

2005 High School Mathematics Contest

- The coefficient of x in the simplified expansion of $\left(\frac{a}{\sqrt{x}} - \frac{\sqrt{x}}{a}\right)^6$ is:
A. 15 B. -15 C. $-\frac{6}{a^9}$ D. $\frac{20}{a^3}$ E. $\frac{15}{a^2}$
- A mathematics test consists of 30 questions. For every correct answer, 4 points are awarded and for every question left blank, 1 point is awarded. No points are awarded for incorrect answers. Given that John attempted 25 questions and scored 97, how many questions did he answer correctly?
A. 20 B. 21 C. 22 D. 23 E. 24
- John and Sam have an equal sum of money. Their father gave \$90 and \$20 to John and Sam respectively. As a result, John had 3 times as much money as Sam. How much did Sam have at first?
A. \$15 B. \$25 C. \$35 D. \$45 E. \$55
- Two runners, Ann and Mary, run along a 400 m circular track. They start from the same point. Ann runs 5 m/s while Mary runs 6 m/s. How many seconds after they started would Mary pass Ann for the first time?
A. 100 B. 200 C. 300 D. 400 E. 500
- The area of a rectangle is 20 cm^2 . Its length is 2 cm more than its breadth. Find its perimeter and give your answer correct to 3 decimal places.
A. 18.284 B. 18.307 C. 18.330 D. 18.353 E. 18.376

6. Find all values of a such that the quadratic equation

$$2x^2 + (2a - 8)x - a = 0$$

has two distinct positive real solutions.

- A. $a < 0$ B. $0 < a < 5$ C. $5 < a < 7$ D. $7 < a < 8$ E. $a > 8$

7. Given the geometric series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$, find the sum of all but the first 6 terms.

- A. $\frac{1}{4374}$ B. $\frac{1}{1458}$ C. $\frac{1}{486}$ D. $\frac{1}{162}$ E. $\frac{364}{729}$

8. If the perimeter of rectangle $ABCD$ is 80 inches, the least value of diagonal AC , in inches, is:

- A. 0 B. $\frac{15\sqrt{2}}{2}$ C. $3\sqrt{17}$ D. $\frac{11\sqrt{3}}{2}$ E. none of these

9. Given $5^x = 125^{y+1}$ and $9^y = 3^{x-9}$, the value of $x + y$ is:

- A. 18 B. 21 C. 24 D. 27 E. 30

10. In a ten-mile race First beats Second by 2 miles and First beats Third by 4 miles. If the runners maintain constant speeds throughout the race, by how many miles does Second beat Third?

- A. 2 B. $2\frac{1}{4}$ C. $2\frac{1}{2}$ D. $2\frac{3}{4}$ E. 3

11. If $\log_{b^2} x + \log_{x^2} b = 1$, $b > 0$, $b \neq 1$, $x \neq 1$, then x equals:
- A. $\frac{1}{b^2}$ B. $\frac{1}{b}$ C. b^2 D. b E. \sqrt{b}
12. The number of positive integers less than 1000 divisible by neither 5 nor 7 is:
- A. 688 B. 686 C. 684 D. 658 E. 630
13. The integer n is obtained by reversing the order of the digits of the 3-digit integer m . If the product of n and m is equal to 368767, then the middle digit of n must be:
- A. 1 B. 3 C. 5 D. 7 E. 9
14. The sides PQ and QR of triangle PQR are respectively of lengths 6 and 8 inches. The median PM is 4 inches. Then PR , in inches, is:
- A. $2\sqrt{2}$ B. $\sqrt{18}$ C. 5 D. $2\sqrt{3}$ E. $2\sqrt{7}$
15. If the graphs of $2y + x + 3 = 0$ and $3y + ax + 2 = 0$ are to meet at right angles, the value of a is:
- A. $\pm\frac{2}{3}$ B. $-\frac{2}{3}$ C. $-\frac{3}{2}$ D. 6 E. -6

16. Medians AD and CE of triangle ABC intersect in M . The midpoint of AE is N . Let the area of triangle MNE be k times the area of triangle ABC . Then k equals:

