UNCW High School Mathematics Contest **Spring 2012**

1. Grandpa and his grandson Gary both had their birthday yesterday. Today, Grandpa's age in years is an even number and 15 times that of Gary. In 4 years time Grandpa's age in years will be the square of Gary's age in years. How many years older than Gary is Grandpa today?

   A. 42 B. 49 C. 56 D. 60 E. 64

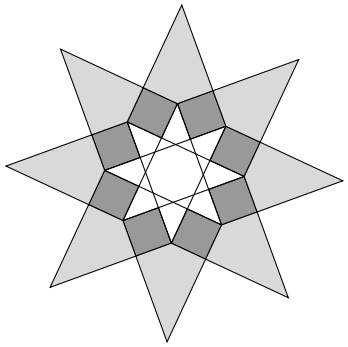
2. A triangle has 2 edges of length 5. What length should be chosen for the third side of the triangle so as to maximize the area within the triangle?

   A. 5 B. 6 C. $5\sqrt{2}$ D. 8 E. $5\sqrt{3}$

3. The equation $x^2 + ax + b = 0$, where a and b are different, has solutions $x = a$ and $x = b$. How many such equations are there?

   A. 0 B. 1 C. 3 D. 4 E. An infinite number

4. The diagram shows a small regular octagram (an eight-sided star) surrounded by eight squares (dark gray) and eight kites (light gray) to make a large regular octagram. Each square has area 1. What is the area of one of the light gray kites?



   A. 2 B. $\sqrt{2}+1$ C. $\frac{21}{8}$ D. $4\sqrt{2}-3$ E. $\frac{11}{4}$

5. What is the minimum value of $x^2 + y^2 + 2xy + 6x + 6y + 4$?

A. -7 B. -5 C. -4 D. -1 E. 4

6. A metal flagpole on level ground broke at a point with the top part toppling over like a hinge and the tip hitting the ground at a point 20 ft. from the base of the flagpole. It was rewelded but broke again at a point 5 ft. lower than the previous break point, with the top hitting the ground at a point 30 ft. from the base of the flagpole. The height of the flagpole in feet is:

A. 50 B. 55 C. 40 D. 45 E. 59

7. Consider the set of integers $\{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$. Select two distinct elements from this set randomly and find the probability that the product of the two numbers is positive.

A. $\frac{13}{36}$ B. $\frac{4}{9}$ C. $\frac{1}{6}$ D. $\frac{1}{2}$ E. $\frac{7}{18}$

8. John travels 5 miles north, then $4\sqrt{2}$ miles northeast, then 2 miles east and finally 1 mile south. How far in miles is he from the starting point?

A. $6 + 4\sqrt{2}$ B. $8 + 4\sqrt{2}$ C. $4\sqrt{6}$ D. 12 E. 10

9. Given that $f(x) = x^3 - 3x^2 + 3x - 5$ and $g(x) = 5 - x^2$, find $f^{-1}(g(-3))$.

A. 1 B. $\frac{1}{4}$ C. -4 D. 0 E. 2

10. The sum of the series $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \cdots$ is:

A. 1 B. $\frac{2}{3}$ C. $\frac{1}{6}$ D. $\frac{5}{6}$ E. $\frac{1}{2}$

11. If $\sqrt{a + \sqrt{a + \sqrt{a + \cdots}}} = a$, where $a > 0$, then a is equal to:

A. 1 B. 4 C. $\frac{3}{2}$ D. $2 + \sqrt{3}$ E. 2

12. Suppose X apples are distributed among 4 people A, B, C, D so that A gets 3 less than a third of the apples, B gets 4 more than a fourth, C gets 5 more than the fifth and D gets the remaining. Find the minimum number of apples so that none of the four people gets a fraction of an apple.

A. 90 B. 100 C. 120 D. 30 E. None of these

13. Find the sum of all the real values of x such that

$$(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3.$$

A. $\frac{5}{2}$ B. $\frac{3}{2}$ C. $\frac{1}{2}$ D. 3 E. 2

14. If $x - y$, x, and $x + y$ are the sides of a right triangle with $y > 0$, then $\frac{x}{y}$ equals:

A. $\frac{1}{4}$ B. 4 C. 2 D. 1 E. Can not be found

15. If a, b, c, d are positive integers and $a + \frac{1}{b+\frac{1}{c+\frac{1}{d}}} = \frac{47}{17}$, then d equals:

A. 4 B. 1 C. 3 D. 2 E. None of these.

16. The distance of the line $2x + y = 5$ from the origin is:

A. $\sqrt{5}$ B. 2 C. $\frac{5}{2}$ D. 3 E. 5

17. A circle is tangent to the x-axis at the point $(2, 0)$ and has a y-intercept at $(0, 4)$. The radius of the circle is:

