# Developing a Discrete Fractional Fourier Transform

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Seminar in Mathematics

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# Goal

**Objective**

*Develop a Discrete Fractional Fourier transform.*

- Fourier transforms are useful analysis tools in many areas of optics and signal processing applications. The Fractional Fourier transform is an extension that can be applied when a time-frequency rotation of a signal is helpful.

# Motivation

The motivation for the Fourier transform comes from the study of Fourier series. In the study of Fourier series, complicated periodic functions are written as the sum of simple waves mathematically represented by sines and cosines.

Due to the properties of sine and cosine it is possible to recover the amount of each wave in the sum by an integral. In many cases it is desirable to use Euler's formula, which states that $e^{2\pi i\theta} = \cos(2\pi\theta) + i\sin(2\pi\theta)$, to write Fourier series in terms of the basic waves $e^{2\pi i\theta}$

**Jean Baptiste Joseph Fourier** (1768 - 1830) was a French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations.

### Example: Heat Equation

*The general form of the heat equation:*

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty, t > 0) \qquad u(x,0) = f(x)$$

*has the solution*

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\omega) e^{-\alpha\omega^2 t} e^{i\omega x} d\omega$$

*by use of the Fourier Transform*

The subsequent development of Fourier's discoveries is now known as **harmonic analysis**.[LY07]

**Definition**

*The* ***Continuous Fourier Transform*** *is a unitary map defined on* $L^2(\boldsymbol{R}) = \{f(x) \mid \int_{\mathbb{R}} |f(x)|^2 dx < \infty\}$ *by the integral transform*

$$F(f(x))(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx$$

- Operational Properties [Asm05]

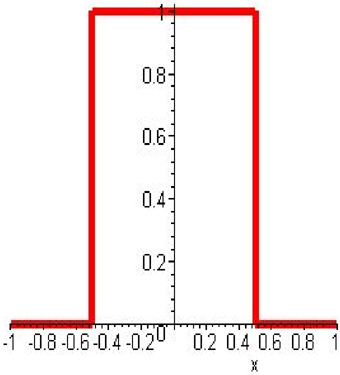
$$F(f(x-k))(\omega) = e^{-ik\omega}\hat{f}(\omega) \qquad F(e^{ikx}f(x))(\omega) = \hat{f}(\omega - k)$$

# Example

An example of applying the Fourier Transform to a function:

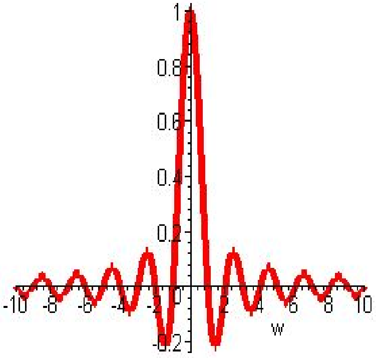
**Piecewise**

$$f(x) = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$$




**Sinc Function**

$$\hat{f}(\omega) = \frac{sin(\pi\omega)}{\pi\omega}$$



# Preliminaries

To be able to use the Fourier Transform in applications we need...

**Definition**

*The* ***Discrete Fourier Transform*** *in* ***Sum Form*** *for* $\mathbf{x} \in \mathbb{C}$

$$X_j = \sum_{k=0}^{N-1} x_k e^{2\pi ijk/N}$$

*in* ***Matrix Form***

$$F_{j,k} = \frac{1}{\sqrt{N}} e^{\frac{2\pi ijk}{N}}$$

*with* $\{j, k = 0 \ldots N-1\}$

# Example Fourier Matrix

Let us define the inverse DFT as the conjugate transpose, $F^* = \overline{F^T}$ and $FF^* = I$. Then, the Fourier matrix with $N = 4$ is:

$$F = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

**Background**

- Transforms one function into another (called the *frequency domain representation* or *DFT* of the original function)

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- Requires an input function that is discrete and whose non-zero values have a limited (finite) duration (i.e. sampling a continuous function like a person's voice)

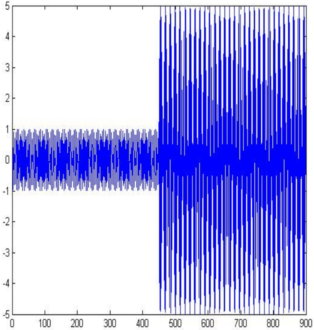
# Discrete Fourier Transform

## Background

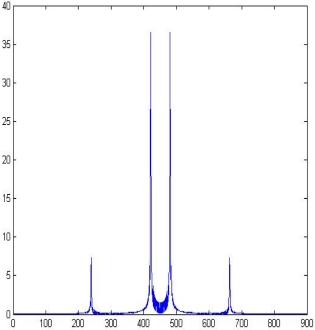
- Transforms one function into another (called the *frequency domain representation* or *DFT* of the original function)
- Requires an input function that is discrete and whose non-zero values have a limited (finite) duration (i.e. sampling a continuous function like a person's voice)
- The short time Fourier transform only evaluates enough frequency components to reconstruct the finite segment that was analyzed [CKO00]