- A. $\frac{1}{6}$ B. $\frac{1}{8}$ C. $\frac{1}{9}$ D. $\frac{1}{12}$ E. $\frac{1}{16}$

17. Triangle ABC is inscribed in a semicircle of radius r so that its base AB coincides with diameter AB . Point C does not coincide with either A or B . Let $s = AC + BC$. Then, for all permissible positions of C :

- A. $s^2 \leq 8r^2$ B. $s^2 = 8r^2$ C. $s^2 \geq 8r^2$ D. $s^2 \leq 4r^2$ E. $s^2 = 4r^2$

18. The difference in the areas of two similar triangles is 18 square feet, and the ratio of the larger area to the smaller is the square of an integer. The area of the smaller triangle, in square feet, is an integer, and one of its sides is 3 feet. The corresponding side of the larger triangle, in feet, is:

- A. 12 B. 9 C. $6\sqrt{2}$ D. 6 E. $3\sqrt{2}$

19. Which of the following sets could NOT be the lengths of the external diagonals of a right rectangular prism?

- A. $\{4, 5, 6\}$ B. $\{4, 5, 7\}$ C. $\{4, 6, 7\}$ D. $\{5, 6, 7\}$ E. $\{5, 7, 8\}$

20. Find the sum of the roots of $\tan^2 x - 9 \tan x + 1 = 0$ that are between $x = 0$ and $x = 2\pi$ radians.

- A. $\frac{\pi}{2}$ B. π C. $\frac{3\pi}{2}$ D. 3π E. 4π

21. $\cot 10 + \tan 5 =$

- A. $\csc 5$ B. $\csc 10$ C. $\sec 5$ D. $\sec 10$ E. $\sin 15$

22. In triangle ABC , $\angle ABC = 120^\circ$, $AB = 5$ and $BC = 12$. If perpendiculars constructed to \overline{AB} at A and to \overline{BC} at C meet at D , then $CD =$

- A. 3 B. $\frac{10}{\sqrt{3}}$ C. 11 D. $\frac{25}{2}$ E. $\frac{22}{\sqrt{3}}$

23. A $10 \times 10 \times 10$ wooden cube is formed by gluing together 10^3 unit cubes. What is the greatest number of unit cubes that can be seen from a single point?

- A. 268 B. 269 C. 270 D. 271 E. 272

24. What is the size of the largest subset, S , of $\{1, 2, 3, \dots, 50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

- A. 6 B. 7 C. 14 D. 22 E. 23

25. For any set S , let $|S|$ denote the number of elements in S , and let $n(S)$ be the number of subsets of S , including the empty set and the set S itself. If A , B , and C are sets for which

$$n(A) + n(B) + n(C) = n(A \cup B \cup C) \text{ and } |A| = |B| = 100,$$

then what is the minimum possible value of $|A \cap B \cap C|$?

- A. 96 B. 97 C. 98 D. 99 E. 100

26. Consider the sequence defined recursively by $u_1 = a$ (any positive number), and $u_{n+1} = -\frac{1}{u_n+1}$, $n = 1, 2, 3, \dots$. For which of the following values of n must $u_n = a$?
- A. 14 B. 15 C. 16 D. 17 E. 18
27. Let n be a positive integer. If the equation $x + y + z = n$ has 66 solutions in non-negative integers x , y and z , then n must be
- A. 8 B. 9 C. 10 D. 11 E. 12
28. A sample consisting of five observations has an arithmetic mean of 10 and a median of 12. The smallest value that the range (largest observation minus smallest) can assume for such a sample is
- A. 2 B. 3 C. 5 D. 7 E. 10
29. One hundred high school students participated in a math contest last year, and their mean score was 100. The number of non-seniors in the contest was 50% more than the number of seniors, and the mean score of the seniors was 50% higher than that of the non-seniors. What was the mean score of the seniors?
- A. 100 B. 112.5 C. 120 D. 125 E. 150
30. Suppose that 4 boys and 16 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row $GBGGGGGBGBGBGGGGGGGGG$ we have $S = 8$. The average value of S (if all possible orders of these 20 people are considered) is closest to
- A. 6 B. 7 C. 8 D. 9 E. 10