The Stochastic KdV - A Review

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1 Wadati's Work

We begin with a review of the results from Wadati's paper 1983 paper [4] but with a sign change in the nonlinear term. The key is an analysis of the stochastic KdV equation given by

$$u_t + 6uu_x + u_{xxx} = \zeta(t), \tag{1}$$

where $\zeta(t)$ represents time-dependent Gaussian white noise. This noise has zero mean and satisfies the statistical average

$$< \zeta(t)\zeta(t') >= 2\epsilon\delta(t-t'). \tag{2}$$

For such time-dependent noise, the stochastic KdV equation can be transformed into an unperturbed KdV equation,

$$U_T + 6UU_X + U_{XXX} = 0, \tag{3}$$

by introducing the Galilean transformation

$$\begin{aligned} u(x,t) &= U(X,T) + W(T), && (4)\\ X &= x + m(t), && (5)\\ T &= t, && (6)\\ m(t) &= -6\int_0^t W(t')dt'. && (7) \end{aligned}$$

This can be seen as follows. Under the above transformations we have from calculus that the derivatives transform as

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T} \\ &= \frac{\partial}{\partial X}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T} \\ &= -6W(T)\frac{\partial}{\partial X} + \frac{\partial}{\partial T}. \end{aligned} \tag{9}$$

Using these transformations, we have

$$\begin{aligned}\zeta(t) &= u_t + 6uu_x + u_{xxx} \\ &= (U+W)_T - 6WU_x + 6(U+W)U_X + U_{XXX} \\ &= U_T + 6UU_X + U_{XXX} + W_T. \end{aligned} \tag{10}$$

Defining

$$\zeta = W_T,$$

or

$$W(t) = \int_0^t \zeta(t')\, dt', \tag{11}$$

leads to the KdV equation in (3).

We next consider the one soliton solution. Let

$$U(X,T) = 2\eta^2 \operatorname{sech}^2(\eta(X - 4\eta^2 T - X_0)) \tag{12}$$

be a solution of Equation (3). Then, the above transformation leads directly to an exact solution of the stochastic KdV equation (1):

$$u(x,t) = 2\eta^2 \operatorname{sech}^2\left(\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\, dt'\right)\right) + W(t). \tag{13}$$

Now we are interested in averaging over and ensemble of solutions. Namely, one considers what happens on average to the soliton under various realizations of the noise. For example, we could consider the average of the solution, denoted by $< u(x,t) >$, and determine how the soliton behaves, such as the effect of the noise on the soliton's amplitude. Thus, we have that

$$< u(x,t) >= 2\eta^2 \left\langle \operatorname{sech}^2\left(\eta\left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\, dt'\right)\right)\right\rangle.$$

Wadati computes this by first turning the hyperbolic function into a series of exponentials. Formally, one can write

$$\begin{aligned}\operatorname{sech}^2 z &= \frac{4}{(e^z + e^{-z})^2} \\ &= \frac{4e^{2z}}{(1+e^{2z})^2} \\ &= -2\frac{d}{dz}\frac{1}{1+e^{2z}} \\ &= -2\frac{d}{dz}\left(\sum_{n=0}^{\infty}(e^{2z})^n\right) \\ &= 2\sum_{n=1}^{\infty}(-1)^{n+1} n e^{2nz}. \end{aligned} \tag{14}$$

Wadati then proceeds by computing.

$$< u(x,t) >= 8\eta^2 \sum_{n=1}^{\infty} (-1)^{n+1} n \left\langle \exp\left[2n\eta \left(x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\,dt' \right)\right]\right\rangle .$$

In order to complete this computation, some useful relations (which will need to be confirmed) are needed:

$$< W(t) >= 0. \tag{15}$$

$$< W(t_1)W(t_2) >= 2\epsilon \min(t_1, t_2). \tag{16}$$

$$< \exp\left(cW(t)\right) >= \exp\left(\frac{1}{2}c^2 < W^2(t) >\right). \tag{17}$$

We can use these to show that

$$\begin{aligned} \left\langle \exp\left(\pm 12n\eta \int_0^t W(t')\,dt' \right)\right\rangle &= \exp\left(72n^2\eta^2 \int_0^t \int_0^t < W(t_1)W(t_2) > \,dt_2 dt_1 \right) \\ &= \exp(48n^2\eta^2\epsilon t^3). \end{aligned} \tag{18}$$

(The average in the exponential is computed later.)