A. 4 B. 2 C. $\frac{5}{2}$ D. 3 E. $\frac{7}{2}$

18. If $f(x) = \frac{x}{3} - 1$, find the value of $f(f(f(27)))$.

A. $\frac{5}{3}$ B. $-\frac{1}{3}$ C. $-\frac{4}{9}$ D. 2 E. $\frac{4}{9}$

19. How many six-digit numbers, with no repeated digits, are there if the last two digits must be 7 or 8?

A. 2940 B. 3360 C. 5880 D. 6720 E. None of these

20. In the expression of $(b-2a)^8$, find the coefficient of a^3b^5.

A. 56 B. -56 C. 448 D. -448 E. -1792

21. Which of the following values for x would satisfy both of the following inequalities?

$$\left(\frac{\sqrt[3]{25}}{10}\right)^{\cos(x)} < 1 \text{ and } \left(\frac{\sqrt[3]{25}}{10}\right)^{\tan(x)} > 1$$

A. 177° B. 37° C. 127° D. 217° E. 307°

22. What is the exact value of $\tan x \cdot \csc x \cdot \sin\left(x - \frac{\pi}{2}\right)$?

A. 2 B. -1 C. 1 D. 0 E. -2

23. If $\sin(x) - \cos(x) = \frac{1}{2}$, what is the value of $\sin^3(x) - \cos^3(x)$?

A. $\frac{1}{8}$ B. $\frac{5}{16}$ C. $\frac{3}{8}$ D. $\frac{5}{8}$ E. $\frac{11}{16}$

24. If $f(x) + 2f(5-x) = x$ for all real numbers x, then $f(1) =$

A. $\frac{3}{7}$ B. $\frac{5}{2}$ C. 9 D. $\frac{2}{5}$ E. $\frac{7}{3}$

25. What is the value of $\left(\dfrac{1+i}{\sqrt{2}}\right)^{2012}$?

A. -1 B. $\dfrac{1+i}{\sqrt{2}}$ C. $\dfrac{\sqrt{2}}{2}$ D. $\dfrac{i-1}{\sqrt{2}}$ E. 1

26. A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of them died from heart disease. Moreover, 312 of them had at least one parent who suffered from heart disease, and 102 of these died from heart disease. Find the probability that a man randomly selected from this group died of heart disease, given that neither of his parents suffered from heart disease.

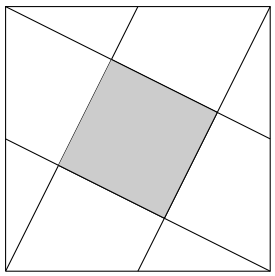
A. 0.115 B. 0.173 C. 0.224 D. 0.327 E. 0.514

27. The sum of seven consecutive integers is 980. How many of them are prime?

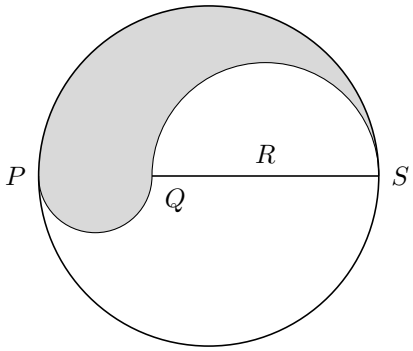
A. 0 B. 1 C. 2 D. 3 E. 4

28. A square with sides of length 2 inches is drawn. Lines are then drawn connecting the midpoint of each side with a corner of the square creating a second smaller square in the interior of the original square as shown in the figure below. Find the area (in square inches) of this smaller (shaded) square.

A. $\frac{4}{5}$
B. 1
C. $\frac{5}{4}$
D. $\frac{4}{3}$
E. $\frac{5}{2}$

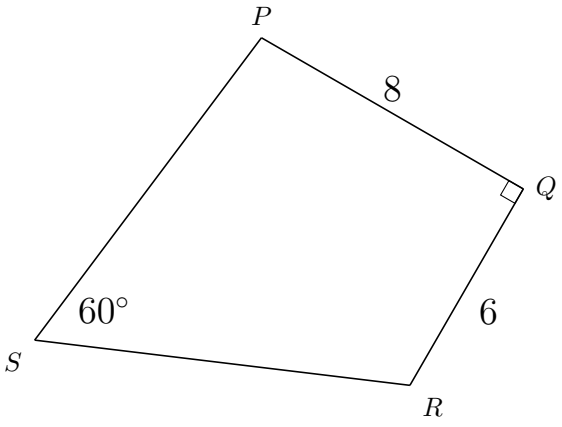


29. $PQRS$ is a diameter of the circle whose radius is r. The lengths PQ, QR and RS are all equal. Semicircles are drawn with diameters PQ and QS to create the shaded region shown in the figure. Find the perimeter of the shaded region.



A. $2\pi r$
B. $\frac{4\pi r}{3}$
C. $\frac{5\pi r}{3}$
D. $\frac{3\pi r}{2}$
E. $\frac{31\pi r}{18}$

30. $PQRS$ is a quadrilateral in which $SP = SR$, the angle at S is 60° and the angle at Q is 90°. Given that the length of PQ is 8 cm and that the length of QR is 6 cm, find the area of the $PQRS$ in square centimeters.



A. $25\sqrt{3} + 24$
B. $\frac{25\sqrt{3}}{2} + 24$
C. $25\sqrt{2} + 24$
D. $48 + \frac{25\sqrt{3}}{2}$
E. $48 + 25\sqrt{3}$