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Frames, Uncertainty principles and Time-Frequency Analysis.

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1. Bases and Frames for $\mathbb{R}^n$

Definition 1.1. *If two sequences in* $\mathbb{R}^n$, $\{\mathbf{e_i}\}_{i=1}^n$ *and* $\{\mathbf{f_i}\}_{i=1}^n$ *have the property that any* $\mathbf{v}$ *in* $\mathbb{R}^n$*,can be written as*

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{e_1} \rangle \mathbf{f_1} + \langle \mathbf{v}, \mathbf{e_2} \rangle \mathbf{f_2} + \langle \mathbf{v}, \mathbf{e_3} \rangle \mathbf{f_3} + \cdots + \langle \mathbf{v}, \mathbf{e_n} \rangle \mathbf{f_n} = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{e_i} \rangle \mathbf{f_i}$$

then we call $\{\mathbf{e_i}\}_{i=1}^n$ *a* **basis** *and* $\{\mathbf{f_i}\}_{i=1}^n$ *a* **dual basis**.

Definition 1.2. *If two sequences in* $\mathbb{R}^n$, $\{\mathbf{e_i}\}_{i=1}^m$ *and* $\{\mathbf{f_i}\}_{i=1}^m$ *with* $m \geq n$ *have the property that any* $\mathbf{v}$ *in* $\mathbb{R}^n$*,can be written as*

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{e_1} \rangle \mathbf{f_1} + \langle \mathbf{v}, \mathbf{e_2} \rangle \mathbf{f_2} + \langle \mathbf{v}, \mathbf{e_3} \rangle \mathbf{f_3} + \cdots + \langle \mathbf{v}, \mathbf{e_m} \rangle \mathbf{f_m} = \sum_{i=1}^{m} \langle \mathbf{v}, \mathbf{e_i} \rangle \mathbf{f_i}$$

then we call $\{\mathbf{e_i}\}_{i=1}^m$ *a* **frame** *and* $\{\mathbf{f_i}\}_{i=1}^m$ *a* **dual frame**.

2. Bases and Frames for Functions $(L^2(\mathbb{R}))$

In 1952 Duffin and Schaeffer **DS** defined frames:

Definition 2.1. *A sequence* $(f_n)_{n\in\mathbb{Z}}$ *of elements of a Hilbert space* H *is called a* **frame** *if there are constants* $A, B > 0$ *such that*

$$A\|f\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad \textit{for all} \quad f \in H.$$

Definition 2.2. *A* **basis** *for the function space* $L^2(\mathbb{R})$ *is a sequence of functions* $\{e_{m,n}(x)\}_{m,n\in\mathbb{Z}}$ *so that for all* $f(x) \in L^2(\mathbb{R})$ *there exist a* **unique** *sequence of numbers* $\{c_{m,n}\}_{m,n\in\mathbb{Z}}$

$$f(x) = \sum_{m,n\in\mathbb{Z}} c_{m,n} e_{m,n}.$$

Example: Let $e(x) = \mathbf{1}_{[\frac{1}{2},-\frac{1}{2})}$ *then*

$$e_{m,n}(x) = e(x-m)e^{2\pi inx}$$

is a basis for $L^2(\mathbb{R})$

Definition 2.3. *A* **frame** *for the function space* $L^2(\mathbb{R})$ *is a sequence of functions* $\{g_{m,n}(x)\}_{m,n\in\mathbb{Z}}$ *so that for all* $f(x) \in L^2(\mathbb{R})$ *there exist a* **not necessarily unique** *sequence of numbers* $\{d_{m,n}\}_{m,n\in\mathbb{Z}}$

$$f(x) = \sum_{m,n\in\mathbb{Z}} d_{m,n} g_{m,n}(x).$$

Example: Let $g(x) = \cos(\pi x)\mathbf{1}_{[\frac{1}{2},-\frac{1}{2})}$ *then*

$$g_{m,n}(x) = g(x-m)e^{2\pi inx}$$

is a basis for $L^2(\mathbb{R})$

3. Fourier transform and the Heisenberg Uncertainty Principle

Definition 3.1. *For $f(x) \in L^2(\mathbb{R})$ define the Fourier transform*

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i w x} dx$$

Engineer interpretation:
w is a frequency and $\hat{f}(w)$ is the amplitude of the frequency.

Physicist interpretation:
w is the momentum variable and $\dfrac{|\hat{f}(w)|^2}{\int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw}$ is the probability distribution for the momentum

Theorem 3.2. *Heisenberg inequality: If $f(x) \neq 0 \in L^2(\mathbb{R})$*

$$\int_{-\infty}^{\infty} w^2 |\hat{f}(w)|^2 dw \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \geq \frac{1}{4\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

4. Time Frequency analysis

Does there exist a function $g(x) \in L^2(\mathbb{R}) so$

$$\int_{-\infty}^{\infty} w^2 |\hat{g}(w)|^2 dw \int_{-\infty}^{\infty} x^2 |g(x)|^2 dx < \infty$$

AND $g(x-k)e^{2\pi i n x}$is a **basis** for $L^2(\mathbb{R})$? Answer: NO

Does there exist a function $g(x) \in L^2(\mathbb{R}) so$

$$\int_{-\infty}^{\infty} w^2 |\hat{g}(w)|^2 dw \int_{-\infty}^{\infty} x^2 |g(x)|^2 dx < \infty$$

AND $g(x-k/2)e^{2\pi i n x}$is a **frame** for $L^2(\mathbb{R})$? Answer: **Yes**