This leads to the following form:

$$< u(x,t) >= 8\eta^2 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{na+n^2 b}, \tag{19}$$

where

$$a = 2\eta(x - x_0 - 2\eta^2 t),$$
$$b = 48\eta^2\epsilon t^3.$$

In principle, this should be sufficient. However, Wadati goes on to develop expressions that analytically give an interpretation to this result and allow for asymptotic estimates of the behavior of the solution to the stochastic KdV. (*We should note at this point that the solution is defined in terms of a divergent series! We will discuss this later.*)

The key to obtaining useful analytic expressions was noted by Wadati. Differentiating the series by a and b leads to the partial differential equation

$$w_b = w_{aa}$$

for $w(a,b) =< u(x,t) >$. Furthermore, we have that

$$w(a,0) = 2\eta^2 \operatorname{sech}^2 \frac{a}{2}.$$

This is an initial-boundary value problem for the heat, or diffusion, equation on the real line. It is solved using Fourier transform methods. Namely, we define the Fourier transform

$$\hat{w}(k,b) = \int_{-\infty}^{\infty} w(a,b)e^{-iak}\,da, \tag{20}$$

and its inverse transform

$$w(a,b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{w}(k,b)e^{iak}\,dk. \tag{21}$$

The heat equation leads to the simple initial value problem

$$\hat{w}_b = -k^2\hat{w}, \tag{22}$$

where

$$\begin{aligned} \hat{w}(k,0) &= 2\eta^2\int_{-\infty}^{\infty} \operatorname{sech}^2\frac{a}{2}e^{-iak}\,da \\ &= 8\eta^2\frac{\pi k}{\sinh \pi k}. \end{aligned} \tag{23}$$

Therefore,

$$\hat{w}(k,b) = 8\eta^2\frac{\pi k}{\sinh \pi k}e^{-bk^2} \tag{24}$$

and the solution is thus found from the inverse Fourier transform as

$$u(x,t) = \frac{4\eta^2}{\pi}\int_{-\infty}^{\infty}\frac{\pi k}{\sinh \pi k}e^{iak-bk^2}\,dk. \tag{25}$$

Wadati indicates that this is simply done using the convolution theorem. Namely, we note that $\hat{w}(k,b) = \hat{f}(k)\hat{g}(k,b)$ for

$$\hat{f}(k) = 8\eta^2\frac{\pi k}{\sinh \pi k}$$

and

$$\hat{g}(k,b) = e^{-bk^2}.$$

The inverse transforms for these are given by

$$f(a) = 2\eta^2\operatorname{sech}^2\frac{a}{2}$$

and

$$g(a,b) = \frac{1}{\sqrt{4\pi b}}e^{-a^2/4b}.$$

The last expression is just the statement the the Fourier transform of a Gaussian is a Gaussian. Convolving these functions, we have

$$\begin{aligned} < u(x,t) > &= w(a,b) \\ &= (f * g)(a) \\ &= \int_{-\infty}^{\infty} f(s) g(a-s)\, ds \\ &= \int_{-\infty}^{\infty} \left(2\eta^2 \operatorname{sech}^2 \frac{s}{2}\right) \left(\frac{1}{\sqrt{4\pi b}} e^{-(a-s)^2/4b}\right) ds \\ &= \frac{\eta^2}{\sqrt{\pi b}} \int_{-\infty}^{\infty} e^{-(a-s)^2/4b} \operatorname{sech}^2 \frac{s}{2}\, ds. \end{aligned} \tag{26}$$

This is the exact solution to which we can compare any simulation results. To date it does not appear that any one has actually done this. Most of the focus of any simulations are with respect to the asymptotic results that Wadati has derived from this solution [3]. Namely, for small times ($b = 48\eta^2 \epsilon t^2 < 1$) one can show that

$$< u(x,t) >= 2\eta^2 \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{\partial^{2n}}{\partial a^{2n}} \operatorname{sech}^2 \frac{a}{2}. \tag{27}$$

For large times ($b = 48\eta^2 \epsilon t^2 > 1$) one can show that

$$< u(x,t) >= \frac{4\eta^2}{\sqrt{\pi}} \left(1 + \sum_{n=1}^{\infty} \frac{(2^{2n}-2) B_n \pi^{2n}}{(2n)!} \frac{\partial^n}{\partial b^n}\right) \frac{e^{-a^2/4b}}{\sqrt{b}}. \tag{28}$$

Most focus on the $t \to \infty$ result that

$$< u(x,t) >\sim \frac{\eta}{\sqrt{3\pi\epsilon}} \frac{1}{t^{3/2}} \exp\left(-\frac{(x - x_0 - 4\eta^2 t)^2}{48\epsilon t^3}\right).$$

Lets look at some results in a special case. The solutions for small [large] times based upon Equation (27) [Equation (28)] is given in Figure 1 [Figure 2].

The amplitudes of the solutions in Figures 1-2 are shown in Figure 3. Note that the large time results probably should start at later times. Then one might be able to see how the two extremes might match smoothly in the intermediate region. *In fact, one might try to use some type of expansion about* $b = 1$.

The exact integral, given by Equation (26), can be numerically integrated. We show a comparison of the exact solution in Figure 4 with that of a simulated result as shown in Figure 5.

