UNCW High School Math Contest Spring 2010

Total 30 questions

Scoring: Each question answered correctly scores 4 points.
Each question left blank scores 1 point.
Each question answered incorrectly scores 0 points.

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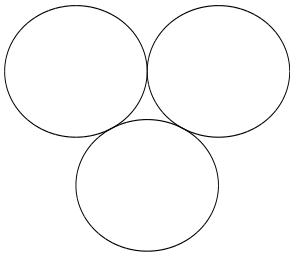
1. The distance from Wilmington to Raleigh is about 120 miles. If Justin has two hours to make the trip, but drives 50 mph for the first 60 miles, how fast must Justin go on the second half of the trip to reach Raleigh in time?

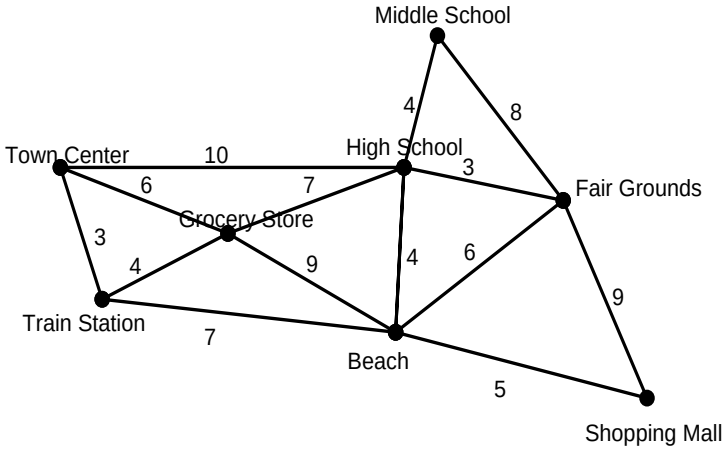
   A. 70 mph
   B. 75 mph
   C. 77.2 mph
   D. 80 mph
   E. None of these

2. What is the area of the gap between these circles of radius 1?



   A. $\sqrt{3}$
   B. $\frac{\pi}{3}$
   C. $\sqrt{3} - \frac{\pi}{2}$
   D. 4
   E. $\frac{\pi}{4}$

3. The graph below shows the distance (in miles) along fourteen roads between eight locations. A bus driver is planning a route that includes all eight locations. The bus should not backtrack, and it should not visit any locations more than once. The bus does not need to return to its starting point. What is the minimum mileage that meets the given conditions?



   A. 27 miles
   B. 30 miles
   C. 32 miles
   D. 35 miles
   E. 37 miles

4. If the roots of $x^2 - bx + c = 0$ are $\sin(\pi/7)$ and $\cos(\pi/7)$, then $b^2 =$?

   A. c
   B. $1 - c$
   C. $1 + c$
   D. $1 + 2c$
   E. $1 + c^2$

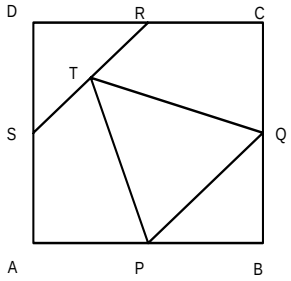
5. What is $(1+i)^{100}$ equal to?

   A. -2^{25}
   B. -2^{50}
   C. 0
   D. 2^{50}
   E. 2^{25}

6. Let $P(x)$ be a polynomial of degree four such that $P(2) = P(-2) = P(-3) = -1$ and $P(1) = P(-1) = 1$. What is $P(0)$?

   A. 2
   B. 3
   C. 4
   D. 5
   E. 6

7. The square ABCD has area $400\,m^2$ (see Figure). The points P,Q,R,S are midpoints of the sides of the square. The point T is the midpoint of the RS segment. Compute the area of the triangle PQT.



   A. 100 m^2
   B. 120 m^2
   C. 140 m^2
   D. 160 m^2
   E. 180 m^2

8. What is the solution of $9^x + 4 \cdot 3^x - 3 = 0$?

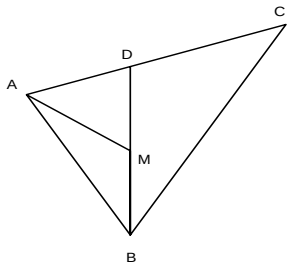
   A. $\log_3(2+\sqrt{7})$
   B. $\frac{-4+\sqrt{28}}{2}$
   C. -0.5
   D. $\frac{\ln(-2+\sqrt{7})}{\ln(3)}$
   E. $\frac{\ln(-5+\sqrt{32})}{\ln(3)}$

9. The diameter of a pizza is twice the diameter of another smaller pizza. The larger pizza is split in 16 equal slices. Which fraction of the smaller pizza corresponds to 3 slices of the larger pizza?

