 1)}, \frac{\zeta \bar{\zeta} - 1}{\zeta \bar{\zeta} + 1} \right) \quad (5.60) \]

The existence of the inverse map shows that the sphere is locally diffeomorphic to the complex plane. If \( \eta \) represents the complex coordinate arising from the stereographic projection from the South pole, then the transition functions on the overlap of the coordinate patches is given by \( \zeta = 1/\eta \) and \( \eta = 1/\zeta \). Thus, \( S^2 \) may be considered as a complex manifold called the **Riemann Sphere**. .

In polar coordinates \( X = R \cos \phi, \ Y = R \sin \phi \), figure 5.8 shows that \( ||\zeta|| = R = \cot(\frac{\theta}{2}) \) and hence

\[ \zeta = \cot(\frac{\theta}{2}) e^{i \phi}. \quad (5.61) \]

The short computation that follows gives the complex metric of the Riemann sphere in terms of the standard metric in spherical coordinates.

\[ d\zeta = -\frac{1}{2} \csc^2(\frac{\theta}{2}) e^{i \phi} d\theta + i \cot(\frac{\theta}{2}) e^{i \phi} d\phi, \]

\[ d\bar{\zeta} = -\frac{1}{2} \csc^2(\frac{\theta}{2}) e^{-i \phi} d\theta - i \cot(\frac{\theta}{2}) e^{-i \phi} d\phi, \]

\[ d\zeta d\bar{\zeta} = \frac{1}{4 \sin^2(\frac{\theta}{2})} (d\theta^2 + \sin^2 \theta d\phi^2). \]

Using equation (5.61) again, we find:

\[ ds^2 = \frac{4 d\zeta d\bar{\zeta}}{(1 + \zeta \bar{\zeta})^2} = d\theta^2 + \sin^2 \theta d\phi^2. \quad (5.62) \]

This shows that the map is conformal. The complex version of the first fundamental form is called the **Fubini-Study** metric of the Riemann sphere.

The formulas for the stereographic projection (5.59) naturally extrapolate to spheres in all dimensions. In the case of the unit circle \( S^1 \), the quantity \( \zeta \) is a real number. Since we are using \( \theta \) as the azimuthal angle, we have the unconventional parametrization of the unit circle \( x = \sin \theta, \ y = \cos \theta \). Then the formulas for \( \pi^{-1} \) and the metric read:

\[ x = \frac{2 \zeta}{\zeta^2 + 1}, \quad y = \frac{\zeta^2 - 1}{\zeta^2 + 1}, \quad d\theta = \frac{2 \zeta}{\zeta^2 + 1} d\zeta. \quad (5.63) \]
Hence, the substitution $\zeta = \cot\left(\frac{\theta}{2}\right)$ transforms integrals of rational functions of sines and cosines into rational functions of $\zeta$. This is the source of the now infamous technique of integration, which for the wrong reasons, is no longer covered in the standard second semester calculus curriculum. It is by this method that one obtains the integral \[5.64\] that often appears in the theory of curves and surfaces. As an added bonus, if in equation \[5.63\] we pick $\zeta = \frac{m}{n}$ to be any rational number on the real line, the inverse stereographic map yields a rational point in the unit circle with entries giving Euclid’s formula for Pythagorean triplets

\[
\left(\frac{2mn}{n^2 + m^2}, \frac{n^2 - m^2}{n^2 + m^2}\right).
\]

Fig. 5.9: Mercator Projection

5.2.13 Example Mercator Projection. Given a surface of revolution \[4.7\] with metric

\[
ds^2 = (1 + f'^2) \, dr^2 + r^2 \, d\phi^2,
\]

\[
= r^2 \left[ \frac{1 + f'^2}{r^2} \, dr^2 + d\phi^2 \right],
\]

we set

\[
d\hat{y} = \frac{\sqrt{1 + f'^2}}{r} \, dr, \quad d\hat{x} = d\phi
\]

\[
\hat{y} = \int \frac{\sqrt{1 + f'^2}}{r} \, dr, \quad \hat{x} = \phi.
\]

This is a conformal map into a plane with Cartesian coordinates $(\hat{x}, \hat{y})$ in which the meridians map to $\hat{x} =$constant, and the horizontal circles on the surface of revolution map to parallels $\hat{y} =$constant. In particular, for a sphere in which $f(r) = \sqrt{a^2 - r^2}$, the substitution $r = a \sin \theta$ yields

\[
\hat{y} = \int \sec \theta \, d\theta = \ln[\tan(\frac{\theta}{2} + \frac{\pi}{4})].
\]

Comparing with equation \[4.22\] we see that in this projection, loxodromes on the sphere map to straight lines on the plane. The projection is more faithful near the equator as shown in figure \[5.9\] The map is depicted with a four-color scheme, which it is always possible because of a famous theorem’
5.2.3 Minimal Surfaces by Conformal Maps

Consider a surface \( \{M, g\} \) with coordinate patch \( x(u, v) \). Let \( \zeta = u + iv \), and \( \overline{\zeta} = u - iv \), so that \( d\zeta = du + idv \), \( d\overline{\zeta} = du - idv \). The composite map gives the patch as

\[
x(\zeta, \overline{\zeta}) = (x^1(\zeta, \overline{\zeta}), x^2(\zeta, \overline{\zeta}), x^3(\zeta, \overline{\zeta})),
\]

The complex derivative with respect to \( \zeta \) of the patch is given by:

\[
x_\zeta = \frac{\partial x}{\partial \zeta} = \frac{1}{2}(x_u - ix_v),
\]

\[
= (\frac{\partial x^1}{\partial \zeta}, \frac{\partial x^2}{\partial \zeta}, \frac{\partial x^3}{\partial \zeta}),
\]

\[
= (x^1_\zeta, x^2_\zeta, x^3_\zeta).
\]

Let \( \phi = 2x_\zeta \), and \( \phi^2 = 4(x_\zeta, x_\zeta) \). Since we have chosen \( \phi \) to depend only on \( \zeta \), \( \partial_{\overline{\zeta}} \phi = 0 \), so \( \phi \) is holomorphic. We have the following theorem:

5.2.14 Theorem A holomorphic patch \( x \) is an isothermal if and only if

\[
\phi^2 = (\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0.
\]

(5.65)

Proof: The proof by direct computation.

\[
\phi^2 = 4(\frac{\partial x}{\partial \zeta}, \frac{\partial x}{\partial \zeta})
\]

\[
= (x_u - ix_v, x_u + ix_v),
\]

\[
= (x_u, x_u) - (x_v, x_v) - 2i(x_u, x_v),
\]

\[
= E - G - 2iF.
\]

The theorem follows because in an isothermal patch, \( E = G \) and \( F = 0 \), so \( \phi^2 = 0 \).

5.2.15 Remark The function \( \phi \) is a complex function, so we must be careful not to confuse \( \phi^2 \) as defined above with \( |\phi|^2 = \overline{\phi} \phi \). The latter is given by

\[
|\phi|^2 = 4(x_\zeta, \overline{x_\zeta}),
\]

\[
= (x_u - ix_v, x_u + ix_v),
\]

\[
= (x_u, x_u) + (x_v, x_v),
\]

\[
= E + G,
\]

(5.66)

so if the patch is isothermal, \( E = G = \frac{1}{2}|\phi|^2 \).

We now have a process to construct minimal surfaces. We need to find a holomorphic patch \( \phi \) with \( \phi^2 = 0 \). To construct such patches we introduce a special class of complex curves.

5.2.16 Definition A complex curve \( \phi : U \subset \mathbb{C} \rightarrow \mathbb{R}^2 \) is called a minimal curve or an isotropic curve if the differential of arc length

\[
ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = 0.
\]

To find such curves we once again we take advantage of the wonderful ingenuity of the leading mathematicians in the 1800’s. Factoring over the complex, we have

\[
(dx^1 + i dx^2)(dx^1 - i dx^2) + (dx^3)^2 = 0,
\]

\[
(dx^1 + i dx^2)(dx^1 - i dx^2) = -(dx^3)^2,
\]

\[
\frac{dx^1 + i dx^2}{dx^3} = -\frac{dx^3}{dx^1 - i dx^2}.
\]
5.2. MINIMAL SURFACES

Setting the left hand side equal to some function $-\tau$ we get a pair of differential equations

\[
\frac{dx^1}{dx^3} + i \frac{dx^2}{dx^3} = -\tau, \\
\frac{dx^1}{dx^3} + i \frac{dx^2}{dx^3} = \frac{1}{\tau}.
\]

If we add the two equations we get:

\[
\frac{dx^1}{dx^3} = \frac{1}{\tau} - \tau, \\
= \frac{1 - \tau^2}{\tau}, \\
\frac{dx^1}{1 - \tau^2} = \frac{dx^3}{2\tau}.
\]

Subtracting the equations instead, we get:

\[
2i \frac{dx^2}{dx^3} = -\tau - \frac{1}{\tau}, \\
= -\frac{1 + \tau^2}{\tau}, \\
\frac{dx^1}{i(1 - \tau^2)} = \frac{dx^3}{2\tau}.
\]

Thus, we are lead to the following equation:

\[
\frac{dx^1}{1 - \tau^2} = \frac{dx^2}{i(1 + \tau^2)} = \frac{dx^3}{2\tau} = F(\tau) \, d\tau
\] (5.67)

for some arbitrary analytic function $F$. At the end, we want real geometric objects, so we integrate and extract the real part.

\[
x^1 = \Re e \int^z (1 - \tau^2) F(\tau) \, d\tau, \\
x^2 = \Re e \int^z i(1 + \tau^2) F(\tau) \, d\tau, \\
x^3 = \Re e \int^z 2\tau F(\tau) \, d\tau.
\] (5.68)

This choice of coordinates satisfies the conditions of holomorphic patch $\phi$ with $\phi^2 = 0$. Although a bit redundant, this can be verified immediately.

\[
\phi = F(1 - \tau^2, i(1 + \tau^2), 2\tau) \\
\phi^2 = F^2[(1 - \tau^2)^2 + i^2(1 + \tau^2)^2 + 4\tau^2] = 0.
\]

Equations [5.68] are the classical Weierstrass coordinates from which one ascertains that a holomorphic function $F(\tau)$ gives rise to a minimal surface.

5.2.17 Example Enneper’s surface. This is the surface corresponding to a Weierstrass patch with the simplest, non-trivial holomorphic function, namely a constant $F(\tau) = a$ which choose to
be $a = 3$. The surface was discovered by Enneper in 1864. Integration of (5.68) with $F(\tau) = 3$ gives,

$$x^1 = Re(3\tau - \tau^3) = 3u + 3uv^2 - u^3,$$
$$x^2 = Re[i(3\tau + \tau^3)] = -3v - 3u^2v + v^3,$$
$$x^3 = Re(3\tau^2) = 3(u^2 - v^2),$$

resulting in the coordinate patch

$$x(u,v) = (3u + 3uv^2 - u^3, -3v - 3u^2v + v^3, 3(u^2 - v^2)). \tag{5.69}$$

If one sets $\tau = re^\phi$ a polar coordinate parametrization is obtained

$$x(r, \phi) = (3r \cos \phi - r^3 \cos 3\phi, -3r \sin \phi - 3r^3 \sin 3\phi, 3r^2 \cos 2\phi). \tag{5.70}$$

Figure 5.10 shows the Enneper, the first with $u$ and $v$ ranging from 0 to 1.2 and the second for $r \in [0,2]$, $\phi \in [\pi, \pi]$. The surface is self intersecting if the range is big enough. In the polar parametrization $K = -(4/9)/(1 + r^2)^{-4}$ and of course $H = 0$. Using an advanced algebraic geometry technique called Gröbner basis, one can eliminate the parameters and show that the surface is algebraic, and can be written implicitly in terms of a ninth degree polynomial. As will be discussed later in equation (5.76) there is an alternate way to write the Weierstrass parametrization. In this alternate formulation, the classical Enneper surface is generated by $f = 1$ and $g = \sigma$. A generalization of the Enneper surface is obtained by taking $g(\sigma) = \sigma^n$, where $n + 1$ is the number of “flaps” bending up. The surface on the right of figure 5.10 corresponds to the case $n = 3$.

Setting $r =$-constant in the polar form of the Enneper surface, we get curves that are not spherical curves, but they are close. On the left of Figure 5.11 we display the intersection of an Enneper surface with a sphere.
This raises the fun problem of finding a curve that is actually spherical and resembles the seam of a baseball. We explore this possibility with a curve of the form

\[ \mathbf{x}(r, \phi) = (a \cos \phi - b \cos 3\phi, a \sin \phi + b \sin 3\phi, c \cos 2\phi), \]

and requiring it to have a constant norm. A short computation using the sum and half-angle formulas for cosine, gives

\[ \|\mathbf{x}\|^2 = a^2 + b^2 - 2ab \cos 4\phi + c^2 \cos^2 2\phi, \]

\[ = a^2 + b^2 - 2ab \cos 4\phi + \frac{c^2}{2}(1 + \cos 4\phi), \]

\[ = a^2 + b^2 + \frac{c^2}{2} + \left(\frac{c^2}{2} - 2ab\right) \cos 4\phi. \]

To make this length constant, independently of \(\phi\), we set \(c^2 = 4ab\), which remarkably, leads to the norm being a perfect square \(\|\mathbf{x}\|^2 = (a + b)^2\). A nice choice leading to all integer coefficients is \(a = 9\), \(b = 4\) which gives \(c = 12\). The graph shown on the right in figure 5.11 shows that this results on a reasonable shape for the seam of a baseball. Personally, I find this more gratifying than the baseball I hand-made as a kid because I could not afford one. I lost that ball in the first pitch when an older boy hit a home run into a pig sty.

5.2.18 **Example** Catenoid. As we would expect, the ubiquitous catenoid can be obtained from the Weierstrass parametrization. Anticipating a coordinate patch with cosh functions, we choose \(F(\tau) = -a/\tau^2\) and then let \(\tau = e^u - e^{-u}\). Integration, followed by an application of the summation formulae for hyperbolic functions, we get:

\[ x^1 = -a \Re \int \left[ \frac{1}{\tau^2} - 1 \right] d\tau, \]

\[ = -a \Re \left[ \left( -1/\tau \right) - \tau \right], \]

\[ = a \Re \left[ e^u + e^{-u} \right] = a \Re \left[ \cosh u \right], \]

\[ = a \Re \left[ \cosh(u + iv) \right], \]

\[ = a \cosh u \cos v. \]

In a similar manner,

\[ x^2 = -a \Re \int i\left[ \frac{1}{\tau^2} + 1 \right] d\tau, \]

\[ = -a \Re \left[ i\left( -1/\tau \right) + \tau \right], \]

\[ = -a \Re \left[ i(e^u - e^{-u}) \right] = -a \Re [i \sinh u], \]

\[ = -a \Re \left[ i \sinh(u + iv) \right], \]

\[ = a \cosh u \sin v. \]

The last integration is a one-liner

\[ x^3 = -2a \Re [\ln \tau] = -2a \Re (w) = -2au. \]

The result is the catenoid

\[ \mathbf{x}(u, v) = (a \cosh u \cos v, a \cosh u \sin v, -2au). \]

5.2.19 **Example** Henneberg’s Surface. This surface was discovered in 1875 by the German mathematician Lebrecht Henneberg. The surface can be obtained from Weierstrass coordinates
by choosing \( F(\tau) = 1 - 1/\tau^4 \) and then letting \( \tau = e^w = e^{u + iv} \). In a manner similar to the computation above, we integrate the equations and follow by applying the summation formulae for hyperbolic functions.

\[
x^1 = \Re \int [(1 - \tau^2)(1 - 1/\tau^4)] \, d\tau,
\]
\[
= \Re \int [1 + 1/\tau^2 - \tau^2 - 1/\tau^4] \, d\tau,
\]
\[
= \Re [(\tau - 1/\tau) - (1/3)(\tau^3 - 1/\tau)]
\]
\[
= 2 \Re [\sinh w - (1/3)\sinh 3w]
\]
\[
= 2 \Re [\sinh(u + iv) - (1/3)\sinh 3(u + iv)]
\]
\[
= 2 \sinh u \cos v - (2/3)(\sinh 3u \cos 3v).
\]

The integral for \( x^2 \) gives,

\[
x^2 = \Re \int i[(1 + \tau^2)(1 - 1/\tau^4)] \, d\tau,
\]
\[
= \Re \int i[1 - \tau^2 + 1/\tau^2 - 1/\tau^4] \, d\tau,
\]
\[
= \Re i[(\tau + 1/\tau) + (1/3)(\tau^3 + 1/\tau^3)]
\]
\[
= 2 \Re i[\cosh w + (1/3)\cosh 3w]
\]
\[
= 2 \Re i[\cosh(u + iv) + (1/3)\cosh 3(u + iv)]
\]
\[
= -2 \sinh u \sin v - (2/3)(\sinh 3u \sin 3v).
\]

The last integration is a bit simpler

\[
x^3 = \Re \int [2\tau(1 - 1/\tau^4)] \, d\tau,
\]
\[
= \Re \int [2\tau - 2/\tau^3] \, d\tau,
\]
\[
= \Re [(\tau^2 + 1/\tau^2)]
\]
\[
= 2 \Re [\cosh 2w]
\]
\[
= 2 \Re [\cosh 2(u + iv)]
\]
\[
= 2 \cosh 2u \cos 2v.
\]

Neglecting the factor of 2 and using a common abbreviation for the hyperbolic functions, the result is:

\[
x(u, v) = (\text{sh} \, u \cos v - \frac{1}{3}(\text{sh} \, 3u \cos 3v), -\text{sh} \, u \sin v - \frac{1}{3}(\text{sh} \, 3u \sin 3v), \text{ch} \, 2u \cos 2v).
\]
5.2. MINIMAL SURFACES

The curve given by \( v = 0 \) is a geodesic in the shape of a semicubical parabola which is the intersection of the surface with the plane \( y = 0 \). This is a reflection of the fact the Hennerberg surface is a Bj"orling surface, that is, a minimal surface that contains a given curve with prescribed normal. It turns out that the Hennerberg surface is the model representing an immersion in \( \mathbb{R}^3 \) of the projective plane.

The curvature for a minimal surface in Weierstrass coordinates can be calculated from equation 4.117 using the fact that \( E = G = \frac{1}{2} |\phi|^2 \), we have,

\[
\phi = F(1 - \tau^2, i(1 + \tau^2), 2\tau),
\]

\[
|\phi|^2 = |F|^2((1 - \tau^2)(1 - \tau^2) + (1 + \tau^2)(1 + \tau^2) + 4\tau\tau)
\]

\[
= |F|^2(2 + 2\tau^2\tau^2 + 4\tau\tau)
\]

\[
= 2|F|^2(1 + |\tau|^2)^2.
\]

The conformal factor is \( E = \lambda^2 = \frac{1}{2} |\phi|^2 \), so the Gaussian curvature is given by

\[
K = -\frac{1}{\lambda^2} \nabla^2(\ln \lambda),
\]

\[
= -\frac{1}{\lambda^2} \frac{\partial^2}{\partial \tau \partial \tau} [\ln(F(1 + |\tau|^2))],
\]

\[
= -\frac{4}{\lambda^2} \frac{\partial^2}{\partial \tau \partial \tau} [\frac{1}{2}(\ln F + \ln F) + \ln(1 + |\tau|^2)],
\]

\[
= -\frac{4}{\lambda^2} \frac{\partial^2}{\partial \tau \partial \tau} \ln(1 + |\tau|^2)],
\]

\[
= -\frac{4}{\lambda^2} \frac{\partial}{\partial \tau} \left( \frac{\tau}{(1 + |\tau|^2)} \right],
\]

\[
= -\frac{4}{\lambda^2} \left( \frac{1 + \tau\tau}{(1 + |\tau|^2)^2} \right) = -\frac{4}{\lambda^2} \left( \frac{1}{(1 + |\tau|^2)^2} \right).
\]

So, the result is,

\[
K = -\frac{4}{|F|^2(1 + |\tau|^2)^4} = -\frac{4}{|F|^2(1 + u^2 + v^2)^4}.
\]  

(5.71)

There is an equivalent formulation of Weierstrass parametrization that also appears frequently in the literature. From equation 5.65, the holomorphic patch \( x \) is isothermal if

\[
(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0.
\]  

(5.72)

Proceeding along the same lines as in the computation of isotropic coordinates, we have

\[
(\phi^1 + i\phi^2)(\phi^1 - i\phi^2) = -(\phi^3)^2,
\]

\[
(\phi^1 + i\phi^2) = -\frac{(\phi^3)^2}{(\phi^1 - i\phi^2)}
\]  

(5.73)

Set

\[
f = (\phi^1 - i\phi^2), \quad g = \frac{\phi^3}{(\phi^1 - i\phi^2)}.
\]  

(5.74)

Here, \( f \) is holomorphic, \( g \) is meromorphic, but \( fg^2 = -(\phi^1 + i\phi^2) \) is holomorphic. Since we also have \( fg = \phi^3 \), we can easily solve for the components of \( \phi \) in terms of \( f \) and \( g \). The result is,

\[
\phi^1 = \frac{1}{2} f(1 - g^2),
\]

\[
\phi^2 = \frac{i}{2} f(1 + g^2),
\]

\[
\phi^3 = fg.
\]  

(5.75)
Since $\phi = x_u - ix_v$, we have the alternative Weierstrass parametrization

\[
x = \mathcal{R}e \int z f(\sigma)(1 - g(\sigma)^2) \, d\sigma,
\]

\[
y = \mathcal{R}e \int z f(\sigma)(1 + g(\sigma)^2) \, d\sigma
\]

\[
z = \mathcal{R}e \int z f(\sigma)g(\sigma) \, d\sigma.
\] (5.76)

For easy reference, we denote this parametrization by the notation $x = \mathcal{W}(f, g)$. To see the relationship to the equations 5.68, consider the case when $g$ is holomorphic and $g^{-1}$ is holomorphic. Then, we can use $g$ as the complex variable, and we set $g = \tau$, so that $dg = \frac{1}{2} f/ dg = \frac{1}{2} f/ dg$. Let $F = \frac{1}{2} f/ dg = \frac{1}{2} f/ dg$.

A remarkable result can be obtained as follows. As a complex variable $g$ can be mapped into the unit sphere $S^2$ by the inverse stereographic projection (See equation 5.60)

\[
\pi^{-1}(g) = \frac{(g + \bar{g}, -i(g - \bar{g}), g\bar{g} - 1)}{(g\bar{g} + 1)}
\] (5.77)

On the other hand, the real and imaginary parts of $\phi = x_u - ix_v$ comprise two orthogonal tangent vectors to the surface. We compute the dot product

\[
(\phi, \pi^{-1}g) = \frac{f}{2(g\bar{g} + 1)}[(1 - g^2)(g + \bar{g}) + (1 + g^2)(g - \bar{g}) + g(|g|^2 - 1)],
\]

\[
= \frac{f}{2(g\bar{g} + 1)}[(g + \bar{g}) + (g - \bar{g}) - g^2(g + \bar{g} - g + \bar{g}) + 2g|g|^2 - 2g],
\]

\[
= \frac{f}{2(g\bar{g} + 1)}[2g - 2g^2\bar{g} + 2g|g|^2 - 2g],
\]

\[= 0.
\]

We conclude that $\pi^{-1}g$ is a unit normal to the surface, and thus we have the following theorem,

**5.2.20 Theorem** If $x(\zeta, \bar{\zeta})$ is an isothermal holomorphic patch with $\phi = 2x_\zeta = x_u - ix_v$, then the function $g = \phi^3/(\phi^3 - i\phi^2)$ in the Weierstrass parametrization $\mathcal{W}(f, g)$ of a minimal surface, is the stereographic projection $\pi$ of the Gauss map. That is,

\[
\pi \circ N = g
\] (5.78)

where as usual, $N = (x_u \times x_v)/\|x_u \times x_v\|$.

Rewriting the expression for the Gaussian curvature 5.71 in terms of $f$ and $g$, we get,

\[
K = - \left[ \frac{4|g'|}{|f|(1 + |g|^2)} \right]^2.
\] (5.79)

We see immediately that the one-parameter family of associated patches $z_t$ given by

\[
z_t = \mathcal{R}e(e^{it}\phi),
\] (5.80)

results on a family of isometric minimal surfaces. This provides another way to view the deformation of a catenoid into a helicoid described earlier in equation 4.115.
5.2. MINIMAL SURFACES

5.2.21 Example Bour surface

The surface $W(f, g)$, with $f = 1$ and $g = \sqrt{\sigma}$ was first discussed in 1861 by Bour, who subsequently was awarded the mathematics prize of the French Academy of Sciences. The integrals are completely elementary,

\[
x^1 = \frac{1}{2} \Re \left( z - \frac{1}{2} z^2 \right), \\
x^2 = \frac{i}{2} \Re \left( z + \frac{1}{2} z^2 \right), \\
x^3 = \frac{2}{3} \Re \left( \frac{1}{3} z^{3/2} \right).
\]

Converting to polar form, we get the parametrization.

\[
x(r, \phi) = \left( \frac{1}{2} r \cos \phi - \frac{1}{4} r^2 \cos 2\phi, -\frac{1}{2} r \sin \phi + \frac{1}{4} r^2 \sin 2\phi, \frac{2}{3} r^{3/2} \cos \left( \frac{2}{3} \phi \right) \right) \quad (5.81)
\]

The surface is rendered in figure 5.13, with $r \in [0, 4]$.

5.2.22 Example Trinoid surface

This surface is part of family of surfaces indexed by $f = \frac{1}{(\sigma k - 1)^2}$, $g = \sigma^{k-1}$, where $k = 2, 3, \ldots$. These remarkable surfaces have neat topologies discovered by Jorge and Meeks in 1983. The binoid case $k = 2$ is just a catenoid. The trinoid corresponds to $k = 3$. The integrals for the trinoid yield the following parametrization

\[
x^1 = \Re \left[ -\frac{2}{9} \ln(z - 1) + \frac{z}{6(z^2 + z + 1)} + \frac{1}{9} \ln(z^2 + z + 1) \right], \\
x^2 = -\frac{1}{9} \Im \left[ \frac{1}{z - 1} + \frac{z + 2}{2(z^2 + z + 1)} - 2\sqrt{3} \tan^{-1}\left( \frac{1}{\sqrt{3}}(2z + 1) \right) \right], \\
x^3 = -\Re \left[ \frac{1}{3(z^3 - 1)} \right],
\]

where $z = u + iv$. The algebra involved in finding the real coordinate patch is messy, so the task is best left to a computer algebra system. Converting to polar coordinates by letting $z = r e^{i\phi}$ improves rendering the surface plot. As shown in the portion of the surface figure 5.13b, the surface has three openings, meaning that the Gauss map misses three points on $S^2$. In the same manner, the $k$-noid is topologically equivalent to a sphere with $k$ points removed.

For convenient reference, we include the following table showing the choices of $f$ and $g$ yielding the listed minimal surfaces.
If I had to pick a minimal surface to represent the “orchid” which is the national flower of my native country, I would pick the Costa surface. This surface was not discovered until 1982, triggering a renewed interest on minimal surfaces with non-trivial topologies. A parametrization was not produced until 1996. The integrals involve Weierstrass elliptic functions, which probably would have been most pleasing to Weierstrass, but requires more advanced knowledge than is typically covered on introductory courses on complex variables. The Weierstrass elliptic function \( \wp(z) \) is implemented in Maple as,

\[
\text{WeierstrassP}(z, g_2, g_3) = \frac{1}{z^2} + \sum_\omega \left( \frac{1}{z - \omega} + \frac{1}{\omega} \right).
\]  

Fig. 5.14: Weierstrass Elliptic Function

The functions \( \wp \) are meromorphic with a pole of order 2 at the origin. They are doubly periodic over an arbitrary lattice \( \{ \omega = 2m\omega_1 + 2n\omega_2 \mid m, n \in \mathbb{Z} \} \) with periods \( \omega_1 \) and \( \omega_2 \). The quantities \( g_2 \) and \( g_3 \) are called the invariants, and are related to \( \omega_1 \) and \( \omega_2 \). In the Maple implementation, the periods are set to \( \omega_1 = \omega_2 = \frac{1}{2} \), which does not result in any real loss of generality. A salient property of the invariants is that they link \( \wp \) to the cubic differential equation

\[
(\wp')^2 = 4(\wp)^3 - g_2\wp - g_3.
\]  

Fig. 5.15: Costa’s Minimal Surface
Following Alfred Grey, as shown in Eric Weisstein’s Wolfram Mathworld [36], we pick \( g_2 = 189.07272 \), \( g_3 = 0 \) and \( A = \sqrt{2\pi g_2} \). As we have done before, we first try simply inserting the choices of \( f \) and \( g \) into \( W(f, g) \), integrating with Maple with the substitution \( z = u + iv \), and then extracting the real part. The result is a bit disconcerting as one obtains expressions with Weierstrass \( \zeta \) function with very large coefficients of the order of \( 10^6 \). The problem becomes immediately clear by observing a plot of the real part \( \wp \) as shown in figure 5.14. The \( W(f, g) \) coordinates are integrated by default from 0 to \( z \), so the problem is caused by the pole of order 2 at the origin. Still, one may persist by proceeding to plot the surface by restricting the values of \( u \) and \( v \) to stay away from the singularities. We chose the ranges to go from 0.02 to 0.98. Surprisingly, Maple takes some time to compute, but it renders the beautiful flowery-shaped curve that appears at the left on figure 5.15. While aesthetically pleasing, the figure is topologically inaccurate. A much more rigorous analysis such as done by A. Grey in deriving the coordinates quoted at the Mathworld site, lead to the widely diffused picture of Costa’s surface that appears on the right. Costa’s surface is topologically equivalent to a torus (genus=1) with three points removed.
Chapter 6

Riemannian Geometry

6.1 Riemannian Manifolds

In the definition of manifolds introduced in section 4.1, it was implicitly assumed a manifold was embedded (or immersed) in \( \mathbb{R}^n \). As such, they inherited a natural metric induced by the standard Euclidean metric of \( \mathbb{R}^n \), as shown in section 4.2. For general manifolds it is more natural to start with a topological space \( M \), and define the coordinate patches as pairs \( \{ U_i, \phi_i \} \), where \( \{ U_i \} \) is an open cover of \( M \) with local homeomorphisms

\[
\phi_i : U_i \subset M \rightarrow \mathbb{R}^n.
\]

If \( p \in U_i \cap U_j \) is a point in the non-empty intersection of two charts, we require that the overlap map \( \phi_{ij} = \phi_i \phi_j^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a diffeomorphism. The local coordinates on patch \( \{ U, \phi \} \) are given by \( (x^1, \ldots, x^n) \), where

\[
x^i = u^i \circ \phi,
\]

and \( u^i : \mathbb{R}^n \rightarrow \mathbb{R} \) are the projection maps on each slot. The concept is the same as in figure 4.2.

Fig. 6.1: Coordinate Charts

but, as stated, we are assuming apriori that \( M \) is embedded (or immersed) in Euclidean space. If in addition the space is equipped with the additional structure of a metric, the space is called a Riemannian manifold. If the signature of the metric is of type \( g = \text{diag}(1,1,\ldots,-1,-1) \), with \( p \) ‘+’ entries and \( q \) ‘-’ entries, we say that \( M \) is a Pseudo-Riemannian manifold of type \( (p,q) \). As we have done with Minkowski’s space, we switch to Greek indices \( x^\mu \) for local coordinates of curved
space-times. We write the Riemannian metric as
\[ ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu. \] (6.1)

We will continue to be consistent with earlier notation and denote the tangent space at a point \( p \in M \) as \( T_p M \), the tangent bundle as \( TM \) and space of vector fields \( \mathcal{X}(M) \). Similarly, we denote the space of differential \( k \)-forms by \( \Omega^k(M) \), and the set of type \( (\tau) \) tensor fields by \( \mathcal{T}_\tau^\tau(M) \).

**Product Manifolds**

Suppose that \( M_1 \) and \( M_2 \) are differentiable manifolds of dimensions \( m_1 \) and \( m_2 \) respectively. Then, \( M_1 \times M_2 \) can be given a natural manifold structure of dimension \( n = m_1 + m_2 \) induced by the product of coordinate charts. That is, if \( (\phi_i, U_i) \) is a chart in \( M_1 \) in a neighborhood of \( p_i \in M_1 \), and \( (\phi_{i_2}, U_{i_2}) \) is a chart in a neighborhood of \( p_2 \in M_2 \), then the map
\[
\phi_i \times \phi_{i_2} : U_i \times U_{i_2} \to \mathbb{R}^n
\]
defined by
\[
(\phi_i \times \phi_{i_2})(p_1, p_2) = (\phi_i(p_1), \phi_{i_2}(p_2))
\]
is a coordinate chart in the product manifold. An atlas constructed from such charts, gives the differentiable structure. Clearly, \( M_1 \times M_2 \) is locally diffeomorphic to \( \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \). To discuss the tangent space of a product manifold, we recall from linear algebra, that given two vector spaces \( V \) and \( W \), the **direct sum** \( V \oplus W \) is the vector space consisting of the set of ordered pairs
\[
V \oplus W = \{(v, w) : v \in V, w \in W\},
\]
together with the vector operations
\[
(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \quad \text{for all } v_1, v_2 \in V, \ w_1, w_2 \in W,
\]
\[
k(v, w) = (kv, kw), \quad \text{for all } k \in \mathbb{R}
\]
People often say that one cannot add apples and peaches, but this is not a problem for mathematicians. For example, 3 apples and 2 peaches plus 4 apples and 6 peaches is 7 apples and 8 peaches. This is the basic idea behind the direct sum. We now have the following theorem.

6.1.1 **Theorem** Let \( (p_1, p_2) \in M_1 \times M_2 \), then there is a vector space isomorphism
\[
T_{(p_1, p_2)}(M_1 \times M_2) \cong T_{p_1} M_1 \oplus T_{p_2} M_2.
\]

**Proof:** The proof is adapted from [17]. Given \( X_1 \in T_{p_1} M_1 \) and \( X_2 \in T_{p_2} M_2 \), let
\[
x_1(t) \text{ be an curve with } x_1(0) = p_1 \text{ and } x'_1(0) = X_1,
\]
\[
x_2(t) \text{ be an curve with } x_2(0) = p_2 \text{ and } x'_2(0) = X_2.
\]
Then, we can associate
\[
(X_1, X_2) \in T_{(p_1, p_2)}(M_1 \times M_2), \text{ which is tangent to the curve } x(t) = (x_1(t), x_2(t)), \text{ at the point } (p_1, p_2).
\]
In the simplest possible case where the product manifold is \( \mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1 \), the vector \( X \) would be the velocity vector \( X = x'(t) \) of the curve \( x(t) \). It is convenient to introduce the inclusion maps
\[
i_{p_1} : M_1 \to (M_1 \times M_2)
i_{p_2} : M_2,
defined by,
\[ i_p(p) = (p, p_q), \text{ for } p \in M_1, \]
\[ i_p(q) = (p_1, q), \text{ for } q \in M_2. \]

The image of the vectors \( X_1 \) and \( X_2 \) under the push-forward of the inclusion maps
\[ i_{p_2}^* : T_{p_1} M_1 \xrightarrow{T(p_1, p_2)(M_1 \times M_2)} T_{p_2} M_2, \]

yield vectors \( \overline{X}_1 \) and \( \overline{X}_2 \), given by,
\[ i_{p_2}^*(X_1) = \overline{X}_1 = (x_1'(t), p_2), \]
\[ i_{p_1}^*(X_2) = \overline{X}_1 = (p_1, x_2'(t)). \]

Then, it is easy to show that,
\[ X = i_{p_2}^*(X_1) + i_{p_1}^*(X_2). \]
Indeed, if \( f \) is a smooth function \( f : M_1 \times M_2 \rightarrow \mathbb{R} \), we have,
\[
X(f) = \frac{d}{dt}(f(x_1(t), x_2(t)))_{t=0},
\]
\[ = \frac{d}{dt}(f(x_1(t), x_2(0)))_{t=0} + \frac{d}{dt}(f(x_1(0), x_2(t)))_{t=0}, \]
\[ = \overline{X}_1(f) + \overline{X}_2(f). \]

More generally, if \( \varphi : M_1 \times M_2 \rightarrow N \)

is a smooth manifold mapping, then we have a typical product rule formula for the Jacobian map,
\[
\varphi_* X = \varphi_*(i_{p_2}^*(X_1)) + \varphi_*(i_{p_1}^*(X_2)),
\]
\[ = (\varphi \circ i_{p_2})_* X_1 + (\varphi \circ i_{p_1})_* X_2 \quad (6.2) \]

This formula will be useful in the treatment of principal fiber bundles, in which case we have a bundle space \( E \), and a Lie group \( G \) acting on the right by a product manifold map \( \mu : E \times G \rightarrow E \).

### 6.2 Submanifolds

A Riemannian submanifold is a subset of a Riemannian manifold that is also Riemannian. The most natural example is a hypersurface in \( \mathbb{R}^n \). If \( (x^1, x^2 \ldots x^n) \) are local coordinates in \( \mathbb{R}^n \) with the standard metric, and the surface \( M \) is defined locally by functions \( x^i = x^i(u^a) \), then \( M \) together with the induced first fundamental form \[ 4.11 \] has a canonical Riemannian structure. We will continue to use the notation \( \nabla \) for a connection in the ambient space and \( \nabla \) for the connection on the surface induced by the tangential component of the covariant derivative
\[
\nabla_X Y = \nabla_X Y + H(X, Y), \quad (6.3) \]
where \( H(X, Y) \) is the component in the normal space. In the case of a hypersurface, we have the classical Gauss equation \[ 4.73 \]
\[
\nabla_X Y = \nabla_X Y + II(X, Y)N \]
\[ = \nabla_X Y + \langle LX, Y \rangle N, \quad (6.4) \]
where $LX = -\nabla_X N$ is the Weingarten map. If $M$ is a submanifold of codimension $n-k$, then there are $k$ normal vectors $N_k$ and $k$ classical second fundamental forms $I_{II}(X,Y)$, so that $H(X,Y) = \sum_k I_{II}(X,Y) N_k$.

As shown by the Theorema Egregium, the curvature of a surface in $\mathbb{R}^3$ depends only on the first fundamental form, so the definition of Gaussian curvature as the determinant of the second fundamental form does not even make sense intrinsically. One could redefine $K$ by Cartan’s second structure equation as it was used to compute curvatures in Chapter 4, but what we need is a more general definition of curvature that is applicable to any Riemannian manifold. The concept leading to the equations of the Theorema Egregium involved calculation of the difference of second derivatives of tangent vectors. At the risk of being somewhat misleading, figure 4.94 illustrates the concept. In this figure, the vector field $X$ consists of unit vectors tangent to parallels on the sphere, and the vector field $Y$ are unit tangents to meridians. If an arbitrary tangent vector $Z$ is parallel-transported from one point on an spherical triangle to the diagonally opposed point, the result depends on the path taken. Parallel transport of $Z$ along $X$ followed by $Y$, would yield a different outcome that parallel transport along $Y$ followed by parallel transport along $X$. The failure of the covariant derivatives to commute is reflection of the existence of curvature. Clearly, the analogous parallel transport by two different paths on a rectangle in $\mathbb{R}^n$ yield the same result. This fact is the reason why in elementary calculus, vectors are defined as quantities that depend on direction and length. As indicated, the picture is misleading, because, covariant derivatives, as is the case with any other type of derivative, involves comparing the change of a vector under infinitesimal parallel transport. The failure of a vector to return to itself when parallel-transported along a closed path is actually measured by an entity related to the curvature called the holonomy of the connection. Still, the figure should help motivate the definition that follows.

6.2.1 Definition On a Riemannian manifold with connection $\nabla$, the curvature $R$ and the torsion $T$ are defined by:

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$  \hspace{1cm} (6.6)

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$  \hspace{1cm} (6.7)

6.2.2 Theorem The curvature $R$ is a tensor. At each point $p \in M$, $R(X,Y)$ assigns to each pair of tangent vectors, a linear transformation from $T_pM$ into itself.

Proof: Let $X,Y,Z \in \mathfrak{X}(M)$ be vector fields on $M$. We need to establish that $R$ is multilinear. Since clearly $R(X,Y) = -R(Y,X)$, we only need to establish linearity on two slots. Let $f$ be a $C^\infty$
function. Then,
\[
R(fX, Y) = \nabla fX \nabla Y Z - \nabla Y \nabla fX Z - \nabla [fX,Y] Z, \\
= f\nabla X \nabla Y Z - \nabla Y (f\nabla X) Z - \nabla [fXY - Y(fX)] Z, \\
= f\nabla X \nabla Y Z - Y(f)\nabla X Z - f\nabla Y \nabla X Z - \nabla [fX,Y] Z, \\
= f\nabla X \nabla Y Z - Y(f)\nabla X Z - f\nabla Y \nabla X Z + \nabla Y (f)\nabla Z + f\nabla Y Z, \\
= f\nabla X \nabla Y Z - f\nabla Y \nabla X Z - f(\nabla XY Z - \nabla Y X Z), \\
= fR(X, Y) Z.
\]

Similarly, recalling that \([X,Y] \in \mathcal{X}\), we get:
\[
R(X, Y)(fZ) = \nabla X \nabla Y (fZ) - \nabla Y \nabla X (fZ) - \nabla [X,Y](fZ), \\
= \nabla X (Y(f)Z) + f\nabla Y Z - \nabla Y (X(f)Z + f\nabla X Z) - [X,Y](fZ) - f\nabla [X,Y](Z), \\
= XY(f)Z + Y(f)\nabla X Z + X(f)\nabla Y Z + f\nabla X \nabla Y Z - YX(f)Z - X(f)\nabla Y Z - Y(f)\nabla X Z - f\nabla Y \nabla X Z - [X,Y](fZ) - f\nabla [X,Y](Z), \\
= fR(X, Y) Z.
\]

We leave it as an almost trivial exercise to check linearity over addition in all slots.

6.2.3 Theorem The torsion \(T\) is also a tensor.

Proof: Since \(T(X, Y) = -T(Y, X)\), it suffices to prove linearity on one slot. Thus,
\[
T(fX, Y) = \nabla fXY - \nabla Y (fX) - [fX, Y], \\
= f\nabla X Y - Y(f)X - f\nabla Y X - fXY + Y(f)X, \\
= f\nabla X Y - Y(f)X - f\nabla Y X - fXY + Y(f)X + fYX, \\
= f\nabla X Y - f\nabla Y X - f[X, Y], \\
= fT(X, Y).
\]

Again, linearity over sums is clear.

6.2.4 Theorem In a Riemannian manifold there exist a unique torsion free connection called the **Levi-Civita connection**, that is compatible with the metric. That is:
\[
[X,Y] = \nabla X Y - \nabla Y X, \\
X(Y, Z) = \nabla X <Y,Z> - <X,\nabla Y Z> + <Y,\nabla X Z>. \\
\]

Proof: The proof parallels the computation leading to equation (4.75) taking cyclic derivatives of the inner product.
\[
\nabla X <Y, Z> = <\nabla X Y, Z> + <Y, \nabla X Z>, \\
\nabla Z <X, Y> = <\nabla Z X, Y> + <X, \nabla Z Y>, \\
\nabla Y <X, Z> = <\nabla Y Z, X> + <Z, \nabla Y X>, \\
\nabla X <Y, Z> - \nabla Z <X, Y> - \nabla Y <Z, X> = <[X,Y], Z> + <[X,Z], Y> - <\nabla Y Z + \nabla Z Y, Z>, \\
= <[X,Y], Z> + <[X,Z], Y> + <[Y,Z], X> - 2 <\nabla Y Z, X>.
\]
Therefore:
\[
\langle \nabla_Y Z, X \rangle = \frac{1}{2} \{ \langle [X,Y], Z \rangle + \langle [X,Z], Y \rangle + \langle [Y,Z], X \rangle \} - X \langle Y, Z \rangle - Z \langle X, Y \rangle - Y \langle Z, X \rangle.
\] (6.10)

Since the inner product is non-degenerate and this equation is true for arbitrary vector fields, it determines \( \nabla_Y Z \) completely in terms of the metric. In disguise, this is the formula for the Christoffel symbols.