# Example

TIME

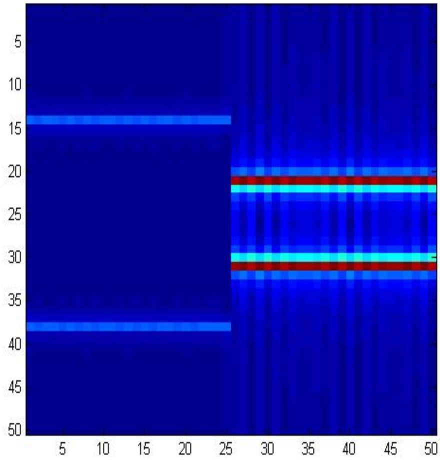



FREQUENCY



# Time-Frequency Plane

Here we have our signal represented in the Time-Frequency Plane



# Continuous Fractional Fourier Transform

- Will rotate a signal in the Time-Frequency Plane/Representation

## Definition

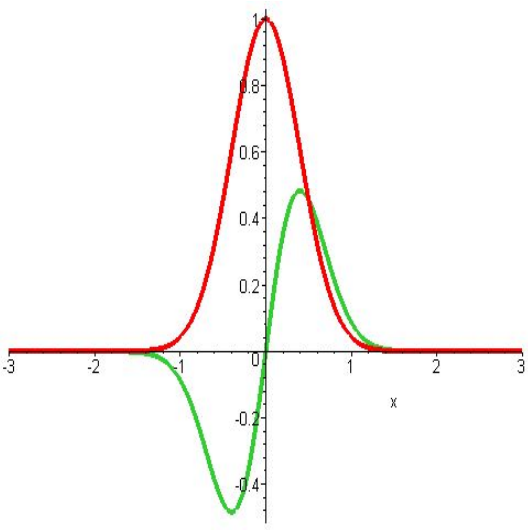
*The **Continuous Fractional Fourier Transform** is:*

$$F_\alpha(f)(\omega) = \sqrt{\frac{1 - i\cot(\alpha)}{2\pi}} e^{\frac{i\cot(\alpha)\omega^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-i\csc(\alpha)\omega t + i\cot(\alpha)t^2}{2}}$$

Note: When $\alpha = \frac{\pi}{2}$ this becomes precisely the definition of the continuous Fourier transform, and for $\alpha = -\frac{\pi}{2}$ it is the definition of the inverse continuous Fourier transform. [LMT08]

# Hermite Functions

Let us show the first two Hermite Functions ($e^{-x^2}$ and $2xe^{-x^2}$)



**Continuous**

- The Gauss-Hermite differential operator is: $H(f) = \frac{d^2 f}{dx^2} + x^2 f$ and the eigenfunctions are continuous Hermite Functions

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**Discrete**

- We want to find a natural finite version of the Gauss-Hermite operator, $\Delta + F^* \Delta F$, and show that the eigenvectors of the G-H operator are also eigenvectors of $F$

# Definitions

- Let us recall a few elementary definitions from Linear Algebra... [Lay06]

**Commuting Matrices**

*For two matrices, A and B: If $AB = BA$, we say that A and B* ***commute*** *with one another*

**Eigenvector**

*An* ***eigenvector*** *of an n x n matrix A is a nonzero vector* $\mathbf{x}$ *such that* $A\mathbf{x} = \lambda\mathbf{x}$ *for some scalar* $\lambda$

# Definitions

**Eigenvalue**

*A scalar $\lambda$ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to $\lambda$*

**Eigenspace**

*The **eigenspace** is the span of the eigenvectors associated with a particular eigenvalue.*

# Definitions cont.

## Diagonalization

*A square matrix $A$ is said to be* ***diagonalizable*** *if $A$ is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix $P$ and some diagonal matrix $D$*

## Orthogonally Diagonalizable

*A matrix $A$ is said to be* ***orthogonally diagonalizable*** *if there are an orthogonal matrix $P$ (with $P^{-1} = P^T$) and a diagonal matrix $D$ such that $A = PDP^T = PDP^{-1}$*

# Commuting Matrices

Let T be the translation matrix, and M be the modulation matrix then, $T = FMF^*$ and $M^* = FTF^*$, $D = I - T$ and $\Delta = D^*D$

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & e^{2\pi i/d^2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & e^{4\pi i/d^2} & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & e^{2\pi i(d^2-2)/d^2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & e^{2\pi i(d^2-1)/d^2} \end{bmatrix}$$