   A. $\frac{1}{3}$
   B. $\frac{3}{8}$
   C. $\frac{1}{2}$
   D. $\frac{3}{4}$
   E. $\frac{5}{8}$

10. In triangle ABC, $AB = 6$, $BC = 9$, DB is the angle bisector of angle ABC, and M is the midpoint of BD. If the area of $\triangle ABC$ is 24, what is the area of $\triangle ABM$?

    A. 3.8
    B. 4
    C. 4.8
    D. 6.2
    E. 9.6



11. Let a, b, c be positive real numbers with $a + b + c = 1$. What is the largest value of $ab + bc + ac - 2abc$?

    A. $\frac{5}{27}$
    B. $\frac{7}{27}$
    C. $\frac{3}{7}$
    D. $\frac{2}{9}$
    E. $\frac{5}{7}$

12. Alice and Bob play a game involving a circle whose circumference is divided by 12 equally-spaced points. The points are numbered clockwise from 1 to 12. Both start on point 12. In each turn of the game, Alice moves 5 points clockwise and Bob moves 9 points counter-clockwise. The game ends when they stop on the same point. How many turns will this take?

    A. 4
    B. 6
    C. 8
    D. 14
    E. 24

13. How many integers between 1 and 401 are divisible by either 5 or 7?

    A. 121
    B. 124
    C. 126
    D. 128
    E. 130

14. If θ is an acute angle and $\sin 2\theta = b$, then $\sin^3\theta + \cos^3\theta$ equals:

    A. $(1-b)\sqrt{b+1}$
    B. $(1-\frac{b}{2})\sqrt{1-b}$
    C. $(1-\frac{b}{2})(b+1)$
    D. $(1-\frac{b}{2})\sqrt{b+1}$
    E. $(1-\frac{b}{2})(1-b)$

15. Suppose that each side of a square tile is one unit in length, and each square tile has an area of one square unit. If the square-tile design sequence follows the pattern of the first three designs, what is the area of the square-tile design in the 50th figure?

    A. 100 square units
    B. 250 square units
    C. 50 square units
    D. 1500 square units
    E. 2500 square units

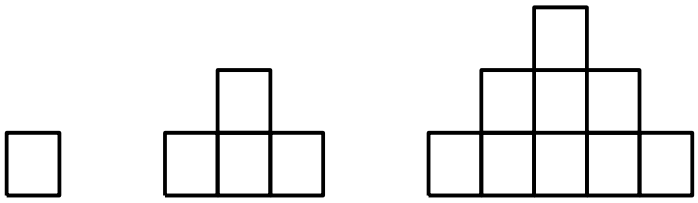


Figure 1 Figure 2 Figure 3

16. Suppose $f(x+y) = f(x) + f(y) + 1$ and $f(1) = 2$, what is $f(3)$?

    A. 5
    B. 7
    C. 8
    D. 9
    E. 6

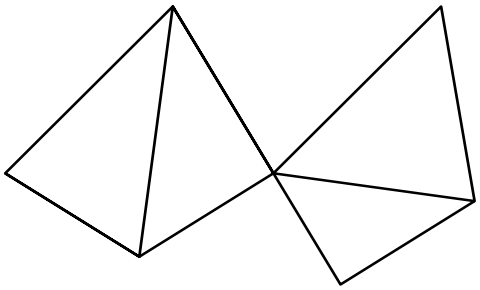
17. How many different positive integers can you form using only the digits 1, 2, 3, and 4 if each digit can be used at most once in any number?

    A. 24
    B. 32
    C. 48
    D. 52
    E. 64

18. Let $x+3$, $2x+1$, and $5x+2$ be consecutive terms of an arithmetic sequence. Find the absolute value of the common difference of the terms.

    A. 1
    B. 2
    C. 5
    D. 3
    E. None of these

19. A graph consists of vertices and edges (segments) that connect the vertices. A set of vertices is called independent if no two vertices in the set are connected by an edge. What is the size of the largest independent set in the graph shown?