As before, if \( \{ e_\alpha \} \) is a frame with dual frame \( \{ \theta^a \} \), we define the connection forms \( \omega \), Christoffel symbols \( \Gamma \) and Torsion components by
\[
\nabla_X e_\beta = \omega^\gamma_\beta(X) e_\gamma,
\]
(6.11)
\[
\nabla e_\alpha e_\beta = \Gamma^\alpha_\alpha_\beta e_\gamma,
\]
(6.12)
\[
T(e_\alpha, e_\beta) = T^\alpha_\alpha_\beta e_\gamma.
\]
(6.13)

As was pointed out in the previous chapter, if the frame is an orthonormal frame such as the coordinate frame \( \{ \partial/\partial x^\mu \} \) for which the bracket is zero, then \( T = 0 \) implies that the Christoffel symbols are symmetric in the lower indices.
\[
T^\gamma_\alpha_\beta = \Gamma^\gamma_\alpha_\beta - \Gamma^\gamma_\beta_\alpha = 0.
\]

For such a coordinate frame, we can compute the components of the Riemann tensors as follows:
\[
R(e_\gamma, e_\delta) e_\beta = \nabla_{e_\gamma} \nabla_{e_\delta} e_\beta - \nabla_{e_\delta} \nabla_{e_\gamma} e_\beta,
\]
\[
= \nabla_{e_\gamma}(\Gamma^\delta_\beta_\alpha e_\alpha) - \nabla_{e_\delta}(\Gamma^\delta_\gamma_\beta e_\alpha),
\]
\[
= \Gamma^\delta_\beta_\gamma e_\alpha + \Gamma^\delta_\beta_\alpha e_\gamma - \Gamma^\alpha_\gamma_\beta e_\delta - \Gamma^\gamma_\beta_\alpha \Gamma^\delta_\mu_\gamma e_\mu,
\]
\[
= [\Gamma^\gamma_\beta_\delta - \Gamma^\gamma_\beta_\gamma \Gamma^\delta_\mu_\gamma - \Gamma^\gamma_\beta_\gamma \Gamma^\delta_\gamma_\mu] e_\mu.
\]
(6.14)

Let \( X = X^\mu e_\mu \) be and \( \alpha = X_\mu e^\mu \) be a covariant and a contravariant vector field respectively. Using the notation \( \nabla_\alpha = \nabla_{e_\alpha} \) it is almost trivial to compute the covariant derivatives. The results are,
\[
\nabla_\beta X = (X^\mu + X^\nu \Gamma^\mu_\beta_\nu) e_\mu,
\]
(6.15)
\[
\nabla_\beta \alpha = (X_\mu, \beta - X^\nu \Gamma^\mu_\beta_\nu) e^\mu.
\]

We show the details of the first computation, and leave the second one as an easy exercise
\[
\nabla_\beta X = \nabla_\beta (X^\mu e_\mu),
\]
(6.16)
\[
= X^\mu_\delta_\beta \mu + X^\nu \Gamma^\mu_\beta_\nu e_\delta,
\]
(6.17)
\[
= (X^\mu_\beta + X^\nu \Gamma^\mu_\beta_\nu) e_\mu.
\]
(6.18)

In classical notation, the covariant derivatives \( X^\mu_\parallel_\beta \) and \( X_\mu_\parallel_\beta \) are given in terms of the tensors components,
\[
X^\mu_\parallel_\beta = X^\mu_\parallel_\beta + X^\nu \Gamma^\mu_\parallel_\beta_\nu,
\]
(6.19)
\[
X_\mu_\parallel_\beta = X_\mu_\parallel_\beta - X_\nu \Gamma^\mu_\parallel_\beta_\nu.
\]
It is also straightforward to establish the Ricci Identities
\[
X^\mu_{\parallel\alpha\beta} - X^\nu R^\mu_{\nu\alpha\beta}, \\
X_{\mu\parallel\alpha\beta} - X_{\mu\parallel\beta\alpha} = -X_{\mu} R^\nu_{\mu\nu\alpha\beta}.
\] (6.20)

Again, we show the computation for the first identity and leave the second to the reader. We take the second derivative. and then reverse the order,

\[
\nabla_\alpha \nabla_\beta X = \nabla_\alpha (X^\mu_\beta e_\mu + X^\nu \Gamma^\mu_{\beta\nu} e_\mu), \\
= X^\mu_\beta e_\mu + X^\nu \Gamma^\mu_{\alpha\beta} e_\delta + X^\nu \Gamma^\mu_{\beta\nu,\alpha} e_\mu + X^\nu \Gamma^\mu_{\beta\nu,\alpha} e_\delta.
\]

Subtracting the two equations, only the last two terms of each survive, and we get the desired result,

\[
2 \nabla_\alpha \nabla_\beta (X^\mu_e) = (X^\nu (\Gamma^\mu_{\beta\alpha} - \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\delta\nu}\Gamma^\delta_{\beta\alpha}) e_\mu).
\]

In the literature many authors use the notation \( \nabla_\beta X^\mu \) to denote the covariant derivative \( X^\mu_{\parallel\beta} \), but id really an (excusable) abuse of notation that arises from thinking of tensors as the components of the tensors. The Ricci identities are the basis for the notion of holonomy, namely, the simple interpretation that the failure of parallel transport to commute along the edges of an rectangle, indicates the presence of curvature. With more effort with repeated use of Leibnitz rule, one can establish more elaborate Ricci identities for higher order tensors. If one assumes zero torsion, the Ricci identities of higher order tensors just involve more terms with the curvature. If the torsion is not zero, there are additional terms involving the torsion tensor; in this case it is perhaps a bit more elegant to use the covariant differential introduced in the next section, so we will postpone the computation until then.

The generalization of the Theorema Egregium to manifolds comes from the same principle of splitting the curvature tensor of the ambient space into the tangential on normal components. In the case of a hypersurface with normal \( N \) and tangent vectors \( X, Y, Z \), we have:

\[
\mathbf{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]

\[
= \nabla_X (\nabla_Y Z + <LY,Z>) - \nabla_Y (\nabla_X Z + <LX,Z>) - \nabla_{[X,Y]} Z,
\]

\[
= \nabla_X \nabla_Y Z + <LX,\nabla_Y Z > + N + \nabla_X <LY,Z> + N + <LY,Z> + LX - \\
\nabla_Y \nabla_X Z <LY,Z> + N - \nabla_Y <LY,Z> + N <LX,Z> + LY - \\
\nabla_{[X,Y]} Z <N,[X,Y]>, Z > N,
\]

\[
= \nabla_X \nabla_Y Z + <LX,\nabla_Y Z > + N + \nabla_X <LY,Z> + N + <LY,Z> + LX - \\
\nabla_Y \nabla_X Z <LY,Z> + N - \nabla_Y <LY,Z> + N <LX,Z> + LY - \\
\nabla_{[X,Y]} Z <N,[X,Y]>, Z > N,
\]

\[
= \nabla_X \nabla_Y Z + <LX,\nabla_Y Z > + N + \nabla_X <LY,Z> + N + <LY,Z> + LX - \\
\nabla_Y \nabla_X Z <LY,Z> + N - \nabla_Y <LY,Z> + N <LX,Z> + LY - \\
\nabla_{[X,Y]} Z <N,[X,Y]>, Z > N,
\]

\[
= R(X,Y)Z + <LY,Z > + LX - <LX,Z > + LY, Z > N + \\
\{(<\nabla_X LY, Z > - <\nabla_Y LX, Z > - <L([X,Y]), Z > N).
\]
CHAPTER 6. RIEMANNIAN GEOMETRY

In particular, if $n = 3$, and at each point in the surface, the vectors $X$ and $Y$ constitute an a basis of the tangent space, we get the coordinate-free Teorema Egregium

$$K = \langle R(X,Y)X, Y \rangle = \langle LX, X \rangle - \langle LY, Y \rangle - \langle LY, X \rangle + \langle LX, Y \rangle = \det(L). \quad (6.22)$$

The expression $[6.21]$ is the coordinate-independent version of the equation of Codazzi.

We expect the covariant definition of the torsion and curvature tensors to be consistent with the formalism of Cartan.

**6.2.5 Theorem** Equations of Structure.

$$\Theta^\alpha = d\theta^\alpha + \omega^\alpha_\beta \wedge \theta^\beta, \quad (6.23)$$

$$\Omega^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta. \quad (6.24)$$

To verify this is the case, we define:

$$T(X,Y) = \Theta^\alpha(X,Y)e_\alpha, \quad (6.25)$$

$$R(X,Y)e_\beta = \Omega^\alpha_\beta(X,Y)e_\alpha. \quad (6.26)$$

Recalling that any tangent vector $X$ can be expressed in terms of the basis as $X = \theta^\alpha(X) e_\beta$, we can carry out a straight-forward computation:

$$\Theta^\alpha(X,Y) e_\alpha = T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$= \nabla_X (\theta^\alpha(Y)) e_\alpha - \nabla_Y (\theta^\alpha(X)) e_\alpha - \theta^\gamma([X,Y]) e_\alpha,$$

$$= X(\theta^\alpha(Y)) e_\alpha + \theta^\alpha(Y) \omega^\alpha_\beta(X) e_\beta - Y(\theta^\alpha(X)) e_\alpha - \theta^\alpha([X,Y]) e_\alpha,$$

$$= \{X(\theta^\alpha(Y)) - Y(\theta^\alpha(X)) - \theta^\alpha([X,Y]) + \omega^\alpha_\beta(X)(\theta^\beta(Y) - \omega^\alpha_\beta(Y)(\theta^\beta(X))\}e_\alpha,$$

where we have introduced a coordinate-free definition of the differential of the one form $\theta$ by

$$d\theta(X,Y) = X(\theta(Y)) - y(\theta(X)) - \theta([X,Y]) \quad (6.27)$$

It is easy to verify that this definition of the differential of a one form satisfies all the required properties of the differential, and that it is consistent with the coordinate version of the differential introduced in Chapter 2. We conclude that

$$\Theta^\alpha = d\theta^\alpha + \omega^\alpha_\beta \wedge \theta^\beta, \quad (6.28)$$

which is indeed the first Cartan equation of structure.

Proceeding along the same lines, we compute:

$$\Omega^\alpha_\beta(X,Y) e_\alpha = \nabla_X \nabla_Y e_\beta - \nabla_Y \nabla_X e_\beta - \nabla_{[X,Y]} e_\beta,$$

$$= \nabla_X (\omega^\alpha_\beta(Y) e_\alpha) - \nabla_Y (\omega^\alpha_\beta(X) e_\alpha) - \omega^\alpha_\beta([X,Y]) e_\alpha,$$

$$= X(\omega^\alpha_\beta(Y)) e_\alpha + \omega^\alpha_\beta(Y) \omega^\gamma_\alpha(X) e_\gamma - Y(\omega^\alpha_\beta(X)) e_\alpha - \omega^\alpha_\beta(X) \omega^\gamma_\alpha(Y) e_\gamma - \omega^\alpha_\beta([X,Y]) e_\alpha,$$

$$= \{(d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta)(X,Y)\}e_\alpha,$$

thus arriving at the second equation of structure

$$\Omega^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta. \quad (6.29)$$

The quantities connection and curvature forms are matrix-valued. Using matrix multiplication notation, we can abbreviate the equations of structure as

$$\Theta = d\theta + \omega \wedge \theta, \quad (6.30)$$

$$\Omega = d\omega + \omega \wedge \omega.$$
Taking the exterior derivative of the structure equations gives some interesting results. Here is the first computation,

\[ d\Theta = d\omega \wedge \theta - \omega \wedge d\theta, \]
\[ = d\omega \wedge \theta - \omega \wedge (\Theta - \omega \wedge \theta), \]
\[ = d\omega \wedge \theta - \omega \wedge \Theta \pm \omega \wedge \omega, \]
\[ = (d\omega + \omega \wedge \omega) \wedge \theta - \omega \wedge \Theta, \]
\[ = \Omega \wedge \theta - \omega \wedge \Theta, \]

so,
\[ d\Theta + \omega \wedge \Theta = \Omega \wedge \theta. \quad (6.31) \]

Similarly, taking \( d \) of the second structure equation we get,

\[ d\Omega = d\omega \wedge \omega + \omega \wedge d\omega, \]
\[ = (\Omega - \omega \wedge \omega) \wedge \omega + \omega \wedge (\Omega - \omega \wedge \omega). \]

Hence,
\[ d\Omega = \Omega \wedge \omega - \omega \wedge \Omega. \quad (6.32) \]

Equations 6.31 and 6.32 are called the first and second Bianchi identities.

The relationship between the full Torsion and Riemann tensor components with the corresponding differential forms are given by:

\[ \Theta^\alpha = \frac{1}{2} T^\alpha_{\gamma\delta} \theta^\gamma \wedge \theta^\delta, \]
\[ \Omega^\alpha_{\beta} = \frac{1}{2} R^\alpha_{\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta. \quad (6.33) \]

In the case of a non-coordinate frame in which the Lie bracket of frame vectors does not vanish, we first write them as linear combinations of the frame

\[ [e_\beta, e_\gamma] = C^\alpha_{\beta\gamma} e_\alpha. \quad (6.34) \]

The components of the Torsion and Riemann tensors are then given by:

\[ T^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} - \Gamma^\alpha_{\beta\gamma} - C^\alpha_{\beta\gamma}, \]
\[ R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\delta\gamma\beta} + \Gamma^\alpha_{\beta\delta\gamma} - \Gamma^\mu_{\beta\gamma} \Gamma^\alpha_{\mu\delta} - \Gamma^\mu_{\beta\delta} \Gamma^\alpha_{\mu\gamma} + \Gamma^\alpha_{\mu\beta} C^\mu_{\gamma\delta} - \Gamma^\alpha_{\sigma\beta} C^\sigma_{\gamma\delta}. \quad (6.35) \]

The Riemann tensor for a torsion-free connection has the following symmetries;

\[ R(X,Y) = -R(Y,X), \]
\[ < R(X,Y)Z,W > = - < R(X,Y)W,Z > \]
\[ R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0. \quad (6.36) \]

In terms of components, the Riemann Tensor symmetries can be expressed as:

\[ R^\alpha_{\beta\gamma\delta} = -R^\alpha_{\beta\delta\gamma} = -R^\alpha_{\gamma\beta\delta}, \]
\[ R^\alpha_{\beta\gamma\delta} = R^\alpha_{\gamma\delta\beta}, \]
\[ R^\alpha_{\beta\gamma\delta} + R^\alpha_{\gamma\delta\beta} + R^\alpha_{\beta\delta\gamma} = 0. \quad (6.37) \]

The last cyclic equation is the tensor version of the first Bianchi Identity with 0 torsion. It follows immediately from setting \( \Omega \wedge \theta = 0 \) and taking a cyclic permutation of the antisymmetric indices.
\{\beta, \gamma, \delta\} of the Riemann tensor. The symmetries reduce the number of independent components in an \( n \)-dimensional manifold from \( n^4 \) to \( n^2(n^2 - 1)/12 \). Thus for 4-dimensional space, there are at most 20 independent components. The derivation of the tensor version of the second Bianchi identity from the elegant differential forms version, takes a bit more effort. In components the formula

\[ d\Omega = \Omega \wedge \omega - \omega \wedge \Omega \]

reads,

\[ R^\alpha_{\beta \kappa \lambda \mu} \theta^\mu \wedge \theta^\kappa \wedge \theta^\lambda = (\Gamma^\rho_{\mu \beta} R^\alpha_{\rho \kappa \lambda} - \Gamma^\alpha_{\mu \rho} R^\rho_{\beta \kappa \lambda}) \theta^\mu \wedge \theta^\kappa \wedge \theta^\lambda, \]

where we used the notation,

\[ \nabla_{\mu} R^\alpha_{\beta \kappa \lambda} = R^\alpha_{\beta \kappa \lambda \mu}. \]

Taking a cycle permutation on the antisymmetric indices \( \kappa, \lambda, \mu \), and using some index gymnastics to show that the right hand becomes zero, the tensor version of the second Bianchi identity for 0 torsion becomes

\[ R^\alpha_{\beta \kappa \lambda \mu} = 0 \quad (6.38) \]

This seems as an appropriate time to generalize the coordinate-free exterior derivative formula 6.27 to arbitrary forms. Let \( \Lambda^k(M) \) be the bundle of alternating covariant tensors of rank \( k \) and denote the sections of the bundle by \( \Omega^k(M) \). As in 2.3, sections of this bundle are called \( k \)-forms on \( M \).

6.2.6 Definition Let \( \omega \) be a \( k \)-form on \( M \), and let \( \{X_1, \ldots, X_{k+1}\} \in \mathcal{X}(M) \). The exterior derivative of \( \omega \) is the \((k+1)\)-form given by, (See, for example, Spivak [31])

\[
d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\
\quad - \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}). \quad (6.39)
\]

where the “hats” mean that these vectors are excluded. Thus, for a 2-form \( \omega \), the formula gives

\[
d\omega(X, Y, Z) = X(\omega(Y, Z)) - Y(\omega(X, Z)) + Z(\omega(X, Y)) \\
\quad - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X).
\]

If one chooses local coordinates as in section 2.3, we find that the operator is consistent with 2.66 and satisfies the properties in 2.67 and 2.71. In particular, the following holds:

6.2.7 Theorem Let \( M, N \) be a manifolds, \( \alpha \in \Omega^k(M) \), \( \beta \in \Omega^l(M) \), and \( \varphi : M \to N \) be a diffeomorphism as above. Then

a. \( d \circ d = 0 \),

b. \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \),

c. \( \varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta \),

d. \( \varphi^*(d\omega) = d(\varphi^*\omega) \).

6.3 Big D

In this section we discuss the notion of a connection on a vector bundle \( E \). Let \( M \) be a smooth manifold and as usual we denote by \( T^*_p(M) \) the vector space of type \( (r,s) \) tensors at a point \( p \in M \). The formalism applies to any vector bundle, but in this section we are primarily concerned with the case where \( E \) is the tensor bundle \( E = T^*_p(M) \). Sections \( \Gamma(E) = \mathcal{F}^*(M) \) of this bundle are called
tensor fields on \( M \). For general vector bundles, we use the notation \( s \in \Gamma(E) \) for the sections of the bundle. The section that maps every point of \( M \) to the zero vector, is called the **zero section**. Let \( \{ e_\alpha \} \) be an orthonormal frame with dual forms \( \{ \theta^\alpha \} \). We define the space \( \Omega^p(M,E) \) tensor-valued \( p \)-form as sections of the bundle,

\[
\Omega^p(M,E) = \Gamma(E \otimes \Lambda^p(M))
\]  

(6.40)

As in equation 2.64, a tensor-valued \( p \) form is a tensor of type \( (r,s+p) \) with components,

\[
T_{\beta_1 \ldots \beta_r}^{\alpha_1 \ldots \alpha_s} e_{\alpha_1} \otimes \ldots e_{\alpha_r} \otimes \theta^{\beta_1} \otimes \ldots \otimes \theta^{\beta_r} \wedge \theta^{\gamma_1} \wedge \ldots \wedge \theta^{\gamma_s}.
\]  

(6.41)

A tensor-valued 0-form is just a regular tensor field \( T \in T^r_s(M) \). The main examples of tensor-valued forms are the torsion and the curvature forms

\[
\Theta = \Theta^{\alpha} e_\alpha, \\
\Omega = \Omega^{\alpha \beta} e_\alpha \otimes \theta^\beta.
\]  

(6.42)

The tensorial components of the Torsion tensor, would then be written as

\[
T_{\beta_\gamma}^{\alpha} e_\alpha \otimes \theta^\beta \otimes \theta^\gamma,
\]

\[
= \frac{1}{2} T^{\alpha \beta_\gamma} e_\alpha \otimes \theta^\beta \wedge \theta^\gamma,
\]

\[
= e_\alpha \otimes (\frac{1}{2} T^{\alpha \beta_\gamma} \theta^\beta \wedge \theta^\gamma).
\]

since the tensor is antisymmetric in the lower indices. Similarly the tensorial components of the curvature, are,

\[
\Omega = \frac{1}{2} R^{\alpha \beta_\gamma \delta} e_\alpha \otimes \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta,
\]

\[
= e_\alpha \otimes \theta^\beta \otimes (\frac{1}{2} R^{\alpha \beta_\gamma \delta} \theta^\gamma \wedge \theta^\delta).
\]

The connection forms

\[
\omega = \omega^{\alpha \beta} e_\alpha \otimes \theta^\beta
\]  

(6.43)

are matrix-valued, but they are not tensorial forms. We seek an operator that behaves like a covariant derivative \( \nabla \) for tensors and exterior derivative \( d \) for forms. Let \( T \) be a type \( (r,s) \) tensor field, and \( \alpha \) a \( p \)-form, so that \( T \otimes \alpha \in \Omega^p(M,E) \) is a tensor-valued \( p \)-form.

### 6.3.1 Linear Connections

Given a vector field \( X \) and a smooth function \( f \), we define a **linear connection** as a map

\[
\nabla_X : \Gamma(T^r_s) \to \Gamma(T^r_s)
\]

with the following properties

1) \( \nabla_X (f) = X(f) \)

2) \( \nabla_X T = fD_X T \)

3) \( \nabla_X (T_1 + T_2) = \nabla_X T_1 + \nabla_X T_2 \), for \( T_1, T_2 \in \Gamma(T^r_s) \),

4) \( \nabla_X (fT) = X(f)T + f\nabla_X T \).

If instead of the tensor bundle we have a general vector bundle \( E \), we replace the tensor fields in the definition above by sections \( s \in \Gamma(E) \) of the vector bundle. The definition induces a derivation on the entire tensor algebra satisfying the additional conditions,
5) \( \nabla_X (T_1 \otimes T_2) = \nabla_X T_1 \otimes T_2 + T_1 \otimes \nabla_X T_2 \).
6) \( \nabla_X \circ C = C \circ \nabla_X \), for any contraction \( C \).

The properties are the same as a Koszul connection, or covariant derivative for tensor-valued 0 forms \( T \). Given an orthonormal frame, consider the identity tensor,

\[
I = \delta^\alpha_\beta e^\alpha \otimes \theta^\beta,
\]

(6.44)

and take the covariant derivative \( \nabla_X \). We get

\[
\nabla_X e^\alpha \otimes \theta^\alpha + e^\alpha \otimes \nabla_X \theta^\alpha = 0,
\]

\[
e^\alpha \otimes \nabla_X \theta^\alpha = -\nabla_X e^\alpha \otimes \theta^\alpha,
\]

\[
= -e_\beta \omega^\beta_\alpha (X) \otimes \theta^\alpha,
\]

\[
e_\beta \otimes \nabla_X \theta^\beta = -e_\beta \omega^\beta_\alpha (X) \otimes \theta^\alpha,
\]

which implies that,

\[
\nabla_X \theta^\beta = -\omega^\beta_\alpha (X) \theta^\alpha.
\]

(6.45)

Thus as before, since we have formulas for the covariant derivative of basis vectors and forms, we are lead by induction to a general formula for the covariant derivative of an \((r,s)\)-tensor given \textit{mutatis mutandis} by the formula 3.33. In other words, the covariant derivative of a tensor acquires a term with a multiplicative connection factor for each contravariant index and a negative term with a multiplicative connection factor for each covariant index.

6.3.1 Definition A connection \( \nabla \) on the vector bundle \( E \) is a map

\[
\nabla : \Gamma(M,E) \to \Gamma(M,E \otimes T^* (M))
\]

which satisfies the following conditions

a) \( \nabla(T_1 + T_2) = \nabla T_1 + \nabla T_2 \), \( T_1, T_2 \in \Gamma(E) \)

b) \( \nabla(fT) = df \otimes T + f \nabla T \)

c) \( \nabla_X T = i_X \nabla T \)

As a reminder of the definition of the inner product \( i_X \), condition (c) is equivalent to the equation,

\[
\nabla T(\theta^1, \ldots, \theta^r, X, X_1, \ldots, X_s) = (\nabla_X)(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s)
\]

In particular, if \( X \) is vector field, then

\[
\nabla X(Y) = \nabla_X Y,
\]

as expected. The operator \( \nabla \) is called the \textbf{Covariant Differential}. Again, for a general vector bundles, we denote the sections by \( s \in \Gamma(E) \) and the covariant differential by \( \nabla s \).

6.3.2 Affine Connections

A connection on the tangent bundle \( T(M) \) is called an \textbf{Affine Connection}. In a local frame field \( e \), we may assume that the connection is represented by a matrix of one-forms \( \omega \)

\[
\nabla e_\beta = e^\alpha \otimes \omega^\alpha_\beta,
\]

\[
\nabla e = e \otimes \omega.
\]

(6.46)

The tensor multiplication symbol is often omitted when it is clear in context. Thus for example, the connection equation is sometimes written as \( \nabla e = e \omega \). In a local coordinate system \( \{x_1, \ldots, x^n\} \), with basis vectors \( \partial_\mu = \frac{\partial}{\partial x_\mu} \) and dual forms \( dx^\mu \), we have,

\[
\omega^\alpha_\beta \mu = \Gamma^\alpha_\beta_\mu dx^\mu.
\]
From equation (6.45) it follows that
\[ \nabla \theta^\alpha = -\omega^\alpha_\beta \otimes \theta^\beta \]
(6.47)
We need to be a bit careful with the dual forms \( \theta^\alpha \). We can view them as a vector-valued 1-form
\[ \theta = e_\theta \otimes \theta^\alpha \]
which has the same components as the identity \( \left( \begin{array}{c} 1 \\ 1 \\ \vdots \end{array} \right) \) tensor. This is a kind of a odd creature. About the closest analog to this entity in classical terms is the differential of arc-length
\[ d\mathbf{x} = i \, dx + j \, dy + k \, dz, \]
which is sort of a mixture of a vector and a form. The vector of differential forms would then be written as a column vector.
In a frame \( \{ e_\alpha \} \), the covariant differential of tensor-valued 0-form \( T \) is given by
\[ \nabla T = \nabla e_\alpha T \otimes \theta^\alpha \equiv \nabla_\alpha T \otimes \theta^\alpha \]
In particular, if \( X = v^\alpha e_\alpha \), we get,
\[ \nabla X = \nabla_\beta X \otimes \theta^\beta = \nabla_\beta (v^\alpha e_\alpha) \otimes \theta^\beta, \]
\[ = (\nabla_\beta (v^\alpha) e_\alpha + v^\alpha \Gamma^\gamma_\beta_\alpha e_\gamma) \otimes \theta^\beta, \]
\[ = (v^\gamma + v^\gamma \Gamma^\alpha_\beta_\gamma) e_\alpha \otimes \theta^\beta, \]
\[ = v^\alpha_\parallel_\beta e_\alpha \otimes \theta^\beta, \]
where we have used the classical symbols
\[ v^\alpha_\parallel_\beta = v^\alpha_\parallel_\beta + \Gamma^\alpha_\beta_\gamma v^\gamma, \]
(6.48)
for the covariant derivative components \( v^\alpha_\parallel_\beta \) and the comma to abbreviate the directional derivative \( \nabla_\beta (v^\alpha) \). Of course, the formula is in agreement with equation (3.26) \( \nabla X \) is a \( \left( \begin{array}{c} 1 \\ 1 \\ \vdots \end{array} \right) \)-tensor.
Similarly, for a covariant vector field \( \alpha = v_\alpha \theta^\alpha \), we have
\[ \nabla \alpha = \nabla (v_\alpha \otimes \theta^\alpha) \]
\[ = \nabla v_\alpha \otimes \theta^\alpha - v_\beta \omega^\alpha_\beta \otimes \theta^\alpha, \]
\[ = (\nabla_\gamma v_\alpha \theta^\gamma - v_\beta \Gamma^\alpha_\beta_\gamma \theta^\gamma) \otimes \theta^\alpha, \]
\[ = (v_{\alpha,\gamma} - \Gamma_\beta^\alpha_\gamma v_\alpha) \theta^\gamma \otimes \theta^\alpha, \]
hence,
\[ v_{\alpha,\parallel_\beta} = v_{\alpha,\gamma} - \Gamma_\beta^\alpha_\gamma v_\alpha. \]
(6.49)
As promised earlier, we now prove the Ricci identities for contravariant an covariant vectors when the torsion is not zero. Ricci Identities with torsion. The results are
\[ X^\mu_\parallel_\alpha_\beta \cdot X^\mu_\parallel_\beta_\alpha = X^\nu R^\mu_\nu_\alpha_\beta - X^\mu_\nu T^\nu_\alpha_\beta, \]
\[ X^\mu_\parallel_\alpha_\beta - X^\mu_\parallel_\beta_\alpha - X^\mu_\parallel_\nu_\nu_\alpha_\beta = -X^\nu_\nu R^\mu_\nu_\alpha_\beta - X^\mu_\nu T^\nu_\alpha_\beta, \]
(6.50)
We prove the first one. Let \( X = X^\mu e_\mu \). We have
\[ \nabla X = \nabla_\beta X \otimes \theta^\beta, \]
\[ \nabla^2 X = \nabla (\nabla_\beta X \otimes \theta^\beta), \]
\[ = \nabla (\nabla_\beta X) \otimes \theta^\beta + \nabla_\beta X \otimes \nabla \theta^\beta, \]
\[ = \nabla_\alpha \nabla_\beta X \otimes \theta^\beta \otimes \theta^\alpha - \nabla_\beta \otimes \omega^\alpha_\beta \otimes \theta^\alpha, \]
\[ = \nabla_\alpha \nabla_\beta X \otimes \theta^\beta \otimes \theta^\alpha - \nabla_\mu X \otimes \Gamma^\mu_\alpha_\beta \theta^\beta \otimes \theta^\alpha, \]
\[ \nabla^2 X = (\nabla_\alpha \nabla_\beta X - \nabla_\mu X \Gamma^\mu_\alpha_\beta) \theta^\beta \otimes \theta^\alpha. \]
On the other hand, we also have $\nabla X = \nabla_\alpha X \otimes \theta^\alpha$, so we can compute $\nabla X$ by differentiating in the reverse order to get the equivalent expression,

$$\nabla^2 X = (\nabla_\beta \nabla_\alpha X - \nabla_\mu X \Gamma^\mu_{\beta\alpha}) \theta^\alpha \otimes \theta^\beta.$$ 

Subtracting the last two equations we get an alternating tensor, or a two-form that we can set equal to zero. For lack of a better notation we call this form $[\nabla, \nabla]$. The notations $\text{Alt}(\nabla^2)$ and $\nabla \wedge \nabla$ also appear in the literature. We get

$$[\nabla, \nabla] = \left[ \nabla_\alpha \nabla_\beta - \nabla_\mu X(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) \right] \theta^\alpha \wedge \theta^\beta,$$

$$\begin{align*}
= [\nabla_\alpha \nabla_\beta - \nabla_\mu X(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) - \nabla_\nu X(\Gamma^\nu_{\alpha\beta} - \Gamma^\nu_{\beta\alpha})] \theta^\alpha \wedge \theta^\beta, \\
= [R(e_\alpha, e_\beta)X + C^\mu_{\alpha\beta} \nabla_\mu X - \nabla_\mu X(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha})] \theta^\alpha \wedge \theta^\beta, \\
= [R(e_\alpha, e_\beta)X + C^\mu_{\alpha\beta} - \nabla_\mu X(\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}) - C^\mu_{\alpha\beta}] \theta^\alpha \wedge \theta^\beta, \\
= \frac{1}{2} (X^\nu R^\mu_{\nu\alpha\beta} - \nabla_\mu X T^\mu_{\nu\alpha\beta}) \theta^\beta \wedge \theta^\alpha.
\end{align*}$$

6.3.3 Exterior Covariant Derivative

Since we know how to take the covariant differential of the basis vectors and covectors and tensor products thereof, an affine connection on the tangent bundle induces an covariant differential on the tensor bundle. It is easy to get a formula by induction for the covariant differential of a tensor-valued 0-form. A given connection can be extended in a unique way to to tensor-valued $p$-forms. Just as with the wedge product of a 0-form $f$ with a $p$-form $\alpha$ for which identify $f \alpha$ with $f \otimes \alpha = f \wedge \alpha$, we write a tensor-valued $p$ form as $T \otimes \alpha = T \wedge \alpha$, where $T$ is a type $(\cdot)^p$ tensor. We define the Exterior Covariant Derivative

$$D : \Omega^p(M, E) \to \Omega^{p+1}(M, E)$$

by requiring that,

$$D(T \otimes \alpha) = D(T \wedge \alpha),$$

$$= \nabla T \wedge \alpha + (-1)^p T \wedge d\alpha. \quad (6.51)$$

it is instructive to show the details of the computation of the exterior covariant derivative of the vector valued one forms $\theta$ and $\Theta$, and the $(\cdot \cdot)_p$ tensor-valued 2-form $\Omega$. The results are

$$D\theta^\alpha = d\theta^\alpha + \omega^\alpha _\beta \wedge \theta^\beta,$$

$$D\Theta^\alpha = d\Theta^\alpha + \omega^\alpha _\beta \wedge \Theta^\beta,$$

$$D\Omega^\alpha _\beta = d\Omega^\alpha _\beta + \omega^\alpha _\gamma \wedge \Omega^\gamma _\beta - \Omega^\alpha _\gamma \wedge \omega^\gamma _\beta \quad (6.52)$$

The first two follow immediately, we compute the third. We start by writing

$$\Omega = \Omega^\alpha _\beta e_\alpha \otimes \theta^\beta,$$

$$= (e_\alpha \otimes \theta^\beta) \wedge \Omega^\alpha _\beta.$$ 

Then,

$$D\Omega = D(e_\alpha \otimes \theta^\beta) \wedge \Omega^\alpha _\beta + (-1)^2 (e_\alpha \otimes \theta^\beta) \wedge d\Omega^\alpha _\beta,$$

$$= (De_\alpha \otimes \theta^\beta + e_\alpha \otimes D\theta^\beta) \wedge \Omega^\alpha _\beta + (e_\alpha \otimes \theta^\beta) \wedge d\Omega^\alpha _\beta,$$

$$= (e_\gamma \otimes \omega_\theta ^\gamma \otimes \theta^\beta + e_\alpha \otimes \omega^\gamma _\gamma \otimes \theta^\gamma) \wedge \Omega^\alpha _\beta + (e_\alpha \otimes \theta^\beta) \wedge d\Omega^\alpha _\beta,$$

$$= (e_\alpha \otimes \theta^\beta) \wedge (d\Omega^\alpha _\beta + \omega^\alpha _\gamma \wedge \Omega^\gamma _\beta - \omega^\alpha _\gamma \wedge \Omega^\gamma _\beta).$$
In the last step we had to relabel a couple of indices so that we could factor out \((e_\alpha \otimes \theta^\beta)\). The pattern should be clear. We get an exterior derivative for the forms, an \(\omega \wedge \Omega\) term for the contravariant index and an \(\Omega \wedge \omega\) term with the appropriate sign, for the covariant index. Here the computation gives

\[
D\Omega^\alpha_\beta = d\Omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \Omega^\gamma_\beta - \omega^\gamma_\alpha \wedge \Omega^\gamma_\beta, \quad \text{or} \\
D\Omega = d\omega + \omega \wedge \Omega - \Omega \wedge \omega.
\] (6.53)

This means that we can write the equations of structure as

\[\Theta = D\theta, \quad \Omega = d\omega + \omega \wedge \omega,\] (6.54)

and Bianchi’s identities as

\[
D\Theta = \Omega \wedge \theta, \\
D\Omega = 0
\] (6.55)

At the risk of sounding repetitious, we reproduce the change of basis formula [3.49]. Let \(e' = eB\) be an orthogonal change of basis. Then

\[
De' = e \otimes dB + DeB, \\
= e \otimes dB + (e \otimes \omega)B, \\
= e' \otimes (B^{-1}dB + B^{-1}\omega B), \\
= e' \otimes \omega',
\]

where,

\[
\omega' = B^{-1}dB + B^{-1}\omega B.
\] (6.56)

Multiply the last equation by \(B\) and take the exterior derivative \(d\). We get.

\[
Bd\omega' = dB + \omega B, \\
Bd\omega' + dB \wedge \omega' = d\omega B - \omega \wedge dB, \\
Bd\omega' + (B\omega' - \omega B) \wedge \omega' = d\omega B - \omega \wedge (\omega' B - \omega B), \\
B(d\omega' + \omega' \wedge \omega') = (d\omega + \omega \wedge \omega)B,
\]

Setting \(\Omega = d\omega + \omega \wedge \omega\), and \(\Omega' = d\omega' + \omega' \wedge \omega'\), the last equation reads,

\[
\Omega' = B^{-1}\Omega B.
\] (6.57)

As pointed out after equation [3.49] the curvature is an tensorial form of adjoint type. The transformation law above for the connection has an extra term, so it is not tensorial. It is easy to obtain the classical transformation law for the Christoffel symbols from equation [6.56]. Let \(\{x^\alpha\}\) be coordinates in a patch \((\phi_\alpha, U_\alpha)\), and \(\{y^\beta\}\) be coordinates on a overlapping patch \((\phi_\beta, U_\beta)\). The transition functions \(\phi_{\alpha\beta}\) are given by the Jacobian of the change of coordinates,

\[
\frac{\partial}{\partial y^\beta} = \frac{\partial x^\alpha}{\partial y^\beta} \frac{\partial}{\partial x^\alpha}, \\
\phi_{\alpha\beta} = \frac{\partial x^\alpha}{\partial y^\beta}.
\]
Inserting the connection components \( \omega^{\alpha\beta} = \Gamma^{\alpha\beta}_{\gamma} dy^\gamma \), into the change of basis formula [6.56] with \( B = \phi_{\alpha\beta} \), we get

\[
\omega^{\prime\alpha\beta} = \frac{\partial y^\alpha}{\partial x^\kappa} d \left( \frac{\partial x^\kappa}{\partial y^\beta} \right) + \frac{\partial y^\alpha}{\partial x^\kappa} \omega^\kappa \frac{\partial x^\lambda}{\partial y^\beta},
\]

\[
\Gamma^{\alpha\beta}_{\gamma} dy^\gamma = \frac{\partial y^\alpha}{\partial x^\kappa} \frac{\partial y^\beta}{\partial y^\gamma} dx^\kappa + \frac{\partial y^\alpha}{\partial x^\kappa} \Gamma^\kappa_{\lambda\sigma} dx^\sigma \frac{\partial x^\lambda}{\partial y^\beta}.
\]

Thus, we retrieve the classical transformation law for Christoffel symbols that one finds in texts on general relativity.

\[
\Gamma^{\alpha\beta}_{\gamma} = \Gamma^\kappa_{\lambda\sigma} \frac{\partial y^\alpha}{\partial x^\kappa} \frac{\partial x^\sigma}{\partial y^\gamma} + \frac{\partial y^\alpha}{\partial x^\kappa} \Gamma^\kappa_{\lambda\sigma} \frac{\partial x^\lambda}{\partial y^\gamma} \frac{\partial x^\sigma}{\partial y^\beta}.
\]  

### 6.3.4 Parallelism

When first introduced to vectors in elementary calculus and physics courses, vectors are often described as entities characterized by a direction and a length. This primitive notion, that two such entities in \( \mathbb{R}^n \) with the same direction and length represent the same vector, regardless of location, is not erroneous in the sense that parallel translation of a vector in \( \mathbb{R}^n \) does not change the attributes of a vector as described. In elementary linear algebra, vectors are described as n-tuples in \( \mathbb{R}^n \) equipped with the operations of addition and multiplication by scalar, and subject to eight vector space properties. Again, those vectors can be represented by arrows which can be located anywhere in \( \mathbb{R}^n \) as long as they have the same components. This is another indication that parallel transport of a vector in \( \mathbb{R}^n \) is trivial, a manifestation of the fact the \( \mathbb{R}^n \) is a flat space. However, in a space that is not flat, such as a sphere, parallel transport of vectors is intimately connected with the curvature of the space. To elucidate this connection, we first describe parallel transport for a surface in \( \mathbb{R}^3 \).

#### 6.3.2 Definition

Let \( u^\alpha(t) \) be a curve on a surface \( x = x(u^\alpha) \), and let \( V = \alpha'(t) = \alpha_* \left( \frac{d}{dt} \right) \) be the velocity vector as defined in [1.25]. A vector field \( Y \) is called parallel along \( \alpha \) if

\[
\nabla_V Y = 0,
\]

as illustrated in figure [6.2]. The notation

\[
\frac{DY}{dt} = \nabla_V Y
\]

is also common in the literature. The vector field \( \nabla_V V \) is called the Geodesic Vector Field, and its magnitude is called the geodesic curvature \( \kappa_g \) of \( \alpha \). As usual, we define the speed \( v \) of the curve by \( \|V\| \) and the unit tangent \( T = V/\|V\| \), so that \( V = vT \). We assume \( v > 0 \) so that \( T \) is defined on the domain of the curve. The arc length \( s \) along the curve is the related the speed by the equation \( v = ds/dt \).

#### 6.3.3 Definition

A curve \( \alpha(t) \) with velocity vector \( V = \alpha'(t) \) is called a geodesic or self-parallel if \( \nabla_V V = 0 \).

\footnote{We use this notation reluctantly, to be consistent with most literature. The notation results in violation of the index notation. We really should be writing \( \phi^\alpha_{\beta} \), since in this case, the transition functions are matrix-valued.}
6.3.4 Theorem A curve \( \alpha(t) \) is geodesic iff

a) \( v = \|V\| \) is constant along the curve and,

b) either \( \nabla_T T = 0 \), or \( \kappa_g = 0 \).

Proof: Expanding the definition of the geodesic vector field:

\[
\nabla V = \nabla_{v T} (v T),
\]

\[
= v \nabla_T (v T),
\]

\[
= \frac{dv}{dt} T + v^2 \nabla_T T,
\]

\[
= \frac{1}{2} \frac{d}{dt}(v^2) T + v^2 \nabla_T T.
\]

We have \( <T, T> = 1 \), so \( 2 < \nabla_T T, T> = 0 \) which shows that \( \nabla_T T \) is orthogonal to \( T \). We also have \( v > 0 \). Since both the tangential and the normal components need to vanish, the theorem follows.

If \( M \) is a hypersurface in \( \mathbb{R}^n \) with unit normal \( n \), we gain more insight on the geometry of geodesics as a direct consequence of the discussion above. Without real loss of generality consider the geometry in the case of \( n = 3 \). Since \( \|\alpha'\|^2 = <\alpha', \alpha'> \) is constant, differentiation gives \( <\alpha', \alpha''> = 0 \), so that the acceleration \( \alpha'' \) is orthogonal to \( \alpha' \). Comparing with equation 4.33 we see that \( T' = \kappa_n n \), which reinforces the fact that the entire curvature of the curve is due to the normal curvature of the surface as a submanifold of the ambient space. In this sense, inhabitants constrained to live on the surface would be unaware of this curvature, and to them, geodesics would appear locally as the straightest path to travel. Thus, for a sphere in \( \mathbb{R}^3 \) of radius \( a \) centered at the origin, the acceleration \( \alpha'' \) of a geodesic only has a normal component, and the normal curvature is \( 1/a \). That is, the geodesic must lie along a great circle.

6.3.5 Theorem Let \( \alpha(t) \) by curve with velocity \( V \). For each vector \( Y \) in the tangent space restricted to the curve, there is a unique vector field \( Y(t) \) locally obtained by parallel transport.