**Lemma**

*We want to prove:* $F(\Delta + F\Delta F^*) = (\Delta + F^*\Delta F)F$

**Proof.**

$$
\begin{aligned}
F\Delta F^* &= F(I - T)(I - T^*)F^* \\
&= F(2I - T^* - T)F^* \\
&= F(2I)F^* - FT^*F^* - FTF^* \\
&= 2I - M - M^*
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
F^*\Delta F &= F^*(2I - T^* - T)F \\
&= 2I - M^* - M
\end{aligned}
$$

$\therefore F(\Delta + F\Delta F^*) = (\Delta + F^*\Delta F)F$ □

**Theorem**

*Let $\mathbf{x}$ be an eigenvector for B with eigenspace dimension 1. Let A commute with B, then $\mathbf{x}$ is also an eigenvector of A.*

**Proof.**

**Prove:** $B\mathbf{x} = \lambda\mathbf{x}$

$\implies AB\mathbf{x} = A\lambda\mathbf{x} = \lambda A\mathbf{x}$

$\implies B(A\mathbf{x}) = \lambda(A\mathbf{x})$

$\implies A\mathbf{x}$ is an eigenvector of B

Since $\lambda$ is unique (i.e. corresponds to one eigenvector) $A\mathbf{x}$ is a scalar multiple of $\mathbf{x} \implies A\mathbf{x} = B\mathbf{x}$ □

## Hermite Functions

Since the Fourier transform commutes with the G-H differential operator, the eigenvectors are also eigenvectors of the Fourier Transform

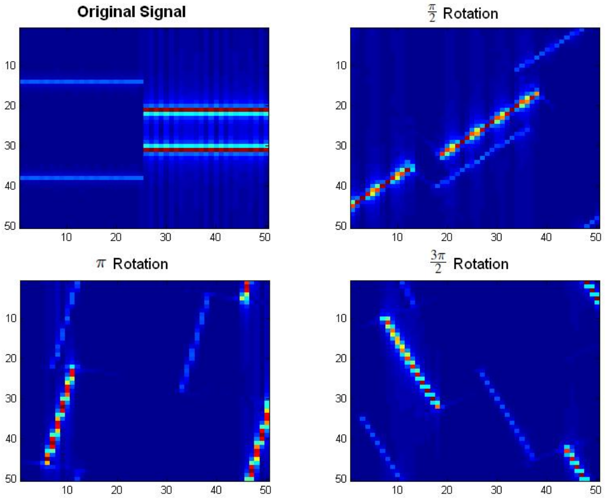
$$\Delta + F^* \Delta F = HE_1H^* \qquad F = HE_2H^*$$

$$HH^* = I$$

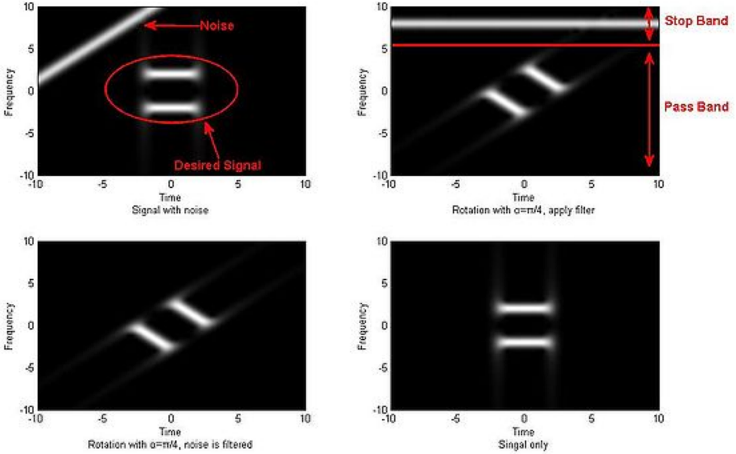
$$F^\alpha = HE^\alpha H^* \longleftarrow Fractional$$

$$i.e. \quad F^{\frac{1}{2}} = HE^{\frac{1}{2}}H^*$$

# Rotation With Fractional Fourier Transform



# Rotation With Fractional Fourier Transform



# Special Thanks

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# References


Naklé H. Asmar.
*Partial Differential Equations with Fourier Series and Boundary Value Problems*.
Pearson, Upper Saddle River, NJ, 2005.

Çağatay Candan, M. Alper Kutay, and Haldun M. Ozaktas.
The discrete fractional Fourier transform.
*IEEE Trans. Signal Process.*, 48(5):1329–1337, 2000.

David C. Lay.
*Linear Algebra and its Applications*.
Pearson, Upper Saddle River, NJ, 2006.

Yury F. Luchko, Héctor Matrínez, and Juan J. Trujillo.
Fractional Fourier transform and some of its applications.
*Fract. Calc. Appl. Anal.*, 11(4):457–470, 2008.

Yue Qian Liu and Hong Yu Yao.
A study of applying fractional Fourier transform to signal processing.
*J. Numer. Methods Comput. Appl.*, 28(3):198–214, 2007.