    A. 2
    B. 3
    C. 4
    D. 1
    E. None of these

20. If $\cos\theta/2 = \sqrt{\frac{x-1}{2x}}$, then $\sin\theta$ equals:

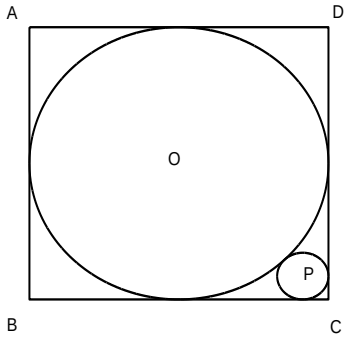
    A. $\frac{\sqrt{x^2-1}}{x}$
    B. $\frac{\sqrt{x^2-1}}{2x}$
    C. $\frac{1}{x}$
    D. $\sqrt{\frac{x+1}{2x}}$
    E. $\sqrt{\frac{x^2-1}{x}}$

21. Suppose that in a survey of 100 people, it was found that 50 take chemistry, 53 take mathematics, 42 take physics, 15 take chemistry and physics, 20 take physics and mathematics, 25 take mathematics and chemistry, and 5 take all three. How many students take mathematics only?

    A. 10
    B. 11
    C. 12
    D. 13
    E. 15

22. A square with sides of length 2 has a inscribed circle O, circle P is tangent to DC, BC and circle O. Compute the area of circle P.

    A. $(17 - 12\sqrt{2})\pi$
    B. $\frac{\sqrt{2}\pi}{2}$
    C. $\pi - 3$
    D. $\frac{\pi}{4}$
    E. None of these



23. $\sin(2\sin^{-1} x)$ equals:

    A. $2x(1 - x^2)$
    B. $2x$
    C. $\frac{2}{x}$
    D. $\frac{x}{\sqrt{1-x^2}}$
    E. $2x\sqrt{1 - x^2}$

24. Five distinct numbers are randomly selected from $1, 2, 3, \ldots, 10$. What is the probability that the median of the numbers selected is 5?

    A. $\frac{1}{6}$
    B. $\frac{5}{21}$
    C. $\frac{1}{7}$
    D. $\frac{2}{5}$
    E. $\frac{3}{5}$

25. For which value of b is there only one intersection between the line $y = x + b$ and the parabola $y = x^2 - 3x + 5$?

    A. 1
    B. 2
    C. 2.5
    D. -2
    E. -1.5

26. Find the values of m so that the difference between the roots of $3x^2 + mx - 12$ is 5.

A. 0
B. ± 4
C. ± 9
D. ± 11
E. None of these

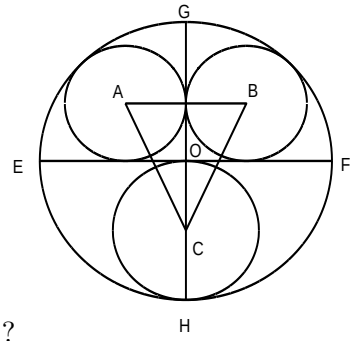
27. The mean of a set of 40 numbers is 64. Two numbers, 88 and 97 are removed. What is the mean of the remaining numbers?

A. 53.5
B. 56.548
C. 59.375
D. 62.5
E. 65.357

28. If $\log_7 3 = a$ and $\log_7 4 = b$, find x in terms of a and b if $9^x = 28$.

A. $x = \frac{1+b}{2a}$
B. $x = \frac{7b}{a^2}$
C. $x = \frac{b-a}{2}$
D. $x = \frac{2a}{b+1}$
E. None of these

29. The circle O has radius 1, GH and EF are two diameters and are perpendicular to each other, circle A is tangent to OE, OG and arc EG; circle B is tangent to OF, OG and arc GF; GH passes through the center of circle C and circle C is tangent to EF and arc EHF. Find the area of triangle ABC.



A. $\sqrt{2}$
B. $4 - 2\sqrt{2}$
C. $\frac{9-7\sqrt{2}}{2}$
D. $\frac{5-3\sqrt{2}}{2}$
E. None of these

30. What is the sum of the roots to the following equation?

$$x^4 + 2x^3 - x + \frac{1}{4} = 0$$

A. 1
B. $\frac{1}{2}$
C. $\sqrt{3}$
D. $2\sqrt{3}$
E. -1