Proof: We choose local coordinates with frame field \( \{e_\alpha = \frac{\partial}{\partial u^\alpha} \} \). We write the components of the vector fields in terms of the frame

\[
Y = y^\beta \frac{\partial}{\partial u^\beta},
\]

\[
V = \frac{du^\alpha}{dt} \frac{\partial}{\partial u^\alpha}
\]

then,

\[
\nabla_T V = \nabla_{u^\alpha e_\alpha} (y^\beta e_\beta),
\]

\[
= u^\alpha \nabla e_\alpha (y^\beta e_\beta),
\]

\[
= \frac{du^\alpha}{dt} \frac{\partial y^\beta}{\partial u^\alpha} + u^\alpha y^\beta \Gamma^\gamma_{\alpha \beta} e_\gamma,
\]

\[
= \frac{dy^\gamma}{dt} + y^\beta \frac{du^\alpha}{dt} \Gamma^\gamma_{\alpha \beta} e_\gamma.
\]

So, \( Y \) is parallel along the curve iff

\[
\frac{dy^\gamma}{dt} + y^\beta \frac{du^\alpha}{dt} \Gamma^\gamma_{\alpha \beta} e_\gamma = 0.
\] (6.59)

The existence and uniqueness of the coefficients \( y^\beta \) that define \( Y \) are guaranteed by the theorem on existence and uniqueness of differential equations with appropriate initial conditions.

At the risk of sounding a bit repetitions, we derive the equations of geodesics by an almost identical computation.
\[ \nabla_V V = \nabla_{\dot{u}^\alpha e_\alpha} [\dot{u}^\beta e_\beta], \]
\[ = \dot{u}^\alpha \nabla_{e_\alpha} [\dot{u}^\beta e_\beta], \]
\[ = \dot{u}^\alpha \left( \frac{\partial \dot{u}^\beta}{\partial u^\alpha} e_\beta + \ddot{u}^\beta \nabla_{e_\alpha} e_\beta \right), \]
\[ = \dot{u}^\alpha \frac{\partial \dot{u}^\beta}{\partial u^\alpha} e_\beta + \dddot{u}^\alpha \dddot{u}^\beta \Gamma^\sigma_{\alpha \beta} e_\sigma, \]
\[ = \dddot{u}^\beta e_\beta + \dddot{u}^\alpha \dddot{u}^\beta \Gamma^\sigma_{\alpha \beta} e_\sigma, \]
\[ = \left[ \dddot{u}^\sigma + \dddot{u}^\alpha \dddot{u}^\beta \Gamma^\sigma_{\alpha \beta} \right] e_\sigma. \]

Thus, the equation for geodesics becomes:
\[ \dddot{u}^\sigma + \Gamma^\sigma_{\alpha \beta} \dddot{u}^\alpha \dddot{u}^\beta = 0. \] (6.60)

The existence and uniqueness theorem for solutions of differential equations leads to the following theorem

**6.3.6 Theorem** Let \( p \) be a point in \( M \) and \( V \) a vector \( T_p M \). Then, for any real number \( t_0 \), there exists a number \( \delta \) and a curve \( \alpha(t) \) defined on \([t_0 - \delta, t_0 + \delta] \), such that \( \alpha(t_0) = p \), \( \alpha'(t_0) = V \), and \( \alpha \) is a geodesic.

For a general vector bundles \( E \) over a manifold \( M \), a section \( s \in \Gamma(E) \) of a vector bundle is called a parallel section if
\[ \nabla s = 0. \] (6.61)

We discuss the length minimizing properties geodesics in section 6.5 and provide a number of examples for surfaces in \( \mathbb{R}^3 \) and for Lorentzian manifolds. Since geodesic curves have zero acceleration, in Euclidean space they are straight lines. In Einstein’s theory of Relativity, gravitation is fictitious force caused by the curvature of space time, so geodesics represent the trajectory of free particles.

### 6.4 Lorentzian Manifolds

The formalism above refers to Riemannian manifolds, for which the metric is positive definite, but it applies just as well to pseudo-Riemannian manifolds for which is not. A 4-dimensional manifold \( \{M, g\} \) is called a Lorentzian manifold if the metric has signature \((+---)\). Locally, a Lorentzian manifold is diffeomorphic to Minkowski’s space which is the model space introduced in section [2.2.3]. Some authors use signature \((-+++)\).

For the purposes of general relativity we introduce the symmetric tensor **Ricci Tensor** by the the contraction
\[ R_{\beta \delta} = R^\alpha_{\beta \alpha \delta}, \] (6.62)
\[ R_{\beta \gamma} \text{ and the scalar curvature } R \]
\[ R = R^\alpha_{\beta}. \] (6.63)

The traceless part of the Ricci tensor
\[ G_{\alpha \beta} = R_{\alpha \beta} - \frac{1}{2} R g_{\alpha \beta}, \] (6.64)
is called **Einstein’s tensor**. The Einstein field equations (without a cosmological constant) are

\[ G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{6.65} \]

where \( T \) is the **stress energy tensor** and \( G \) is the gravitational constant. As I first learned from one of my professors Arthur Fischer, the equation states that curvature indicates the presence of matter, and matter tells the space how to curve.

A space time which satisfies

\[ R_{\alpha\beta} = 0 \tag{6.66} \]

is called Ricci-Flat.

### 6.4.1 Example: Vaidya Metric

This example of a curvature computation in four dimensional space-time is due to W. Israel. It appears in his 1978 notes on Differential Forms in General Relativity, but the author indicates the work arose 10 years earlier from a seminar at the Dublin Institute for Advanced Studies. The Vaidya metric

\[ ds^2 = 2drdu + \left[1 - \frac{2m(u)}{r}\right]du^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{6.67} \]

where \( m(u) \) is an arbitrary function. The geometry described by the Vaidya solution to Einstein equations, represents the gravitational field in the exterior of a radiating, spherically symmetric star. The variable \( u \) is the retarded time and \( m(u) \) is the mass. If \( m(u) \) is constant, this is equivalent to a Schwarzschild metric. The computation will be carried out using Cartan’s structure equations. The first step in the computation involves picking out a basis of one forms. The idea is to pick out the forms so that in the new basis, the metric has constant coefficients. One possible choice of 1-forms is

\[ \begin{align*}
\theta^0 &= du, \\
\theta^1 &= dr + \frac{1}{2}\left[1 - \frac{2m(u)}{r}\right]du \\
\theta^2 &= r d\theta \\
\theta^3 &= r \sin \theta d\phi.
\end{align*} \tag{6.68} \]

In terms of these forms, the line element becomes

\[ ds^2 = g_{\alpha\beta}\theta^\alpha\theta^\beta = 2\theta^0\theta^1 - (\theta^2)^2 - (\theta^3)^2. \]

where

\[ g_{01} = g_{10} = -g_{22} = -g_{33} = 1, \]
while all the other \( g_{\alpha\beta} = 0 \). In all our previous Gaussian curvature computations the metric has been diagonal. This is an instructive example of non-diagonal metric which in this frame the metric and its inverse have components:

\[
g_{\alpha\beta} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}.
\] (6.69)

Since the coefficients of the metric are constant, the components \( \omega_{\alpha\beta} \) of the connection will be antisymmetric. This means that

\[
\omega_{00} = \omega_{11} = \omega_{22} = \omega_{33} = 0.
\]

We thus conclude that

\[
\begin{align*}
\omega^1_0 &= g^{10} \omega_{00} = 0 \\
\omega^0_1 &= g^{01} \omega_{11} = 0 \\
\omega^2_2 &= g^{22} \omega_{22} = 0 \\
\omega^3_3 &= g^{33} \omega_{33} = 0.
\end{align*}
\]

To compute the connection, we take the exterior derivative of the basis 1-forms. The result of this computation is

\[
\begin{align*}
d\theta^0 &= 0 \\
d\theta^1 &= -d\left[\frac{m}{r^2} du\right] = \frac{m}{r^2} dr \wedge du = \frac{m}{r^2} \theta^1 \wedge \theta^0 \\
d\theta^2 &= dr \wedge d\theta = \frac{1}{r} \theta^1 \wedge \theta^2 - \frac{1}{2r} \left[1 - \frac{2m}{r}\right] \theta^0 \wedge \theta^2 \\
d\theta^3 &= \sin \theta \, dr \wedge d\phi + r \cos \theta \, d\theta \wedge d\phi \\
&= \frac{1}{r} \theta^1 \wedge \theta^3 - \frac{1}{2} \left[1 - \frac{2m}{r}\right] \theta^0 \wedge \theta^3 + \frac{1}{r} \cot \theta \, \theta^2 \wedge \theta^3.
\end{align*}
\] (6.70)

For convenience, we write below the first equation of structure \([6.23]\) in complete detail.

\[
\begin{align*}
d\theta^0 &= \omega^0_0 \wedge \theta^0 + \omega^0_1 \wedge \theta^1 + \omega^0_2 \wedge \theta^2 + \omega^0_3 \wedge \theta^3, \\
d\theta^1 &= \omega^1_0 \wedge \theta^0 + \omega^1_1 \wedge \theta^1 + \omega^1_2 \wedge \theta^2 + \omega^1_3 \wedge \theta^3, \\
d\theta^2 &= \omega^2_0 \wedge \theta^0 + \omega^2_1 \wedge \theta^1 + \omega^2_2 \wedge \theta^2 + \omega^2_3 \wedge \theta^3, \\
d\theta^3 &= \omega^3_0 \wedge \theta^0 + \omega^3_1 \wedge \theta^1 + \omega^3_2 \wedge \theta^2 + \omega^3_3 \wedge \theta^3.
\end{align*}
\] (6.71)

Since the the \( \omega \)'s are one forms, they must be linear combinations of the \( \theta \)'s. Comparing Cartan's first structural equation with the exterior derivatives of the coframe, we can start with the initial guess for the connection coefficients below:

\[
\begin{align*}
\omega^1_0 &= 0, & \omega^1_1 &= \frac{m}{r^2} \theta^0, & \omega^1_2 &= A \, \theta^2, & \omega^1_3 &= B \, \theta^3, \\
\omega^2_0 &= -\frac{1}{2} \left[1 - \frac{2m}{r}\right] \theta^2, & \omega^2_1 &= \frac{1}{r} \theta^2, & \omega^2_2 &= 0, & \omega^2_3 &= C \, \theta^3, \\
\omega^3_0 &= -\frac{1}{2} \left[1 - \frac{2m}{r}\right] \theta^3, & \omega^3_1 &= \frac{1}{r} \theta^3, & \omega^3_2 &= 1 \, \cot \theta \, \theta^3, & \omega^3_3 &= 0.
\end{align*}
\] (6.72-6.74)

Here, the quantities \( A, B, \) and \( C \) are unknowns to be determined. Observe that these are not the most general choices for the \( \omega \)'s. For example, we could have added a term proportional to \( \theta^1 \) in
the expression for $\omega^1_1$, without affecting the validity of the first structure equation for $d\theta^1$. The strategy is to interactively tweak the expressions until we set of forms completely consistent with Cartan’s structure equations.

We now take advantage of the skewsymmetry of $\omega_{\alpha\beta}$, to determine the other components. The find $A$, $B$ and $C$, we note that

$$
\begin{align*}
\omega^1_2 &= g^{10}\omega_{02} = -\omega_{20} = \omega^2_0, \\
\omega^1_3 &= g^{10}\omega_{03} = -\omega_{30} = \omega^3_0, \\
\omega^2_3 &= g^{22}\omega_{23} = \omega_{32} = -\omega^3_2,
\end{align*}
$$

From equations \[6.72\], \[6.73\], and \[6.74\], we find that

$$A = -\frac{1}{2}\left[1 - \frac{2m}{r}\right], \quad B = -\frac{1}{2}\left[1 - \frac{2m}{r}\right], \quad C = -\frac{1}{r}\cot \theta. \quad (6.75)$$

Similarly, we have

$$
\begin{align*}
\omega^0_0 &= -\omega^1_1, \\
\omega^0_2 &= \omega^2_1, \\
\omega^0_3 &= \omega^3_1,
\end{align*}
$$

hence,

$$
\begin{align*}
\omega^0_0 &= -\frac{m}{r^2}\theta^0, \\
\omega^0_2 &= -\frac{1}{r}\theta^2, \\
\omega^0_3 &= \frac{1}{r}\theta^3.
\end{align*}
$$

By inspection we see that our choices for the $\omega$’s are consistent with first structure equations, so by uniqueness, these must be the right values.

There is no guesswork in obtaining the curvature forms. All we do is take the exterior derivative of the connection forms, and pick out the components of the curvature from the second Cartan equations \[6.24\]. Thus, for example, to obtain $\Omega^1_1$, we proceed as follows.

$$
\begin{align*}
\Omega^1_1 &= d\omega^1_1 + \omega^1_1 \wedge \omega^1_1 + \omega^1_2 \wedge \omega^2_1 + \omega^1_3 \wedge \omega^3_1, \\
&= d\left[\frac{m}{r^2}\theta^0\right] + 0 - \frac{1}{2r^2}\left[1 - \frac{2m}{r}\right] \omega^3_3 \wedge \omega^3_1 + (\theta^2 \wedge \theta^2 + \theta^3 \wedge \theta^3), \\
&= -\frac{2m}{r^3}dr \wedge \theta^0, \\
&= -\frac{2m}{r^3}\theta^1 \wedge \theta^0.
\end{align*}
$$

The computation of the other components is straight forward and we just present the results.

$$
\begin{align*}
\Omega^1_2 &= -\frac{1}{r^2}\frac{dm}{du}\theta^2 \wedge \theta^0 - \frac{m}{r^3}\theta^1 \wedge \theta^2, \\
\Omega^1_3 &= -\frac{1}{r^2}\frac{dm}{du}\theta^3 \wedge \theta^0 - \frac{m}{r^3}\theta^1 \wedge \theta^3, \\
\Omega^2_1 &= \frac{m}{r^3}\theta^2 \wedge \theta^0,
\end{align*}
$$
\[ \Omega^3_1 = \frac{m}{r^3} \theta^3 \wedge \theta^0, \]
\[ \Omega^2_3 = \frac{2m}{r^3} \theta^2 \wedge \theta^3. \]

The other components of the curvature are determined by observing that like the connection forms, the curvature forms are also antisymmetric. We can also read the components of the full Riemann curvature tensor from the definition

\[ \Omega^\alpha{}_{\beta} = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta. \]  \hspace{1cm} (6.76)

Thus, for example, we have
\[ \Omega^1_1 = \frac{1}{2} R^1{}_{1\gamma\delta} \theta^\gamma \wedge \theta^\delta, \]

hence
\[ R^1_{101} = -R^1{}_{110} = \frac{2m}{r^3}; \text{ other } R^1{}_{1\gamma\delta} = 0. \]

Using the antisymmetry of the curvature forms, we see, that for the Vaidya metric \( \Omega^1_0 = \Omega^0_0 = 0, \Omega^2_0 = -\Omega^1_2, \) etc., so that
\[ R_{00} = R^2_{020} + R^3_{030} = R^1_{220} + R^1_{330} \]

Substituting the relevant components of the curvature tensor read from the \( \Omega \)’s we find that
\[ R_{00} = \frac{2}{r^2} \frac{dm}{du} \]  \hspace{1cm} (6.77)

while all the other components of the Ricci tensor vanish. As stated earlier, if \( m \) is constant, we get the Ricci flat Schwarzschild metric.
6.5 Geodesics

Geodesics were introduced in the section on parallelism. The equation of geodesics on a manifold given by equation 6.60 involves the Christoffel symbols. Whereas it is possible to compute all the Christoffel symbols starting with the metric as in equation 4.75, this is most inefficient, as it is often the case that many of the Christoffel symbols vanish. Instead, we show next how to obtain the geodesic equations by using variational principles

\[ \delta \int L(u^\alpha, \dot{u}^\alpha, s) \, ds = 0, \]  

(6.78)

to minimize the arc length. Then we can pick out the non-vanishing Christoffel symbols from the geodesic equation. Following the standard methods of Lagrangian Mechanics, we let \( u^\alpha \) and \( \dot{u}^\alpha \) be treated as independent (canonical) coordinates and choose the Lagrangian in this case to be

\[ L = g_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta. \]  

(6.79)

The choice will actually result in minimizing the square of the arc length, but clearly this is an equivalent problem. It should be observed that the Lagrangian is basically a multiple of the kinetic energy \( \frac{1}{2} m v^2 \). The motion dynamics are given by the Euler-Lagrange equations.

\[ \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{u}^\gamma} \right) - \frac{\partial L}{\partial u^\gamma} = 0. \]  

(6.80)

Applying this equation keeping in mind that \( g_{\alpha\beta} \) is the only quantity that depends on \( u^\alpha \), we get:

\[
0 = \frac{d}{ds} \left[ g_{\alpha\beta} \delta^\gamma_\delta \dot{u}^\delta + g_{\alpha\beta} \dot{u}^\alpha \delta^\gamma_\delta \right] - g_{\alpha\beta,\gamma} \dot{u}^\alpha \dot{u}^\beta \\
= \frac{d}{ds} \left[ g_{\gamma\beta} \dot{u}^\beta + g_{\alpha\gamma} \dot{u}^\alpha \right] - g_{\alpha\beta,\gamma} \dot{u}^\alpha \dot{u}^\beta \\
= g_{\gamma\beta} \ddot{u}^\beta + g_{\alpha\gamma} \dot{u}^\alpha + g_{\gamma\beta,\alpha} \dot{u}^\alpha \ddot{u}^\beta + g_{\alpha\gamma,\beta} \dot{u}^\alpha \dot{u}^\beta - g_{\alpha\beta,\gamma} \dot{u}^\alpha \dot{u}^\beta \\
= 2g_{\gamma\beta} \ddot{u}^\beta + [g_{\gamma\beta,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}] \dot{u}^\alpha \dot{u}^\beta \\
= \delta^\gamma_\beta \ddot{u}^\beta + \frac{1}{2} g^{\gamma\sigma} [g_{\gamma\beta,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}] \dot{u}^\alpha \dot{u}^\beta \\
\]

where the last equation was obtained contracting with \( \frac{1}{2} g^{\gamma\sigma} \) to raise indices. Comparing with the expression for the Christoffel symbols found in equation 6.60, we get

\[ \ddot{u}^\sigma + \Gamma^\sigma_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta = 0 \]

which is exactly the equations of geodesics 6.60.

6.5.1 Example Geodesics of Sphere

Let \( S^2 \) be a sphere of radius \( a \) so that the metric is given by

\[ ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta \, d\phi^2. \]

Then the Lagrangian is

\[ L = a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \, \dot{\phi}^2. \]

The Euler-Lagrange equation for the \( \phi \) coordinate is

\[ \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0 \]

\[ \frac{d}{ds} (2a^2 \sin^2 \theta \dot{\phi}) = 0 \]
and therefore the equation integrates to a constant

\[ \sin^2 \theta \dot{\phi} = k. \]

Rather than trying to solve the second Euler-Lagrange equation for \( \theta \), we evoke a standard trick that involves reusing the metric. It goes as follows:

\[
\begin{align*}
\sin^2 \theta \frac{d \phi}{ds} &= k, \\
\sin^2 \theta d \phi &= k ds, \\
\sin^4 \theta d \phi^2 &= k^2 ds^2, \\
\sin^4 \theta d \phi^2 &= k^2 (a^2 d \theta^2 + a^2 \sin^2 \theta d \phi^2), \\
(sin^4 \theta - k^2 a^2 \sin^2 \theta) d \phi^2 &= a^2 k^2 d \theta^2.
\end{align*}
\]

The last equation above is separable and it can be integrated using the substitution \( u = \cot \theta \).

\[
\begin{align*}
\frac{d \phi}{ds} &= \frac{a k}{\sin \theta \sqrt{\sin^2 \theta - a^2 k^2}} d \theta, \\
&= \frac{a k}{\sin^2 \theta \sqrt{1 - a^2 k^2 \csc^2 \theta}} d \theta, \\
&= \frac{a k}{\sin^2 \theta \sqrt{1 - a^2 k^2 (1 + \cot^2 \theta)}} d \theta, \\
&= \frac{a k \csc^2 \theta}{\sqrt{1 - a^2 k^2 (1 + \cot^2 \theta)}} d \theta, \\
&= \frac{a k \csc^2 \theta}{\sqrt{1 - a^2 k^2 - \cot^2 \theta}} d \theta, \\
&= \frac{-1}{\sqrt{c^2 - u^2}} du, \quad \text{where} \quad c^2 = \frac{1 - a^2 k^2}{a^2 k^2}. \\
\phi &= -\sin^{-1}(\frac{1}{c} \cot \theta) + \phi_0.
\end{align*}
\]

Here, \( \phi_0 \) is the constant of integration. To get a geometrical sense of the geodesics equations we have just derived, we rewrite the equations as follows:

\[
\begin{align*}
\cot \theta &= c \sin(\phi_0 - \phi), \\
\cos \theta &= c \sin \theta (\sin \phi_0 \cos \phi - \cos \phi_0 \sin \phi), \\
a \cos \theta &= (c \sin \phi_0)(a \sin \theta \cos \phi) - (c \cos \phi_0)(a \sin \theta \sin \phi), \\
z &= Ax - By, \quad \text{where} \quad A = c \sin \phi_0, \quad B = c \cos \phi_0.
\end{align*}
\]

We conclude that the geodesics of the sphere are great circles determined by the intersections with planes through the origin.

**6.5.2 Example** Geodesics in Orthogonal Coordinates.

In a parametrization of a surface in which the coordinate lines are orthogonal, \( F = 0 \). Then first fundamental form is

\[ ds^2 = E du^2 + G dv^2, \]
and we have the lagrangian,

\[ L = E \dot{u}^2 + G \dot{v}^2. \]

The Euler-Lagrange equations for the variable \( u \) are:

\[
\frac{d}{ds}(2E \ddot{u} - Eu \dot{u}^2 - Gu \dot{v}^2) = 0,
\]

\[
2E \ddot{u} + (2Eu \ddot{u} + 2Ev \ddot{v}) \dot{u} - Eu \dot{u}^2 - Gu \dot{v}^2 = 0,
\]

\[
2E \ddot{u} + Eu \dot{u}^2 + 2Ev \dot{u} \dot{v} - Gu \dot{v}^2 = 0.
\]

Similarly for the variable \( v \),

\[
\frac{d}{ds}(2G \ddot{v} - Eu \dot{u}^2 - Gu \dot{v}^2) = 0,
\]

\[
2G \ddot{v} + (2Gu \ddot{u} + 2Gv \ddot{v}) \dot{v} - Eu \dot{u}^2 - Gu \dot{v}^2 = 0,
\]

\[
2G \ddot{v} - Ev \dot{u}^2 + 2Gu \dot{u} \dot{v} + Gv \dot{v}^2 = 0.
\]

So, the equations of geodesics can be written neatly as,

\[
\ddot{u} + \frac{1}{2E}[Eu \dot{u}^2 + 2Ev \dot{u} \dot{v} - Gu \dot{v}^2] = 0,
\]

\[
\ddot{v} + \frac{1}{2G}[Gv \dot{v}^2 + 2Gu \dot{u} \dot{v} - Ev \dot{u}^2] = 0.
\]

(6.81)

### 6.5.3 Example Geodesics of Surface of Revolution

The first fundamental form of a surface of revolution \( z = f(r) \) in cylindrical coordinates as in 4.7 is

\[ ds^2 = (1 + f'^2) \, dr^2 + r^2 \, d\phi^2, \]  

(6.82)

Of course, we could use the expressions for the equations of geodesics we just derived above, but since the coefficients are functions of \( r \) only, it is just as easy to start from the Lagrangian,

\[ L = (1 + f'^2) \, r^2 \dot{\phi}^2. \]

Since there is no dependency on \( \phi \), the Euler-Lagrange equation on \( \phi \) gives rise to a conserved quantity.

\[
\frac{d}{ds}(2r^2 \dot{\phi}) = 0,
\]

\[
r^2 \dot{\phi} = c \]  

(6.83)

where \( c \) is a constant of integration. If the geodesic \( \alpha(s) = \alpha(r(s), \phi(s)) \) represents the path of a free particle constrained to move in the surface, this conserved quantity is essentially the angular momentum. A neat result can be obtained by considering the angle \( \sigma \) that the tangent vector \( V = \alpha' \) makes with a meridian. Recall that the length of \( V \) along the geodesic is constant, so let’s set \( \|V\| = k \) From the chain rule we have

\[
\alpha'(t) = x_r \frac{dr}{ds} + x_\phi \frac{d\phi}{ds}.
\]

Then

\[
\cos \sigma = \frac{\langle \alpha', x_\phi \rangle}{\|\alpha'\| \cdot \|x_\phi\|} = \frac{G \frac{d\phi}{ds}}{k \sqrt{G}} = \frac{1}{k} \sqrt{G} \frac{d\phi}{ds} = \frac{1}{k} \dot{\phi}.
\]
We conclude from [6.83] that for a surface of revolution, the geodesics make an angle $\sigma$ with meridians that satisfies the equation

$$r \cos \sigma = \text{constant.} \quad (6.84)$$

This result is called Clairaut’s relation. Writing equation 6.83 in terms of differentials, and reusing the metric as we did in the computation of the geodesics for a sphere, we get:

$$r^2 \, d\phi = c \, ds,$$
$$r^4 \, d\phi^2 = c^2 \, ds^2,$$
$$= c^2 [(1 + f^2) \, dr^2 + r^2 \, d\phi^2],$$
$$(r^4 - c^2 r^2) \, d\phi^2 = c^2 [(1 + f^2) \, dr^2,$$
$$r \sqrt{r^2 - c^2} \, d\phi = c \sqrt{1 + f^2} \, dr,$$

So

$$\phi = \pm c \int \frac{\sqrt{1 + f^2}}{r \sqrt{r^2 - c^2}} \, dr. \quad (6.85)$$

If $c = 0$, then the first equation above gives $\phi =$constant, so the meridians are geodesics. The parallels $r =$constant are geodesics when $f'(r) = \infty$ in which case the tangent bundle restricted to the parallel is a cylinder with a vertical generator.

In the particular case of a cone of revolution with a generator that makes an angle $\alpha$ with the $z$-axis, $f(r) = \cot(\alpha)r$, equation 6.85 becomes:

$$\phi = \pm c \int \frac{\sqrt{1 + \cot^2 \alpha}}{r \sqrt{r^2 - c^2}} \, dr$$

which can be immediately integrated to yield:

$$\phi = \pm \csc \alpha \sec^{-1} (r/c) \quad (6.86)$$

As shown in figure 6.4, a ribbon laid flatly around a cone follows the path of a geodesic. None of the parallels, which in this case are the generators of the cone, are geodesics.

6.5.4 Example Morris-Thorne (MT) Wormhole

In 1987, Michael Morris and Kip Thorne from the California Institute of Technology proposed a tantalizing simple model for teaching general relativity by alluding to interspace travel in a geometry
of traversible wormhole. We constrain the discussion purely to geometrical aspects of the model and not the physics of stress and strains of a "traveller" traversing the wormhole. The MT metric for this spherically symmetric geometry is:

\[ ds^2 = -c^2 dt^2 + dl^2 + \left(b_0^2 + l^2\right) \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right), \]

where \( b_0 \) is a constant. The obvious choice for a coframe is

\[ \theta^0 = c \, dt \quad \theta^2 = \sqrt{b_0^2 + l^2} \, d\theta \]

\[ \theta^1 = dl \quad \theta^3 = \sqrt{b_0^2 + l^2} \, \sin \theta \, d\phi. \]

We have \( d\theta^0 = d\theta^1 = 0 \). To find the connection forms we compute \( d\theta^2 \) and \( d\theta^3 \), and rewrite in terms of the coframe. We get

\[ d\theta^2 = \frac{l}{\sqrt{b_0^2 + l^2}} \, dl \wedge d\theta = -\frac{l}{\sqrt{b_0^2 + l^2}} \, d\theta \wedge dl, \]

\[ = -l \frac{b_0^2 + l^2}{\theta^2} \wedge \theta^1, \]

\[ d\theta^3 = \frac{l}{\sqrt{b_0^2 + l^2}} \, \sin \theta \, dl \wedge d\phi + \cos \theta \sqrt{b_0^2 + l^2} \, d\theta \wedge d\phi, \]

\[ = -\frac{l}{b_0^2 + l^2} \, \theta^3 \wedge \theta^1 - \frac{\cot \theta}{\sqrt{b_0^2 + l^2}} \, \theta^3 \wedge \theta^2. \]

Comparing with the first equation of structure, we start with simplest guess for the connection forms \( \omega \)'s. We set

\[ \omega^2_1 = \frac{l}{b_0^2 + l^2} \, \theta^2, \]

\[ \omega^3_1 = \frac{l}{b_0^2 + l^2} \, \theta^3, \]

\[ \omega^3_2 = \frac{\cot \theta}{\sqrt{b_0^2 + l^2}} \, \theta^3. \]

Using the antisymmetry of the \( \omega \)'s and the diagonal metric, we have \( \omega^2_1 = -\omega^1_2, \omega^3_1 = -\omega^1_3, \) and \( \omega^2_3 = -\omega^3_2 \). This choice of connection coefficients turns out to be completely compatible with the entire set of Cartan’s first equation of structure, so, these are the connection forms, all other \( \omega \)'s are zero. We can then proceed to evaluate the curvature forms. There is no guess-work here. An straight-forward calculus computation which results in some pleasing cancellations, yields:

\[ \Omega^1_2 = d\omega^1_2 + \omega^2_1 \wedge \omega^2_1 = -\frac{b_0^2}{\left(b_0^2 + l^2\right)^2} \, \theta^1 \wedge \theta^2, \]

\[ \Omega^1_3 = d\omega^1_3 + \omega^2_1 \wedge \omega^2_3 = -\frac{b_0^2}{\left(b_0^2 + l^2\right)^2} \, \theta^1 \wedge \theta^3, \]

\[ \Omega^2_3 = d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = \frac{b_0^2}{\left(b_0^2 + l^2\right)^2} \, \theta^2 \wedge \theta^3. \]

Thus, from equation 6.35 other than permutations of the indices, the only independent components of the Riemann tensor are

\[ R_{2323} = -R_{1212} = R_{1313} = \frac{b_0^2}{\left(b_0^2 + l^2\right)^2}. \]
and the only non-zero component of the Ricci tensor is

\[ R_{11} = -2 \frac{b_0^2}{(b_0^2 + l^2)^2}. \]

Of course, this space is a 4-dimensional continuum, but since the space is spherically symmetric, we may get a good sense of the geometry by taking a slice with \( \theta = \pi/2 \) at a fixed value of time. The resulting metric \( ds^2 \) for the surface is

\[ ds^2 = dl^2 + (b_0^2 + l^2) d\phi^2. \]  

(6.88)

Let \( r^2 = b_0^2 + l^2 \). Then \( dl^2 = (r^2/l^2) dr^2 \) and the metric becomes

\[ ds^2 = \frac{r^2}{r^2 - b_0^2} dr^2 + r^2 d\phi^2, \]

(6.89)

\[ = \frac{1}{1 - \frac{b_0^2}{r^2}} dr^2 + r^2 d\phi^2. \]

(6.90)

Comparing to 4.25 we recognize this to be a Catenoid of revolution, so the equations of geodesics are given by 6.85 with \( f(r) = b_0 \cosh^{-1}(r/b_0) \). Substituting this value of \( f \) into the geodesic equation, we get

\[ \phi = \pm c \int \frac{1}{\sqrt{r^2 - b_0^2} \sqrt{r^2 - c^2}} dr. \]

(6.91)

There are three cases. If \( c = b_0 \), the integral gives immediately \( \phi = \pm (c/b_0) \tanh^{-1}(r/b_0) \). We consider the case \( c > b_0 \). The remaining case can be treated in a similar fashion. Let \( r = c/\sin \beta \). Then \( \sqrt{r^2 - c^2} = r \cos \beta \) and \( dr = -r \cot \beta \ d\beta \), so, assuming the initial condition \( \phi(0) = 0 \), the the substitution leads to the integral

\[ \phi = \pm c \int_0^s \frac{1}{r \cos \beta \sqrt{\frac{\sin^2 \beta}{c^2} - b_0^2}} \frac{(-r \cos \beta)}{\sin \beta} d\beta, \]

\[ = \pm c \int_0^s \frac{1}{\sqrt{c^2 - b_0^2 \sin^2 \beta}} d\beta, \]

\[ \begin{align*}
&= \pm \int_0^s \frac{1}{\sqrt{1 - k^2 \sin^2 \beta}} d\beta, \quad (k = b_0/c) \\
&= F(s, k),
\end{align*} \]

(6.92)

(6.93)

where \( F(s, k) \) is the well known incomplete elliptic integral of the first kind.

![Geodesics on Catenoid](image1)

Fig. 6.5: Geodesics on Catenoid.
6.5. GEODESICS

Elliptic integrals are standard functions implemented in computer algebra systems, so it is easy to render some geodesics as shown in figure 6.5. The plot of the elliptic integral shown here is for $k = 0.9$. The plot shows clearly that this is a 1-1, so if one one wishes to express $r$ in terms of $\phi$ one just finds the inverse of the elliptic integral which yields a Jacobi elliptic function. Thomas Muller has created a neat Wolfram-Demonstration that allows the user to play with MT wormhole geodesics with parameters controlled by sliders.
6.6 Gauss-Bonnet Theorem

This section is dedicated to the memory of the Professor S.-S. Chern. I prelude the section with a short anecdote that I often narrate to my students. In June 1979, an international symposium on Differential Geometry was held at the Berkeley campus in honor of the retirement of Professor Chern. The invited speakers included an impressive list of the most famous differential geometers at the time. At the end of the symposium, Chern walked on the stage of the packed auditorium to give thanks and to answer some questions. After a few short remarks, a member of the audience asked Chern what he thought was the most important theorem in differential geometry. Without any hesitation he answered, “there is only one theorem in differential geometry, and that is Stoke’s theorem.” This was followed immediately by a question about the most important theorem in analysis. Chern gave the same answer: “there is only one theorem in analysis, Stoke’s theorem. A third person then asked Chern what was the most important theorem in Complex Variables. To the amusement of the crowd, Chern responded, “There is only one theorem in complex variables, and that that is Cauchy’s theorem. But if one assumes the derivative of the function is continuous, then this is just Stoke’s theorem.” Now, of course it is well known that Goursat proved that the hypothesis of continuity of the derivative is automatically satisfied when the function is holomorphic. But the genius of Chern was always his uncanny ability to extract the essential of what makes things work, in the simplest terms.

The Gauss-Bonnet theorem is rooted on the theorem of Gauss (4.71), which combined with Stoke’s theorem, provides a beautiful geometrical interpretation of the equation. This is undoubtedly part of what Chern had in mind at the symposium, and also when wrote in his Euclidean Differential Geometry Notes (Berkeley 1975 ) [4] that the theorem has “profound consequences and is perhaps one of the most important theorems in mathematics.”

Let \( \beta(s) \) by a unit speed curve on an orientable surface \( M \), and let \( T \) be the unit tangent vector. There is Frenet frame formalism for \( M \), but if we think of the surface intrinsically as 2-dimensional manifold, then there is no binormal. However, we can define a “geodesic normal” taking \( G = J(T) \), where \( J \) is the simplectic form 5.50. Then the geodesic curvature is given by the Frenet formula

\[
T' = \kappa_g G.
\]  

(6.94)

6.6.1 Proposition Let \( \{e_1, e_2\} \) be an orthonormal on \( M \), and let \( \beta(s) \) be a unit speed curve as above, with unit tangent \( T \). If \( \phi \) is the angle that \( T \) makes with \( e_1 \), then

\[
\kappa_g = \frac{\partial \phi}{\partial s} - \omega^1_2(T).
\]  

(6.95)

Proof: Since \( \{T, G\} \) and \( \{e_1, e_2\} \) are both orthonormal basis of the tangent space, they must be related by a rotation by an angle \( \phi \), that is

\[
\begin{bmatrix}
T \\
G
\end{bmatrix} = \begin{bmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2
\end{bmatrix},
\]  

(6.96)

that is,

\[
T = (\cos \phi)e_1 + (\sin \phi)e_2,
\]

\[
G = -(\sin \phi)e_1 + (\cos \phi)e_2.
\]  

(6.97)
6.6. GAUSS-BONNET THEOREM

Since $T = \beta'$, we have

\[
\beta'' = \nabla_T T = -(\sin \phi) \frac{\partial \phi}{\partial s} e_1 + \cos \phi \nabla_T e_1 + (\cos \phi) \frac{\partial \phi}{\partial s} e_2 + \sin \phi \nabla_T e_2,
\]

\[
= -(\sin \phi) \frac{\partial \phi}{\partial s} e_1 + (\cos \phi) \omega_1^2(T) e_2 + (\cos \phi) \frac{\partial \phi}{\partial s} e_2 + (\sin \phi) \omega_1^1(T) e_1,
\]

\[
= \left[ \frac{\partial \phi}{\partial s} - \omega_2^1(T) \right] [-(\sin \phi) e_1] + \left[ \frac{\partial \phi}{\partial s} - \omega_2^1(T) \right] [(\cos \phi) e_2]
\]

\[
= \left[ \frac{\partial \phi}{\partial s} - \omega_2^1(T) \right] [-(\sin \phi) e_1 + (\cos \phi) e_2]
\]

\[
= \left[ \frac{\partial \phi}{\partial s} - \omega_2^1(T) \right] G,
\]

\[
= \kappa_g G.
\]

comparing the last two equations, we get the desired result.

This theorem is related to the notion discussed in figure 6.2 to the effect that in a space with curvature, the parallel transport of a tangent vector around a closed curve, does not necessarily result on the same vector with which one started. The difference in angle $\Delta \phi$ between a vector and the parallel transport of the vector around a closed curve $C$ is called the Holonomy of the curve.

The Holonomy of the curve is given by the integral

\[
\Delta \phi = \int_C \omega_2^1(T) \, ds.
\] (6.98)

6.6.2 Definition Let $C$ be a smooth closed curve on $M$ parametrized by arc length with geodesic curvature $\kappa_g$. The line integral $\oint_C \kappa_g \, ds$ is called the total geodesic curvature. If the curve is piecewise smooth, the total geodesic curvature is the sum of the integrals of each piece.

The simplest example is that of circle of radius $R$. The geodesic curvature is the constant $1/R$, so the total geodesic curvature is $(1/R)2\pi R = 2\pi$.

If we integrate formula 6.95 around a smooth simple closed curve $C$ which is the boundary of a region $R$ and use Stoke’s Theorem, we get

\[
\oint_C \kappa_g \, ds = \oint_C d\phi - \oint_C \omega_2^1 \, ds,
\]

\[
= \oint_C d\phi - \int_R d\omega_2^1.
\]

For a smooth simple closed curve, $\int_C d\phi = 2\pi$. Using the Cartan-form version of the Teorema Egremion 4.105, we get immediately

\[
\int_R K \, dS + \oint_C \kappa_g \, ds = 2\pi.
\] (6.99)

If the boundary of the region consists of $k$ piecewise continuous functions as illustrated in figure 6.6 the change of the angle $\phi$ along $C$ is still $2\pi$, but the total change needs to modified by adding the exterior angles $\alpha_k$. Thus we obtain a fundamental result called the Gauss-Bonnet formula,

6.6.3 Theorem

\[
\int_R K \, dS + \oint_C \kappa_g \, ds + \sum_k \alpha_k = 2\pi.
\] (6.100)

Every interior $\iota_k$ angle is the supplement of the corresponding exterior $\alpha_k$ angle, so the Gauss Bonnet formula can also be written as

\[
\int_R K \, dS + \oint_C \kappa_g \, ds + \sum_k (\pi - \iota_k) = 2\pi.
\] (6.101)
The simplest manifestation of the Gauss-Bonnet formula is for a triangle in the plane. Planes are flat surfaces, so $K = 0$ and the straight edges are geodesics, so $\kappa_g = 0$ on each of the three edges. The interior angle version of the formula then just reads $3\pi - \iota_1 - \iota_2 - \iota_3 = 2\pi$, which just says that the interior angles of a flat triangle add up to $\pi$. Since a sphere has constant positive curvature, the sum of the interior angles of a spherical triangle is larger than $\pi$. That amount of this sum over $2\pi$ is called the spherical excess. For example, the sum of the interior angles of a spherical triangle that is the boundary of one octant of a sphere is $3\pi/2$, so the spherical excess is $\pi/2$.

6.6.4 Definition The quantity $\int \int K \, dS$ is called the Total Curvature

6.6.5 Example A sphere of radius $R$ has constant Gaussian Curvature $1/R^2$. The surface area of the sphere is $4\pi R^2$, so the total Gaussian curvature for the sphere is $4\pi$.

6.6.6 Example For a torus generated by a circle of radius $a$ rotating about an axis with radius $b$ as in example (4.40), the differential of surface is $dS = a(b + a \cos \theta) \, d\theta d\phi$, and the Gaussian curvature is $K = \cos \theta / [a(b + a \cos \theta)]$, so the total Gaussian curvature is

$$\int_0^{2\pi} \int_0^{2\pi} \cos \theta \, d\theta d\phi = 0.$$ 

We now relate the Gauss-Bonnet formula to a topological entity.

6.6.7 Definition Let $M$ be a 2 dimensional manifold. A Triangulation of the surface is subdivision of the surface into triangular regions $\{\triangle_k\}$ which are the images of regular triangles under a coordinate patch, such that:
6.6. **GAUSS-BONNET THEOREM**

1) \( M = \bigcup_k \triangle_k \).
2) \( \triangle_i \cap \triangle_j \) is either empty, or a single vertex or an entire edge.
3) All the triangles are oriented in the same direction.

For an intuitive visualization of the triangulation of a sphere, think of inflating a tetrahedron or an octahedron into a spherical balloon. We state without proof:

**6.6.8 Theorem** Any compact surface can be triangulated.

**6.6.9 Theorem** Given a triangulation of a compact surface \( M \), let \( V \) be the number of vertices, \( E \) the number of edges and \( F \) the number of faces. Then the quantity

\[
\chi(M) = V - E + F, \tag{6.102}
\]

is independent of the triangulation. In fact the quantity is independent of any “polyhedral” subdivision. This quantity is a topological invariant called the **Euler Characteristic**.

**6.6.10 Example**

1. A balloon-inflated tetrahedron has \( V = 4 \), \( E = 6 \), \( F = 4 \), so the Euler characteristic of a sphere is 2.

2. A balloon-inflated octahedron has \( V = 6 \), \( E = 12 \), \( F = 8 \), so the we get the same number 2.

3. The diagram on the right of Figure 6.7 represents a topological torus. In the given rectangle, opposites sides are identified in the same direction. The number of edges without double counting are shown in red, and the number of vertices not double counted are shown in black dots. We have \( V = 6 \), \( E = 18 \) \( F = 12 \). So the Euler characteristic of a torus is 0.

4. In one has a compact surface, one can add a “handle”, that is, a torus, by the following procedure. We excise a triangle in each of the two surfaces and glue the edges. We lose two faces and the number of edges and vertices cancel out, so the Euler characteristic of the new surface decreases by 2. The Euler characteristic of a pretzel is -4

5. The Euler characteristic of an orientable surface of genus \( g \), that is, a surface with \( g \) holes is given by \( \chi(M) = 2 - 2g \).

**6.6.11 Theorem** Gauss-Bonnet Let \( M \) be a compact, orientable surface. Then

\[
\frac{1}{2\pi} \int_M K \ dS = \chi(M). \tag{6.103}
\]

**Proof:** Triangulate the surface so that \( M = \bigcup_{k=1}^F \triangle_k \). We start the the Gauss-Bonnet formula

\[
\int \int_M K \ dS = \sum_{k=1}^F \int_{\triangle_k} K \ dS = -\sum_{k=1}^F \left[ \int_{\delta \triangle_k} \kappa_g \ ds + \pi + (\iota_{k1} + \iota_{k2} + \iota_{k3}) \right].
\]

Where \( F \) is the number of triangles and the \( \iota_k \)'s are the interior angles of triangle \( \triangle_k \). The line integrals of the geodesic curvatures all cancel out since each edge in every triangle is traversed twice, each in opposite directions. Rewriting the equation, we get

\[
\int \int_M K \ dS = -\pi F + S
\]
where \( S \) is the sum of all interior angles. Since the manifold is locally Euclidean, the sum of all interior angles at a vertex is \( 2\pi \), so we have

\[
\int \int_M K \, dS = -\pi F + 2\pi V
\]

There are \( F \) faces. Each face has three edges, but each edge is counted twice, so \( 3F = 2E \), and we have \( F = 2E - 2F \). Substituting in the equation above, we get,

\[
\int \int_M K \, dS = -\pi (2E - 2F) + 2\pi V = 2\pi (V - E + F) = \chi(M).
\]

This is a remarkable theorem because it relates the bending invariant Gaussian curvature to a topological invariant. Theorems such as this one, which cut across disciplines are the most significant in mathematics. It was of course Chern who proved a generalization of the Gauss-Bonnet theorem to general orientable Riemannian manifolds of even dimensions [5].
Chapter 7

Groups of Transformations

7.1 Lie Groups

At the IX International Colloquium on Group Theoretical Methods in Physics held in 1980 at Cocoyoc, Mexico, one of the invited addresses was delivered by the famous mathematician Bertram Konstant, who years later would be awarded the Wigner Medal. In his opening remarks, Konstant made the following intriguing statement, “In the 1800’s, Felix Klein and Sophus Lie decided to divide mathematics among themselves. Klein took the Discrete, and Lie took the Continuous. I am here to tell you that they were both working on the same thing.”

In this chapter we present an elementary introduction to Lie groups and corresponding Lie algebras. A simple example of a Lie group is the circle group $U(1) = \{ z \in \mathbb{C} : |z|^2 = 1 \}$ which, as a manifold, corresponds to the unit circle $S^1$. In polar form, an element of this group can be written in the form $e^{i\theta}$. The group multiplication $z \to e^{i\theta}z$ corresponds to a rotation of the vector $z$ by an angle $\theta$. Using the matrix representation \[ e^{1\theta} = \cos \theta + i \sin \theta \] for complex numbers and Euler’s formula, we see that the group is isomorphic to $SO(2, \mathbb{R})$ as shown in example 3.4. This example captures the essence of what we seek, that is, a group that is also a manifold and that can be associated with some matrix group. A $U^1$ bundle consists of a base manifold $M$ with a structure that locally looks like a cross product of an open set in $M$ with $U^1$. Generalizations in which the fibers are Lie groups leads to a structure called a principal fiber bundle. Lie Groups, Lie algebras and principal fiber bundles provide the mathematical foundation in modelling symmetries in classical and quantum physics. The subject is much too rich to give a comprehensive treatment here, but we hope the material will serve as a starting block for further study.

7.1.1 Definition A Lie Group $G$ is a group that is also a smooth manifold. It is assumed that the usual group multiplication and inverse operations,

\[ \mu : G \times G \to G, \]

\[ (g_1, g_2) \mapsto g_1g_2, \]

\[ \iota : G \to G, \]

\[ g \mapsto g^{-1}, \]

are $C^\infty$.

7.1.2 Definition A Lie Subgroup is a subset $H \subset G$ of a Lie group $G$, that is itself a Lie Group.

7.1.3 Examples
1. The (real) General Linear group is the set of \( n \times n \) matrices,

\[ \text{GL}(n, \mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) : \det(A) \neq 0 \}. \quad (7.1) \]

Topologically, \( \text{GL}(n, \mathbb{R}) \) is equivalent to \( \mathbb{R}^{n^2} \) and has the structure of an \( n^2 \)-dimensional differentiable manifold. The map \( \det : \text{GL}(n, \mathbb{R}) \to \mathbb{R} \) is continuous. The inverse image of 0 under this map is a closed space and \( \text{GL}(n, \mathbb{R}) \) is the complement, so \( \text{GL}(n, \mathbb{R}) \) is an open subset of \( \mathbb{R}^{n^2} \) and thus it is not compact. \( \text{GL}(n, \mathbb{R}) \) is not connected being the union of two disjoint open sets defined by whether \( \det(A) > 0 \) or \( \det(A) < 0 \). The connected component \( \text{GL}^+(n, \mathbb{R}) \) corresponding to \( \det(A) > 0 \), contains the identity. Furthermore, if \( \det(A) > 0 \) and \( \det B > 0 \), we have \( \det(AB) > 0 \) and \( \det(A^{-1}) > 0 \), so \( \text{GL}^+(n, \mathbb{R}) \) is a (non-compact) Lie subgroup.

2. The subset of \( \text{GL}^+(n, \mathbb{R}) \) with the restriction \( \det(A) = 1 \) is called the Special Linear group \( \text{SL}(n, \mathbb{R}) \). This is also a non-compact subgroup.

3. The complex General Linear group is the set of \( n \times n \) matrices,

\[ \text{GL}(n, \mathbb{C}) = \{ A \in M_{n \times n}(\mathbb{C}) : \det(A) \neq 0 \}. \quad (7.2) \]

The subgroup of matrices \( A \in \text{GL}(n, \mathbb{C}) \) with \( \det(A) = 1 \) is called the Special Linear group \( \text{SL}(n, \mathbb{C}) \).

4. The real Orthogonal Group is the set of \( n \times n \) real matrices

\[ \text{O}(n, \mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) : A^{-1} = A^T \}. \quad (7.3) \]

The condition \( A^{-1} = A^T \) is equivalent to \( AA^T = A^TA = I \). We have the following

a) \( A^T = I^{-1} = I \), so \( I \in \text{O}(n, \mathbb{R}) \).

b) If \( A, B \in \text{O}(n, \mathbb{R}) \), then \( (AB)(AB)^T = ABB^TA^T = AA^T = I \), so \( AB \in \text{O}(n, \mathbb{R}) \).

c) If \( A \in \text{O}(n, \mathbb{R}) \), then \( A^{-1}(A^{-1})^T = A^T(A^T)^T = A^TA = I \).

Hence \( \text{O}(n, \mathbb{R}) \) is a Lie subgroup of the \( \text{GL}(n, \mathbb{R}) \). The map \( T(A) = AA^T \) is continuous and \( \text{O}(n, \mathbb{R}) = T^{-1}(I) \), so \( \text{O}(n, \mathbb{R}) \) is closed.

If we denote by \( e_k \) the \( k \)th column vector of \( A \), then the matrix element of \( A^T \) in the \( j \)th row and \( k \)th column is given by

\[ (e_j)^T e_k = (e_j, e_k) = \delta_{jk}. \]

Thus, the columns (and rows) of an orthogonal matrix constitute a set of orthonormal vectors. There are \( n \) column vectors, so under the Euclidean norm of \( \mathbb{R}^{n^2} \), we have \( \|A\|^2 = n \). That is, elements of \( \text{O}(n, \mathbb{R}) \) lie on a sphere \( S^{n^2-1} \), so the set is bounded. By the Heine-Borel theorem, \( \text{O}(n, \mathbb{R}) \) is compact. By equation (7.1) and the subsequent Theorem, we can characterize the orthogonal group as the set of linear transformations that preserves the standard metric

\[ g = \text{diag}(+1, +1, \cdots + 1) \] in \( \mathbb{R}^n \).

5. If a matrix \( A \) is orthogonal, then the condition \( AA^T = I \) implies that

\[ \det(AA^T) = \det(A) \det(A^T), \]

\[ = \det(A)^2 = \det(I), \]

so \( \det(A) = \pm 1 \). We define the (real) Special Orthogonal group \( \text{SO}(n, \mathbb{R}) \) to be the subset of \( \text{O}(n, \mathbb{R}) \) of orthogonal matrices \( A \), with \( \det A = 1 \). \( \text{SO}(n, \mathbb{R}) \) is a compact Lie subgroup of dimension \( \frac{1}{2}n(n-1) \).
6. The Unitary group is the set if \( n \times n \) matrices

\[
U(n) = \{ A \in M_{n \times n}(\mathbb{C}) : A^{-1} = A^\dagger \},
\]

(7.4)

where \( A^\dagger \) is the Hermitian adjoint. This is the complex analog of the orthogonal group. The condition \( A^{-1} = A^\dagger \) is equivalent to \( AA^\dagger = I \), which implies that \( \det(A) = \pm 1 \). The subgroup of unitary matrices \( A \) with \( \det(A) = 1 \) is called the Special Unitary group \( SU(n) \); it is a compact group of dimension \( n^2 - 1 \).

7. Let \( \{ M, g \} \) be the pseudo-Riemannian manifold \( \mathbb{R}^n \) with metric \( g = \text{diag}(1, 1, \cdots -1, -1) \) of type \( (p,q) \). The group of transformations preserving this metric is called the \( O(p,q) \). If in addition, the matrices \( A \in O(p,q) \) are required to have \( \det(A) = 1 \), the group is called \( SO(p,q) \). These groups are not compact. The special case \( L = O(1,3) \) is the group of transformations preserving the Minkowski metric. This group is called the Lorentz group which, is central to relativistic physics.

8. In a completely analogous manner, let \( \{ M, g \} \) be the pseudo-Riemannian complex manifold \( \mathbb{C}^n \) with a hermitian metric \( g = \text{diag}(1, 1, \cdots -1, -1) \) of type \( (p,q) \). The group of transformations preserving this metric is called \( U(p,q) \). If in addition, the matrices \( A \in U(p,q) \) are required to have \( \det(A) = 1 \), the group is called \( SU(p,q) \). These groups are also not compact. The special case \( SU(2,2) \) is isomorphic to the Poincaré group and is of interest in Twistor theory.

9. Let \( \mathbb{R}^{2n} \) with local coordinates \( (q^1, \ldots q^n, p_1, \ldots p_n) \). We may think of the local coordinates as representing position and momenta. Let \( \Omega \) be the non-degenerate skew-symmetric two-form

\[
\Omega = dq^i \wedge dp_i, \; i = 1 \ldots n.
\]

(7.5)

The (real) Symplectic group \( Sp(2n, \mathbb{R}) \) is the group of transformations preserving the symplectic form \( \Omega \). More specifically, The tensor components of \( \Omega \) in standard basis are given by

\[
\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}
\]

(7.6)

where \( I_n \) is the identity \( n \times n \) matrix. Then the symplectic group is defined by

\[
Sp(2n, \mathbb{R}) = \{ A \in M_{2n \times 2n}(\mathbb{R}) : A^T \Omega A = \Omega \}.
\]

(7.7)

The symplectic group is an essential structure in the differential geometry description of Lagrangian and Hamiltonian mechanics. In the simplest case in which \( n = 1 \), the components of the canonical symplectic form is the complex structure introduced in equation [5.50] in the context of conformal maps. It is immediately clear that \( Sp(2, \mathbb{R}) \) consists of all \( 2 \times 2 \) matrices \( A \) with \( \det(A) = 1 \), so \( Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R}) \).

### 7.1.1 One-Parameter Groups of Transformations

In this section we formalize the notion of flows of vector fields mentioned in remark [1.1] The concept of flows of vector fields permeates all of physics. The classical example of magnetic fields illustrates this well. Consider for example the Earth’s magnetic field. At any point around the planet one associates a vector and a direction for the magnetic field at that point. If one picks any such point and follows in a infinitesimal trajectory along the earth magnetic field vector, one arrives at new point with corresponding field vector. Iterating the process, one obtains an integral curve on which the vector field restricted to that curve is tangential to the curve. Doing this at all points in a neighborhood of the point then gives rise to a family non-intersecting integral curves that we usually call the magnetic field lines. Magnetic field lines traced by iron filings around a
laboratory-grade magnetic sphere, give geometrical rendition of the direction of the magnetic field at any point. The converse notion of flows is also intuitively clear as shown in figure 7.1. If one has a non-intersecting family of curves on a neighborhood of a point on a manifold, one would expect that the tangent vectors to the family of curves would constitute a vector field. Students acquainted with more advanced classical physics will know that state space of a dynamical systems is equipped with a real value function $H$ called the Hamiltonian, which effectively represents the energy at each point. Hamiltonian mechanics is then formulated in terms of a symplectic structure that associates with $H$, a Hamiltonian vector field $X_H$ whose integral curves correspond the solutions of the equations of motion. For a rigorous and elegant treatment of this subject, see Abraham-Marsden [19].

7.1.4 One-Parameter Group of Diffeomorphisms

Let $U \subset M$ be an open subset of an $n$-dimensional manifold, $p \in U$, and let $I_\epsilon = (-\epsilon, \epsilon)$ with $\epsilon > 0$ be an open interval in $\mathbb{R}$. A one-parameter group of diffeomorphisms is a smooth map, 

$$\varphi_t : I_\epsilon \times U \to M$$

$$(t, p) \mapsto \varphi_t(p), \ |t| < \epsilon,$$

with the following properties. Suppose that $t, s \in \mathbb{R}$, with $|t|, |s| < \epsilon$, $|s + t| < \epsilon$, and $\varphi_s(p), \varphi_t(p), \varphi_{s+t}(p) \in U$, then

a) $\varphi_s \circ \varphi_t = \varphi_{s+t}$,

b) $\varphi_0(p) = p$ for all $p \in U$.

The map $\varphi_t$ is clearly a local diffeomorphism with inverse function given by $(\varphi_t)^{-1} = \varphi_{-t}$. Now consider a vector field $X$ such that $X_p = \varphi_t'(p)$ at each point $p = \varphi_t(p)$, as shown in figure 7.1. If $f : M \to \mathbb{R}$ is a smooth function, then the action $X$ on $f$ as a linear derivation is given by the push-forward formula 1.25

$$X_p(f) = \varphi_t'(p)(f),$$

$$= (\varphi_t) \ast \left( \frac{d}{dt} \right)(f)|_p,$$

$$= \frac{d}{dt} (f \circ \varphi_t(p))|_{t=0}$$

Conversely, if $X$ is vector field given in local coordinates $x^\mu$ in a neighborhood of a point $p$, given by,

$$X = v^\mu \frac{\partial}{\partial x^\mu}, \quad \mu = 1 \ldots n,$$

then, for a curve $\varphi_t(p)$ with initial condition that $\varphi_0(p) = p$, to have $X_p$ as a tangent vector, it must be the case that,

$$\varphi_t' = \frac{dx^\mu}{dt} \frac{\partial}{\partial x^\mu} = X.$$

Thus, we are lead to a system of first order ordinary differential equations,

$$\frac{dx^\mu}{dt} = v^\mu(x^1 \ldots x^n).$$

subject to the condition $x^\mu \circ \varphi_0(p) = v^\mu(p)$ By the existence and uniqueness theorem of solutions of such systems, for sufficiently small $\epsilon$, there exists a unique integral curve that satisfies the equation on $I_\epsilon = (-\epsilon, \epsilon)$. We conclude that smooth vector fields can be viewed as generators of infinitesimal groups of diffeomorphisms. The one-parameter group of diffeomorphisms $\varphi_t$ is also called the Flow of the vector field $X$. 

---

Fig. 7.1: Integral Curve
At this point it is worthwhile to review the notion of the push-forward of a vector field first introduced in \[1.11\]. Let \( \varphi : M \to N \) be a smooth manifold mapping and \( X \in \mathcal{X}(M) \). If \( g : N \to \mathbb{R} \) is a smooth function, the push-forward of \( X \) is a vector field \( Y = \varphi_* X \in \mathcal{X}(N) \) defined by

\[
(\varphi_* X)(g) = X(g \circ \varphi).
\]

More precisely, if \( p \in M \),

\[
Y(g)(\varphi(p)) = [X(g \circ \varphi)](p),
\]

so that,

\[
Y(g) \circ \varphi = X(g \circ \varphi) \quad \text{(7.8)}.
\]

Suppose that in addition, \( \varphi \) is a diffeomorphism, let \( \xi_t \) be a one-parameter group associated with a vector field \( X \in \mathcal{X}(M) \). Then we can push-forward to a one-parameter subgroup \( \psi_t \) associated with \( \varphi_* X \) in \( N \) given by

\[
\psi_t = \varphi \circ \xi_t \circ \varphi^{-1},
\]

as illustrated in the commuting diagram,

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & N \\
\xi_t \downarrow & & \psi_t \downarrow \\
M & \xrightarrow{\varphi} & N.
\end{array}
\]

In other words, under a diffeomorphism \( \varphi \), the integral curves of a vector field \( X \) are mapped to the integral curves of \( \varphi_* X \).

Diffeomorphisms also allow us to use the inverse of the push-forward to pullback vectors, and push-forward functions with the inverse of the pullback. Specifically, if \( f \in \mathcal{F}(M) \) and \( Y \in \mathcal{X}(N) \),

\[
\varphi^* Y(f) = (\varphi^{-1})_* Y(f) = Y(f \circ \varphi^{-1}),
\]

which we can write,

\[
\varphi^* Y(f) = Y(\varphi_* f).
\]

When \( \varphi \) is a diffeomorphism, one may extend the pullback to tensor fields. Suppose \( M \) is an \( n \)-dimensional manifold. Since \( M \) locally looks like \( \mathbb{R}^n \), on a coordinate patch on a neighborhood of point \( p \) we can pick an orthonormal basis \( \{e_1, \ldots, e_n\} \) for the tangent space, with dual basis \( \{\theta^1, \ldots, \theta^n\} \). Referring back to section \[2.2.1\], a tensor field \( T \in \mathcal{T}_s(M) \) is a section of the bundle of tensor products of the tangent and cotangent spaces. In the given basis the tensor is an expression of the form,

\[
T = T^{ij_1,\ldots,i_j} (e_i \otimes \cdots \otimes e_i \otimes \theta^{i_1} \otimes \cdots \otimes \theta^{i_j}) \quad \text{(7.13)}
\]

The tensor components are defined as,

\[
T^{ij_1,\ldots,i_j} = T(\theta^{i_1}, \ldots, \theta^{i_j}, e_{j_1}, \ldots, e_{j_n}). \quad \text{(7.14)}
\]

Let \( t \) be a tensor field on \( N \). The generalization of the pull-back \[2.70\] is given by,

\[
(\varphi^* t)_p(\theta^{i_1}, \ldots, \theta^{i_r}, e_{j_1}, \ldots, e_{j_n}) = t_{\varphi(p)}(\varphi_*^{-1} \theta^{i_1}, \ldots, \varphi_*^{-1} \theta^{i_r}, \varphi_* e_{j_1}, \ldots, \varphi_* e_{j_n}) \quad \text{(7.15)}
\]

Once again, the fancy equation can be demystified as just a generalized version of the chain rule. If the diffeomorphism is given in local coordinates by \( y^\mu = f^\mu(x^\nu) \), so that \( e_k = \partial/\partial x^k \) and \( \theta^k = dx^k \), equation \[7.15\] is the classical transformation law for tensors.
\[ (\varphi^* t)_{j_1, \ldots, j_r}^{i_1, \ldots, i_s} = \frac{\partial y^{i_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{i_s}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{j_r}}{\partial y^{j_s}} t_{k_1, \ldots, k_s}. \] (7.16)

The matrices \( \partial x^\nu / \partial y^\mu \) are allowed because \( \varphi \) is a diffeomorphism, so the Jacobians are invertible. Of course, we can pull-back (or push-forward) vectors and tensors if \( \phi \) is a local diffeomorphism of \( M \) into itself.

### 7.1.2 Lie Derivatives

#### 7.1.5 Definition

Let \( X \) and \( Y \) be vector fields in \( M \), and \( \varphi_t(p) \) be the one-parameter family of diffeomorphisms generated by \( X \). The **Lie Derivative** of a vector \( Y \) with respect to \( X \) at \( p \) is defined as,

\[
\mathcal{L}_X Y = \frac{d}{dt} (\varphi_t^{-1})_* Y \bigg|_{t=0} = \lim_{t \to 0} \frac{(\varphi_t^{-1})_* Y_{\varphi_t(p)} - Y_p}{t}.
\] (7.17)

The definition says that rate of change of \( Y \) along the integral curve of \( X \) is obtained by dragging back \( Y \) with the pullback from a point near \( p \), and measuring the infinitesimal change compared to \( Y \) at \( p \). If \( f \in \mathcal{F}(M) \) is a smooth function, we define

\[
\mathcal{L}_X f = \frac{d}{dt} (\varphi_t)^* f \bigg|_{t=0} = \lim_{t \to 0} \frac{f \circ \varphi_t(p) - f(p)}{t},
\] (7.18)

\[
= X_p(f),
\] (7.19)

\[
= df(X)_{|p}.
\] (7.20)

which as expected, is just the directional derivative.

#### 7.1.6 Theorem

\( \mathcal{L}_X Y = [X, Y] \).

**Proof:** Consider the function \( f \circ \varphi_t \). The Taylor expansion about the point \( p \) gives

\[
f \circ \varphi_t = (f \circ \varphi_t(p))(0) + t \frac{d}{dt} [f \circ \varphi_t(p)] \bigg|_{t=0} + \mathcal{O}(t^2),
\]

\[
= f(p) + tX_p + \mathcal{O}(t^2).
\]

Let \( f \in \mathcal{F}(M) \). The definition of Lie derivative is equivalent to

\[
\mathcal{L}_X Y = \lim_{t \to 0} \frac{Y_p - (\varphi_t)_* Y}{t}.
\]
Applying this to \( f \) at \( p \), we get,

\[
\mathcal{L}_{X_p} Y(p) = \lim_{t \to 0} \frac{Y_p(f) - Y_{\varphi_t^{-1}(p)}(f \circ \varphi_t)}{t} = \lim_{t \to 0} \frac{Y_p(f) - Y_{\varphi_t^{-1}(p)}(f(p) + tX_p)}{t} = \lim_{t \to 0} \left[ \frac{Y_p(f) - Y_{\varphi_t^{-1}(p)}(f)}{t} - Y_{\varphi_t^{-1}(p)}(X)(f) \right],
\]

\[
= X_p(Y(f)) - Y_p(X(f)),
\]

\[
\mathcal{L}_{X_p} Y(p) = [X,Y]_p(f).
\]

### 7.1.7 Theorem
If \( \varphi : M \to N \) is a diffeomorphism and \( f \in \mathcal{F}(M) \) is any smooth function on \( M \), then,

\[
\mathcal{L}_{\varphi_*} X(\varphi_* f) = \varphi_*(\mathcal{L}_X f) \tag{7.21}
\]

**Proof:**

\[
\mathcal{L}_{\varphi_*} X(\varphi_* f)|_{\varphi(p)} = \varphi_* X|_{\varphi(p)}(f \circ \varphi^{-1}),
\]

\[
= X(f \circ \varphi^{-1} \circ \varphi)(p),
\]

\[
= X(f)(p),
\]

\[
= X(f) \circ \varphi^{-1}|_{\varphi(p)},
\]

\[
\mathcal{L}_{\varphi_*} X(\varphi_* f) = \varphi_*(\mathcal{L}_X f).
\]

### 7.1.8 Theorem
If \( \varphi : M \to N \) is a diffeomorphism and \( X, Y \in \mathcal{X}(M) \) are vector fields on \( M \), then,

\[
\mathcal{L}_{\varphi_*} X(\varphi_* Y) = \varphi_*(\mathcal{L}_X Y); \quad \text{that is,}
\]

\[
[\varphi_* X, \varphi_* Y] = \varphi_* [X, Y]. \tag{7.22}
\]

**Proof:** Let \( g \in \mathcal{F}(N) \). By equation [7.8](#), we have to show that \([\varphi_* X, \varphi_* Y](g) \circ \varphi = [X, Y](g \circ \varphi)\).

We have,

\[
[\varphi_* X, \varphi_* Y](g) \circ \varphi = [\varphi_* X, \varphi_* Y],
\]

\[
= (\varphi_* X(\varphi_* Y(g)) \circ \varphi - (\varphi_* Y(\varphi_* X(g)) \circ \varphi,
\]

\[
= X(\varphi_* Y(g) \circ \varphi) - Y(\varphi_* X(g) \circ \varphi),
\]

\[
= X(Y(g \circ \varphi)) - Y(X(g \circ \varphi)),
\]

\[
= [X, Y](g \circ \varphi).
\]

Since the formula [7.15](#) shows that one can pull-back tensors in the case of a diffeomorphism, the Lie derivative can be extended to a linear derivation on the full tensor algebra.

### 7.1.9 Definition
Let \( X \) be a vector fields in \( M \), and \( \varphi_t(p) \) be the one-parameter family of diffeomorphisms generated by \( X \), let \( T \in \mathcal{T}_x \) be a tensor field. The **Lie Derivative** of the tensor
T with respect to X at p is defined as,

\[ \mathcal{L}_X T = \frac{d}{dt}(\phi_t^{-1})_p T \bigg|_{t=0} \]

or,

\[ \frac{d}{dt} \phi_t T = \phi_t^* \mathcal{L}_X T. \] (7.23)

The operator \( \mathcal{L}_X : \mathcal{T}_s^r \to \mathcal{T}_s^r \) is clearly linear and satisfies Leibnitz rule. We have,

\[ \begin{align*}
    \mathcal{L}_X f &= X(f), \\
    \mathcal{L}_X [Y] &= [X,Y], \\
    \mathcal{L}_X (fY) &= \mathcal{L}_X f Y + f \mathcal{L}_X Y, \\
    \mathcal{L}_X (T_1 \otimes T_2) &= \mathcal{L}_X T_1 \otimes T_2 + T_1 \otimes \mathcal{L}_X T_2.
\end{align*} \]

7.1.10 Theorem. Let \( f \in \mathcal{F}, X, Y \in \mathcal{X}, \) and \( T_1, T_2 \) be tensors. Then

a) \( \mathcal{L}_X f = X(f), \)

b) \( \mathcal{L}_X [Y] = [X,Y], \)

c) \( \mathcal{L}_X (fY) = \mathcal{L}_X f Y + f \mathcal{L}_X Y, \)

d) \( \mathcal{L}_X (T_1 \otimes T_2) = \mathcal{L}_X T_1 \otimes T_2 + T_1 \otimes \mathcal{L}_X T_2. \)

Leibnitz rule also extends to contractions (See 2.22). So, if \( \omega \) is a one-form, we have,

\[ \mathcal{L}_X \langle \omega | Y \rangle = \langle \mathcal{L}_X \omega | Y \rangle + \langle \omega | \mathcal{L}_X Y \rangle, \]

so, this gives the Lie derivative of a one form as,

\[ \begin{align*}
    \langle \mathcal{L}_X \omega | Y \rangle &= \mathcal{L}_X \langle \omega | Y \rangle - \langle \omega, \mathcal{L}_X Y \rangle, \\
    \mathcal{L}_X \omega(Y) &= \mathcal{L}_X \omega(Y) + \omega(\mathcal{L}_X Y). \end{align*} \] (7.24)

Now that we know the Lie derivative of functions, vector fields, and one forms, it is a straight-forward exercise to use induction on tensor products, to find the formula for the Lie derivative of any tensor field \( T \in \mathcal{T}_r^s. \) The formula is,

\[ \mathcal{L}_X [T(\omega^1, \ldots, \omega^r, X_1, \ldots X_s)] = \mathcal{L}_X T(\omega^1, \ldots, \omega^r, X_1, \ldots X_s) \]

\[ + \sum_{i=1}^r T(\omega^1, \ldots, \mathcal{L}_X \omega_i, \ldots, \omega^r, X_1, \ldots X_s) \]

\[ + \sum_{i=1}^s T(\omega^1, \ldots, \omega^r, X_1, \ldots, \mathcal{L}_X X_i, \ldots X_s) \] (7.25)

7.1.11 Theorem If \( X \in \mathcal{X}(M) \) and \( \omega \in \Omega^k(M), \eta \in \Omega^l(M), \) then,

\[ \begin{align*}
    d\mathcal{L}_X \omega &= \mathcal{L}_X d\omega, \\
    \mathcal{L}_X (\omega \wedge \eta) &= \mathcal{L}_X \omega \wedge \mathcal{L}_X \eta. \end{align*} \] (7.26)

Proof: Let \( \phi_t \) be the one-parameter flow of \( X. \) By definition,

\[ \mathcal{L}_X \omega(p) = \frac{d}{dt}(\phi_t^* \omega(p))|_{t=0}. \]

The theorem follows immediately from the fact that \( d \) is linear and so, it commutes with \( d/dt, \) plus the already established formulas \( \phi_t^* d\omega = d\phi_t^* \omega, \) and \( \phi_t^* (\omega \wedge \eta) = \phi_t^* \omega \wedge \phi_t^* \eta. \)

We recall the definition if the interior product 2.23. Let \( M \) be a manifold, \( X, X_1, \ldots, X_k \in \mathcal{X}(M) \) and \( \omega \in \Omega^{k+1}(M), \) then,

\[ i_X \omega(X_1, \ldots, X_k) = \omega(X, X_1, \ldots, X_k). \] (7.28)

By convention, we set \( i_X f = 0. \)
### 7.1.12 Theorem

Let \( \varphi : M \to N \) be a diffeomorphism. Then,

\[
\varphi^*(i_X \omega) = i_{\varphi^* X} \varphi^* \omega.
\]  

**Proof:** Let \( \{Y = \varphi^* X, Y_i = \varphi^* X_i, i = 1 \ldots k\} \in \mathcal{X}(M) \). We have

\[
i_Y \varphi^* \omega(Y_1, \ldots, Y_k) = \varphi^* \omega(Y, Y_1, \ldots, Y_k),
\]

\[
= \omega(\varphi_*, Y_1, \ldots, \varphi_* X),
\]

\[
= \omega(X, X_1, \ldots, X_k),
\]

\[
i_{\varphi_* X} \varphi^* \omega(Y_1, \ldots, Y_k) = \varphi^* i_X \omega(Y_1, \ldots, Y_k),
\]

If \( \varphi_i \) is a local diffeomorphism on \( M \) the theorem can be restated as,

\[
\varphi_i^* \circ i_X = i_{\varphi_i^* X} \circ \varphi_i^*.
\]  

### 7.1.13 Theorem

Let \( i_X : \Omega^k(M) \to \Omega^{k-1}(M) \), be the interior product and let \( \alpha \in \Omega^k(M), \beta \in \Omega^l(M), f \in F(M) \). Then

a) \( i_X (\alpha \wedge \beta) = i_X \alpha \wedge \beta + (-1)^k \alpha \wedge i_X \beta \),

b) \( i_X \alpha = f i_X \alpha \),

c) \( i_X df = \mathcal{L}_X f = X(f) \)

**Proof:** The proof of part (a) involves some combinatorics arising from the definition of the wedge product [2.62]. We leave out the somewhat messy details. Part (b) follows immediately from the multilinearity of \( \alpha \) and the definition of the interior product. Part (c) is trivial. We have \( i_X df = df(X) = X(f) \). The interior product [7.28], the intrinsic exterior derivative [6.39] and the Lie derivative [7.25] are related by the following formula.

### 7.1.14 Theorem (H. Cartan)

\[
d \circ i_X + i_X \circ d = \mathcal{L}_X.
\]  

**Proof:** We prove (a) by induction. First, we verify that the formula is true for zero-form \( f \) and a 1-form \( \omega \).

\[
(d i_X + i_X d) f = d i_X f + i_X df,
\]

\[
= i_X df,
\]

\[
= df(X) = X(f) = \mathcal{L}_X f.
\]

Let \( \omega \) be a one form and \( X, Y \in \mathcal{X}(M) \).

\[
(d i_X + i_X d) \omega(Y) = (d (i_X \omega))(Y)) + i_X(d \omega)(Y),
\]

\[
= d(\omega(X))(Y) + d\omega(X,Y),
\]

\[
= Y(\omega(X)) + X(\omega(y)) - \omega([X,Y]),
\]

\[
= X(\omega(Y)) - \omega(\mathcal{L}_X Y),
\]

\[
= \mathcal{L}_X \omega(Y).
\]

Suppose the proposition is true for \( k \)-forms \( \omega_i \). A \( k + 1 \)-form can be written as \( df_i \wedge \omega_i \) that for simplicity we write as \( df \wedge \omega \). By induction hypothesis,

\[
d i_X \omega + i_X d \omega = \mathcal{L}_X \omega
\]
CHAPTER 7. GROUPS OF TRANSFORMATIONS

Using Leibnitz rule, and the properties \( d \circ d = 0 \), \( i_X f = 0 \), \( i_X df = \mathcal{L}_X f \), we compute,

\[
(d i_X + i_X d)(df \wedge \omega) = d(i_X(df \wedge \omega) + i_X d(df \wedge \omega)), \\
= d(\mathcal{L}_X f \wedge \omega + \mathcal{L}_X f \wedge df - (\mathcal{L}_X f \wedge \omega) - (i_X df \wedge d \omega - df \wedge i_X d \omega)), \\
= \mathcal{L}_X df \wedge \omega + \mathcal{L}_X f \wedge df - \mathcal{L}_X df \wedge \omega + df \wedge (i_X \omega + i_X df), \\
= \mathcal{L}_X df \wedge \omega + df \wedge \mathcal{L}_X \omega, \quad \text{(using the induction hypothesis and } \mathcal{L}_X d = d \mathcal{L}_X), \\
= \mathcal{L}_X (df \wedge \omega),
\]

which is what we wanted to establish. The following diagram, which is reminiscent of chain-complexes in singular homology, helps to visualize this most elegant result, sometimes called Cartan’s magic formula.

\[
\begin{array}{cccccc}
\cdots & d & \Omega^{k-1} & \mathcal{L}_X & d & \Omega^k & \mathcal{L}_X & d & \Omega^{k+1} & \mathcal{L}_X & d & \cdots \\
\end{array}
\]

Fig. 7.2: Cartan’s Magic Formula

7.1.15 Poincaré Lemma If \( M \) is manifold which is smoothly contractible to a point, a closed form \( \omega \) is exact.

**Proof:** We present a proof in the special case in which the manifold is a ball \( B^n \in \mathbb{R}^{n+1} \) centered at the origin. This is the alternative proof that appears in Abraham-Marsden [19]. For all \( p \in B^n \), and \( 0 < t \leq 1 \), let \( \phi_t \) be the one-parameter group of diffeomorphisms defined by \( \phi_t(p) = tp \). This kind of map is an example of a deformation retract. When \( t = 1 \), the map is the identity map and as \( t \to 0 \), the sphere continuously shrinks to a point. Let \( X_t \) be the vector field \( X_t(p) = p/t \). We have,

\[
\frac{d}{dt} \phi_t(p) = X_t(\phi_t(p)) = p,
\]

so \( X_t \) is the tangent vector field of the one-parameter family of curves. Let \( \omega \) be a closed \( k \)-form in \( B^n \). By the definition of the Lie derivative and Cartan’s magic formula [7.31] we have

\[
\frac{d}{dt}(\phi^*_t \omega) = \phi^*_t(\mathcal{L}_X \omega), \\
= \phi^*_t(di_X(\omega) + i_X(\omega)), \\
= \phi^*_t(di_X(\omega)),
\]

Integrating from a small value \( \epsilon > 0 \) to 1, we get,

\[
\phi^*_t \omega|_\epsilon = \int_\epsilon^1 \phi^*_t(di_X(\omega)) \, dt, \\
= d \int_\epsilon^1 \phi^*_t(i_X(\omega)) \, dt,
\]

Taking the limit as \( \epsilon \to 0 \) and recalling that at \( t = 1 \), \( \phi_t \) is the identity map, we get

\[
\omega = d \beta, \quad \beta = \int_0^1 \phi^*_t(i_X(\omega)) \, dt. \quad (7.32)
\]
Although the theorem is proved, it is helpful to find a more explicit formula for the $(k-1)$-form $\beta$, using the definitions of the push-forward and the interior product. Let $\{e_1, \ldots, e_{k-1}\}$ be part of a set of basis vectors and $p$ the position vector at the point $p$. Then

$$\beta_p(e_1, \ldots, e_{k-1}) = \int_0^1 \phi_t^* i_p \omega(e_1, \ldots, e_{k-1}) \, dt,$$

$$= \int_0^1 \omega(\phi_t(p), \phi_* e_1, \ldots, \phi_* e_{k-1}) \, dt,$$

$$= \int_0^1 \omega_t(p, te_1, \ldots, te_{k-1}) \, dt,$$

so,

$$\beta_p = \int_0^1 t^{k-1} \omega_t(p, e_1, \ldots, e_{k-1}) \, dt. \quad (7.33)$$

The more general theorem is computationally more complicated but the idea is essentially the same. One seeks a one parameter group of diffeomorphisms $\phi_t : M \times [0, 1]$ such that $\phi_1$ is the identity and $\phi_0$ is constant map. One then finds a linear map $h : \Omega^k \to \Omega^{k-1}$ such that the form $h \omega = \beta$ satisfies,

$$\phi_t^* \omega - \phi_0^* \omega = d(h\omega) + h(d\omega)$$

This general form of the theorem is roughly for the novice. In Abraham-Marsden, the form $h \omega = \beta$ just pops out of nowhere, so without the alternative proof, the whole process appears rather mysterious. Spivak provides some motivation for finding $h \omega$ by first treating the case where $\omega$ is a one form. In his book Calculus on Manifolds he carries out the full computation when $M$ is a star-shaped region in $\mathbb{R}^n$. Either way, the computation is more difficult. The Poincaré Lemma is often be stated by saying that in a manifold $M$, a closed form is locally exact; that is, given a closed form on an open set $U \subset M$, then for each point $p \in U$, there exists a ball $B \subset M$ centered at $p$ in which the form is exact.

### 7.1.16 Theorem

$$[\mathcal{L}_X, i_Y] \equiv \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X = i_{[X,Y]}, \quad (7.34)$$

$$[\mathcal{L}_X, \mathcal{L}_Y] \equiv \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]}, \quad (7.35)$$

**Proof:** Let $\omega$ be a $k$-form and $Y, X, X_1, \ldots, X_{k-1}$ be vector fields. The proof is by direct computation using the formula for Lie derivatives \[7.25\] and the definition of the interior product \[7.28\].

$$(i_Y \mathcal{L}_X \omega)(X_1, \ldots, X_{k-1}) = (\mathcal{L}_X \omega)(Y, X_1, \ldots, X_{k-1}),$$

$$= \mathcal{L}_X(\omega(Y, X_1, \ldots, X_{k-1})) - \sum_{i=1}^{k-1} \omega(Y, X_1, \ldots, [X_i, X_{i+1}], \ldots X_{k-1})$$

$$- \omega([X, Y], X_1, \ldots, X_{k-1}), \quad \text{(term not included in the sum)}$$

$$= \mathcal{L}_X i_Y (X_1, \ldots, X_{k-1}) + \sum_{i=1}^{k-1} i_Y (X_1, \ldots, [X_i, X_{i+1}], \ldots X_{k-1})$$

$$- i_{[X,Y]} (X_1, \ldots, X_{k-1}),$$

$$= (\mathcal{L}_X i_Y - i_{[X,Y]}) \omega(X_1, \ldots, X_{k-1}).$$

We leave the second part as an exercise, using the formula of Cartan \[7.31\].
7.2 Lie Algebras

7.2.1 Definition A vector space $g$ over a field $F$ (we are interested in fields $F = \mathbb{R}$ or $\mathbb{C}$) is called a **Lie algebra** if there exists an operation $[\ ,\ ] : g \times g \rightarrow g$ (called the Lie Bracket), such that,

a) $[\ ,\ ]$ is $F$-bilinear,

b) $[X,Y] = -[Y,X]$, for all $X,Y \in g$,

c) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$,  **Jacobi Identity**

7.2.2 Definition A **Lie Subalgebra** $\mathfrak{h}$ of a Lie algebra $g$ is a subspace that is closed under the Lie Bracket.

7.2.3 Theorem The vector space $\mathfrak{X}(M)$ together with the operation $\mathcal{L}_X Y = [X,Y]$ gives the space the structure of a Lie Algebra.

The Jacobi identity follows directly from equation 7.35 applied to vector fields, but a much more elementary proof follows directly from the definition of the bracket of two vector fields and the property of vector fields being linear derivations on the space of functions. The details of the proof have already appeared in theorem 4.4.2.

Lie subalgebras are intricately connected with the theory of submanifolds $N \subset M$. The can be used to generalized the idea of integral curves. Let $M$ be an $n$ dimensional manifold, and $p$ be a point $p \in M$. A $k$-dimensional distribution at $p$ is a subset $D_p \subset T_p M$. Let $\{X_1, X_2, \ldots, X_k\}$ be a set of linearly independent vector fields in a neighborhood $U$ of $p$ which constitutes a basis for $D_q$, $q \in U$. If these can be chosen in a smooth way, $D$ is called a $C^\infty$ distribution.

7.2.4 Definition Let $\{X_1, X_2, \ldots, X_k\} \mid p \in U$ span a distribution $D$. The distribution is **integrable** if the vectors form a subalgebra of the Lie algebra of vectors fields in $M$. That is, there exist $C^\infty$ functions $C^i_{\ i,j}$, such that

$$[X_i, X_j] = C^k_{\ i,j} X_k.$$ 

7.2.5 Definition A distribution $D$ arises from a **foliation** of $M$, if for each point $p \in M$, there exists a $k$-dimensional local submanifold $N$ of $M$, containing $p$ with

$$i_*(T_p N) = D_p, \text{ for all } p \in U$$

where $i : N \rightarrow M$ is the inclusion map.

Locally, foliations look like layers of local submanifolds, called the Leaves of the foliation. Perhaps the most famous foliation is the Reeb foliation of $S^3$ that locally looks like onion layers formed a sock pushed infinitely into itself. A standard method to treat the field equations in general relativity is to think of a four dimensional Lorentzian manifold as a $3+1$ manifold, foliated by 3-dimensional spatial surfaces evolving in time.

The main result on distributions is,

7.2.6 Theorem (Frobenius) A distribution $D$ is integrable if and only if, it arises from a foliation. An alternative formulation of the Frobenius integrability theorem can be stated in terms of differential forms. Let $\Omega^*(M)$ be the graded ring on smooth differential forms. The subring $\mathcal{I}(D)$ of all the forms $\omega$ that annihilate $D$, namely, for all $X_1, \ldots, X_k$ in $D$

$$\omega(X_1, \ldots, X_k) = 0,$$ 

generate an ideal $\mathcal{I}(D)$. If $\{e_1, \ldots, e_k\}$ is an orthonormal frame in $D$, then the dual forms $\{\theta^1, \ldots, \theta^k\}$ span $\mathcal{I}(D)$. The differential forms version of the Frobenius says that $D$ is integrable if for every
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ω ∈ ℐ(𝐷), we have 𝑑ω ∈ ℐ(𝐷); that is, the differential ideal is closed under exterior derivatives. More specifically, there exist forms 𝛼^j_k, such that

\[ 𝑑𝜃^j = 𝛼^j_k ∧ 𝜃^k ∈ ℐ(𝐷). \]

The connection between the two versions of the theorem is achieved through the structure constant formula,

\[ 𝑑𝜃^j = \frac{1}{2} C^j_{jk} 𝜃^j ∧ 𝜃^K \]

which prove in equation (7.59). If we extend the orthonormal frame spanning 𝐷 to an orthonormal frame \{𝑒_1, \ldots, 𝑒_k, 𝑒_{k+1}, \ldots, 𝑒_n\}, with dual forms \{𝜃^1, \ldots, 𝜃^k, 𝜃^{k+1}, \ldots, 𝜃^n\} then the integral submanifolds are defined by the (pfaffian) system,

\[ 𝜃^m = 0, \quad m = k + 1, \ldots, n. \]

The theorem guarantees the existence of local coordinates \{𝑥^1, \ldots, 𝑥^n\} about \( p ∈ U \), with tangent vectors \{∂/∂𝑥^1, \ldots, ∂/∂𝑥^k\}_p spanning 𝐷_p, and with the dual forms

\[ \{dx^{k+1}, \ldots, dx^n\}_p \]

annihilating 𝐷. The forms \{𝜃^{k+1}, \ldots, 𝜃^n\} can then be written as linear combinations of the coordinate one-forms above. The sets

\[ N_p = \{x^{k+1}|_p = a^{k+1}, \ldots, x^n|_p = a^n\}, \]

where the a’s are constants, are integral submanifolds of the distribution 𝐷_p.

We do not present a proof of this very important theorem in this rather perfunctory treatment of the topic. Instead, we refer the reader to classic textbooks such as [19] or [31]. The theorem is the starting point to the deep subject of Lie groups of symmetries of partial differential equations and prolongation theory.

Given any Lie group \( G \), we can construct an associated Lie algebra \( g \). Given a group element \( g ∈ G \), we define the Left and the Right translation maps \( L_g, R_g : G → G \) by,

\[ L_g(g_0) = gg_0, \quad \text{(7.37)} \]
\[ R_g(g_0) = g_0g, \quad \text{for all } g_0 ∈ G. \quad \text{(7.38)} \]

Hereafter, for every statement we make about Left translation, there is a corresponding statement about right translation. The map \( L_g \) is a diffeomorphism with inverse given by \((L_g)^{-1} = L_{g^{-1}}\).

Fig. 7.3: Left Invariant Vector Field.
7.2.7 Definition A vector field $X \in \mathcal{X}(G)$ is called **Left Invariant** if for all $g \in G$, we have,

$$L_g X = X.$$  \hspace{1cm}(7.39)

More specifically, if $X_{g_0}$ is the tangent vector at $g_0$, then,

$$L_{g_0} X_{g_0} = X_{L_{g_0}(g_0)} = X_{g_0},$$

is the tangent vector at $gg_0$.

7.2.8 Definition Let $\mathfrak{g} = \mathcal{L}(G)$ be the set of all left-invariant vector fields in $G$. Then $\mathfrak{g}$ has the structure of a Lie algebra.

**Proof:** Let $X, Y \in \mathfrak{g}$. Then by equation 7.22 we have,

$$L_g [X, Y] = [L_g X, L_g Y],$$

$$= [X, Y],$$

so $[X, Y] \in \mathfrak{g}$. Thus, the set of left-invariant vector fields is closed under Lie brackets and hence it is a Lie subalgebra of the Lie algebra of all vector fields $\mathcal{X}(G)$.

Let $T_e G$ be the tangent space at the identity of a Lie Group. For every tangent vector $X_e \in T_e G$ we can generate a vector field $X$ by simply defining the value of the vector field at any point $g \in G$, to be the tangent vector,

$$X_g = L_{g^{-1}} X_e.$$  \hspace{1cm}(7.40)

The vector field $X$ so defined is almost tautologically left invariant. Indeed, if $g_0 \in G$,

$$L_{g_0} X_{g_0} = L_{g_0} (L_{g_0^{-1}} X_e),$$

$$= L_{g_0^{-1}} X_e.$$

For each left invariant vector field $X$ with value $X_g$, there is a unique tangent vector $X_e = L_{(g^{-1})} X_g$ at the identity, so we have the following theorem,

7.2.9 **Theorem** The tangent vector space at the identity $T_e G$ of a Lie group is isomorphic to the Lie algebra of left-invariant vector fields $\mathfrak{g} = \mathcal{L}(G)$.

We now consider the behavior of left invariant vector fields under mappings. Let $\phi : G \rightarrow H$ be a homomorphism between two Lie groups $G$ and $H$ with identity elements $e$ and $e'$ respectively, and push-forward map $\phi_\ast : T_e G \rightarrow T_{e'} H$. Consider a tangent vector $X_e \in T_e G$ generating a left

\[ \begin{array}{ccc}
X_e & \in & T_e G \\
\downarrow & & \downarrow \\
g & \in & G \\
\downarrow & & \downarrow \\
L_g & \phi & \rightarrow H \\
\downarrow & & \downarrow \\
G & \phi & \rightarrow H \\
\end{array} \]

Fig. 7.4: Lie Algebra Homomorphism

invariant vector field $X$. So, if $g \in G$, $L_g X = X$. Denote by $Y$ the left invariant vector field in $H$ whose value at the identity is $Y_{e'} = \phi_\ast X_e$. As shown in figure 7.4 we have,

$$\phi \circ L_g = L_{\phi(g)} \circ \phi.$$
Then, for the push-forward of the vector field $X$ at $g$, we have,

$$
\phi_*X_g = \phi_*L_g X_e,
$$

$$
= (\phi \circ L_g)_* X_e,
$$

$$
= (L_{\phi(g)} \circ \phi)_* X_e,
$$

$$
= L_{\phi g} \phi_* X_e,
$$

$$
= L_{\phi g} Y_{e'},
$$

$$
= Y_{\phi(g)}.
$$

Hence, the push-forward of a left-invariant vector field $X$ is a left-invariant vector field $Y$. Since the push-forward preserves brackets, the map $\phi_*$ is a Lie algebra homomorphism, that is,

$$
\phi_*(aX^1 + bX^2) = a\phi_*X^1 + b\phi_*X^2,
$$

$$
\phi_*[X^1, X^2] = [\phi_*X^1, \phi_*X^2]. \tag{7.41}
$$

### 7.2.1 The Exponential Map

Let $\phi : \mathbb{R} \to G$ be a smooth Lie group homomorphism, and let $A = \phi'(0) \in T_e G$ be a tangent vector at the identity. Such homomorphism is called an one-parameter subgroup of $G$. Let $G = GL(n, \mathbb{R})$. Consider the case where $G = GL(n, \mathbb{R})$. This is a matrix group, so a tangent vector at the identity is a matrix; this is the reason why we changed the notation from $X$ to $A$. Since

$$
\phi(s + t) = \phi(s) \circ \phi(t),
$$

evaluating the derivative at $s = 0$ gives,

$$
\frac{d}{dt} \phi(t) = \phi'(0) \cdot \phi(t),
$$

$$
= A\phi(t), \quad \text{with,}
$$

$$
\phi(0) = I. \tag{7.42}
$$

Here, the dot is matrix multiplication. By analogy to the one dimensional case, the solution of this differential equation is,

$$
\phi(t) = e^{At}, \tag{7.43}
$$

where the exponential of a matrix is defined as the power series,

$$
e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots + \frac{A^n}{n!} + \ldots. \tag{7.44}
$$

We see that the one-parameter family of matrices $\phi(t) = e^{At}$ is the integral curve of the vector $A$. This leads to the following definition,

### 7.2.10 Definition

For any Lie group $G$, we define the exponential map

$$
\exp : \mathfrak{g} \to G \tag{7.45}
$$

as follows. Let $A \in T_e G$, and let $\phi : \mathbb{R} \to G$ be the unique homomorphism such that $\phi'(0) = A$. Then,

$$
\exp(A) \equiv e^A = \phi(1)
$$

Clearly,

$$
e^{(s+t)A} = e^{sA} e^{tA},
$$

$$
e^{-tA} = (e^{tA})^{-1}$$
The map \( t \mapsto e^{tA} \) is a local diffeomorphism from \( T_e G \) to \( G \). The maximal extension of the integral curve is the one-parameter subgroup of \( G \) indicated at the beginning of this subsection. The converse is also true. Any one-parameter subgroup of \( G \) is generated by a map \( t \mapsto e^{tA} \), where \( A \in T_e G \). Since \( g \cong L(G) \), there is a one-to-one correspondence between the Lie algebra of a Lie group and the one-parameter subgroups. Roughly speaking, the exponential map yields a neighborhood of \( e \in G \), which is filled by one-parameter subgroups emanating from \( e \) by the integral curves of tangent vectors \( A \in T_e G \). In fact if the diagram in \([7.4]\) and the result in equation \([7.41]\) are applied to the homomorphism \( \phi : \mathbb{R} \to T \) with the condition that, 

\[
\phi_*(\frac{d}{dt}|_0) = A,
\]

then \( \phi \) is a Lie algebra homomorphism. The left-invariant vector field generated by \( A \) is is the vector field tangent to the unique integral curve given by the map \( t \mapsto e^{tA} \).

We define,

\[
\ln(e^A) = A, \quad e^{\ln(1+A)} = 1 + A,
\]

wherever the formal power series converges. If \( A \) and \( B \) are two matrices near 0, we can study the behavior of

\[
\ln(e^A e^B) = \ln[(1 + A + \frac{A^2}{2^1} + \ldots)(1 + B + \frac{B^2}{2^1} + \ldots)].
\]

If we only retain the first order terms, we get,

\[
\ln(e^A e^B) \approx \ln(1 + A + B) = A + B.
\]

If we wish to compute the quadratic terms, we formally multiply the power series for the exponentials, and then use the formal series expansion for \( \ln(1 + X) = X - \frac{1}{2}X^2 + \ldots \). However we need to be careful with the fact that matrix multiplication does not commute. The result is,

\[
\ln(e^A e^B) = \ln \left( 1 + A + B + \frac{A^2}{2} + AB + \frac{B^2}{2} + \ldots \right),
\]

\[
= \left( A + B + \frac{A^2}{2} + AB + \frac{B^2}{2} \right) - \frac{1}{2} \left( A + B + \frac{A^2}{2} + AB + \frac{B^2}{2} \right)^2 + \ldots
\]

\[
= A + B + \frac{A^2}{2} + AB + \frac{B^2}{2} - \frac{1}{2} (A + B + \ldots)^2 + \ldots,
\]

\[
= A + B + \frac{1}{2}[A,B] + \ldots
\]

The full expansion is called the Campbell-Baker-Hausdorff (CBH) formula. The terms up to third order are,

\[
\ln(e^A e^B) = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] + \ldots
\]

All the terms of order two or higher are expressible in terms of brackets, so

\[
[A,B] = 0 \Rightarrow e^A e^B = e^{A+B}.
\]

Exponentiating the CBH formula \([7.46]\) and keeping only the terms up to order 2, we can establish the following formulas,

\[\text{Theorem} \]

\[
e^{tA} e^{tB} = \exp \left\{ t(A + B) + \frac{1}{2}t^2[A,B] + O(t^3) \right\},
\]

\[
e^{-tA} e^{-tB} e^{tA} e^{tB} = \exp \left\{ t^2[A,B] + O(t^3) \right\},
\]

\[
e^{tA} e^{tB} e^{-tA} = \exp \left\{ tB + t^2[A,B] + O(t^3) \right\}.
\]
7.2. LIE ALGEBRAS

The first of these formulas follows immediately from the CBH formula. The other two require more manipulation of exponential and logarithmic expansions along the same lines as in the computations leading to 7.46. A complete proof of this theorem can be found in [31]. Next, we prove that the exponential map is natural with respect to the push-forward.

7.2.12 Theorem Let \( \phi : G \to H \) be a smooth homomorphism between two Lie groups \( G \) and \( H \). Then the exponential loop in following diagram commutes,

\[
\begin{array}{ccc}
T \mathbb{R} & \xrightarrow{\alpha_*} & T_e G \\
\downarrow \exp & & \downarrow \exp \\
\mathbb{R} & \xrightarrow{\alpha} & G \\
\end{array}
\]

That is,

\[ \phi \circ \exp = \exp \circ \phi_* \quad \text{(7.48)} \]

Proof: Let \( A \in T_e G \) and let \( \alpha : \mathbb{R} \to G \) be a one parameter subgroup of \( G \) given by \( \alpha(t) = e^{tA} \). Then, \( A = \alpha'(0) = \alpha_*(\frac{d}{dt}|_0) \). Define \( \psi : \mathbb{R} \to H \) by the composition \( \psi = \phi \circ \alpha \). That is,

\[ \psi(t) = \phi(e^{tA}). \]

We have,

\[
\phi(s + t) = \phi(e^{(s+t)A}),
\]

\[
= \phi(e^{sA}e^{tA}),
\]

\[
= \phi(e^{sA})\phi(e^{tA}),
\]

\[
= \phi(s)\phi(t),
\]

so \( \phi \) is a one parameter subgroup of \( H \). The rest of the proof amounts to untangling the definition of the push-forward as shown in equation 1.25. Suppose that \( B \in T_e H \) such that \( \psi(t) = e^{tB} \). That means that \( B = \psi'(0) \). We consider the action of this tangent vector on an arbitrary smooth function \( f : H \to \mathbb{R} \). We have,

\[
B(f) = \psi'(0)(f) = \psi_*(\frac{d}{dt}|_0)(f) = \frac{d}{dt}(f \circ \psi)|_0 = \frac{d}{dt}(f \circ \phi \circ \alpha)|_0 = \alpha'(0)(f \circ \phi) = A(f \circ \phi) = \phi_*A(f).
\]

We conclude that \( B = \phi_*A \). Thus, setting \( t = 1 \), \( \psi(t) = e^{tB} \) can be rewritten

\[ \phi(e^A) = e^{\phi_*A}, \quad \text{or} \quad \phi(\exp(A)) = \exp(\phi_*A) \]

which is what we wanted to prove.
7.2.2 The Adjoint Map

7.2.13 Definition Let $G$ be a Lie group and let $g_0 \in G$. For each $g \in G$, consider the conjugation automorphism $C_g = L_g R_g^{-1} : G \to G$ given by $g_0 \mapsto gg_0g^{-1} = L_g R_g^{-1}(g_0)$. We define the Adjoint Map $Ad_g : g \to g$ by the linear transformation,

$$Ad_g = C_{g^*} = (L_g R_g^{-1})^*.$$  (7.49)

7.2.14 Definition Denote by $Aut(V)$ automorphism group of all invertible linear transformations of some vector space $V$ over $\mathbb{R}$ (or $\mathbb{C}$). If $V$ has dimension $n$, then $Aut(V)$ is isomorphic to the matrix group $GL(n, \mathbb{R})$. A homomorphism,

$$\phi : G \to GL(n, \mathbb{R})$$

from a Lie group to $GL(n, \mathbb{R})$ is called a (real) representation of order $n$. If the homomorphism is 1-1, the representation is called faithful. If $W$ is subspace of $V$, and $\phi(g)v \in W$ for all $v \in W$ and $g \in G$, we say that $W$ is an invariant subspace. A representation with no non-trivial invariant subspace is called irreducible. Same idea applies to Lie algebras.

Since $Ad_{g'} = Ad_g \circ Ad_{g'}$, the Adjoint is a homomorphism, $Ad : G \to Aut(g)$ is called the Adjoint representation. The kernel is the center of the group $G$.

By equation [7.48] if $X \in g$ we have $exp(C_\ast X) = C(exp(X))$, that is

$$e^{Ad_g(X)} = g(e^X)g^{-1}. \quad (7.50)$$

Consider the case $G = GL(n, \mathbb{R})$. We evaluate the Adjoint on a one-parameter subgroup of $G$. Let $Y \in g = gl(n, \mathbb{R})$ and $t \mapsto e^{tY}$ be one such one parameter subgroup. Then for a matrix $g \in GL(n, \mathbb{R})$, the conjugation map gives

$$C_g(e^Y) = ge^{tY}g^{-1}.$$ 

Taking the derivative with respect to $t$ and evaluating at $t = 0$, we get

$$Ad_g(Y) = gYg^{-1}. \quad (7.51)$$

Now, we evaluate along the derivative of the Adjoint map at $t = 0$, along another one-parameter subgroup $t \mapsto e^{tX}$,

$$Ad_{e^{tX}}Y = e^{tX}Ye^{-tX},$$

$$= (1 + tX + \frac{1}{2!}t^2X^2 + \ldots)Y(1 - tX + \frac{1}{2!}t^2X^2 + \ldots),$$

$$= 1 + t[X,Y] + O(t^2),$$

$$\frac{d}{dt}Ad_{e^{tX}}Y|_{t=0} = [X,Y]$$

We denote the quantity on the left hand side above by the notation $ad_X Y$

$$ad_X Y = [X,Y]. \quad (7.52)$$

Equivalently, $ad_X \in End(g)$ is an endomorphism given by the map $Y \mapsto [X,Y]$. The reemergence of the of an operator yielding the Lie bracket is indicative that there is a Lie derivative floating around. The details are easy to clarify. The map $\varphi(t) = g(t) = e^{tX}$ is a local diffeomorphism generated by the flow of $X \in T_\epsilon G$. The Lie derivative of a vector field $Y$ is given by

$$\mathcal{L}_X Y = \frac{d}{dt}((\varphi^{-1}_t)_*Y)\bigg|_{t=0}.$$
The push forward of the Conjugation map $C_g = R_g^{-1} L_g$ action on $Y$ gives,

$$
Ad_{g(t)}(Y) = (R_g^{-1})_* L_{g(t)} Y
= (R_g^{-1})_* Y, \quad \text{since } Y \text{ is left invariant},
$$

$$
Ad_{e^t X} Y = (R_g^{-1})_* Y,
$$

$$
\frac{d}{dt} Ad_{e^t X} Y|_{t=0} = \frac{d}{dt} (R_g^{-1})_* Y|_{t=0},
$$

which gives,

$$
ad_X Y = \mathcal{L}_X Y.
$$

We conclude that,

$$
Ad_{\exp X} = \exp(ad_X),
\quad Ad_{e^x} = e^{ad_X} = 1 + ad_X + \frac{1}{2!} (ad_X)^2 + \ldots,
$$

(7.53)

where the first term represents of course the identity element in the algebra.

If the Lie algebra is finite dimensional, we can define the **Killing Form** as the form $B$ given by,

$$
B(X,Y) = \text{Tr}(ad_X \circ ad_Y) = \text{Tr}(ad_X ad_Y).
$$

(7.54)

The Killing form is not a differential form, but rather, a symmetric bilinear entity that plays the role of a metric in the Lie algebra. In the adjoint representation, the Killing form is adjoint invariant, meaning,

$$
B(ad_X Y, Z) + B(Y, ad_X Z) = 0
$$

In terms of $ad_X$, the Jacobi identity in definition 7.2 becomes,

$$
[ad_X, ad_Y] Z = ad_{[X,Y]} Z,
$$

(7.55)

which shows explicitly that $ad$ is a Lie algebra homomorphism.

The adjoint map can be used to prove an interesting formulation of the CBH formula first proved in 1899 by Poincaré [13]. Let,

$$
\beta(w) = \frac{w}{1 - e^{-w}} = \sum_{n=0}^{\infty} \frac{B^+_n w^n}{n!}
$$

be the generating function for the Bernoulli numbers. Define,

$$
g(z) = \beta(\ln z) = \frac{z \ln z}{z - 1}.
$$

Then, the Campbell-Baker-Hausdorff formula can be written in the form,

$$
\ln(e^{A} e^{B}) = A + \int_{0}^{1} g(e^{ad_A e^{t ad_B}}) B dt.
$$

(7.56)

The formula is complicated to use for explicit evaluation of the terms in the expansion, but nonetheless is a neat result because it makes it manifestly clear that the expansion depends only on the brackets. Following Hall [13], we illustrate how to get the first three terms. Set $z = v+1$ and expand $g(v+1)$ in a Maclaurin series,

$$
g(v+1) = \frac{v+1}{v} \ln(v+1),
= \frac{v+1}{v} (v - \frac{1}{2} v^2 + \frac{1}{3} v^3 + \ldots),
= (v+1)(1 - \frac{1}{2} v + \frac{1}{3} v^2 + \ldots),
= 1 + \frac{1}{2} v - \frac{1}{3} v^2 + \ldots.
$$
Next, we set $e^{ad_A t} e^{ad_B} - 1 = v$ and evaluate $g$ up to second order in $ad_A, ad_B$. We ignore terms that contain $B$ on the right of $ad_B$, since $ad_B B = 0$.

\[
v = [I + ad_A + \frac{1}{2}(ad_A)^2 + \ldots][I + t ad_B + \frac{1}{2} t^2 (ad_B)^2 + \ldots] - I,
\]

\[
v^2 = (ad_A)^2 + t ad_A ad_B + \ldots,
\]

\[
g(v + 1) = I + \frac{1}{2} (ad_A + \frac{1}{2} (ad_A)^2) - \frac{1}{12} ((ad_A)^2 + t ad_B ad_A) + \ldots
\]

\[
\int_0^1 g(v + 1) \, dt = I + \frac{1}{2} ad_A + \frac{1}{12} (ad_A)^2 - \frac{1}{12} ad_B ad_A + \ldots
\]

Hence,

\[
\ln(e^A e^B) = A + B + \frac{1}{2} [A, B] + \frac{1}{12} [A, [A, B]] - \frac{1}{12} [B, [A, B]] + \ldots
\]

### 7.2.3 The Maurer-Cartan Form

Let $G$ be a Lie group and $\alpha : \mathbb{R} \rightarrow G$ be a curve with $\alpha(0) = A$. Then $\beta(t) = A^{-1} \alpha(t)$ is a curve with $\beta(0) = e$ and $X = A^{-1} \alpha'(0) \in T_e G$. Since $X \in \mathfrak{g}$, we have defined a Lie algebra valued form $\omega$. Alternatively, let $g \in G$. then $L_{g^{-1}} \ast : T_g G \rightarrow T_e G$. If $X \in T_g G$, $\omega$, the **Maurer-Cartan** form is the Lie algebra valued one-form defined by,

\[
\omega(X) = L_{g^{-1}} \ast X.
\]

The Maurer-Cartan form $\omega$ is left-invariant, meaning,

\[
L^g_* \omega = \omega.
\] (7.57)

That is, if the form at point $g_0$ is given by $\omega_{g_0}$, then,

\[
L^g_* \omega_{g_0} = \omega_{g^{-1} g_0}.
\]

On the other hand (the right), we have,

### 7.2.15 Theorem

\[
R^g_* \omega = Ad_{g^{-1}} \omega.
\] (7.58)

**Proof:** Let $X \in \mathfrak{g}$ generate a left invariant vector field in $G$ via the flow of the exponential map $X \mapsto e^{tX}$. Let $g_0 \in G$. By definition, $\omega_{g_0}(X_{g_0}) = X$. Then

\[
(R^g_* \omega_{g_0})(X_{g_0}) = \omega_{g_0} (R^g_* X_{g_0}),
\]

\[
= \frac{d}{dt} [\omega_{g_0} (p e^{tX} g)]_{t=0},
\]

\[
= \frac{d}{dt} [(g_0 g)^{-1} (g_0 e^{tX} g)]_{t=0},
\]

\[
= \frac{d}{dt} [g^{-1} e^{tX} g]_{t=0},
\]

\[
= Ad_{g^{-1}} X,
\]

\[
= Ad_{g^{-1}} \omega_{g_0}(X_{g_0})
\]

Suppose $\{e_\alpha\}$ is a basis for the Lie algebra of left-invariant vector fields $\mathfrak{g}$. The bracket of two vectors in the Lie algebra must be expressible as a linear combinations the basis vectors, so there exist constants $C^\gamma_{\alpha \beta}$ such that

\[
[e_\alpha, e_\beta] = C^\gamma_{\alpha \beta} e_\gamma.
\] (7.59)
The quantities \( C_{\gamma \alpha \beta} \) are called the **structure constants**. Since \( [e_\alpha, e_\beta] = -[e_\beta, e_\alpha] \), the structure constants are antisymmetric on the lower indices. The frame \( \{e_\alpha\} \) is called a Maurer-Cartan frame. Let \( \{\omega^\alpha\} \) be the dual basis, so that \( \omega^\alpha(e_\beta) = \delta_\beta^\alpha \). By definition, applying the Maurer-Cartan form to \( e_\alpha \) returns the value of \( e_\alpha \) at the identity. This gives an almost tautological expression for the components of the Maurer-Cartan form in terms of the Maurer-Cartan coframe,

\[
\omega = e_\alpha(e) \otimes \omega^\alpha
\]

Applying the definition for the differential of a one-form \( \text{6.27} \), we get,

\[
d\omega^\alpha(e_\beta, e_\gamma) = e_\beta(\omega^\alpha(e_\gamma)) - e_\gamma(\omega^\alpha(e_\beta)) - \omega^\beta([e_\beta, e_\gamma]),
\]

\[
= e_\beta(\delta_\gamma^\alpha) - e_\gamma(\delta_\alpha^\beta) - C^\gamma_{\alpha \beta} \omega^\alpha(e_\gamma),
\]

\[
= -C^\alpha_{\beta \gamma}.
\]

Using the antisymmetry of the wedge product and the antisymmetry of the structure constants in the lower indices, we can rewrite the last equation as,

\[
d\omega^\alpha = -\frac{1}{2} C^\alpha_{\beta \gamma} \omega^\beta \wedge \omega^\gamma.
\](7.60)

This equation of structure is called the **Maurer-Cartan** equation. It is often written in commutator notation, keeping in mind that \( \omega \) is a Lie algebra valued one-form,

\[
d\omega + \frac{1}{2} [\omega, \omega] = 0
\](7.61)

If \( G \) is a matrix group, real or complex, the Maurer-Cartan form can be written as in equation \( \text{3.41} \),

\[
\omega = A^{-1} dA.
\](7.62)

For a matrix group, \( A = g \) plays the role of the attitude matrix introduced in section \( \text{3.4} \).

We can express the components of the Killing form in terms of the structure constants. First, we compute

\[
(ad_{e_\alpha} ad_{e_\beta}) e_\gamma = ad_{e_\alpha}([e_\beta, e_\gamma]),
\]

\[
= [e_\alpha, [e_\beta, e_\gamma]],
\]

\[
= [e_\alpha, C^\sigma_{\beta \gamma} e_\sigma],
\]

\[
= C^\sigma_{\alpha \sigma} C^\sigma_{\beta \gamma} e_\rho.
\]

Taking the trace means we perform a contraction with the dual form \( \omega^\gamma \) which is the same as setting \( \gamma = \rho \) on the coefficients on the right. We get

\[
B_{\alpha \beta} = C^\sigma_{\alpha \sigma} C^\sigma_{\beta \gamma}.
\](7.63)

Notice that the components of the Killing form result from the contraction of the antisymmetric indices of the structure constants; this yields the simplest symmetric tensor that can be constructed from the structure constants. Cartan used the Killing form to characterize an important attribute of Lie algebras called semisimple. A non-Abelian Lie algebra is **simple** if it has no, non-trivial proper ideals. A **semisimple** Lie algebra can be decomposed as the direct sum of simple Lie algebras. The Cartan criterion states that the Lie algebra is semisimple if and only if the Killing form is non-degenerate. Of course, the structure constants depend on the choice of the basis. But, since the form is symmetric, it can be diagonalized by an orthogonal matrix, so it can be classified by the eigenvalues. If the Lie group is compact and semisimple, the Killing form is positive definite and in diagonal form, all the entries are positive. The most salient achievement of E. Cartan was
to provide a complete classification of semisimple Lie algebras. Included in this classification are all the Lie algebras associated with the special Lie groups mentioned above in this chapter.

It is also easy to express the adjoint representation in terms of the structure constants. All is really needed is to define matrices $T_\alpha$ whose components are,

$$[T_\alpha]_\sigma^\gamma = C^{\gamma}_{\alpha\sigma}.$$ 

We then get yet another manifestation of the Jacobi identity in the form,

$$[T_\alpha, T_\beta] = C^{\gamma}_{\alpha\beta} T_\gamma.$$ (7.64)

This is the component version of equation and it shows that the structure constants themselves, generate the adjoint representation.

### 7.2.4 Cartan Subalgebra

To get at the physics applications of Lie algebras we need to dip our toes into representation theory. Actually, what we should say is that it is the physics that lead to the development of representation theory by giants like Cartan and Weyl.

#### 7.2.16 Definition  A Cartan Subalgebra $h \in g$ is a nilpotent subalgebra that is its own normalizer. The dimension of the Cartan subalgebra is called the rank of $g$.

Every pair of elements of $h$ commute, and every element of $g$ that commutes with all elements of $h$ is in $h$. In this sense, a Cartan subalgebra is the maximum number of commuting generators of the Lie algebra. The idea is to simultaneously diagonalize the basis of $h$. The eigenvalues are used to label the states of the system.

Using the Killing form, one finds an orthonormal basis $\{h_i\}$ for $h$ and extends to a basis of $g$

$$\{h_1, h_2, \ldots, h_k, g_1, g_{-1}, g_2, g_{-2}, \ldots, g_{\frac{n-k}{2}}, g_{-\frac{n-k}{2}}\}$$

with the following properties:

1. $[h_1, h_j] = 0$.
2. $[h, g] = \lambda(h)g$ for all $h \in h$ and $0 \neq g \in g$. Or, in terms of the basis,
   $$[h_i, g_j] = \lambda^{(j)}_i g_j,$$
3. $[g_j, g_{-j}] \in h$.

The first property is a re-statement that each pair basis vectors in $h$ commute. The second property is a kind of generalized eigenvector equation for $ad_h$. For each $g_j$ we associate a position vector $r^{(j)} = (\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_k^{(j)})$. These are called the Roots and the set of all roots is called the root space. The plot of the root vectors in $\mathbb{R}^k$ is a set of arrows that exhibits certain reflection symmetries; the set is called the Root Diagram. Root spaces lead to Cartan’s classification of semisimple Lie algebras. The classification can also be visualized by a scheme called Dynkin diagrams. The basis elements $g_j$ and $g_{-j}$ are called the Raising and Lowering operators. We will show in the next chapter how this abstract machinery, leads to real concrete results in physics. Lie symmetries of physical systems are interconnected with the deep subject of representation theory.
7.3 Transformation Groups

The importance of this chapter for the purpose of applications to physics is that in many physical models, Lie groups are manifested as transformation groups that act on the system. In the simplest case we have linear transformations in $\mathbb{R}^n$ which in a particular basis, can be represented by matrix multiplication of an element of the general linear group $GL(n, \mathbb{R})$ with a vector. The group of rotations in $\mathbb{R}^3$ constitutes a symmetry group in the dynamical system of rigid body motion, and the Lorentz group is the essential symmetry group of space-time. The Lie algebra of a Lie group is basically a first order symmetry approximation. If one thinks of Lie group such as the rotation group as a symmetry group acting on a manifold, then an element of the Lie algebra represents an infinitesimal transformation near the identity of the group. Bases vectors of the Lie algebras of the corresponding Lie group transformation are then interpreted as generators of infinitesimal transformations. The exponential map provides a bridge between full elements of the group and infinitesimal transformations represented by elements of the Lie Algebra and elements of the Lie group. The Lie algebra is determined by the structure constants and the Maurer-Cartan equations.

7.3.1 Definition Let $M$ be an $n$-dimensional manifold and $G$ a Lie group. $G$ is called a Lie Transformation Group on (the right of) $M$, if there exists a smooth map $\mu : M \times G \rightarrow M$

$$(p,g) \mapsto p \cdot g = R_g(p), \quad \text{for} \ (p,g) \in M \times G,$$

such that for each $p \in M$,
1) $p \cdot e = p$
2) $(p \cdot g_1) \cdot (g_2) = p \cdot (g_1 \cdot g_2)$, \quad for every $g_1, g_2 \in G$.

Of course, a one-parameter group of diffeomorphisms in the sense of definition[7.1.1] is a special case of a Lie transformation group, with $G = \mathbb{R}$. Another example would be the action of the rotation group $SO(3, \mathbb{R})$ on a sphere $S^2$. We are often interested in linear representations of the group acting on some vector space, in which the action respects the linear structure. An example of this, would be adjoint representation of the action of a Lie group into itself.

7.3.2 Definition Let $G$ be a transformation group on $M$ with identity $e$, and let $p \in M$ be any point in the manifold. We say the transformation is,

1. Effective, if the Kernel $K = \{g \in G : p \cdot g = p\} = \{e\}$. In other words, if $g \neq e$, there exists a point $p$, such that $p \cdot g \neq p$. The Kernel of a group is a normal subgroup. If the Kernel is not trivial, the action of $G$ on $M$ is not effective, but the action of $G/K$ is.

2. Free, if $g \neq e$, then $p \cdot g \neq p$. In other words, if $g \neq e$, then there is no fixed point for $g$. If there is a fixed point $p$, we define the Isotropy subgroup of $p$ as, $Iso(p) = \{g \in G : p \cdot g = p\}$.

3. Transitive, if $p \neq q$, then there exists a $g$ such that $p \cdot g = q$. The set of all $q$ such that $p \cdot g = p$ for some $g$, is called the Orbit of the point $p$, and it is denoted by $Gp$. The map $p \mapsto Gp$ which sends $p$ to $p \cdot g$ defines a diffeomorphism $G/Iso(p) \overset{\sim}{\rightarrow} Gp$.

Unless otherwise stated, we assume that when we say that a Lie group acts on a manifold, the action is on the right, and the action is transitive and effective. Let $X$ be an element of the Lie algebra $g$ of a Lie group $G$ acting on $M$, and let $p \in M$. The one-parameter subgroup given by the exponential map

$$X \mapsto e^{\lambda X}$$

generates a curve on $M$ given by

$$\varphi_t(p) = p e^{\lambda X} = R_{e^{\lambda X}}(p), \quad (7.65)$$
with \( \varphi_0(p) = p \) with tangent vector \( \sigma(X) \) at \( p \) given by,

\[
\sigma(X)|_p = \frac{d}{dt}(pe^{tx})|_{t=0}
\]  

(7.66)

If \( U \) is a neighborhood of \( p \), the map \( \varphi_t(p) \) above constitutes a local one-parameter group of transformations of \( M \) associated with the vector field \( \sigma(X) \), often called a fundamental vector field. The map

\[
\sigma : \mathfrak{g} \rightarrow \mathfrak{X}(M)
\]

given by

\[
X \mapsto \sigma(X)
\]

is a Lie algebra homomorphism, that is,

\[
\sigma([X,Y]) = [\sigma(X),\sigma(Y)].
\]

7.3.3 Theorem

\((R_g)_*(\sigma(X)) = \sigma(Ad_{g^{-1}}X).\)  

(7.67)

Proof: Let \( \varphi_t \) but the one-parameter group of diffeomorphisms associated with \( \sigma(X) \) at \( p \) and \( \psi_t \) be the one-parameter group of diffeomorphisms associated with \((R_g)_*(\sigma(X))\) at \( pg \). The map \( R_g : M \rightarrow M \) is a local diffeomorphism, so as in equation 7.9 we have the commuting diagram,

\[
\begin{array}{ccc}
M & \xrightarrow{R_g} & M \\
\varphi_t \downarrow & & \psi_t \downarrow \\
M & \xrightarrow{R_g} & M.
\end{array}
\]

Thus, we get,

\[
\psi_t = R_g \circ \varphi_t \circ R_{g^{-1}},
\]

\[
= R_g \circ R_{e^{tx}} \circ R_{g^{-1}},
\]

\[
= R_{g^{-1}e^{tx}g}.
\]

The one-parameter group of diffeomorphisms \( \{ g^{-1}e^{tx}g \} \) is generated by \( Ad_{g^{-1}}X \), so the vector field associated with \( \psi_t \) is \( \sigma(Ad_{g^{-1}}X) \). We summarize this in the following diagram,

\[
\begin{array}{ccc}
\mathfrak{X}(M) & \xrightarrow{(R_g)_*} & \mathfrak{X}(M) \\
\sigma \uparrow & & \sigma \uparrow \\
\mathfrak{g} & \xrightarrow{Ad_{g^{-1}}} & \mathfrak{g}.
\end{array}
\]
It might instructive to present a second proof in the style of theorem 7.58

\[(R_g)_\sigma(X) = \frac{d}{dt}[R_g(\lambda_X^t)]_{t=0},\]
\[= \frac{d}{dt}[(\lambda_X^t)]_{t=0},\]
\[= \frac{d}{dt}[(\lambda_X^t)]_{t=0},\]
\[= \frac{d}{dt}[(\lambda_X^t)]_{t=0},\]
\[= \sigma_{pg}\text{Ad}_{g^{-1}}X.\]

Here, we have used equation 7.50 in the next to last step. This formula is consistent with the formula for the pull-back of the Maurer-Cartan form 7.58 by the following computation,

\[R^*_g\omega(\sigma(X)) = \omega(R_g\sigma(X)),\]
\[= \omega(\sigma(\text{Ad}_{g^{-1}})X),\]
\[= \text{Ad}_{g^{-1}}\omega(\sigma(X)).\]

For the time being, this important equation might appear as a pure formality, but as we will see later, it is instrumental in the definition of connections on principal fiber bundles. The formula tells us how to push forward along the orbit of right-translations, a vector field on \(M\) generated by the exponential map of a vector in the Lie algebra.
In this section we present a pedestrian view of some of the common Lie algebras and classical Lie groups that appear in mathematical physics.

8.1 Orthogonal Groups

8.1.1 Rotations in $\mathbb{R}^2$

Let $z = x + iy$ be a complex number, and consider the map introduced in section 5.2.2

\[ (x + iy) \xrightarrow{\alpha} \begin{bmatrix} x & y \\ -y & x \end{bmatrix}. \]

The map is clearly a vector space isomorphism between the complex numbers and a subset of the set $2 \times 2$ matrices. The map can be written as

\[ (x + iy) \xrightarrow{\alpha} xI + yJ, \]

where $I$ is the identity matrix and $J$ is the symplectic matrix, with $J^2 = -I$. Define

\[ U(1) = \{ z \in \mathbb{C} : |z| = 1 \} \]

to be the group of unimodular complex numbers. If $z \in U(1)$, then we can write $z$ in the form $z = e^{i\theta}$. The map $\theta \mapsto e^{i\theta}$ is not 1-1 because replacing $\theta$ by $\theta + 2\pi$ gives the same number. The Kernel of the map is the set $\{2\pi\mathbb{Z}\}$, that is, the integer multiples of $2\pi$. $U(1)$ acts on $\mathbb{C}$ by multiplication, which results on a rotation by $\theta$. The action passes to the circle $S^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$. By Euler’s formula, $e^{i\theta} = \cos \theta + i\sin \theta$, so $\alpha$ restricts to a map from $U(1)$ to the special orthogonal group $SO(2, \mathbb{R})$ consisting of $2 \times 2$ rotation matrices,

\[ e^{i\theta} \xrightarrow{\alpha} R_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}. \]

It is an elementary exercise to verify that

\[ e^{i\theta_1} \cdot e^{i\theta_2} \xrightarrow{\alpha} R_{\theta_1} \cdot R_{\theta_2}. \]

The exercise amounts multiplying the rotation matrices and recognizing the summation formulas for sine and cosine. Thus the map is a smooth Lie group homomorphism. Since the map is also a diffeomorphism, the we have a Lie group isomorphism $U(1) \cong SO(2, \mathbb{R})$. 

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Now, consider the exponential map $\phi : \mathbb{R} \to SO(2, \mathbb{R})$ given by,

$$t \xrightarrow{\phi} e^{tA} = \begin{bmatrix} \cos t\theta & \sin t\theta \\ -\sin t\theta & \cos t\theta \end{bmatrix}. \quad (8.1)$$

A matrix $e^{tA}$ is orthogonal if $(e^{tA})^T = (e^{tA})^{-1} = e^{-tA}$, so this implies that $A^T = -A$. Per our previous discussion, the matrix $A$ is a representative of the Lie algebra, so the Lie algebra of the orthogonal group consists of antisymmetric matrices. For the special orthogonal group with matrices with $\det e^A = 1$ The formula 5.45,

$$\det e^A = e^{\text{Tr}A}$$

implies that elements $A$ of the Lie algebra also have 0 trace. $A = \phi'(0)$, with $\phi(0) = I$. so,

$$A = \frac{d}{dt} \begin{bmatrix} \cos t\theta & \sin t\theta \\ -\sin t\theta & \cos t\theta \end{bmatrix}_{t=0} = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} = \theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (8.2)$$

This means that the matrix $J$ is a basis for the Lie algebra $\mathfrak{so}(2, \mathbb{R})$. This example is as simple as it gets, but there are some good lessons to be learned. As it was discussed earlier, the significance of the Lie algebra element $A$ and the basis $J$ is that they constitute the generators of an infinitesimal rotation near the identity. This becomes clear if one looks at the rotation matrix with $\theta$ small. Then, $\cos \theta \approx 1$ and $\sin \theta \approx \theta$. To first order, the rotation matrix is given by $R_\theta \approx I + \theta J = I + A$. We verify that the exponential map gives the group element from the infinitesimal generator. The computation is almost identical to the proof of Euler’s formula by Maclaurin series. The key is that $J^2 = -I, J^3 = -J$, etc., so that

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \frac{1}{5!}A^5 + \ldots$$

$$= I + \theta J + \frac{1}{3!}(\theta J)^2 + \frac{1}{4!}(\theta J)^3 + \frac{1}{5!}(\theta J)^4 + \frac{1}{6!}(\theta J)^5 \ldots$$

$$= (I - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \ldots)I + (\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots)J.$$

Hence

$$e^{\theta J} = (\cos \theta)I + (\sin \theta)J,$$ \quad (8.4)

which is the original rotation matrix. The irreducible representations of $U(1)$ are the trivial representation and the $2 \times 2$ matrix representations given by the maps,

$$\phi_n(e^{i\theta}) = \begin{bmatrix} \cos 2n\theta & \sin 2n\theta \\ -\sin 2n\theta & \cos 2n\theta \end{bmatrix}, \quad n \in \mathbb{Z}^+,$$

so basically, the irreducible representations are the homomorphisms of the group into itself.

### 8.1.2 Rotations in $\mathbb{R}^3$

The Lie group $SO(3, \mathbb{R})$ consists of $3 \times 3$ orthogonal matrices with determinant equal to 1. The Lie algebra $\mathfrak{so}(3, \mathbb{R})$ is the set of $3 \times 3$ antisymmetric matrices with zero trace. The zero trace condition is superfluous since the diagonal elements of an antisymmetric matrix are zero. In consideration of the case above for $\mathfrak{so}(2, \mathbb{R})$, we choose as basis for $\mathfrak{so}(3, \mathbb{R})$, the matrices.

$$\alpha_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \alpha_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \alpha_z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$
The exponentials $R_x = e^{\alpha_x \theta_1}$, $R_y = e^{\alpha_y \theta_2}$, $R_z = e^{\alpha_z \theta_3}$ represent rotations about the $x$, $y$ and $z$ axes respectively. In explicit form, these matrices are,

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos \theta_2 & 0 & -\sin \theta_2 \\ 0 & 1 & 0 \\ \sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix}, \quad R_z(\theta) = \begin{bmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Any rotation in $\mathbb{R}^3$ can be obtained by a composition of rotations about $x$, $y$ and $z$ axes. However, the standard in physics is to utilize the Euler angles $\{\phi, \theta, \psi\}$ introduced by Euler to study the motion of rigid bodies. The general Euler angle rotation is obtained by the composition of three rotations carried as follows (See figure 8.1).

Fig. 8.1: Euler Angles

1. Perform a rotation $R_z(\phi)$ by an angle $\phi$ around the $z$ axis. We label the new axes as $\{\xi, \eta, z\}$.
2. Follow by a rotation $R_\xi(\theta)$ by an angle $\theta$ around the new $x$ axis, which in step (1) we labelled $\xi$. We label the new axes as $\{\xi, \eta', z'\}$.
3. Finish with a rotation $R_{z'}(\psi)$ by an angle $\psi$ around the new $z$-axis, which in step (2) we labelled $z'$. The final axes are labelled $\{x', y', z'\}$.

The rotation matrices are

$$R_z(\phi) = \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix}, \quad R_\xi(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}, \quad R_{z'}(\psi) = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (8.5)$$

A straight-forward matrix multiplication yields the full rotation $R = R_z(\phi) \cdot R_\xi(\theta) \cdot R_{z'}(\psi)$,

$$R = \begin{bmatrix} \cos \psi \cos \phi - \sin \theta \sin \phi & \cos \phi \sin \psi + \cos \theta \sin \phi & \sin \phi \sin \theta \\ -\sin \psi \cos \phi + \cos \theta \cos \phi & \cos \psi \cos \phi + \cos \theta \sin \phi & \sin \phi \cos \theta \\ \sin \theta \sin \phi & \sin \theta \cos \phi & \cos \theta \end{bmatrix}. \quad (8.6)$$

Since $R$ is the product of orthogonal matrices, the matrix is also orthogonal and $R^{-1} = R^T$. If we consider the unit 2-sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$, the rotation gives a map $R : S^2 \rightarrow S^2$. Any rotation of $S^2$ can be viewed as a composition $R$ of the three Euler angle rotations, or as a single rotation around an axis pointing towards the image of the north pole along the axis $z'$. In principle, we should be able to prescribe a direction and an angle as the data to find a matrix representing a rotation by the given angle around the given direction. Finding this data in terms of the Euler angles requires a bit of work.

We present now a representation of rotations in terms of $2 \times 2$ complex matrices. As discussed in section 1.4, orthogonal transformations in $\mathbb{R}^n$ are isometries, so they can be described as the group of transformations that preserve length. In $\mathbb{R}^3$ the length is given by $g(X, X) = x^2 + y^2 + z^2$ under the standard metric. At the risk of getting a little ahead of ourselves, let’s consider the metric
\( \eta = \text{diag}(+---) \) for Minkowski’s space as in 2.36. Let \( x^\mu = (t, x, y, z) \) be the components of a vector in \( M_{1,3} \). Consider the map from \( M(1,3) \) to a \( 2 \times 2 \) Hermitian matrix given by,

\[
x^\mu \rightarrow x^{AB} = \begin{bmatrix} t + z & x - iy \\ x + iy & t - z \end{bmatrix}, \quad \mu = 0, 1, 2, 3
\]  

(8.7)

The index notation for the matrix \( X = (X^{AB}) \) is meant to elucidate the property of Hermitian matrices for which, the complex conjugate \( X^{\dot{A}\dot{B}} \) equals the transpose \( X^{B\dot{A}} \). The bar index notation was used in early work on spinors by Veblen and Taub. Bar indices were later changed to prime indices \( X^\mu \sigma^A \), \( \bar{\sigma}^A \) in some seminal work by R. Penrose in the context of twistors. When convenient, we will invoke the Penrose notation. The map is chosen so that,

\[
\|x^\mu\|^2 = \det X = \det(x^{AB}).
\]

We can write the matrix in terms of a basis,

\[
x^{AB} = \begin{bmatrix} t + z & x - iy \\ x + iy & t - z \end{bmatrix} = t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

(8.8)

where,

\[
s_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad s_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

(8.9)

Here, \( s_0 = I \) and \( \{s_i, i = 1, 2, 3\} \) are the Pauli matrices. For now we constrain to the spatial part of the matrix by restricting to indices \( i, j \ldots = 1, 2, 3 \). Since \( \det(X) \) is equal to the metric we wish to preserve, we seek unitary matrices \( Q \in SU(2) \) such that \( \det(X) \) is invariant under a similarity transformation,

\[
\tilde{X} = QXQ^\dagger
\]

(8.10)

The structure of the Lie algebra \( \mathfrak{su}(2) \) can be obtained in a completely analogous manner as was done for the orthogonal groups. If \( t \mapsto e^{tA} \) is a one-parameter subgroup of \( SU(2) \), then the inverse of \( e^{tA} \) is equal to its Hermitian adjoint, that is,

\[
(e^{tA})^{-1} = (e^{tA})^\dagger,
\]

\[
e^{-tA} = e^{-tA}.
\]

Taking the derivative at \( t = 0 \), we find that \( A^\dagger = -A \). The formula \( \det(e^A) = e^{\text{Tr}A} \) shows that if \( \det(e^A) = 1 \) then \( \text{Tr}A = 0 \). We conclude that the Lie algebra,

\[
\mathfrak{su}(2) = \{A \in GL(2, \mathbb{C}) : A^\dagger = -A, \quad \text{Tr}(A) = 0\},
\]

(8.11)

consists of all traceless anti-Hermitian \( 2 \times 2 \) matrices. By inspection, the set of Pauli matrices \( \sigma = \{s_1, s_2, s_3\} \) is a basis for the Lie algebra. A quick computation gives,

\[
\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I, \quad \det(\sigma_i) = -1.
\]

and structure constants,

\[
[\sigma_i, \sigma_j] = 2i\epsilon_{ij}^k \sigma_k,
\]

(8.12)

where \( \epsilon_{ij}^k \) is the Levi-Civita permutation symbol.
8.1. ORTHOGONAL GROUPS

The factor $2i$ in the formula for the structure constants creates a minor conflict between physicists and mathematicians, but this is historically unavoidable. For example, from the second permutation symbol identity in $\epsilon_{ijk}$, it follows immediately that, because of the $i$ factor, the components of the Killing form are $-4\delta_{ij}$. Thus, for a physicist, a Lie algebra is compact if the Killing form is negative definite, which is the opposite of what was stated earlier. In quantum mechanics, it is customary to denote the set of Pauli matrices by a vector-like notation $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, in which case, the Spin operator in the spin $1/2$ representation, is written as $J = \frac{1}{2}\sigma$. The multiplication table of Pauli matrices exhibits a cyclic permutation feature as shown in the adjacent figure. We can immediately verify that $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$ and $\sigma_3\sigma_1 = i\sigma_2$. Thus, as shown in the diagram, the product of two Pauli matrices gives $i$ times the third matrix if the product is taken clockwise, and $-i$ times the third matrix if traversed counterclockwise. At the center if the triangular diagram there is an $i$ as part of a reminder in the pneumonic, not to forget this factor. Since the squares of the Pauli matrices give the identity, to get an analog of Euler’s formula in matrix form as in equation 8.4, we use the set $\{i\sigma_1, i\sigma_2, i\sigma_3\}$. Like $J$, these matrices all have squares equal to $-I$. At this point it might be in order to inject the observation that the algebra of Pauli matrices is very closely related to the set $H$ of Quaternions. A Quaternion is an entity of the form $q = a1 + bi + cj + dk$, where the basis elements satisfy, $i^2 = j^2 = k^2 = ijk = -1$. It quickly follows that, $ij = k$, $jk = i$, and $ki = j$. A matrix representation of these is obtained by setting $\{i, j, k\} = \{-i\alpha_1, -i\alpha_2, -i\alpha_3\}$. This is one of those places where the factor of $i$ in the structure constants for Pauli matrices causes differences with mathematical standards. Back to the Lie algebra, the quantities $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ represent the infinitesimal transformation that generate the elements of the group. Thus, for example, to generate a rotation by an angle $\phi$ around the $z$-axis, we set,

$$Q_{\phi} = e^{i\phi/2}\sigma_3.$$  

(8.13)

Proceeding exactly as in the computation leading to equation 8.4 we find

$$Q_{\phi} = \cos\frac{\phi}{2}I + i\cos\frac{\phi}{2}\sigma_3,$$

$$= \begin{bmatrix}
\cos\frac{\phi}{2} & 0 \\
0 & \cos\frac{\phi}{2}
\end{bmatrix}.$$  

(8.14)

(8.15)

The result is a diagonal matrix since $\sigma_3$ is diagonal, and hence, so is any power of $\sigma_3$. Yet another computation of the similarity transformation $\tilde{X} = Q_{\phi}XQ_{\phi}^\dagger$, where,

$$X = \begin{bmatrix} z \\
x + iy
\end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} \tilde{z} \\
\tilde{x} + i\tilde{y}
\end{bmatrix}$$

yields,

$$\tilde{x} = x\cos\phi + y\sin\phi,$$
$$\tilde{y} = -x\sin\phi + y\cos\phi,$$
$$\tilde{z} = z.$$
given by,

\[ Q_\theta = e^{\frac{i}{2}\theta \sigma_z} \]
\[ = \cos \frac{\theta}{2} I + i \cos \frac{\theta}{2} \sigma_1, \quad (8.16) \]
\[ = \begin{bmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{bmatrix} \quad (8.17) \]
\[ = \begin{bmatrix} \cos \theta_2 & i \sin \theta_2 \\ i \sin \theta_2 & \cos \theta_2 \end{bmatrix} \quad (8.18) \]

The third Euler angle rotation looks just like the one in equation 8.15 with \( \phi \) replaced by \( \psi \),

\[ Q_\psi = \cos \frac{\psi}{2} I + i \cos \frac{\psi}{2} \sigma_3, \quad (8.19) \]
\[ = \begin{bmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{bmatrix} \quad (8.20) \]

The composite rotation is given by

\[ Q = Q_\psi Q_\theta Q_\phi = \begin{bmatrix} e^{i(\psi+\phi)/2} \cos \frac{\theta}{2} & ie^{i(\psi-\phi)/2} \sin \frac{\theta}{2} \\ ie^{-i(\psi-\phi)/2} \sin \frac{\theta}{2} & e^{-i(\psi+\phi)/2} \cos \frac{\theta}{2} \end{bmatrix}. \quad (8.21) \]

The entries of the this matrix are called the **Cayley-Klein** parameters. It should be noted that if \( Q \) represents a rotation, then \(-Q\) represents exactly the same rotation, since the minus sign cancels out in the similarity transformation. In this sense, \( SU(2) \) is called a double cover of \( SO(3, \mathbb{R}) \).

Let \( n = (n^1, n^2, n^3) \) be a unit vector, and as before, set \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \). We call a unit **Pauli-Bloch vector**, denoted by \( n \cdot \sigma \), the expression given by the matrix,

\[ n \cdot \sigma = n^k \sigma_k = \begin{bmatrix} n^3 & n^1 - in^2 \\ n^1 + in^2 & -n^3 \end{bmatrix} \]

It is very easy to verify that given two vectors \( a \) and \( b \), we have

\[ (a \cdot \sigma)(b \cdot \sigma) = (a \cdot (b) I + (a \times a) \cdot \sigma) \]

Although we do not need the result above here, it is very neat that the formula which is in essence the product of two quaternions, incorporates both, the dot and the cross products. With the notation above, the equation,

\[ e^{\frac{i}{2} \theta (\hat{n} \cdot \hat{\sigma})} = \cos \theta \frac{I}{2} + i (n \cdot \sigma) \cos \theta \frac{I}{2} \quad (8.22) \]

gives a generalization of Euler’s formula extended to quaternions via Pauli matrices. This is a beautiful result which was the goal that led Hamilton to introduce quaternions in 1843. The formula represents a rotation by an angle \( \theta \) about an axis in the direction of \( n \). For a discussion of the dynamics of rigid bodies using Euler angles, see the the book Classical Mechanics by Goldstein [11].

**Hopf Fibration**

The matrices that represent elements of \( SU(2) \) can be written in terms of the the Caley-Klein parameters

\[ Q = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \]

where,

\[ \alpha = e^{i(\psi+\phi)/2} \cos \frac{\theta}{2}, \quad (8.23) \]
\[ \beta = e^{i(\psi-\phi)/2} \sin \frac{\theta}{2} \quad (8.24) \]
8.1. ORTHOGONAL GROUPS

With apologies to the purists, we leave out a factor of $i$ in the $\beta$ parameter in this section. This is done for the sake of better consistency with earlier formalism that we need for this discussion. Since the $\det Q = 1$, we have,

$$|\alpha|^2 + |\beta|^2 = 1 \quad (8.25)$$

This can be corroborated immediately since,

$$|\alpha|^2 + |\beta|^2 = \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} = 1$$

We write $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$ in the form,

$$\alpha = x^1 + ix^2, \quad \beta = x^3 + ix^4.$$  

As a vector space, $\mathbb{C}^2 \cong \mathbb{R}^4$, so equation $8.25$ gives parametric equations for a unit sphere $S^3 \subset \mathbb{R}^4$.

$$\pi(\alpha, \beta) = (2\alpha\bar{\beta}, |\alpha|^2 - |\beta|^2) \subset \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3. \quad (8.26)$$

Indeed, for all $\alpha$ and $\beta$, the image of this map is in $S^2$ because

$$|\pi(\alpha, \beta)|^2 = 4|\alpha|^2|\beta|^2 + (|\alpha|^2 - |\beta|^2)^2,$$

$$= (|\alpha|^2 + |\beta|^2)^2,$$

$$= 1$$

Any other point $(\alpha', \beta')$ that maps to the same point $\pi(\alpha, \beta)$ must satisfy $(\alpha', \beta') = \gamma(\alpha, \beta)$ for some complex number with $|\gamma|^2 = 1$. When this happens, we say that these points are in the same equivalence class. Then, the projective space defined as, $\mathbb{CP}^1 = \mathbb{C}^2/\sim$ represents the space of lines through the origin of $\mathbb{C}^2$. The complex projective plane $\mathbb{CP}_1$ has the structure of a compact complex manifold of complex dimension 1. Geometrically, it can be viewed as sphere $S^2$ in which antipodal points are identified. In quantum physics, and quantum computing, this is called the Bloch sphere. Points in $\mathbb{CP}^1$ can be described by inhomogeneous coordinates $(\alpha, \beta)$ as representative of the equivalence classes, or by inhomogeneous coordinates $\zeta = \alpha/\beta$. Since a complex line through the origin is really a plane, the intersection of one such lines with $S^3$ is a circle $S^1$. In other words, if $p \in \mathbb{CP}^1 \cong S^2$ then $\pi^{-1}(p)$ is a circle $S^1 \subset S^3$. The circles $S^1$ are called the fibers of the bundle,

$$\pi : S^3 \rightarrow \mathbb{S}^1 \rightarrow S^2 \quad (8.27)$$

To “unpack” this bundle in familiar terms, we first compute the map in Cayley-Klein coordinates,

$$2\alpha\bar{\beta} = 2[e^{i(\psi + \phi)/2} \cos \frac{\theta}{2}]e^{-i(\psi - \phi)/2} \sin \frac{\theta}{2},$$

$$= 2e^{i\phi} \cos \frac{\theta}{2} \sin \frac{\theta}{2},$$

$$= e^{i\phi} \sin \theta,$$

$$|\alpha|^2 - |\beta|^2 = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2},$$

$$= \cos \theta.$$  

Let $\omega = 2\alpha\bar{\beta}$. Identifying $\mathbb{C} \times \mathbb{R}$ with $\mathbb{R}^3$, that is, taking $x = \Re(\omega), y = \Im(\omega)$, we get a point in $S^2$ in spherical coordinates

$$x = \cos \phi \sin \theta,$$

$$y = \sin \phi \sin \theta,$$

$$z = \cos \theta.$$
Let $\zeta$ be the complex number in equation 5.60, whose inverse image under the stereographic projection $\pi_s$ gives the coordinates on the sphere,

\[(x, y, z) = \left( \frac{\zeta + \bar{\zeta}}{\zeta \bar{\zeta} + 1}, \frac{\zeta - \bar{\zeta}}{i(\zeta \bar{\zeta} + 1)}, \frac{\zeta \bar{\zeta} - 1}{\zeta \bar{\zeta} + 1} \right)\]

Setting $\zeta = \alpha/\beta$ to be the inhomogeneous coordinates of $\mathbb{C}P_1$ and simplifying the double fraction using the fact that $|\alpha|^2 + |\beta|^2 = 1$, we get the Hopf map 8.26. That is, in point in the image of the Hopf map is just,

$$\pi_s^{-1} \left( \frac{\alpha}{\beta} \right).$$

As stated above, the fiber of a point in $S^2$ is a circle $S^1$ in $S^3$. The fibers of points on a circle in $S^2$ parallel to the equator, are linked circles that lie on a torus - these are called Villarceau Circles. Geometrically, the Villarceau circles are obtained by the intersection of a torus and plane tangential to antipodal images of the generating circle. Hopf discovered the fibration in 1931, but

![Fig. 8.2: Hopf Fibration](image)

I only learned about Hopf fibrations in 1975 from studying Taub’s solution to Einstein’s equation. Taub’s metric has topology $\mathbb{R} \times S^3$ and spatial $SU(2)$ symmetry. The Taub metric is of the form

$$ds^2 = -d\tau^2 + \eta_{ij}(\tau) \omega^i \otimes \omega^j$$

where $\omega$ is essentially the Maurer-Cartan form of $su(2)$. This is an example of a gravitational instanton.

However, the first time I saw a pictorial representation of the fibration was a magnificent hand drawing made by Roger Penrose discussing Robinson Congruences in the context of twistor theory; a reproduction of this drawing appears in [27], for example. One has to marvel at the earlier masters who were able to visualize this complex structure. For us lesser humans, nowadays it takes little effort to render the images with a CAS system, by lifting a parallel circle in $S^2$ to $S^3$ followed by a stereographic projection from $S^3$ to $\mathbb{R}^3$. The nested toroidal $S^1$ links in figure 8.2 are the fibers of three circular parallels in the base space $S^2$. The reader will find a beautiful explanation of Villarceau circles in a paper by Hirsch [15].

There is a generalization of Hopf fibrations with $S^1$ fibers to all complex projective spaces, $\mathbb{C}P^n$. Consider the space $\mathbb{C}^n$ with coordinates $Z = (z_1, z_2, \ldots, z_n)$. The equation $|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1$ describes a sphere $S^{2n+1} \subset \mathbb{C}^n$. The space $\mathbb{C}P^n$ of complex lines through the origin. Let $a, b \in S^{2n+1}$. Define an equivalence relation $a \sim b$, if there exists $c \in S^1$ such that $a = bc$. The idea is that a complex line through the origin is a 2-real dimensional plane, which intersects the sphere on
a circle. All points in such circles are in the same equivalence class. \( \mathbb{CP}^n \) can be identified with the space \( \mathbb{C}^n / \sim \). The generalized fibration is,

\[
S^{2n+1} \xrightarrow{S^1} \mathbb{CP}^n
\]  
(8.29)

The corresponding fibration in the real case is

\[
S^n \xrightarrow{\mathbb{Z}_2} \mathbb{RP}^n,
\]  
(8.30)

since a line through the origin intersects the sphere \( S^n \) in two antipodal points. The group elements are the identity and the antipodal map.

Even neater, there is a Hopf fibration associated with the projective plane over each of the only existing normed division algebras, namely, the Real numbers, the Complex Numbers, the Quaternions and the Octonions. The constructions of the bundles is done by exactly the same process using stereographic projections. The Projective plane Hopf fibrations in exact sequence form are:

\[
\begin{align*}
S^0 &\rightarrow S^1 \rightarrow S^1 \cong \mathbb{RP}^1, \\
S^1 &\rightarrow S^3 \rightarrow S^2 \cong \mathbb{CP}^1, \\
S^3 &\rightarrow S^7 \rightarrow S^4 \cong \mathbb{HP}^1, \\
S^7 &\rightarrow S^{15} \rightarrow S^8 \cong \mathbb{OP}^1.
\end{align*}
\]

The Hopf fibration is a seminal discovery in Algebraic Topology because through a formalism called the long, exact, homotopy sequence of a fibration, it became possible to establish the existence of the first, non-vanishing, high dimension homotopy groups of spheres. Thus, example, we find the \( \pi^3(S^2) \cong \pi^2(S^2) = \mathbb{Z} \).

The Hopf fibration associated with \( \mathbb{CP}^1 = \mathbb{P}^1(\mathbb{C}) \) describes a singly-charge Dirac monopole, and the fibration associated with \( \mathbb{HP}^1 \) describes a Yang-Mills instanton.

**Angular Momentum**

As indicated in the preface to this book, we present a simplified and limited version of basic Quantum Mechanics for the benefit of those mathematics students who have no formal training in the subject. Quantum Mechanics was developed in 1926 by Schröedinger and Heisenberg. The axiomatic description here is a summary of the framework as envisioned by Dirac and von-Neumann. The axioms really are axioms in the sense of Euclid; they can not be proved. The axioms are not self-evident, but they are founded on experience and physical intuition.

- **Postulate QM1** The state of a particle is can be described by a wave function \( \psi(t, x, y, z) \) in some complex Hilbert space \( \mathcal{H} \) with inner product \( \langle \cdot | \cdot \rangle \). The quantity,

\[
\psi^* \psi d^3r
\]

represents a probability density of finding the particle within a volume element \( d^3r \). The total probability is,

\[
P = \int_{\mathcal{H}} \psi^* \psi d^3r = 1.
\]

- **Postulate QM2** Measurable (or observable) quantities such as energy and momentum, are represented by a linear Hermitian operator \( L \) acting on \( \psi \). The measurement of the state is
given by the expectation value,

\[ \langle \psi \rangle = \langle \psi^* | L | \psi \rangle, \]

\[ = \int_\mathcal{H} \psi^* L \psi d^3 r. \]

- **Postulate QM3** From the spectral theorem for Hermitian operators, the possible outcomes of the observables are the eigenvalues of the operator. The eigenvalues of real and eigenstates corresponding to different eigenvalues are orthogonal.

The position operator is multiplication. The linear momentum and energy operators are obtained intuitively by starting with a classical solution to the wave equation. In dimension one, the quantity,

\[ \psi = Ae^{i(kx-\omega t)} \]

is such a solution with speed \( v = k/\omega \). The energy \( E \) and momentum \( p \) are related to the wave number \( k \) and the angular frequency \( \omega \) by the equations,

\[ p = \hbar k, \quad E = \hbar \omega, \]

so that,

\[ \psi = Ae^{i(px-Et)}. \]

Taking partial derivatives with respect to \( x \) and \( t \) respectively, we get,

\[ \frac{\partial}{\partial x} \psi = \frac{i}{\hbar} p \psi, \]

\[ \frac{\partial}{\partial t} \psi = -\frac{i}{\hbar} E \psi, \]

By ansatz, the choice for the operators is,

\[ \hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}, \]

\[ \hat{E} = i\hbar \frac{\partial}{\partial t}, \]

Generalizing momentum to dimension 3, the operators become,

\[ \mathbf{p} = \frac{\hbar}{i} \nabla, \]

\[ \hat{E} = i\hbar \frac{\partial}{\partial t}. \]

The operator for total energy operator called the Hamiltonian \( H \), is the Kinetic energy \( KE = \mathbf{p}^2/(2m) \) plus the potential energy \( PE = V \). Thus, Schrödinger was led to the quantum mechanics equation,

\[ H \psi = \hat{E} \psi, \]

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = i\hbar \frac{\partial \psi}{\partial t}. \]  \hspace{1cm} (8.31)

For a free particle for which the energy does not depend on time, the stationary states are described by the discrete set of eigenfunctions and energy eigenvalues of the equation,

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi_n = E_n \Psi_n, \]
where
\[ \Psi_n = e^{-(i/\hbar)E_n t} \psi_n(r), \]
and \( \psi_n \) depends only on the spatial coordinates \( r = (x_1, x_2, x_3) = (x, y, z) \). Let \( p = (p_x, p_y, p_y) \) be the components of the momentum operator. The following basic commutation relations hold,
\[ [x_i, x_j] = 0, \]
\[ [p_i, p_j] = i\hbar \delta_{ij}, \]
\[ [p_i, x_j] = i\hbar \epsilon_{kij} x_k. \] (8.32)

The first commutation relation above is trivial since in number multiplication, the order of the factors does not alter the product. The commutation relation for two momenta follows from the symmetry of indices of second order partial derivatives. The third commutation relation can done for each pair of indices. In most elementary quantum mechanics books, one example is worked out and the rest are taken on faith or left as an exercise. Here is one example. Let \( f \) be an arbitrary function; compute,
\[ [p_x, x](f) = -i\hbar \left( \frac{\partial}{\partial x} (xf) - x \frac{\partial f}{\partial x} \right), \]
\[ = -i\hbar (f + xf_x - xf_x), \]
\[ = -i\hbar x f, \]
\[ [p_x, x] = -i\hbar x. \]

At this stage, having gained experience in manipulating indices, it is just as easy to do all the cases at once,
\[ [p_i, x_j]f = -i\hbar \left( \frac{\partial}{\partial x_i} (x_j f) - x_j \frac{\partial f}{\partial x_i} \right), \]
\[ = -i\hbar \left( \frac{\partial x_j}{\partial x_i} f + x_j \frac{\partial f}{\partial x_i} - x_j \frac{\partial f}{\partial x_i} \right), \]
\[ = -i\hbar \delta_{ij} f, \]
\[ [p_i, x_j] = -i\hbar \delta_{ij}. \]

The definition of the angular momentum operator in quantum mechanics is given by simple extension of the classical formula,
\[ L = r \times p. \] (8.33)

The explicit components of angular momentum are,
\[ L_x = yp_z - zp_y, \]
\[ L_y = zp_x - xp_y, \]
\[ L_z = xp_y - yp_x. \] (8.34)

In index notation,
\[ L_i = \epsilon_{ik} x_j p_k. \] (8.35)

Let us derive the following commutation relations involving angular momentum,
\[ [L_i, x_j] = i\hbar \epsilon_{ij}^k x_k, \]
\[ [L_i, p_j] = i\hbar \epsilon_{ij}^k p_k, \] (8.36)
For the first equation, we can demonstrate an instance,

\[
[L_x, y]f = (yp_z - zp_y)(yf) - y(yp_z - zp_y)f,
\]

\[
= y^2 p_z f - zp_y (yf) - y^2 p_z f + yzp_y f
\]

\[
= -z[p_y, y]f,
\]

\[
[L_x, y] = i\hbar z.
\]

But, since we have already introduced the Levi-Civita symbol \( \varepsilon_{i k m} \) we can use the momentum commutators \( 8.34 \) and have fun doing all the cases at once. Here is the computation,

\[
[L_i, x_j]f = [\varepsilon_{i k m} x_k p_m, x_j]f,
\]

\[
= \varepsilon_{i k m} x_k p_m (x_j f) - x_j (\varepsilon_{i k m} x_k p_m) f,
\]

\[
= x_k \varepsilon_{i k m} [p_m, x_j] f
\]

\[
= -i\hbar x_k \varepsilon_{i k m} \delta_{m j},
\]

\[
= -i\hbar \varepsilon_{i k m} \delta_{m j},
\]

\[
= i\hbar \varepsilon_{i k m} \delta_{m j}.
\]

The formula for the commutator \( [L_i, p_j] \) is very similar and we leave it as an exercise. Instead, we go for the gold of the commutators.

8.1.1 Proposition

\[
[L_i, L_j] = i\hbar \varepsilon_{i j k} L_k. \tag{8.37}
\]

As above, we show that the concept is easy by doing the following case

\[
[L_x, L_y] = L_x (zP_x - xP_z) - (zP_x - xP_z) L_x,
\]

\[
= (L_x zP_x - zP_x L_x) - (L_x xP_z - xP_z L_x),
\]

\[
= (L_x z - zL_x) P_x - x(L_x P_z - P_z L_x), \quad \text{because } P_x L_x = L_x P_x, \quad L_x x = xL_x,
\]

\[
= -i\hbar p_x + i\hbar x y_P,
\]

\[
= i\hbar (x p_y - y p_x),
\]

\[
= i\hbar L_z.
\]

To do all cases at once, one needs the product of permutation symbol identities \( 2.42 \). This is a great exercise in index gymnastics but hides the simplicity of the two other independent cases that can be done as above.

Equation \( 8.37 \) is the primary reason why the topic of angular momentum is included in this section. We will work in units of \( \hbar \), that is, we set \( \hbar = 1 \). Then comparing with the commutator relations \( 8.12 \) for the Pauli matrices, we see that apart from a factor of 2, we see that the components of \( \mathbf{L} \) are generator of the lie algebra \( \mathfrak{su}(2) \). Following the QM postulates, we seek a Hilbert space on which angular momentum acts as a linear operator, to find a function space that provides a representation for the algebra. Naturally, we seek such functions over a two sphere as the base space. The standard process one finds in most classic books on quantum mechanics might look a bit mysterious to those who see it for the first time, but as we will see, it is just an implementation of the Cartan subalgebra for the angular momentum representation of the algebra.

In the case of \( \mathfrak{su}(2) \) there is only one generator, in the Cartan subalgebra, which we choose to be \( L_z \), so, the rank of the algebra is one. We look for a representation in which \( L_z \) is diagonal. We also seek a Casimir Operator, namely, an operator that commutes with all basis elements of \( \mathfrak{g} \)
The number of Casimir operators in a semisimple Lie algebra is equal to the rank of the algebra. The candidate for the Casimir operator for $\mathfrak{su}(2)$ is,

$$L^2 = L_x^2 + L_y^2 + L_z^2.$$  \hfill (8.38)

We show that the operator commutes with all three generators, that is,

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$$

Let’s show for instance, that $[L^2, L_z] = 0$. We need to establish that

$$[L_x^2 + L_y^2 + L_z^2, L_z] = 0.$$

First, it is easy to verify by direct computation of both sides that in general,


Applying this identity to the square of the components of $L$, we get,

$$[L_x^2, L_z] = L_x[L_x, L_z] + [L_x, L_z]L_x,$$

$$= -iL_xL_y - iL_yL_x,$$

$$[L_y^2, L_z] = L_y[L_y, L_z] + [L_y, L_z]L_y,$$

$$= iL_yL_x + iL_xL_y,$$

$$[L_z^2, L_z] = 0.$$

Adding the last three equations yields,

$$[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z],$$

$$= -iL_xL_y - iL_yL_x + iL_yL_x + iL_xL_y,$$

$$= 0.$$

We introduce the Ladder operators,

$$L_+ = L_x + iL_y,$$

$$L_- = L_x - iL_y.$$ \hfill (8.39)

We will see below, that the Ladder operators are the Raising and Lowering operators of the algebra. The process of finding the irreducible representations starts with establishing a number of commutation relationships. First, it is easy to verify that,

$$L_+L_- = L_x^2 + L_y^2 + L_z,$$

$$L_-L_+ = L_x^2 + L_y^2 - L_z.$$ \hfill (8.40)

Indeed, we have,

$$L_+L_- = (L_x + iL_y)(L_x - iL_y),$$

$$= L_x^2 + L_y^2 + i(L_yL_x - L_xL_y),$$

$$= L_x^2 + L_y^2 + i[L_y, L_x],$$

$$= L_x^2 + L_y^2 + L_z,$$
and similarly for $L_+ L_-$. It is also easy to verify that,

$$[L_+, L_-] = 2L_z,$$
$$[L_z, L_+] = L_+,$$
$$[L_z, L_-] = -L_-$$

These commutators define the Cartan Subalgebra. Since in this case, the subalgebra is one dimensional, the roots are the vectors in $\mathbb{R}$ given by $r^{(1)} = (1)$ and $r^{(-1)} = (-1)$. We associate the $0$ root with $L_z$. The root diagram called $A_1$ is a simple as it gets; it consists of of two unit vectors at the origin in $\mathbb{R}$. Putting the commutator results above together leads to the following formula,

$$L^2 = L_+ L_- + L_z^2 - L_z,$$
$$= L_- L_+ + L_z^2 + L_z.$$  \hspace{1cm} (8.42)$$

The result follows directly from equation 8.40. We show the steps for the first of these.

$$L_+ L_- = L_z^2 + L_y^2 + L_z,$$
$$= L^2 - L_z^2 + L_z,$$
$$L^2 = L_+ L_- + L_z^2 - L_z.$$ The formalism can then be used to obtain the ubiquitous expression for the momentum operator of a single particle in spherical coordinates. The computation starts with inverting the Jacobian matrix in 2.31

$$\frac{\partial}{\partial x} = \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi},$$
$$\frac{\partial}{\partial y} = \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi},$$
$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta}.$$  \hspace{1cm} (8.43)$$

After some algebraic manipulations, we get,

$$L_z = -i \frac{\partial}{\partial \phi},$$
$$L_+ = e^{i \phi} \left( \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right),$$
$$L^2 = - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \right].$$ \hspace{1cm} (8.46)$$

Comparing with equation 3.9 we recognize $L^2$ as the angular part of the spherical Laplacian. Perhaps the most direct way get the function space is to solve Laplace’s equation in spherical coordinates by separation of variables, that is, assuming that the solution is of the form $\psi = \Theta(\theta) \Phi(\phi)$. The complete process of obtaining the solutions is best suited for a course on partial differential equations or a course in electrodynamics, so we just present the result. The equation is manifestly self-adjoint, hence the eigenvalues are real and eigenvectors corresponding to different eigenvalues are orthogonal. The eigenvalue equations are,

$$L_z Y_{l,m}(\theta, \phi) = m Y_{l,m}(\theta, \phi)$$
$$L^2 Y_{l,m}(\theta, \phi) = l(l + 1) Y_{l,m}(\theta, \phi).$$ \hspace{1cm} (8.47)$$

Here, eigenvalues $l(l + 1)$ are the orbital quantum numbers $l$ are positive integers. The solution of Laplace’s equation in spherical coordinates are the well known spherical harmonics $Y_{l,m}$,

$$Y_{l,m} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2l + 1}{2} \frac{(l - m)!}{(l + m)!}} P_l^m(\cos \theta) e^{im\phi},$$ \hspace{1cm} (8.48)$$
where $P_{lm}$ are the associated Legendre polynomials given by Rodrigues’ formula,

$$P_{lm}(\cos \theta) = \frac{(-1)^l}{2^l l!} \sin^m \theta \left( \frac{d}{d(\cos \theta)} \right)^{m+l} (\sin^2 \theta).$$  

(8.49)

For each $l$ there are $2l + 1$ possible values for $m$ given by the integers $m = -l, \ldots, l$.

The eigenfunctions $\psi$ can also be obtained by a method which is almost entirely algebraic. First, the eigenvalue equation for the $z$-component of angular momentum is,

$$L_z \psi = \lambda \psi,$$

$$-i \frac{\partial}{\partial \psi} = \lambda \psi.$$

The solution is

$$\psi = f(\theta) e^{-\lambda \theta}.$$

For the function to be periodic on a sphere, we must have $\lambda = m \in \mathbb{Z}$. The normalized eigenfunctions of the $\phi$ component are,

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im \phi}, \quad m = 0, \pm 1, \pm 2, \ldots.$$

Since $L^2 - L_z^2 = L_x^2 + L_y^2$ is physically a positive operator, the absolute value of the eigenvalues of $L_z$ for a given $L^2$ are bounded. Let $l$ be the integer corresponding to the largest value of $L_z$ for particular value of $L^2$.

We switch to the bracket notation and denote the eigenstates by

$$\psi = |l, m\rangle.$$

Let $\psi_m$ be an eigenstate of $L_z$ and recall from equation 8.41 that $[L_z, L_\pm] = L_\pm$. Apply $L_z$ to the state $L_\pm \psi$. We get,

$$L_z(L_\pm |l, m\rangle) = (L_\pm L_z + [L_z, L_\pm])|l, m\rangle,$$

$$= (L_\pm L_z + L_\pm)|l, m\rangle,$$

$$= (m \pm 1)(L_\pm |l, m\rangle).$$

Thus, if $\psi_m$ is an eigenfunction with eigenvalue $m$, $L_\pm \psi_m$ are eigenfunctions with eigenvalues $m \pm 1$. The ladder $L_\pm$ operators lower or raise the $m$ quantum number without changing the eigenvalue of $L^2$. Hence, if $m = l$ is the maximum value for a particular state of $L^2$, we have.

$$L_+ \psi_l = 0$$

Now, apply $L_-$ to this state. Using formula 8.42 this yields,

$$L_+ L_- \psi_l = (L^2 - L_z^2 - L_z) \psi_l = 0.$$

But we are seeking states which are simultaneous eigenfunctions of the commuting operators $L^2$ and $L_z$, so the eigenvalue of $L^2$ must be $l(l + 1)$. For those who have studied the solution Laplace’s equations by separation of variables and infinite series, this would correspond to the step where one sets the eigenvalue of Legendre’s differential equation to $l(l + 1)$ to cause the infinite series to terminate, and thus yield polynomial solutions. In this manner, we have recovered the eigenvalues in 8.47 almost entirely algebraically,

$$L_z |l, m\rangle = m |l, m\rangle,$$

$$L^2 |l, m\rangle = l(l + 1) |l, m\rangle.$$  

(8.50)
Eigenstates are basis vectors for the Hilbert space, so they should be normalized. Thus, we require
\[ \langle l, m | l, m \rangle = 1. \]
Suppose we have constants \( C^\pm_{l,m} \) such that,
\[ \psi = L^\pm |l, m\rangle = C^\pm_{l,m} |l, m \pm 1\rangle. \]
Then, \( \langle \psi | \psi \rangle = |C^\pm_{l,m}|^2 \). On the other hand, we have,
\[ L^\pm |l, m\rangle = (L^2 - L_z^2 \mp L_z) |l, m\rangle, \]
so we choose,
\[ C^\pm_{l,m} = \sqrt{l(l+1) - m(m \pm 1)}. \]
We conclude that the effect of applying the ladder operators to normalized eigenstates is
\[ L^\pm |l, m\rangle = \sqrt{l(l+1) - m(m \pm 1)} |l, m\rangle. \quad (8.51) \]
Recalling that \( L^+ \) is a linear first order differential operator, and noting that \( L^+ |l, l\rangle = 0 \), we get a linear first order differential equation for \( \Theta(\theta) \) that is very easy to solve. Then, carefully banging the solution with lowering operators leads to Rodrigues’ formula for the Associated Legendre polynomials. The matrix elements of the representation are complicated. They are described by unitary \( (2l+1) \)-dimensional unitary matrices called Wigner D-Matrices. If \( R(\psi, \theta, \phi) \) is a rotation by Euler angles, and \( |l, m\rangle \) are spherical harmonic eigenstates, the matrix elements are given by,
\[ D^l_{mm'}(\psi, \theta, \phi) = \langle l, m' | R(\psi, \theta, \phi) | l, m \rangle, \quad (8.52) \]
\[ = e^{-im' \phi} d^l_{mm'}(\theta) e^{-im \phi}, \quad (8.53) \]
where,
\[ d^l_{mm'}(\theta) = \langle l, m' | e^{i\theta L^x} | l, m \rangle = D^l_{mm'}(0, \theta, 0) \quad (8.54) \]
We content ourselves in these notes in wetting the appetite of the reader to dig into more details in any senior/first-year graduate level text in quantum mechanics.

### 8.2 Lorentz Group

The appropriate symmetry group in special relativity is the Lorentz group. This is the group of transformations that leaves invariant the metric \( \eta = \text{diag}(+) \) in Minkowski’s space \( M_{1,3} \). We will denote Lorentz transformations by the notation,
\[ x'^{\mu} = L^\mu_{\nu} x^{\nu}. \quad (8.55) \]
The metric is invariant if, \( <Lx, Lx> = <x, x> \), that is,
\[ \eta_{\mu\nu} L^\mu_{\alpha} L^\nu_{\beta} = \eta_{\alpha\beta}. \]
Transformations for which \( |L^\mu_{\nu}| = 1 \), and \( L^0_0 > 0 \), so that past and future are not interchanged constitute the proper, orthochronous Lorentz group \( SO^+(1,3) \). The Lorentz transformation laws for tensors \( T \) is the same as in the Riemannian metric case as shown in equation \[ 7.16 \]
\[ T^{\beta_1, \ldots, \beta_s}_{\alpha_1, \ldots, \alpha_s} = \frac{\partial x^{\beta_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x^{\beta_s}}{\partial x^{\nu_s}} \frac{\partial x^{\nu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x^{\nu_s}}{\partial x^{\alpha_s}} T^{\mu_1, \ldots, \mu_s}_{\nu_1, \ldots, \nu_s}. \quad (8.56) \]
As usual, the metric is used to raise and lower indices and thus convert between covariant and contravariant tensors. Another way to obtain a covariant tensor from a contravariant one, is by use of the permutation symbol, but we need to be a little careful. As noted in the paragraph elaborating on the Hodge star operator the Levi-Civita symbol does not transform like a tensor, but rather like tensor density of weight \((-1)\). Instead, if \(g\) is any Riemannian or pseudo-Riemannian metric such as \(\eta\) we define 
\[
\epsilon_{\mu\nu\kappa\lambda} \epsilon_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma},
\]
\[
\epsilon_{\mu\nu\kappa} \epsilon_{\alpha\beta} = 2! \delta_{\alpha\beta} \delta_{\mu
u},
\]
\[
\epsilon_{\mu\beta\gamma} = 3! \delta_{\mu} \delta_{\alpha}.
\]
An appropriate contraction of a tensor \(T\) with the Levi-Civita tensor gives another tensor called the dual tensor \(\tilde{T}\). Thus, for example, the dual of an antisymmetric tensor \(T_{\mu\nu}\) is
\[
\tilde{T}^{\alpha\beta} = \frac{1}{2\sqrt{\det g}} \epsilon^{\alpha\beta\mu\nu} T_{\mu\nu}.
\]
A tensor as above for which \(T_{\mu\nu} = \tilde{T}_{\mu\nu}\) is called self-dual. Such self-dual tensors play a special role in the representation of the Lorentz group.

### 8.2.1 Infinitesimal Transformations

There are 6 infinitesimal generators for the Lie algebra \(\mathfrak{so}(1,3)\). We can take these generators to be,
\[
j_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad j_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad j_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]
\[
k_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]
The three generators \(\{j_1, j_2, j_3\}\) correspond to the subgroup \(SO(3)\) of spatial rotations and thus span the subalgebra \(\mathfrak{so}(3)\). The exponential map of these generators yield matrices in which the spatial \(3 \times 3\) blocks are the same rotation matrices as in \(8.1.2\). There are three other generators which we call \(\{k_1, k_2, k_3\}\) involving the time parameter. These generate the boosts. The \(k\) generators are not manifestly antisymmetric, but that is because the signature of the metric. The exponential map of the boost generators yield hyperbolic blocks. For example, the \(k_1\) infinitesimal transformation in the \(t\) and \(x\) coordinates,
\[
\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \theta \\ \theta & 0 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}
\]
yields a Lorentz transformation of the form,
\[
\begin{bmatrix} \cosh \theta & \sinh \theta & 0 & 0 \\ \sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
The transformation does represent a rotation, but it is imaginary. Here
\[
\cosh \theta = \frac{1}{\sqrt{1 - \beta^2}}, \quad \sinh \theta = -\frac{\beta}{\sqrt{1 - \beta^2}}
\]
with \(\beta = v/c\). This is the way these transformations appear in a first course in special relativity.

Any infinitesimal Lorentz transformation \(x^\mu' = L^\mu_\nu x^\nu\) has the form,
\[
L^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu, \quad \text{or},
L^\mu\nu = \delta^\mu\nu + \omega^\mu\nu.
\]

To preserve the metric, we must have,
\[
\eta^\mu_\nu L^\mu_\alpha L^\nu_\beta = \eta^\alpha_\beta + \omega^\alpha_\beta + \omega^\beta_\alpha = \eta^\alpha_\beta.
\]

This means that \(\omega^\alpha_\beta\) is antisymmetric, as expected from the analysis of the similar situation with the exponential map of the rotation group. In any representation of the Lorentz group, the infinitesimal transformations take the form,
\[
I + \frac{1}{2} \omega^\mu\nu M^\mu_\nu, \quad \text{(8.59)}
\]
where \(M^\mu_\nu\) are the antisymmetric matrices representing the six infinitesimal transformations. These matrices obey the integrability commutation relations,
\[
[M^\mu_\nu, M^\sigma_\tau] = M^\mu_\nu M^\sigma_\tau - M^\sigma_\tau M^\mu_\nu, \quad \text{or},
= g^\nu_\sigma M^\mu_\tau - g^\mu_\sigma M^\nu_\tau + g^\mu_\tau M^\nu_\sigma - g^\nu_\tau M^\mu_\sigma. \quad \text{(8.60)}
\]

Apart from a factor of \(i\hbar\), this is the relativistic Angular Momentum Tensor, symbolically written, \(r \wedge p\), where \(r\) and \(p\) are the position, and the 4-momentum respectively. The notation means that
\[
M^\mu_\nu = x^\mu p^\nu - p^\mu x^\nu
\]
The spatial components \(j_k = \epsilon^{ij} M_{ij}\) are the generators of the subgroup \(SO(3)\), and the spatial-temporal components \(k_i = M_{0i}\) generate the boosts. The Lie algebra commutator relations are given by
\[
[J_i, J_j] = i\epsilon^{ij} J_k, \quad [K_i, K_j] = -i\epsilon^{ij} J_k, \quad [J_i, K_j] = i\epsilon^{ij} J_k, \quad \text{(8.61)}
\]
where,
\[
J = \frac{1}{2} \sigma.
\]
For a spin \(\frac{1}{2}\), the boosts are generated by,
\[
K = \pm \frac{1}{2} \sigma,
\]
giving two inequivalent representations.

These matrices constitute a representation of the Lie algebra of the Lorentz group called the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) spin representation. The group elements are given by the exponential map,
\[
L^\mu_\nu = \exp \left[ \frac{i}{2} \omega^\alpha_\beta (M^\alpha_\beta)^\mu_\nu \right]. \quad \text{(8.62)}
\]
8.2. LORENTZ GROUP

8.2.2 Spinors

In a manner analogous to the construction of the 2-1 isomorphism between $SU(2)$ and $SO(3)$, starting with the map \text{[8.7]} we seek a representation of Lorentz transformations of a vector $x^\mu \in M_{1,3}$ in terms of transformations of the $2 \times 2$ Hermitian matrix $X = X^{\dot{A}\dot{B}}$. Since $\det X$ is equal to the norm of a vector that we wish to preserve, the condition is equivalent to invariance under unimodular transformations

$$X' = QXQ^\dagger,$$

where $Q \in SL(2, \mathbb{C})$.

We introduce Spin space as a pair $\{S_2, \epsilon_{AB}\}$, where $S_2$ is a 2-dimensional complex vector space and $\epsilon_{AB}$ is the symplectic form with components,

$$\epsilon_{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$  \hfill (8.64)

The matrix elements of the symplectic form are the same as the Levi-Civita symbol in dimension 2. It is assumed that spinors obey the transformation law,

$$\phi'_A = \phi_B Q^B_A,$$  \hfill (8.65)

where $Q \in SL(2, \mathbb{C})$. An element $\phi_A \in S_2$ is called a covariant 2-spinor of rank 1. Associated with $S_2$ there are three other spaces, the dual $S_2^*$, the complex conjugate $\overline{S_2}$, and the complex conjugate dual $\overline{S_2}^*$. We will use the following index convention,

$$\phi_A \in S_2, \quad \phi^A \in S_2^*,$$

$$\phi_{\dot{A}} \in \overline{S_2}, \quad \phi^{\dot{A}} \in \overline{S_2}^*.$$  

We introduce dual and conjugate versions of the symplectic form $\epsilon_{\dot{A}\dot{B}}$, $\epsilon^{AB}$ etc., all of which have the same matrix values. We use these to map manipulate spinor indices according to the rules,

$$\phi_A = \epsilon_{AB} \phi^B, \quad \phi^B = \phi_A \epsilon^{AB},$$

$$\phi_A = \epsilon_{AB} \phi^{\dot{B}}, \quad \phi^{\dot{B}} = \phi_A \epsilon^{\dot{A}\dot{B}}$$  \hfill (8.66)

namely, we Lower on the Left and Raise on the Right. The two operations are inverse of each other as we can easily verify by lowering, then raising and index,

$$\phi^A = (\epsilon_{BC} \phi^C) \epsilon^{BA},$$

$$= \epsilon_{BC} \epsilon^{BA} \phi^C,$$

$$= \delta^{BA} \phi^C = \phi^A.$$  

Here we have used the permutation symbol identity which is the same as for the Levi-Civita symbol,

$$\epsilon_{BC} \epsilon^{BA} = \delta^{AC}.$$  

Higher rank spinors can be constructed as elements of tensor products of Spin spaces. Because of the antisymmetry of the symplectic form, one needs to be more careful when raising and lowering spinor indices. For instance, Consider, $\phi^A \psi_A \in S_2 \otimes S_2^*$. Then,

$$\phi^A \psi_A = \phi^A \epsilon_{AB} \psi^B,$$

$$= \epsilon_{AB} \phi^A \psi^B,$$

$$= -\epsilon_{BA} \phi^A \psi^B,$$

$$= -\phi_B \psi^B,$$

$$= -\phi_A \psi_A.$$
We conclude that exchanging the position of a repeated spinor index introduces a minus sign. In particular, for any rank one spinor \( \phi^A \), we have
\[
\phi^A \phi_A = \phi_A \phi^A = 0.
\]

It follows that the full contraction such as \( \phi^{ABC} \phi_{ABC} \) of a spinor with odd number of indices with itself is zero. If a spinor is symmetric on any two indices, then contracting on those two indices gives 0. Contraction on two indices reduces the rank of a spinor by two. If a spinor is completely symmetric, it is not possible to reduce the the rank by contraction since any such contraction is zero. The only completely antisymmetric spinor must be of rank two since the indices can only attain the values 1 or 2 and there are only to permutations possible. In fact, a completely antisymmetric spinor must be a multiple of \( \epsilon_{AB} \). Thus, for example, for any spinor \( \phi_A \), we have
\[
\epsilon_{AB} \phi^C + \epsilon_{CA} \phi_B + \epsilon_{BC} \phi_A = 0
\]
because this combination of spinor is antisymmetric and of rank 3. Another good example is the relation,
\[
\phi_{AC} \phi^C_B = -\phi_A^C \phi_B
\]
Which is true since the position index \( C \) was exchanged. Hence the quantity on the left is an antisymmetric spinor of rank two and it must be a multiple of \( \epsilon_{AB} \). It is easy to check that the correct multiplicative factor is given by,
\[
\phi_{AC} \phi^C_B = -\frac{1}{2} \phi_{CD} \phi^{CD} \epsilon_{AB}.
\]

We can also have spin-tensors, the main example being the “connecting spinor” \( \sigma^{\mu A B} \) in equation 8.9 which has one covariant tensor index and two spinor indices. For convenience, we list the components here again,
\[
\sigma^{\mu A B} = \begin{cases}
[1 & 0] \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix},
[0 & 1] \begin{bmatrix}
i & 0 \\
1 & 0
\end{bmatrix},
[0 & -1] \begin{bmatrix}1 & 0 \\
0 & -1
\end{bmatrix}
\end{cases}
\]
It is elementary to verify that the inverse matrices \( \sigma_{\mu A B} \), that is, the matrices such that,
\[
\sigma^{\mu A B} \sigma_{\nu A B} = \delta_{\nu}^\mu,
\]
are given by,
\[
\sigma^{\mu A B} = \begin{cases}
[1 & 0] \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix},
[0 & -1] \begin{bmatrix}1 & 0 \\
0 & 1
\end{bmatrix},
[i & 0] \begin{bmatrix} i & 0 \\
0 & 0
\end{bmatrix},
[1 & 0] \begin{bmatrix}1 & 0 \\
i & 0
\end{bmatrix}
\end{cases}
\]
The result is consistent with rasing the tensor index with the metric and lowering the spinor indices with the symplectic form. With these matrices, the reverse of equation 8.8 is,
\[
x^\mu = \frac{1}{2} \sigma^{\mu A B} x^{AB}.
\] (8.68)

A few words about index conventions are in order. The convention used here most closely resembles that of the early developers Veblen and Taub (See for example [33]). The main difference here is in choosing \( \sigma^{\mu A B} \) to be consistent with Pauli matrices; this is closer to the choice in the Penrose prime notation. In following established protocol, I have reluctantly adapted the notation with both indices up, which is inconsistent with the index summation convention for matrix multiplication of index-free expressions such as \( \sigma_1 \sigma_2 \). My preference would have been to choose \( \sigma^{\mu A B} \) to correspond to the Pauli matrices. Instead, lowering the second index results on the following matrices,
\[
\sigma_{\mu A B} = \begin{cases}
[0 & -1] \begin{bmatrix}1 & 0 \\
0 & -1
\end{bmatrix},
[1 & 0] \begin{bmatrix} i & 0 \\
0 & i
\end{bmatrix},
[0 & 1] \begin{bmatrix} 0 & -1 \\
i & 0
\end{bmatrix}
\end{cases}
\]
It is straightforward to verify the following spinor identities,
\[
\sigma_\mu^A \sigma_\nu^B = 2 \delta^A_C \delta^B_D,
\]
\[
\sigma_\mu^A \sigma_\nu^B = 2 \epsilon^{AC} \epsilon^{BD},
\]
\[
\sigma_\mu^A \sigma_\nu^B + \sigma_\nu^A \sigma_\mu^B = 2 g_{\mu\nu} \delta^A_C.
\]
(8.69)

Remembering that spin matrices are Hermitian, and switching the position of summation index \(B\), the last equation above can be rewritten as,
\[
\sigma_\mu^A \sigma_\nu^B \bar{\psi}^C + \sigma_\nu^A \sigma_\mu^B \bar{\psi}^C = -2 g_{\mu\nu} \delta^A_C.
\]

This equation is in the right summation index format for matrix multiplication, so it can be written in an index-free form as,
\[
\sigma^\mu \sigma^\nu + \sigma^\nu \sigma^\mu = -2 g^{\mu\nu} I.
\]
(8.70)

In view of the remarks above, the reader should be cautioned that the matrices in this neat equation are not the standard Pauli matrices in the coordinate representation we have chosen. Equation (8.70) is most important in formulating Dirac’s equation, as shown below.

As discovered by Dirac, to obtain a relativistic extension of Schrödinger’s equation in the spin \(1/2\) representation, which is invariant under Lorentz transformations, one must introduce a second spinor field \(\psi^A\). The field equations are,
\[
\sigma^\mu_\nu A \bar{\psi}^B \left( \frac{\hbar}{i} \frac{\partial}{\partial x^\mu} \right) \bar{\psi}^B = -i mc \phi^A,
\]
\[
\sigma^\mu_\nu A \bar{\phi}^B \left( \frac{\hbar}{i} \frac{\partial}{\partial x^\mu} \right) \bar{\phi}^B = -i mc \psi^A,
\]
(8.73)

where the \(\sigma\) spin-tensors satisfy equation (8.70). The more familiar 4-spinor Dirac equation is obtained by rewriting (8.73) in matrix form,
\[
\frac{\hbar}{i} \frac{\partial}{\partial x^\mu} \left[ \begin{array}{cc} 0 & -i \sigma^\mu_\nu A \bar{\psi}^B \\ i \sigma^\mu_\nu A \bar{\phi}^B & 0 \end{array} \right] = mc \left[ \begin{array}{c} \phi^A \\ \bar{\psi}^A \end{array} \right],
\]
which is the standard Dirac equation for a free particle,
\[
(\gamma^\mu p_\mu - mc) \Psi = 0.
\]
(8.74)

Here,
\[
\Psi = \left[ \begin{array}{c} \phi^A \\ \bar{\psi}^A \end{array} \right] \in S_2 \times \bar{S}_2^*.
\]

is a Dirac 4-spinor, and \(\gamma\) has the \(4 \times 4\) matrix representation,
\[
\gamma = \left[ \begin{array}{cc} 0 & -i \sigma^\mu_\nu A \bar{\psi}^B \\ i \sigma^\mu_\nu A \bar{\phi}^B & 0 \end{array} \right].
\]
Comparing with equation 8.70, we see that the $\gamma$'s satisfy the so-called **Clifford Algebra** relation,

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu \nu} I.$$  \hfill (8.75)

We would like to establish some interesting connection between some spinor and tensorial quantities. Consider for instance a null vector $l_\mu \in M(1, 3)$. Since the length of the vector is zero, the corresponding matrix $l_{A\bar{B}}$ has determinant equal to zero; so the first row, is a multiple of the second and hence the matrix is of the form,

$$l_\mu = \sigma_\mu^{A\bar{B}} \phi_A \bar{\psi}_B.$$

If we choose,

$$\phi_A = \begin{bmatrix} \zeta \\ 1 \end{bmatrix},$$

then, up to a constant, the components of the null vector are given by,

$$l_\mu = \sigma_\mu^{A\bar{B}} \phi_A \bar{\phi}_B.$$

Using the matrix components of the sigma matrices as in 8.9, a short computation gives,

$$l_\mu = \frac{1}{2}(\zeta \bar{\zeta} + 1, \zeta + \bar{\zeta}, -i(\zeta + \bar{\zeta}), \zeta \bar{\zeta} - 1)$$

Normalizing, the vector becomes,

$$l_\mu = \begin{bmatrix} 1, \frac{\zeta + \bar{\zeta}}{\zeta \bar{\zeta} + 1}, \frac{(\zeta + \bar{\zeta})}{i(\zeta \bar{\zeta} + 1)}, \frac{\zeta \bar{\zeta} - 1}{\zeta \bar{\zeta} + 1} \end{bmatrix}.$$  \hfill (8.76)

The spatial part precisely the inverse image of complex numbers $\zeta$ under the stereographic projection 5.60 viewed as inhomogeneous coordinates $\zeta = \phi^1/\phi^2$ on the Riemann sphere $S^2 \cong \mathbb{CP}^1$. The norm of the spatial part is one, so the norm in $M_{1,3}$ is zero, as it should be. One may view the sphere as the intersection of the null cone with the hyperplane $t = 1$ (or -1), so this is essentially the celestial sphere. Spinors transform by elements of $SL(2, \mathbb{C})$ which is the universal covering group of the group of Möbius transformations. Möbius transformations are conformal maps, so this gives a connection with minimal surfaces.

Next, we note that self-dual tensors of rank 2 are associated with symmetric spinors. The connection is made by first defining the spinor,

$$\sigma_{\mu\nu}^{A\bar{B}} = \sigma_{[\mu}^{A\bar{C}} \sigma_{\nu]\bar{B}C},$$

$$= \frac{1}{2}(\sigma_{\mu}^{A\bar{C}} \sigma_{\nu\bar{B}C} - \sigma_{\nu}^{A\bar{C}} \sigma_{\mu\bar{B}C}),$$

where the contraction $\sigma_{\mu\nu}^{A\bar{B}} = 0$ is clearly 0. By direct computation in the coordinate system we have chosen. The values of the traceless matrices are,

$$\sigma_{01}^{A\bar{B}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \sigma_{02}^{A\bar{B}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_{03}^{A\bar{B}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\sigma_{12}^{A\bar{B}} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \sigma_{23}^{A\bar{B}} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad \sigma_{13}^{A\bar{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It follows that $\sigma_{\mu\nu}^{AB}$ is symmetric. The values are,

$$\sigma_{01AB} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_{02AB} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad \sigma_{03AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\sigma_{12AB} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \sigma_{23AB} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \sigma_{13AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
8.2. LORENTZ GROUP

We find from the last set of equations that,

\[ \sigma_{12} = \sigma^{12} = -i\sigma_{03}, \]
\[ \sigma_{13} = \sigma^{13} = i\sigma_{02}, \]
\[ \sigma_{01} = -\sigma^{01} = i\sigma_{23}, \]

and hence

\[ \hat{\sigma}^{\mu\nu} = \frac{1}{2\sqrt{\det g}} \epsilon^{\mu\nu\sigma\tau} \sigma_{\sigma\tau}. \]  \hspace{1cm} (8.78)

is self dual. From this, it follows that if \( F^{AB} \) is a symmetric spinor, then

\[ F_{\mu\nu} = \sigma_{\mu\nu AB} F^{AB} \]  \hspace{1cm} (8.79)

is a self-dual tensor. It can also be verified by computation in the chosen coordinate system that.

\[ \sigma_{\mu\nu AB} \sigma^{\nu\tau AB} = -2\delta^\sigma_\mu \delta^\tau_\nu. \]  \hspace{1cm} (8.80)

From this, the equation above can be inverted,

\[ F^{CD} = \frac{1}{8} F_{\sigma\tau} \sigma^{\sigma\tau CD} \]  \hspace{1cm} (8.81)

yielding a symmetric spinor from a self-dual tensor. This establishes a 1-1 correspondence between self-dual tensors of rank two, and symmetric spinors of rank two. Tensors that transform like symmetric spinors of rank 2\( n \) are said to be irreducible under a spin \( n \) representation. This is particularly relevant for self-dual Maxwell tensors. More specifically, if,

\[ F = F_{\mu\nu} dx^\mu \wedge dx^\nu \]

is the Maxwell 2-form, the corresponding Maxwell spinor is,

\[ F_{A\dot{A}B\dot{B}} = \sigma_{A\dot{A} B\dot{B}} F_{\mu\nu} \]

Here, (almost) following Penrose, we have used undotted and dotted letters with the same character, to avoid the proliferation of different letters. The Maxwell spinor can be written as,

\[ F_{A\dot{A}B\dot{B}} = \frac{1}{2} (F_{A\dot{A} B\dot{B}} - F_{B\dot{B} A\dot{A}}), \]

where, \( \phi_{AB} \) and \( \bar{\phi}_{\dot{A}\dot{B}} \) are symmetric spinors given by,

\[ \phi_{AB} = \frac{1}{2} F_{A\dot{C} B\dot{C}} \]
\[ \bar{\phi}_{\dot{A}\dot{B}} = \phi_{AB} \]

This gives a spinor decomposition of the tensor into self-dual and anti-self-dual parts.

The matrices \( \sigma_{\mu\nu}^{\ A\ B} \) also serve to connect infinitesimal Lorentz and spinor transformations. Given an infinitesimal Lorentz transformations as in equation \[8.59\] the corresponding infinitesimal spinor transformation is.

\[ (I + \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu})^A \ B = \delta^A \ B + \frac{1}{4} \omega^{\mu\nu} (\sigma_{\mu\nu})^A \ B. \]

In other words, the spinor representation of the angular momentum tensor is,

\[ (M_{\mu\nu})^A \ B = \frac{1}{2} \sigma_{\mu\nu}^{\ A\ B}. \]  \hspace{1cm} (8.82)
Index manipulation of Dirac 4-spinors $\Psi^\mu$ is done by an extension $\epsilon_{\mu\nu}$ of the symplectic form to a $4 \times 4$ matrix which is also antisymmetric. In an accepted abuse of language, some authors refer to $\epsilon_{\mu\nu}$ as the metric spinor. As with 2-spinors, we lower on the left and raise on the right,

$$
\Psi_\mu = \epsilon_{\mu\nu} \Psi^\nu ,
$$

$$
\Psi^\mu = \Psi_\nu \epsilon^{\nu\mu}.
$$

In the coordinate basis we have chosen, $\epsilon_{\mu\nu}$ has components

$$
\epsilon_{\mu\nu} = \begin{bmatrix} i \epsilon^{AB} & 0 \\ 0 & -i \bar{\epsilon}^{AB} \end{bmatrix}.
$$

The spinor-index version of the Clifford algebra relation 8.75 is

$$
\gamma^\mu\beta \gamma^{\nu\beta} + \gamma^{\nu\alpha} \gamma^\mu\beta \gamma_\gamma = 2 g^{\mu\nu} \delta^\alpha_\gamma.
$$

In a completely analogous manner equation 8.77, we construct self-dual 4-spinor $\gamma^\mu\nu\alpha\beta$ matrices , which we write in matrix form as,

$$
\gamma^{\mu\nu} = \gamma^\mu\nu - \gamma^{\nu\mu} = [\gamma^\mu, \gamma^\nu] \quad (8.85)
$$

Then, the infinitesimal momentum tensor 8.82 in the 4-spinor representation becomes,

$$
(M_{\mu\nu})^\alpha_\beta = \frac{1}{2} (\gamma_{\mu\nu})^\alpha_\beta.
$$

The matrices $\{I, \gamma^\mu, \gamma^{\mu\nu}\}$ and their duals, span the Clifford algebra. The dual of $I$ is the celebrated matrix

$$
\gamma^5 = \frac{i}{4!} \epsilon_{\mu\nu\sigma\tau} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\tau.
$$

The commutation relations 8.60 insure that

$$
[\gamma^{\mu\nu}, \gamma^\alpha] = \eta^{\mu\alpha} \gamma^\nu - \eta^{\nu\alpha} \gamma^\mu.
$$

The dual of the six $\gamma^{\mu\nu}$ matrices do not yield independent matrices, and the dual of $\gamma^\mu$ is essentially $\gamma^\mu \gamma^5$, so the algebra is spanned by $\{I, \gamma^\mu, \gamma^{\mu\nu}, \gamma^{\mu\nu} \gamma^5, \gamma^5\}$, and it has $1+4+6+4+1 = 16$ dimensions. We have defined the gamma matrices in a particular matrix representation, but the general relations that describe the algebra are basis-independent.

### 8.3 NP-Formalism

Spinors provide an interesting formalism introduced in 1962 by Newman and Penrose [23] for the study of general relativity. Let $\text{cal}M, g_{\mu\nu}$ be a space-time with Lorentzian metric of signature $(+, -,-,-)$. Introduce a null tetrad $h^a_\mu = \{n_\mu, l_\mu, -m_\mu, -\bar{m}_\mu\}$, and dual frame forms,

$$
\theta^a = h^a_\mu dx^\mu,
$$

associated with the frame,

$$
e_a = h^a_\mu \partial_\mu, \quad h^\mu_a = \{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}.
$$

In terms of the tetrad, the space-time metric is given by,

$$
ds^2 = \eta_{ab} \theta^a \theta^b,
$$

8.91}
where $\eta$ is the quasi-orthonormal flat metric,

$$
\eta_{ab} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{bmatrix},
$$

(8.92)

used to manipulate tetrad indices. Here, the metric tensor has the form

$$
g_{\mu\nu} = 2l(\mu n_{\nu}) - 2m(\mu \bar{m}_{\nu}).
$$

(8.93)

The Cartan structure equations are,

$$
d\theta^a + \omega^a_{\ b} \wedge \theta^b = 0, \quad \omega_{ab} = -\omega_{ba},
$$

$$
d\omega^a_{\ b} + \omega^{\alpha}_{\ c} \wedge \omega^c_{\ b} = \Omega^a_{\ b}, \quad \Omega_{ab} = -\Omega_{ba},
$$

$$
d\Omega^a_{\ b} - \Omega^a_{\ c} \wedge \omega^c_{\ b} + \omega^{\alpha}_{\ c} \wedge \Omega^c_{\ b} = 0.
$$

(8.94)

In this formalism the connection components are called the **Ricci Rotation Coefficients**, which are defined by,

$$
\gamma^a_{\ bc} = h^a_{\ b\mu} h^{\mu}_{\ c} h^{\alpha}_{\ \mu}.
$$

(8.95)

The Riemann tetrad components satisfy,

$$
\omega^a_{\ b} = \gamma^a_{\ bc} \theta^c,
$$

$$
\Omega^a_{\ b} = \frac{1}{2} R^{a}_{\ bcd} \theta^c \wedge \theta^d.
$$

(8.96)

Einstein’s equations in tetrad form are,

$$
R_{ab} - \frac{1}{2} \eta_{ab} R = T_{ab}
$$

(8.97)

As is well known in the literature, the Riemann tensor admits the decomposition,

$$
R_{abcd} = C_{abcd} + \frac{R}{12} g_{abcd} + E_{abcd},
$$

(8.98)

where, $C_{abcd}$ is the conformal Weyl tensor, and,

$$
g_{abcd} = \eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc},
$$

$$
E_{abcd} = g_{abf} (dS^f_{\ c}),
$$

$$
S_{ab} = R_{ab} - \frac{1}{2} \eta_{ab} R.
$$

(8.99)

The transition between tensor and spinor quantities is made by contraction with the connecting spinors $\tau^*_{AB}$ which satisfy,

$$
g_{\mu\nu} \tau^*_{A\ C} \tau^*_{B\ D} = \epsilon_{AB} \epsilon_{CD}.
$$

The spinor dual one-forms $\theta^{AB}$ are defined by,

$$
\theta^{AB} = \tau^{*}_{\mu AB} dx^\mu = \theta^a \tau^*_{a AB},
$$

where the tetrad connecting spinors are chosen such that,

$$
ds^2 = \det \theta^{AB},
$$

$$
= \begin{bmatrix}
\theta^0 & \theta^2 \\
\theta^3 & \theta^1
\end{bmatrix}.
$$
We now introduce a spin dyad $\zeta^A_a = (\phi^A, \psi^A)$, with $\phi^A \psi_A = 1$. The null tetrad can be written as,

\begin{align*}
  l_\mu &= \tau^A_B \phi_A \tilde{\phi}_B, \\
  m_\mu &= \tau^A_B \phi_A \psi_B, \\
  n_\mu &= \tau^A_B \psi_A \tilde{\phi}_B, \\
  \bar{m}_\mu &= \tau^A_B \psi_A \tilde{\psi}_B,
\end{align*}

(8.100)

The spin coefficients corresponding the 24 Ricci rotation coefficients are given by,

\[ \Gamma_{ab\mu} = \zeta_A^a \zeta_B^b \zeta^{\mu A} \zeta^{\mu B}. \]

(8.101)

The 12 complex spin coefficients may be arranged in three groups of four, according to the scheme,

\[ A_\mu = \Gamma_{00 \mu} = \{\kappa, \rho, \sigma, \tau\}, \]
\[ B_\mu = \Gamma_{01 \mu} = \{\epsilon, \alpha, \beta, \gamma\} = \Gamma_{10 \mu}, \]
\[ C_\mu = \Gamma_{11 \mu} = \{\pi, \lambda, \mu, \nu\}, \]

(8.102)

The Spin Connection $\Gamma_{AB\mu}$ which gives rise to the covariant derivative of spinors, is related to the spin coefficients by,

\[ \Gamma_{A B \mu} = \mathcal{A}_{a \mu} \mathcal{C}_a^A \mathcal{C}_B^B. \]

Following equation 8.77, we define,

\[ \sigma_{ab}^A = \mathcal{A}_{[a} \mathcal{C}_{b]} \mathcal{C}_a^A \mathcal{C}_B^B, \]

(8.103)

which we use to construct the connection and curvature spinors,

\[ \Gamma_{A B} = \omega_{ab} \sigma_{ab}^A B, \]
\[ \Omega_{A B} = \phi_{ab} \sigma_{ab}^A B. \]

(8.104)

We can now write the spinor version of the equations of structure, which not surprisingly, have a very similar appearance,

\[ d \theta^{AB} + \Gamma_{AC}^B \wedge \theta^{CB} + \Gamma_{BC}^A \wedge \theta^{AC} = 0, \]
\[ d \Gamma_{AB} + \Gamma_{AC}^D \wedge \Gamma_{DB}^C + \Gamma_{BC}^D \wedge \Gamma_{DA}^C = \Omega_{AB}, \]
\[ d \Omega_{AB} - \Omega_{AC}^D \wedge \Gamma_{DB}^C + \Gamma_{BC}^D \wedge \Omega_{DA}^C = 0. \]

(8.105)

As already discussed, if the coframes $\theta^a$ undergo a Lorentz transformation $\hat{\theta}^a = L^a_b \theta^b$, the spinor coframe undergoes a similarity transformation $Q \times \bar{Q} \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$,

\[ \hat{\theta}^{AB} = Q_{a C} \theta^{CD} \bar{Q}_{DB}. \]

In matrix notation, the connection and curvature spinors transform according to,

\[ \hat{\Gamma} = Q \Gamma Q^{-1} + Q dQ^{-1}, \]
\[ \hat{\Omega} = Q \Omega Q^{-1}. \]

(8.106)

The spin connection $\Gamma_{A B}$ and the curvature spinor $R_{A B c d}$ satisfy equations analogous to 8.96,

\[ \Gamma_{A B} = \Gamma_{A B c} \theta^c = \Gamma_{A C B D} \theta^{C D}, \]
\[ \Omega_{A B} = \frac{1}{2} R_{A B c d} \theta^c \wedge \theta^d, \]
\[ = \frac{1}{2} R_{A B C D E F} \theta^{C E} \wedge \theta^{D F}. \]

(8.107)
8.3. NP-FORMALISM

We also have a decomposition of curvature spinor into irreducible components

\[ R_{A\dot{B}B\dot{C}C\dot{D}D} = \Psi_{ABCD} \bar{\epsilon}^\dot{A}^\dot{B}^\dot{C}^\dot{D} + \bar{\Psi}_{\dot{A}B\dot{C}D} \epsilon_{ABCD} + \frac{1}{12} R(\epsilon_{AC\dot{B}D} \bar{\epsilon}^\dot{A}^\dot{C}^\dot{B}^\dot{D} + \epsilon_{AB\dot{C}D} \bar{\epsilon}^\dot{A}^\dot{B}^\dot{C}^\dot{D} + \epsilon_{CD} \Phi_{AB\dot{C}D} \bar{\epsilon}^\dot{A}^\dot{B}) \]  

(8.108)

where,

\[ \Phi_{AB\dot{C}D} = \Phi_{(AB)(\dot{C}D)} = \Phi_{AB\dot{C}D}, \]  

(8.109)

is the traceless Ricci spinor, and,

\[ \Psi_{ABCD} = \Psi_{(ABCD)} \]  

(8.110)

is the completely symmetric Weyl conformal spinor. Finally, the spinorial version of Einstein’s equation takes the form,

\[ \Phi_{AB\dot{C}D} = \frac{1}{4} (T_{A\dot{C}B\dot{D}} + T_{B\dot{C}A\dot{D}}), \quad T = T^a_a = R. \]  

(8.111)

Equations 8.111 and 8.94 are called the Newman-Penrose (N-P) equations. When written in detail in terms of the 12 spin coefficients, the (N-P) result in a formidable set of systems of coupled first order differential equations consisting of 4 metric equations, 18 spin coefficient equations and 8 Bianchi identities. Fortunately, the spin coefficients have geometric interpretations that motivate imposing conditions on the N-P equations that makes them tractable. The Weyl spinor leads an elegant classification of so-called, algebraically special space-times. The classification was originally done by Petrov, but it is now commonly known as the Cartan-Petrov-Pirani-Penrose classification. One starts by writing the completely symmetric conformal spinor as,

\[ \Psi_{ABCD} = \alpha_{(AB\dot{C}D)} \]  

(8.112)

Each of the rank one spinors is associated with a null vector. The classification is as follows,

1. Type I. Algebraically general - 4 distinct null directions.
2. Type II. Two null directions coincide.
3. Type D. There are two pairs of null directions that coincide.
4. Type III. Three principal directions coincide.
5. Type N. All principal directions coincide - also called Type Null.

The 1962 seminal paper by Newman and Penrose [23] is noted by the elegant proof in terms of the spin coefficients, of the Goldberg-Sachs theorem. The theorem states that a non-flat, vacuum space-time is algebraically special, if, and only if, it contains a null geodesic congruence that is shear-free; that is, there is a null vector with \( \kappa = 0 \) and \( \sigma = 0 \).

The literature on applications of the N-P formalism is huge. A Google search on “Newman-Penrose Spin Coefficients” yields over 54,000 results.

8.3.1 Eth Operator

One of the most pulchritudinous results arising from the NP formalism, was the serendipitous discovery by Newman and Penrose of a characterization of asymptotically half-flat space-times in terms of an operator called eth. The operator acts on a space consisting of spin-weighted functions on a sphere. A function \( \eta \) on a sphere has spin weight \( s \) if it transforms as,

\[ \eta' = e^{is\psi} \eta, \]  

(8.113)
under a rotation about the north pole. The spin-weight is constrained to be a $\frac{1}{2}$-integer. Here we are only interested in the nature of the operator in its relation to representations, but to provide some historical context, we say a few words about half-flat space times. The simplest way to introduce the notion of half-flat is to consider the good cut differential equation,

$$\partial^2 Z(\zeta, \tilde{\zeta}) Z(\zeta, \tilde{\zeta}) = \sigma^0(Z, \zeta, \tilde{\zeta}),$$

(8.114)

where the $\eth$ operator is defined by,

$$\eth\eta = 2P^{1-s} \frac{\partial}{\partial \zeta} (P^s \eta),$$

$$\eth\eta = 2P^{1+s} \frac{\partial}{\partial \tilde{\zeta}} (P^{-s} \eta).$$

(8.115)

Here, $P$ is the conformal factor in the Fubini-Study metric, $P = \frac{1}{2}(1 + \zeta \tilde{\zeta})$

$\sigma^0(Z, \zeta, \tilde{\zeta})$ is some complex valued function (which in context represents the asymptotic value of the shear spin index $\sigma$). The idea in the NP formalism is that if one could solve this equation, then one could construct an (asymptotically) half-flat space-time. The definition of a half-flat space-time starts with space-time analytically continued into the complex. Then the components of the spinor image of the Weyl tensor,

$$C_{abcd} \mapsto \Psi_{ABCD} \tilde{\epsilon}^{\dot{A}\dot{B}} \tilde{\epsilon}^{\dot{C}\dot{D}} + \tilde{\Psi}^{\dot{A}\dot{B}\dot{C}\dot{D}} \epsilon_{ABCD}$$

(8.116)

are now independent; here indicated by replacing $\tilde{\Psi}$ by $\tilde{\Psi}$. The two components are the self-dual, and the anti-self-dual parts of the spinor. The space is half-flat, if it is Ricci flat, and,

$$\tilde{\Psi}^{\dot{A}\dot{B}\dot{C}\dot{D}} = 0$$

(8.117)

The reason a sphere $S^2$ enters into the picture can be motivated by a simple geometrical argument. If space time were spherical, then null rays would converge to a single point at infinity. Conformal null infinity in this case can be viewed as the intersection of a hyperplane with the 4-sphere at the north pole. However, if the space is Lorentzian and asymptotically flat, then conformal infinity looks like a hyperplane intersecting with a hyperbolic surface, which is a cone with topology $S^2 \times \mathbb{R}$.

In spherical coordinates, the $\eth$ operator acting on a function $\eta$ of spin weight, takes the tantalizing form

$$\eth\eta = -(\sin \theta)^s \left[ \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right] (\sin \theta)^{-s} \eta,$$

$$\eth\eta = -(\sin \theta)^{-s} \left[ \frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right] (\sin \theta)^{s} \eta.$$

(8.118)

We have,

$$(\eth\eta)' = e^{i(s+1)\phi} \eta,$$

$$(\eth\eta)' = e^{i(s-1)\phi} \eta.$$

(8.119)

so these act as raising and lowering operators of spin weight. One can also verify that,

$$(\eth\eth - \eth\eth) \eta = 2s\eta.$$

(8.120)

The eigenfunctions of $\eth\eth = \eth^2 \eta = 0$ are called spin-weighted spherical harmonics and are denoted by $sY_{lm}(\theta, \phi)$, where $|s| < l$. Some authors have pointed out that these entities were
previously known to Gelfand. In the case $s = 0$, the operator $\bar{\sigma}^2$ is just the Laplacian, so the eigenfunctions are spherical harmonics. Since $\bar{\sigma}$ and $\bar{\bar{\sigma}}$ raise and lower the spin weight respectively, the spin-weighted spherical harmonics can be obtained by successive applications of the operators to spherical harmonics. The elegant formulas were derived by Goldberg, et-al [10].

$$\bar{\sigma}^2 Y_{lm} = -(l-s)(l+s+1)sY_{lm},$$
$$\bar{\sigma} Y_{lm} = \sqrt{(l-s)(l+s+1)s+1}Y_{s+1 lm},$$
$$\bar{\bar{\sigma}} Y_{lm} = -\sqrt{(l+s)(l-s+1)s+1}Y_{s-1 lm}.$$  

Applying these equations iteratively, one can show that

$$\lambda Y_{lm} = \begin{cases} \sqrt{\frac{(l-s)!}{(l+s)!}} \bar{\sigma}^s Y_{lm} & \text{if } 0 \leq s \leq l, \\ \sqrt{\frac{(l+s)!}{(l-s)!}} (-1)^s \bar{\bar{\sigma}}^{-s} Y_{lm} & \text{if } -l \leq s \leq 0 \end{cases} \quad (8.121)$$

In the context of representation theory of the rotation group, the main result is that Wigner $D_{nm^{'}}^l$ matrix elements can be neatly expressed after a messy computation, by the neat formula [10],

$$D_{-ms}^l(\phi, \theta, -\psi) = (-1)^m \sqrt{\frac{4\pi}{2l+1}} sY_{lm}(\theta, \phi) e^{i\phi} \quad (8.122)$$

In 1985, T. Dray [7] proved that with a appropriate choice of spin gauge, spin weighted spherical harmonics were the same as the monopole harmonics introduced by Wu and Yang as solutions of a semiclassical electron in the field of a Dirac monopole. This is not surprising since, as we will see in the next chapter, the Dirac monopole is associated with a connection on a $U(1)$ Hopf bundle over $S^2$, whereas, the transformation law for spin-weighted function is basically a gauge transformation in such a bundle.

### 8.4 SU(3)

The $SU(3)$ group was introduced by Gell-Mann in 1961, as a candidate for a symmetry gauge group to accommodate quark “flavors”. In the language a particle physics in this theory, Hadrons are made up of three quarks with flavors called: up, down and strange $(u, d, s)$, at a time when particles with “color” attributes of charm, top, or bottom $(c, t, b)$ were unknown. If one denotes a flavor state by,

$$|\psi_f\rangle = \begin{bmatrix} u \\ d \\ s \end{bmatrix},$$

The isospin action by $g \in SU(3)$ is simply given by $|\psi_f\rangle \mapsto g|\psi_f\rangle$. The Lie algebra $su(3)$ consists of $3 \times 3$ traceless, Hermitian matrices. The dimension of the special unitary group $SU(n)$ is $n^2 - 1$, so $su(3)$ has 8 generators. Gell-Mann chose for these generators, the closest extension of Pauli matrices. The 8 Gell-Mann matrices are, (see Georgi [9])

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix},$$

$$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \quad (8.123)$$
Anticipating a factor of $2i$ as in the case of Pauli matrices, we use the standard convention of denoting the structure constants by,

$$[\lambda_j, \lambda_k] = 2if^i_{jk}\lambda_i.$$  \hfill (8.124)  

Effectively, the matrices are normalized so that,

$$\text{Tr}(\lambda_j\lambda_k) = \delta_{jk}$$

It is customary to set

$$T_j = \frac{1}{2}\lambda_j$$

The structure constants turn out to be completely antisymmetric, so, modulo permutations, there are only 8 independent ones. The upper $2 \times 2$ block of $\{\lambda_1, \lambda_2, \lambda_3\}$ are just the Pauli matrices, so this set constitutes an $\mathfrak{su}(2)$ subalgebra, and we have

$$f^i_{jk} = \epsilon^i_{jk}$$

whenever all indices are less than or equal to 3. The isotopic spin $SU(2)$ algebra generated by the exponential map of these generators, result in rotations of the flavor state $|\psi_f\rangle$ that leave $s$ invariant. Defining,

$$\tau_+ = \sqrt{3}\lambda_8 + \lambda_3,$$

$$\tau_- = \sqrt{3}\lambda_8 - \lambda_3,$$

one can also identify two more $\mathfrak{su}(2)$ subalgebras generated by $\{\lambda_4, \lambda_5, \tau_+\}$ and $\{\lambda_6, \lambda_7, \tau_-\}$ respectively. The rest of the the non-zero structure constants are easily computed using a CAS system. The results are,

$$f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2},$$

$$f_{156} = f_{367} = -\frac{1}{2},$$

$$f_{458} = f_{687} = \frac{1}{2}\sqrt{3}.$$  

A concise formula for the structure constants is given by,

$$f_{ijk} = -\frac{i}{4}\text{Tr}(\lambda_i[\lambda_j, \lambda_k])$$  \hfill (8.125)  

Another useful fact is the formula for the anti-commutators,

$$\{\lambda_j, \lambda_k\} = \frac{3}{4}\delta_{jk}I + 2d^i_{jk}\lambda_i,$$

where,

$$d_{ijk} = \frac{1}{4}\text{Tr}(\lambda_i[\lambda_j, \lambda_k]).$$

The Killing form, modulo the usual problematic factor of $2i$,

$$g_{jk} = f^i_{jm}f^m_{ki} = 3\delta_{jk}$$

is non-degenerate and positive definite, as expected from a semi-simple, compact group. The Cartan subalgebra is spanned by $h_3 = T_3$ and $h_8 = T_8$ which commute with all the other generators. Thus, the subalgebra is of rank 2 and there are 2 Casimir operators. Since the generators of the Cartan subalgebra are already in diagonal form, is easy to find the eigenvectors and corresponding weights \[9\],

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto (\frac{1}{2}, \frac{1}{2\sqrt{3}}), \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto (0, -\frac{1}{\sqrt{3}}).$$
To find the roots we need generators that take one weight to another. From lessons learned from $\mathfrak{su}(2)$ we take as raising and lowering operators,\[ \frac{1}{\sqrt{2}}(T_1 \pm iT_2), \quad \frac{1}{\sqrt{2}}(T_4 \pm iT_5), \quad \frac{1}{\sqrt{2}}(T_6 \pm iT_7). \]

A straightforward computation of the commutators with the generators of the Cartan subalgebra gives,

\[ [T_3, (T_1 \pm iT_2)] = \pm((T_1 \pm iT_2), T_3), \]
\[ [T_8, (T_1 \pm iT_2)] = 0, \]

so, we have found two roots, $(\pm 1, 0)$. Continuing the computation for the next set of ladder operators, we get,

\[ [T_3, (T_4 \pm iT_5)] = \pm \frac{1}{2}((T_4 \pm iT_5), T_3), \]
\[ [T_8, (T_4 \pm iT_5)] = \pm \frac{\sqrt{3}}{2}((T_4 \pm iT_5), T_8), \]

so the next pair of roots are $(\pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2})$. Finally, the commutators for the third set of ladder operators, leads to,

\[ [T_3, (T_6 \pm iT_6)] = \mp \frac{1}{2}((T_6 \pm iT_6), T_3), \]
\[ [T_8, (T_6 \pm iT_6)] = \pm \frac{\sqrt{3}}{2}((T_6 \pm iT_6), T_8), \]

giving the last pair of roots $(\mp \frac{1}{2}, \pm \frac{\sqrt{3}}{2})$. The complexification of $\mathfrak{su}(3)$ is $\mathfrak{sl}(3, \mathbb{C})$. In Cartan’s classification of semisimple Lie algebras, the root system for $\mathfrak{sl}(n + 1, \mathbb{C})$ is called $A_n$. In the root diagram $A_2 = \mathfrak{su}(3)$, the roots form a regular hexagon with two roots at the center, as shown in figure 8.3.

We refer the reader to the famous book by Georgi [9] for a full discussion of representations of groups in Physics.
Chapter 9

Bundles and Applications

9.1 Fiber Bundles

The late 1970’s was an exciting time to be a graduate student at Berkeley. At the time, the University had a powerhouse of some of the top, world-class mathematicians in differential geometry and related fields, including, Chern, Kobayashi, Wolf, Gilkey and Weinstein; a number of renowned general relativity researchers such as Taub, Marsden, and Sachs; as well as a battery of visiting faculty and invited speakers at the frontiers of research. Prior to 1975, with the exception of Professor Sachs, who had a dual appointment in the physics department, I don’t think I ever saw, as an undergraduate or as graduate student, a physics professor enter the math building or a math professor walk the hallways of the physics building. It just so happened, that on 1975, Belavin, Polyakov, Schwartz, and Tyupkin, published a paper on pseudoparticle solutions to the Yang-Mills equations [3]. This so-called BPST instanton, drew widespread attention in the physics community. The instantons are extremals under a variational principle of the Yang-Mills Lagrangian, which generalizes the electromagnetic Lagrangian \(2.123\) to non-Abelian Lie algebras. The paper included a tantalizing discussion of topological properties such as homotopy classes and a footnote referring to a particular equation as a Pontrjagin class. In 1977, Schwarz (apparently the correct spelling) used the Atiyah-Singer index theorem to show that the number of instantons and zero fermion modes is given by some topological invariant \((8k - 3)\) [30]. Perhaps the inclusion of such heavy-duty mathematics made some of the particle physicists a bit uncomfortable. I say this because in 1977, when I. M. Singer was offered a position at Berkeley, his seminars on the Penrose Twistor programme and Gauge theory got flooded with non-Abelian gauge physicists including Mandelstam. The relevance of twistor theory to gauge fields was first established by R. Ward in a brilliant short paper [35] in which he showed that certain complex vector bundles related to \(\mathbb{CP}^3\), in twistor theory, could be used to generate self-dual gauge fields. This also drew the attention of algebraic geometers such as R. Hartshorne, working on moduli spaces of vector bundles. The presence of Singer at Berkeley attracted a slew of prominent visitors such as M. Atiyah, S. Yau, and later, A. Lichnerowicks. Atiyah and Singer became major contributors to the mathematical formulation of Yang-Mills Theory; in particular, in a paper published in 1978, [2], Atiyah, Hitchin and Singer computed the dimension of the moduli space of irreducible, self-dual connections for Yang-Mills equations in 4-space for all compact gauge groups. The dimension of this space for \(SU(2)\) is the topological invariant \((8k - 3)\) derived by Schwarz.

Among the physicists attending Singer’s lectures was a young researcher named A. Hanson with whom I partnered to become the note-takers for the seminar series. The section on Yang-Mills in these notes, is partly distilled from my 1977-78 notes with Hanson on the lectures by Singer. A couple of years later, Hanson, along with T. Eguchi and P. Gilkey, who introduced me to algebraic topology, published a comprehensive work, in which among other things, they announced the discovery of a
new metric for a gravitational instanton [8].

We have already encountered several examples of fiber bundles of interest in physics. The most fundamental of these are the tangent and cotangent bundles and tensor product thereof. We have also discussed the Hopf bundle which is endowed with a non-trivial topology. We present now a more formal approach to fiber bundles, still keeping in mind that to try to make the material more accessible to physicists, we occasionally might sacrifice a bit of rigor in favor of simplicity.

9.1.1 Definition A smooth fiber bundle is a set \( \xi = \{ E, M, \pi, F \} \) consisting of manifolds \( E, M \) and \( F \) and a \( C^\infty \) map \( \pi : E \to M \) from \( E \) onto \( M \) satisfying the following properties:

1. The map is Locally trivial. This means that for every point \( p \in M \), there exists an open set \( U \) containing \( p \), together with a diffeomorphism \( \phi : \pi^{-1}(U) \to U \times F \). We call \( F_p = \pi^{-1}(\{p\}) \) the fiber at the point \( p \). We allow the simpler notation \( F_p = \pi^{-1}(p) \) for the fiber at \( p \) and

\[
F|_U = \pi^{-1}(U) = \bigcup_{p \in U} F_p
\]

for the fiber space over the set \( U \). This part of the definition says that locally, the bundle looks like a cross-product; that is, \( \pi^{-1}(U) = U \times F \). The pair \( \{U, \phi\} \) is called a coordinate neighborhood or a coordinate patch, or a local trivialization of the bundle.

2. The fiber space is glued in a smooth manner. More specifically, if \( \{U_i, \phi_i\} \) and \( \{U_j, \phi_j\} \) are two coordinate neighborhoods with nonempty intersection and \( p \in U_i \cap U_j \), then

\[
\phi_{ij} = \phi_i^{-1} \phi_j : U_i \times F \to U_j \times F
\]

is a diffeomorphism. The quantities \( \phi_{ij} \) are called transition functions. The sets \( \{U_i\} \) constitute an open cover of \( M \). (See figure 9.1).

![Fig. 9.1: Fiber Bundle](image)

The manifold \( M \) is called the base space, \( E \) is called the total space or the bundle space, and \( \pi \) the projection map. The notation

\[
F \to E \xrightarrow{\pi} M
\]

or simply

\[
\pi : E \to M
\]

is also used, probably because it is easier to typeset. There is a common abuse of language in calling \( E \) the bundle space, since the bundle is really the set \( \xi \). A Trivial Bundle is one that is a simple cross product, \( E = M \times F \). In that sense, we could view \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \) as bundle with base space \( M = \mathbb{R} \) and fiber \( F = \mathbb{R} \). There would be no advantage to view \( \mathbb{R}^2 \) this way, but it is a bundle anyway. The total space is the union of all the fibers, which locally does look like a
cross product, but globally it might have a non-trivial topology, as in the case of the Hopf bundle. It is possible to start the treatment of fiber bundles in for topological spaces, in which case, the projection map would be a continuous function, the coordinate maps homeomorphisms, and the base space would have the quotient topology. Many authors prefer to start with a formal treatment of topological bundles and then, using the formalism to introduce differentiable structures to construct smooth bundles. In fact, it would be more natural to treat this subject in terms of categories and morphisms and functors, but we will resist the temptation.

9.1.2 Example Let $M = S^1$, $I = [-1, 1]$ and $E = S^1 \times I$. This trivial bundle is just a cylinder with a circular base space and each fiber a copy of the interval $[-1, 1]$ Topologically, we can construct

the bundle by looking at the base space as an interval, say $[-\pi, \pi]$ with the end points identified. We glue the bundle by identifying the fibers at the endpoints, as suggested by the arrows pointing in the same direction in the picture in the left, on figure 9.2. On the other hand, if we identify the vertical edges in opposite direction as shown in rectangle on the right, we get a strip with a twist, which is topologically equivalent to a Möbius band. A smooth parametrization of this surface and a picture rendering the surface is given in 5.1.

9.1.3 Definition A section $s$ of a bundle $\xi = \{E, M, \pi, F\}$ is a smooth map $s : M \to E$ such that

$$\pi \circ s = id_M$$

This is the same concept introduced in the context of the tangent bundle for which the sections are called vector fields (See figure 1.1). We use the same notation $\Gamma(E)$ for the set of all smooth sections. For the tangent bundle, the fibers are copies of $\mathbb{R}^n$, so they are vector spaces of dimension $n$. In such a case we call the bundle a Vector Bundle. The tangent bundle $T(\mathbb{R})^n$ is a trivial bundle. If $p \in U \subset \mathbb{R}^n$, then $\pi^{-1}U \cong U \times \mathbb{R}^n$ and the coordinate patch maps are,

$$\phi : \left(p, a^1 \frac{\partial}{\partial x^1}, \ldots, a^n \frac{\partial}{\partial x^n}\right) \mapsto (p, a^1, \ldots, a^n).$$

This slightly more formal point of view is consistent with our earlier definition of a tangent vector. A vector bundle has more structure that a run-of-the-mill fiber bundle. If $(p, f)$ is an point in the bundle in the intersection of two coordinate patches $\{U_i, \phi_i\}$ and $\{U_j, \phi_j\}$, the transition functions satisfy,

$$\phi_{ij} = (p, \varphi_{ij}(p)f),$$

where $\varphi_{ij}(p) \in GL(n, \mathbb{R})$ gives a linear isomorphism on the fibers. For the tangent bundle, this is yet another way of saying that if we change coordinates near $p$, the components of a tangent vector at $g$ change by an action of an element of $GL(n, \mathbb{R})$ represented by the Jacobian. If the fibers are $k$-dimensional with a $GL(k, \mathbb{R})$ action on the fibers, we just say that the the vector bundle is $k$-dimensional, even though the real total dimension is $(n + k)$. a vector bundle of dimension 1 is called a line bundle. The normal bundle of the sphere $S^2$ in $\mathbb{R}^3$ would be an example of a line bundle.

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1Again, we adopt this notation with reluctance, although is common in the literature. The quantities $\varphi_{ij}(p)$ are matrix-valued, so they really should be written as $\varphi'_{ij}(p)$.
Since the fiber of vector bundles are vector spaces and every vector space has a 0 vector, there is a special section \( s \) such that \( s(p) = (p, 0) \); this is called the **Zero Section**. This is a trivial global section in all vector bundles. The set of all sections of a vector bundle has a natural structure of a vector space, in which the zero-section is the zero-vector. There is no problem finding non-singular global sections of the tangent bundle \( T(R^n) \). However, for submanifolds \( M \) of \( R^n \), there might be topological obstructions to the existence of vector fields that are non-zero everywhere. For example, the reader may be acquainted with the theorem that one can not “comb” a hairy sphere \( S^2 \). As proved by Poincaré, the obstruction in this case is the Euler characteristic which would need to vanish, but for the sphere, \( \chi(S^2) = 2 \).

**9.1.4 Definition**  
*A Covering Space* of a space \( M \) is a bundle \( \xi = \{ E, M, \pi, F \} \) in which every point \( p \in M \), has a neighborhood \( U \subset M \), such that \( \pi^{-1}(U) \), is the disjoint union of sets \( V_k \), each homeomorphic to \( U \). In other words, the fibers of \( U \) are discrete. Here, \( \pi \) is a local homeomorphism and \( M \) has the quotient topology.

**9.1.5 Example**  
Most likely, the first examples of covering spaces students encounter in a course in algebraic topology are associated with the circle \( S^1 \). The map \( \pi : S^1 \rightarrow S^1 \) given by

\[
\pi(e^{i\theta}) = e^{in\theta}, \quad n \in Z^+
\]

gives an \( n \)-sheet covering of \( S^1 \) for each positive integer \( n \). One can envision this covering as a rubber band folded into \( n \) loops around a cylinder. The continuous homomorphism \( \pi : R \rightarrow S^1 \) given by

\[
\pi(t) = e^{it}
\]

is a covering of \( S^1 \) by the real line. A covering that is simply connected as it is the case here, is called a **Universal Covering Space**. This map is the starting point in establishing that the fundamental group of \( S^1 \), also called the first homotopy group \( \pi_1 \), is given by

\[
\pi_1(S^1) \cong Z
\]

A differentiable manifold structure is not required in this example. All that is needed is that \( S^1 = R/Z \) is a topological group, \( R \) is simply connected, and \( Z \) is a discrete subgroup of \( R \). The homotopy equivalence classes are the loops that have the same winding number. The additive group structure is given adding the number of loops of two group elements.

**9.1.6 Example**  
The projective space \( RP^n \) of lines in \( R^n \) is defined by the quotient space of \( S^n \) obtained by identifying antipodal points; that is, the two antipodal points of intersection of the sphere with a line through the origin. The covering space of \( RP^n \) is the real version of the Hopf bundle

\[
\pi : S^n \rightarrow RP^n,
\]

with fiber \( Z_2 \). The group covering transformations consist of the identity and the antipodal map. In this example

\[
\pi_1(RP^n) \cong Z_2.
\]

I have to thank professor P. Gilkey for motivating me to overcome the fear of algebraic topology machinery, with his wonderful illustration of the above, for the case \( n = 3 \). He showed up the first day of classes with a toy consisting of two equilateral triangles, one large, one small. The triangles were connected at corresponding vertices with three separate untangled strings. He set the large triangle on the table and flipped the small triangle in the air by half revolution. He asked for a
volunteer in a class of 5 students to untangle the strings only allowing parallel translations of the small triangle. Clearly, it was not doable. He reset the toy and then flipped the small triangle a full revolution. I got lucky and untangled the strings almost immediately. He proceeded to illustrate by combinations of half-turns and full-turns, until it was almost self-evident, that there were only two possible outcomes. Of course, the key question of the day was, how does one prove that? He explained that the deformations of the shape of the strings by parallel motion were examples of homotopies; that a rotation in space left two antipodal points on a sphere fixed; defined projective space; and concluded that what we had here was a manifestation that the first homotopy group of the projective space had two generators. Four weeks later we had learned enough tools to prove the assertion. I should add that the case $n = 3$

$$\pi : S^3 \to \mathbb{RP}^3$$

is equivalent to the fibration

$$\pi : SU(2) \to SO(3)$$

described in detail by associating a rotation in $\mathbb{R}^3$ with a hermitian matrix with Caley-Klein parameters $Q$ as shown in equation 8.21. $S^3$ is the group manifold of $SU(2)$ and $\mathbb{RP}^3$ is the group manifold of $SO(3)$. As was stated then, $SU(2)$ is a 2-1 covering of $SO(3)$.

**9.1.7 Definition** Let $\xi = \{E, M, \pi, F\}$ and $\xi' = \{E', M', \pi', F'\}$ be smooth vector bundles. A **Vector Bundle Map** is a pair of smooth maps $f_M : M \to M'$ and $f : E \to E'$, such that

1. The diagram 9.3 commutes,

\[\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow \pi & & \downarrow \pi' \\
M & \xrightarrow{f_M} & M'
\end{array}\]

Fig. 9.3: Bundle Map: $f_M \circ \pi = \pi' \circ f$.

2. The map induced by $f$ on the fibers is a linear map.

The meaning of this commuting diagram is that fibers are mapped to fibers. Indeed, if $F_p = \pi^{-1}(p)$ is a fiber at $p \in M$, then,

$$(\pi' \circ f)(F_p) = (f_M \circ \pi)(F_p),$$

$$= (f_M \circ \pi)(\pi^{-1}(p)),$$

$$= f_M(p),$$

so that a point in the fiber of $p \in U \subset M$ lands on a point in the fiber of $f_M(p)$. Thus if makes sense to say that the map induced on the fibers needs to be a linear map of vector spaces. More specifically, if $(p, v) \in E$, then $f(p, v) = (f_M(p), T(p) \cdot v)$, where $T : U \to L(F, F')$ gives a linear map $T(p)$ from the fiber $F_p$ to the fiber $F'_{f_M(p)}$. If these linear maps are vector space isomorphisms and $f_M$ is a diffeomorphism, the bundle map is called a **Vector Bundle Isomorphism**. In this case, the two bundles are essentially the same.

**9.1.8 Definition** Pull-Back Bundle Let $\pi : E \to M$ be a vector bundle with fiber $F$ and let $f : M' \to M$ be a smooth map. We can define a pull-back bundle denoted by $f^*\pi : f^*E \to M'$
by assigning the fiber $F_{p'}$ to each point $p' \in M'$, the corresponding fiber $F_{f(p')}$ at $p = f(p')$. More precisely, if $v \in E$, then

$$f^*E = \{(p', v) \in M' \times E \mid f(p') = \pi(v)\}, \quad f^*\pi(p', v) = p'$$

(9.2)

It clear that $f^*E \subset M' \times E$ is locally trivial as it should be. If $\{U_i, \phi_i\}$ is a cover of $\pi : E \to M$ by coordinate patches, then $\{f^{-1}(U_i)\}$ is a cover of $M'$ with transition functions $f^*\phi_{ij}(p') = \phi_{ij}(p)$. if,

$$M'' \xrightarrow{\phi} M' \xrightarrow{f} M$$

then,

$$(f \circ g)^*E \cong f^*(g^*E)$$

This should elicit memories of the properties of the pull-back of differential forms 2.70.

One of the main application of pull-back bundles is the interesting relation to homotopy.

9.1.9 Definition Two maps $f, g : M' \to M$ are called Homotopic if there exist a map $F : M' \times [0, 1] \to M$, such that

a) $F(x', 0) = f(x)$,

b) $F(x', 01) = g(x)$.

If we let $F_t(x') = F(x, t)$, $t \in [0, 1]$ then the homotopy describes a smooth deformation from $f = F_0$ to $g = F_1$. The main theorem in this regard is that if $f$ and $g$ are homotopic, then the pull-back bundles $f^*E \cong g^*E$ are isomorphic.

9.1.10 Example Suppose $M' = M = \mathbb{R}^n$. Let $f$ be the identity and $g$ be the constant map $g = 0$. Then

$$F_t(x) = F(x, t) = tx,$$

defines a homotopy from $g$ to $f$. A space $M$ in which the identity map is homotopic to a constant map at some point in $M$ is called Contractible. If $M$ is contractible, $f$ is the identity map and $g = p_0$ the map that retracts $M$ to a point, then the theorem says that $E = f^*E \cong g^*E = M \times F$, which proves that if $M$ is contractible, then $E$ is trivial.

We remark that we have already used this homotopy in the the proof of the Poincaré Lemma 7.1.2.

9.2 Principal Fiber Bundles

A smooth principal fiber bundle (PBF) is essentially a fiber bundle in which the fibers are Lie groups, or manifolds on which there is a free and transitive action by a Lie group. In that sense, one can view the base space as the parameter space for a family of fibers $F$ where at any point $p$ in the base space, the fiber $F_p$ is diffeomorphic to a Lie group. More formally, we have the following definition

9.2.1 Definition A smooth principal fiber bundle is a set $\xi = \{E, M, \pi, F, G\}$, where $F \to E \xrightarrow{\pi} M$ is a fiber bundle, and $G$ is a Lie group acting on $E$ freely along the fibers.

As a reminder, $G$ acts on $E$ on the right, if there exists a smooth map $\mu : E \times G \to E$ given by $(b, g) \to bg$. The action is free along the fibers if the only element that acts as an identity is the the identity and the action is transitive if any two points on the same fiber are connected by an element of the group. Thus, if the action is free and transitive, the fibers are diffeomorphic to the orbits of the group. This means that the bundles

$$\begin{align*}
E & \xrightarrow{id} E, \\
\pi & \downarrow \quad pr \downarrow \\
M & \xrightarrow{\sim} E/G
\end{align*}$$
are isomorphic. Here, \( pr \) is the natural projection of \( E \) onto its cosets.

### 9.2.2 Example Bundle of Frames

One of the most important principal fiber bundles is the bundle of frames. We hope that the following discussion does not obscure simplicity for the sake of formalism. The bundle of frames is the structure one gets by attaching to each point on a manifold, the space of all possible frame fields of tangent vectors. The structure group is \( Gl(n, \mathbb{R}) \), which acts on the fibers each point on the manifold, by matrix multiplication that changes one frame into another. Parallel transport of vectors and frames on the manifold, correspond to choosing a horizontal subspace of the tangent space of the bundle. Choosing a horizontal subspace in the bundle is then equivalent to choosing a connection.

### 9.2.3 Definition

Let \( M \) be a smooth manifold of dimension \( n \), and let \( \{ (\phi_i, U_i) \} \) be an atlas of coordinate charts covering \( M \). As usual, we label the coordinates of \( p \in U_i \) as \( (x^1, \ldots, x^n) \). The **Frame Bundle** \( B(M) \) is defined as the bundle \( \pi : B(M) \to M \), where

\[
B(M) \equiv \{(p, e_1, e_2, \ldots, e_n) : p \in M, \beta = (e_1, e_2, \ldots) \text{ is a basis for } T_p(M) \\}
\]

The projection map \( \pi \) is given by

\[
\pi : F(M) \to M, \\
(p, e_1, \ldots, e_n) \xrightarrow{\pi} p.
\]

Let \( (\phi, U) \) be a coordinate chart with coordinates \( (x^1, \ldots, x^n) \), with \( \overline{U} = \pi^{-1}U \). As shown in figure 9.4, we can lift the chart in \( M \)

\[
\phi : U \to \mathbb{R}^n
\]

to a chart in \( B(M) \),

\[
\overline{\phi} : \overline{U} \to \mathbb{R}^n \times \text{Gl}(n, \mathbb{R}) \simeq \mathbb{R}^{n+n^2},
\]

as follows. Let \( \{ \partial_i = \frac{\partial}{\partial x^i} \} \) be the standard basis for the tangent space \( T_p M \), associated with the coordinates \( p = (x^1, \ldots, x^n) \). Let \( b \in B(M) \) be a point with coordinates

\[
b = (p, e_1, \ldots, e_n)
\]

on the fiber \( F_p \). Then, as shown in equation 3.1 there exists a matrix \( A \in \text{Gl}(n, \mathbb{R}) \), with

\[
e_i = A^j_i \partial_j
\]

\[
\pi^{-1}(U_i) \xrightarrow{\overline{\phi}_i} \mathbb{R}^n \times \text{Gl}(n, \mathbb{R})
\]

![Fig. 9.4: Bundle of Frames](image)
Matrix multiplication by \( A \) is on the right, but, as in 3.1, we choose to write the equation as above, to make it clear that \( \partial_i \) is not acting as a differential operator on the components of \( A \). Thus, \( b \in F_p \) can be written as
\[
b = (p, A^i_j \frac{\partial}{\partial x^j}).
\]
The bundle coordinate patch on \( U \) is defined by
\[
\overline{\phi}_i (p, e_1, \ldots, e_n) = (p, A^i_j)
\]
Here, we identify the matrix \( A \in \text{Gl}(n, \mathbb{R}) \) with a point in \( \mathbb{R}^{n^2} \), where the coordinates in \( \mathbb{R}^{n^2} \) are given by the entries in the column vectors of \( A \). The standard coordinates of \( \text{Gl}(n, \mathbb{R}) \) are given by the matrices \( X^i_j \) that have a 1 entry in the \( i \)th row, \( j \)th column, and a 0 entry everywhere else.

The right action \( \mu : B(M) \times \text{Gl}(n, \mathbb{R}) \to B(M) \) along the fibers is given by
\[
\mu : (b, g) \to bg = (p, f_i = e_j g^i_j); \quad g \in \text{Gl}(n, \mathbb{R}).
\]
Given two overlapping charts in \( B(M) \)
\[
\overline{\phi}_i : \pi^{-1}(U_i) \to U_i \times \text{Gl}(n, \mathbb{R}),
\]
\[
\overline{\phi}_j : \pi^{-1}(U_j) \to U_j \times \text{Gl}(n, \mathbb{R}).
\]
with transition functions \( \overline{\phi}_{ij} \) on \( U_i \cap U_j \), given by \( \overline{\phi}_{ij} = (\overline{\phi}_i)^{-1} \overline{\phi}_j \). The group action on the fibers satisfies,
\[
(bg_1)g_2 = b(g_1 g_2).
\]
The atlas \( (\overline{\phi}_i, U_i) \) thus gives \( B(M) \) the structure of a differentiable manifold. A local section of the bundle \( s_U \in \Gamma(B(m)) \) represents a smooth choice of a family of frames at points in \( U \), with \( \pi \circ s_U = \text{id} \).

**9.2.4 Example** Let \( G \) be Lie group and \( H \subset G \) a compact subgroup. Then \( \pi : G \to G/H \), where \( \pi \) is the projection map onto the orbit space, is an \( H \)-bundle. This is one of the early results in the theory of fiber bundles, first proved by H. Samelson in 1941.

**9.2.5 Example** The projective space fibration
Recall that the complex projective space \( \mathbb{C}P^n \) is the quotient space \( \mathbb{C}^n/\sim \), where \( a, b \in \mathbb{C}^n \) are equivalent, \( a \sim b \), if there exists a \( c \in S^1 \) such that \( a = bc \). That is, \( \mathbb{C}P^n \) is the space of lines through the origin. Then,
\[
S^{2n+1} \xrightarrow{S^1} \mathbb{C}P^n
\]
is a principal \( U(1) \) bundle. The special case for \( n = 1 \) is the ubiquitous Hopf map 8.26.

**9.2.6 Example** The \( SO(n) \) bundle
The group of special orthogonal matrices
\[
SO(n) = \{ A \in M_{n \times n}(\mathbb{R}) : A^{-1} = A^T, \det A = 1 \}
\]
acts transitively on the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). The subgroup that leaves the north pole fixed, that is, the isotropy subgroup of the point \( e_1 = (1, 0, \ldots, 0) \), is the set of matrices of the form
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}, \quad B \in SO(n-1).
\]
Thus, \( SO(n)/SO(n-1) \) is diffeomorphic to \( S^{n-1} \). This, with the projection map
\[
\pi : SO(n) \xrightarrow{SO(n-1)} S^{n-1}
\]
9.2.7 Example The $U(n)$ bundle

The group of unitary matrices

$$U(n) = \{ A \in M_{n \times n}(\mathbb{C}) : A^{-1} = A^\dagger \}$$

acts transitively on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. The subgroup that leaves the north pole fixed, that is, the isotropy subgroup of $e_1 = (1, 0, 0, \ldots, 0)$, is the set of matrices of the form,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}, \quad B \in U(n-1).$$

Thus, $U(n)/U(n-1)$ is diffeomorphic to $S^{2n-1}$. This, with the projection map

$$\pi : U(n) \xrightarrow{U(n-1)} S^{2n-1}$$

constitutes a $U(n-1)$ bundle. The bundle structure above is also true if the unitary groups is replaced by special unitary groups, which corresponds to requiring the matrices $A$ to have $\det A = 1$, or equivalently, to picking an orientation of the frames.

9.2.8 Example The Grassmannian

Let $F$ denote, either of the vector spaces $\mathbb{R}$ or $\mathbb{C}$. The space of orthonormal $k$-frames in $F^n$ is called the Stiefel manifold $V_k(F^n)$. We can characterize the Stiefel manifolds as the set of $n \times k$ matrices,

$$V_k(F^n) = \{ A \in M_{n \times k} : AA^\dagger = I_k \}, \quad (9.3)$$

where $I_k$ is the $k \times k$ identity matrix. We interpret $I_k$ as a matrix $I_k = [e_1, e_2, \ldots, e_k]$ of orthonormal basis vectors. Viewed as representing linear transformations $L(F^k, F^n)$, the matrices $A$ have rank $k$.

Analogous to the construction of projective spaces, we say that two matrices $A, B, \in V_k(F^n)$ are equivalent, $A \sim B$, if there exists a $k \times k$ orthogonal (or unitary in the complex case) matrix $C$ such that $A = BC$. The Grassmannian is defined as

$$Gr(k, F^n) = V_k(F^n)/\sim. \quad (9.4)$$

The Grassmannian is the space of $k$-planes in $F^n$. In $\mathbb{R}^n$ the projection map

$$\pi : Gr(k, \mathbb{R}^n) \xrightarrow{O(k)} V_k(\mathbb{R}^n), \quad (9.5)$$

is a principal bundle with fiber group $O(k)$. The group $O(n)$ acts transitively on $V_k(\mathbb{R})$, and the isotropy subgroup of the $n \times k$ matrix

$$A = \begin{bmatrix} I_k \\ 0 \end{bmatrix},$$

consists of the matrices in $O(n)$ of the form

$$\begin{bmatrix} I_k & 0 \\ 0 & B \end{bmatrix},$$

where $B \in O(n-k)$. Thus, we have a diffeomorphism,

$$O(n)/O(n-k) \cong V_k(\mathbb{R}^n),$$

and we can write the Grassmannian bundle as,

$$\pi : Gr(k, \mathbb{R}^n) \xrightarrow{O(k)} O(n)/O(n-k). \quad (9.6)$$
Equivalently, we have
\[ Gr(k, \mathbb{R}^n) \cong \frac{O(n)}{O(k) \times O(n-k)} \]  
(9.7)

In the complex and quaternionic vector spaces, we have
\[ Gr(k, \mathbb{C}^n) \cong \frac{U(n)}{U(k) \times U(n-k)}, \]
\[ Gr(k, \mathbb{H}^n) \cong \frac{Sp(n)}{Sp(k) \times Sp(n-k)}, \]  
(9.8)

The Grassmannian of lines in \( \mathbb{R}^3 \) is the projective space \( \mathbb{RP}^2 \) and the Grassmannian of planes is the same since to every line through the origin, there corresponds a unique orthogonal plane. Hence, the simplest Grassmannian that is not a projective space is the space \( Gr(2, \mathbb{R}^4) \) of planes in \( \mathbb{R}^4 \), or equivalently, the space of projective lines in \( \mathbb{CP}^3 \). The space \( \mathbb{CP}^3 \) of projective lines in \( \mathbb{C}^4 \) with a Hermitian metric of signature \(+ + --\) is the base space for Twistor theory; in this case the symmetry group preserving the metric is \( SU(2, 2) \) which is the double cover of the Conformal group. It has been cited by R. Penrose \[27\], that the geometrical foundation for the theory can be traced back to the work of Plücker and Klein on subspaces of planes. In view of this historical setting, we present a brief summary of the parametrization of two dimensional subspaces of \( \mathbb{R}^4 \) used by Plücker.

Given two vectors \( v = (v^1, v^2, v^3, v^4) \) and \( w = (w^1, w^2, w^3, w^4) \), we can determine a plane by a linear map \( L(\mathbb{R}^2, \mathbb{R}^4) \) with matrix representation given by a \( 4 \times 2 \) matrix \( A \) of rank 2, whose column vectors are \( v^T \) and \( w^T \). That is,
\[ A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \end{bmatrix}, \]

The Grassmannian \( Gr(2, \mathbb{R}^4) \) is the space of equivalence classes of such matrices. Plücker coordinates \( p_{ij} \) are defined by the determinants of pairs of rows \( i \) and \( j \).

\[ p_{ij} = v_i w_j - v_j w_i, \]
\[ = \begin{bmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{bmatrix}. \]

We may view the antisymmetric matrix \( P = (p_{ij}) \) as a element of the Lie algebra \( \mathfrak{so}_4 \), and the quantities \( (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) \) as homogenous coordinates in \( \mathbb{RP}^5 \). A short computation using the antisymmetry property in the definition of \( p_{ij} \), yields,

\[ p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0 \]

The square of this entity is equal to \( \det(P) \). This equation, up to a constant, represents a quadric hypersurface \( Q \) in \( \mathbb{RP}^5 \). The permutation signs of the equation give a hint that there is a duality lurking somewhere, namely the orthogonal subspaces. Introducing independent coordinates

\[ p_{12} = X + R, \quad p_{34} = X - R, \]
\[ p_{13} = S - Y, \quad p_{24} = S + Y, \]
\[ p_{14} = Z + T, \quad p_{23} = Z - T, \]

the equation becomes,

\[ (X^2 + Y^2 + Z^2) - (R^2 + S^2 + T^2) = 0 \]
By re-scaling the homogenous coordinates we can write the quadric $Q$ as,

$$X^2 + Y^2 + Z^2 = 1, \quad R^2 + S^2 + T^2 = 1.$$  

Thus, $Q$ has the topology of a torus $S^2 \times S^2$.

**9.2.9 Example $S^3$**

The $U(n)$ bundle with $n = 2$ gives $U(2)/U(1) \cong S^3$.

The Grassmannian $Gr(1, \mathbb{C}^2) \cong \mathbb{CP}^1 \cong S^3$, gives the bundle

$$U(2)/U(1) \cong U(2)/U(1) \times U(1),$$

$$S^3 \rightarrow S^2.$$

This view of the Hopf bundle is the foundation for the argument that a three-sphere $S^3$ is homeomorphic to the union of two solid tori whose intersection are their common boundaries with topology $S^1 \times S^1$. As a static picture, figures such as in 8.2 are as good as it gets in trying to visualize the union of the two tori.

**9.2.10 Definition Associated Vector Bundle**

Let $\xi = \{E, M, \pi, F, G\}$ be a PFB. Suppose there is a vector space $V$ on which $G$ acts on the left. Let $(e, v) \in E \times V$ and $g \in G$. We can define an action on the cross product $E \times V$ by

$$(e, v)g \rightarrow (eg, gv).$$

Denote $(E \times V)/G$ by $E \times_G V$. Then the natural projection map,

$$p : E \times_G V \rightarrow M$$

defines a vector bundle called the associated vector bundle.

**9.2.11 Example** If $E = B(M)$ is the bundle of frames, and $V = \mathbb{R}^n$, the associated vector bundle is the tangent bundle.

### 9.3 Connections on PFB’s

We have noted earlier in that in Riemannian manifold, there exists a unique, torsion-free connection (See theorem [6.9]). The metric on the manifold allows us to define orthonormal frames, and the connection gives a prescription on how to parallel transport tangent vectors and frames. We now present the bundle viewpoint of connections. We will do this quite generally, but the reader should keep in mind the bundle of frames as the model space. There is a learning curve for the mathematical formalism, but the idea is very intuitive. We illustrate this with bundle of frames. Given a point $p$ on a manifold $M$, the fiber of the bundle of frames $F(M)$ consists of the point and all the frames at that point. The action of the general linear group along the fibers, transforms one frame at $p$ onto another frame at $p$. Since the right action of the group is transitive and effective, there is a natural way to identify a vertical direction in the tangent space at a point $b$ on the bundle, namely, a vertical tangent vector at $b$ on the frame bundle, corresponds to a frame at the point $p$ in the base manifold. If the frames are restricted to be orthonormal, the picture is the same, but one has to reduce the group to the orthogonal group. Thus, the action of the group tells us how move frames along the fibers, but it does not tell us how to move a frame to the fiber of a nearby point on the manifold. This requires more structure, namely a connection. In the case of Riemannian
manifold, the natural structure is provided by the Levi-Civita connection, which quantifies how to parallel transport a frame along any particular curve. Lifting the curve and the moving frame on that curve to the bundle would then yield a section of the bundle that we could identify as a horizontal direction. Thus, the basic idea of connection on a principal fiber bundle amounts to choosing horizontal direction for the tangent space of the bundle.

Let \( \xi = \{ E, M, \pi, F, G \} \) be a principal fiber bundle and let the coordinate chart \( U_i \) give a local trivialization of the bundle

\[
\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times G,
\phi_i(b) = (\pi(b), \varphi_i(p))
\]

On the overlap \( U_i \cap U_j \) of two coordinate charts with

\[
\varphi_{ij} : U_i \cap U_j \rightarrow G,
p \mapsto \varphi_{ij}(p)
\]

the transition functions \( \varphi_{ij} = \varphi_i^{-1} \cdot \varphi_j \) give a map

\[
s_i : U_i \rightarrow \pi^{-1}(U_i),
s_j : U_j \rightarrow \pi^{-1}(U_j),
\]

are sections of the bundle over sets with \( U_i \cap U_j \neq \emptyset \), then on the overlap the sections are related by

\[
s_j(p) = s_i(p) \varphi_{ij}(p). \tag{9.9}
\]

9.3.1 Definition Let \( \xi = \{ E, M, \pi, F, G \} \) be a principal fiber bundle and let \( b \in F_p \) be a point on the bundle over the fiber \( F_p \). A tangent vector \( Y \in T_b E \) is called Vertical if

\[
\pi_* Y = 0.
\]

The vector space \( V_b \) of all such vectors is called the Vertical Subspace of \( T_b E \). If \( X \in \mathfrak{g} \), that action of the group \( G \) on \( E \) induces a fundamental vector field \( Y = \sigma(X) \) as defined by equation \[7.66\]. Such a vector field would then yield a vertical vector at any point \( b \in F_p \). We now introduce the following,

9.3.2 Definition An connection \( \Gamma \) on a principal fiber bundle is a choice of a subspace \( H_b \) of \( T_b E \), such that

a) For each \( b \in E \), we have \( T_b E = V_b \oplus H_b \),
b) \( R_{g_*} H_b = H_{bg} \),
c) \( H_b \) is a \( C^\infty \) distribution.
The vector space $H_b$ is called the **Horizontal Subspace** of $T_bE$ at $b$, and tangent vectors in this space are called **Horizontal**. Condition (a) says that any tangent vector $Y \in T_bE$ can be split as a sum of a vertical and a horizontal component

$$Y = Y_V + Y_H$$

Condition (b) says that the distributions $b \to H_b$ is right-invariant under the action of $G$. A connection on a principal fiber bundle as defined above is called an **Ehresmann Connection**.

Given a connection $\Gamma$ we define a Lie-Algebra valued one form

$$\omega_b : T_bE \to \mathfrak{g}$$

that for each tangent vector $Y \in T_bE$, it assigns the unique vector vector $X \in \mathfrak{g}$ whose fundamental vector field $\sigma(X)$ is the vertical component of $Y$. In the language of distributions, $H_b$ is the kernel of the map,

$$H_b = \{ Y \in T_bE : \omega(Y) = 0 \}$$

Since the map $\omega$ is onto for every $b \in E$, the kernel $H_b$ is a linear subspace of $T_bE$ with dimension equal to the dimension of $M$ with $\omega(Y_H) = 0$. To be clear, we are saying,

$$\omega(Y) = \begin{cases} X & \text{if } Y = \sigma(X) \\ 0 & \text{if } Y \text{ is horizontal} \end{cases}$$

Here we assume that the space $H_b$ annihilated by $\omega$ is a $C^\infty$ distribution in the sense of Frobenius 7.36.

Motivated by the formula for the pull-back of the Maurer-Cartan form 7.58 and by equation 7.67 we can equivalently characterize an an Ehresmann connection, by the conditions stated in the following,

![Fig. 9.5: Ehresmann Connection](image)

**9.3.3 Theorem** An **Ehresmann Connection** on a principal fiber bundle $\xi = \{ E, M, \pi, F, G \}$, is a smooth, Lie Algebra valued form $\omega \in \Omega(E)$ that satisfies the following conditions:

a) $\omega(\sigma(X)) = X$, for all $X \in \mathfrak{g}$,

b) $R_g^*\omega(Y) = \text{Ad}_{g^{-1}}\omega(Y)$, for all $g \in G$, and all tangent vectors $Y$ on $E$.

Part (a) is immediate from the definition. Part (b) essentially follows from the fact that the fundamental vector field associated with $R_g^*\omega$ is $\text{Ad}_{g^{-1}}\omega$. $Y$ can be split uniquely into a horizontal and a vertical component. If $Y$ is horizontal, then both sides of the equation are zero. If $Y$ is vertical,
it is the fundamental vector field $\sigma(X)$ for some $X \in \mathfrak{g}$. Then

$$(R^g_\omega)_b(Y) = \omega_{bg}(R_g Y),$$

$$= \omega_{bg}(\sigma(Ad_g^{-1} X)) \text{ by equation 7.67}$$

$$= Ad_g^{-1} X$$

$$(R^g_\omega)_b(Y) = Ad_g^{-1}(\omega_{bg}(Y)). \quad (9.10)$$

We now show how to pull down a connection $\omega$ on a principal fiber bundle $E$ down to a family of local connections on the manifold. Here we adapt the procedure from Kobayashi and Nomizu [17], with apologies to the authors for spoiling the elegance of their proof, for the sake of clarity provided by more details. Let $\{U_i\}$ be an open cover of $M$ with coordinate charts $\{(\phi_i, U_i)\}$. For each $i$ let $s_i(p)$ be the section over $U_i$ defined by

$$s_i(p) = \phi_i^{-1}(p, e), \quad p \in U_i, \quad e = id \in G$$

Given a connection $\omega$ in the bundle $E$, define local connections on $M$ using the pullback of the section maps. Thus, over each pair of overlapping charts $U_i \cap U_j \neq \emptyset$, we define,

$$\omega_i = s_i^* \omega, \quad \omega_j = s_j^* \omega$$

Since the transition functions map

$$\phi_{ij} : U_i \cap U_j \to G,$$

we can pull-back forms. In particular, let $\theta$ be the left invariant Maurer-Cartan form on $G$. We define a $\mathfrak{g}$-valued form on $U_i \cap U_j$ by

$$\phi_{ij}^* : T G \to T(U_i \cap U_j),$$

$$\theta_{ij} = \phi_{ij}^* \theta$$

For applications to gauge theory, following result is very important,

9.3.4 Theorem On $U_i \cap U_j \neq \emptyset$, the local forms $\omega_i$ and $\theta_{ij}$ on $M$ satisfy the condition

$$\omega_j = (Ad_{\phi_{ij}^{-1}}) \omega_i + \theta_{ij} \quad (9.11)$$

Proof: Given a point $p \in U_i \cap U_j$, let $X_p \in T_p(U_i \cap U_j)$ be a tangent vector to a curve $x(t)$, with $x(0) = p$ and $X = x'(t)$. Then the transition equation for the sections reads

$$s_j(x(t)) = s_i(x(t)) \phi_{ij}(x(t)).$$

The push-forward

$$s_{j*}(X_p) : T_p(U_i \cap U_j) \to T_{s_j(p)}E$$

is the image of $(s_{i*}(X), \phi_{ij*}(X))$ under the isomorphism

$$T_{s_i(p)}E \otimes T_{\phi_{ij}(p)}G \cong T_{s_j(p)}E$$

More specifically, taking the derivative $d/dt$ and evaluating a $t = 0$ as done with the product rule formula 6.2 we get

$$\frac{d}{dt}|_{t=0} [s_j(x(t))]|_{t=0} = \frac{d}{dt}|_{t=0} [s_i(x(t)) \phi_{ij}(x(t))]|_{t=0},$$

$$\frac{d}{dt}|_{t=0} [s_i(x(t)) \phi_{ij}(x(t))]|_{t=0} + \frac{d}{dt}|_{t=0} [s_i(x(t)) \phi_{ij}(x(t))]|_{t=0},$$

$$s_{j*}(X) = \frac{d}{dt}|_{t=0} [R_{\phi_{ij}(p)}] s_i(x(t))|_{t=0} + \frac{d}{dt}|_{t=0} [s_i(x(t)) \phi_{ij}(x(t))]|_{t=0},$$

$$= R_{\phi_{ij}(p)} s_i(X) + s_{i*}(p) \phi_{ij*}(X))$$
We now apply $\omega$ to both side remembering the general definition of the pullback $s^*\omega(X) = \omega(s_*X)$. We get

$$\omega(s_{ij*}(X)) = \omega(R_{\varphi_{ij*}}(p) s_{i*}(X)) + \omega(s_{i*}(p) \varphi_{ij*}(X)),
$$

$$s_{ij*}^\omega(X) = R_{\varphi_{ij*}}^\omega(s_{i*}(X)) + \omega(s_{i*}(p) \varphi_{ij*}(X)),
$$

$$\omega_j(X) = R_{\varphi_{ij*}}^\omega(\omega_1(X)) + \omega(s_{i*}(p) \varphi_{ij*}(X)),
$$

$$\omega_j(X) = (Ad_{\varphi_{ij}}^\omega)\omega_1(X) + \omega(s_{i*}(p) \varphi_{ij*}(X)),
$$

where in the first term on the right, we have used the condition 9.10 for an Ehresmann connection. The second term on the right is a bit trickier. We see from the diagram below

\[
\begin{array}{c}
T_p \left( U_i \cap U_j \right) \xrightarrow{\varphi_{ij*}} T_{\varphi_{ij*}(p)} G \\
\downarrow \\
p \in (U_i \cap U_j) \xrightarrow{\varphi_{ij}} \varphi_{ij}(p) \in G,
\end{array}
\]

that $\varphi_{ij}(x(t))$ is a curve in $G$, whose differential map sends the tangent vector $X$ at $p$ to a tangent vector in $G$ at $\varphi_{ij}(p)$

$$X_p \xrightarrow{\varphi_{ij*}} \varphi_{ij*}(X)|_{\varphi_{ij}(p)}$$

On the other hand, one can think of $s_*(p)$ as a map from $G$ to $E$ given by

$$s_i : G \to E,
$$

$$g \xrightarrow{s_i(p)} s_i(p)g.$$

Thus $s_{ij*}(p) \varphi_{ij*}(X)$ is the push-forward of $\varphi_{ij*}(X)$ to $TE$ by the Jacobian map $s_{ij*} : TG \to TE$. If $Y \in g$ is the left-invariant vector \footnote{Specifically, $Y = L_{\varphi_{ij*}(p)}^{-1}(\varphi_{ij*}(X)) \varphi_{ij*}(p)$. The notation is a bit cluttered but the concept is rather simple. The vector $Y$ generates a one parameter subgroup of $G$ whose tangent vector at $\varphi_{ij}(p)$ coincides with $\varphi_{ij*}(X)$. The integral curve of $Y$ induces a fundamental vector field $\sigma(Y)$ on the fiber $F_p$} in $G$ such that $Y = \varphi_{ij*}(X)$ at $\varphi_{ij}(p)$, then, we have

$$\theta(Y) = \theta(\varphi_{ij*}X) = Y$$

The image of $Y$ under $s_{ij*}(p)$ corresponds to a fundamental vector $\sigma(Y)$, so, by the definition of an Ehresmann connection

$$\omega(s_{ij*}(p) \varphi_{ij*}(X)) = \omega(\sigma(Y)),$$

$$= Y,$$

$$= \theta(\varphi_{ij*}(X)),$$

$$= \varphi_{ij*}^\circ \theta(X),$$

$$= \theta_{ij}(X)$$

The converse of the theorem is obtained by reversing the argument above. This concludes the proof.

If $G$ is a matrix group, and on the overlap of the charts $U_i \cap U_j$ denote the transition functions $\varphi_{ij}(p)$ by a matrix transformation $B$, then equation 9.11 reads

$$\omega_j = B^{-1}\omega_1 B + B^{-1}dB,$$

which we immediately recognize as the transformation law for an affine connection in the manifold. Since this holds for any pair of overlapping patches, we see that the connection in the bundle gives rise to a family of connections on $M$ which piece together as we desire on any overlap.
References
Bibliography