2 Code

The code used to compare the simulated and numerical integration of Equation (26).

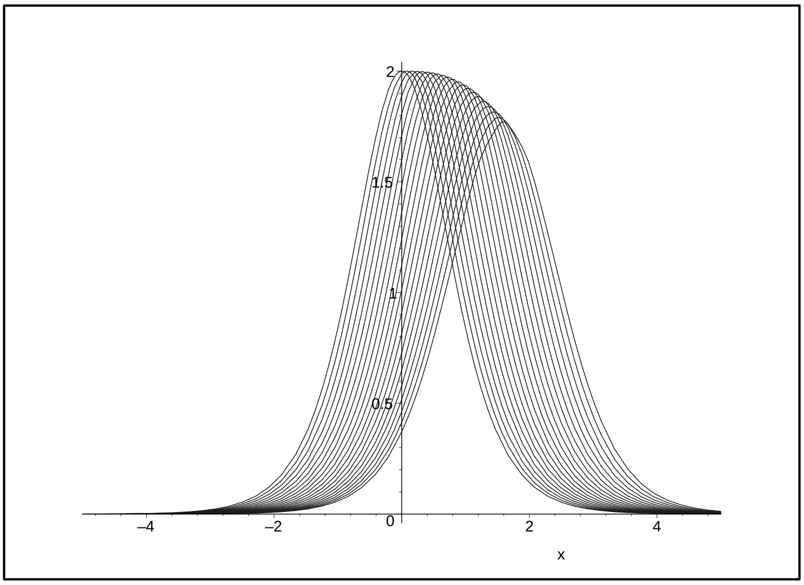

Figure 1: The solution for small times based upon Equation (27).

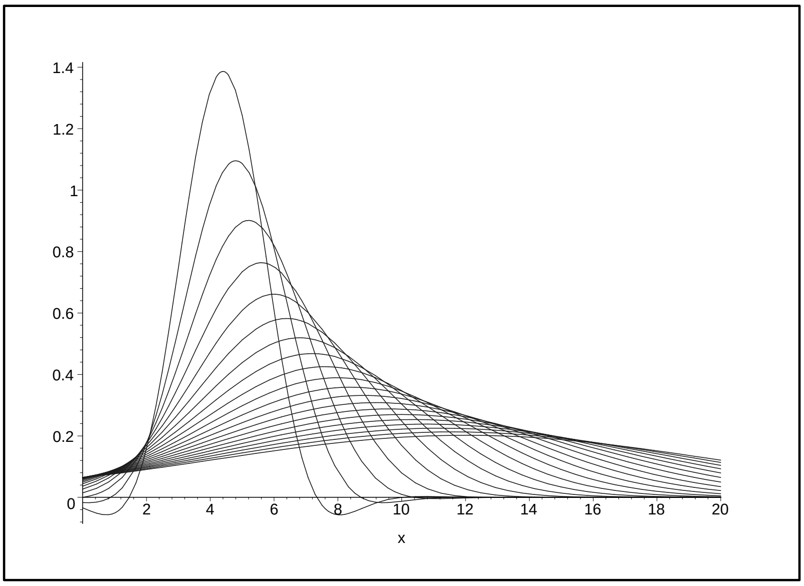

Figure 2: The solution for large times based upon Equation (28).

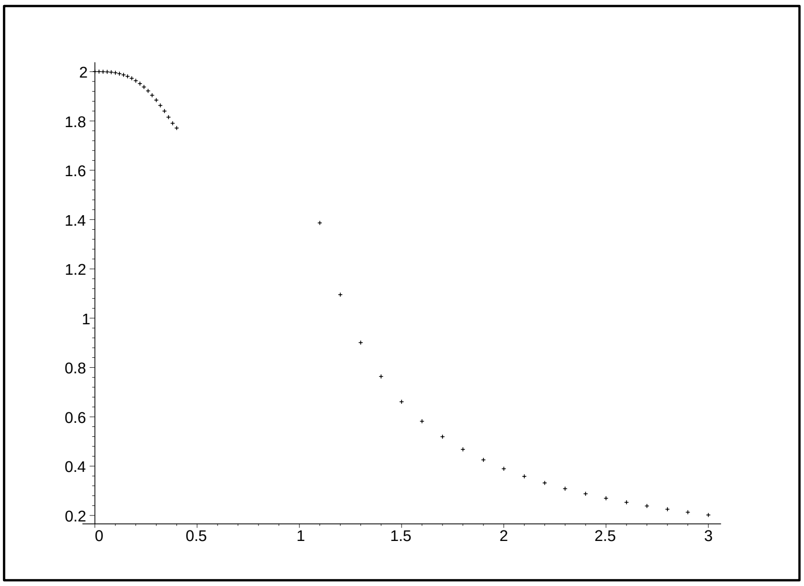

Figure 3: The amplitude of the solutions in the last two figures.

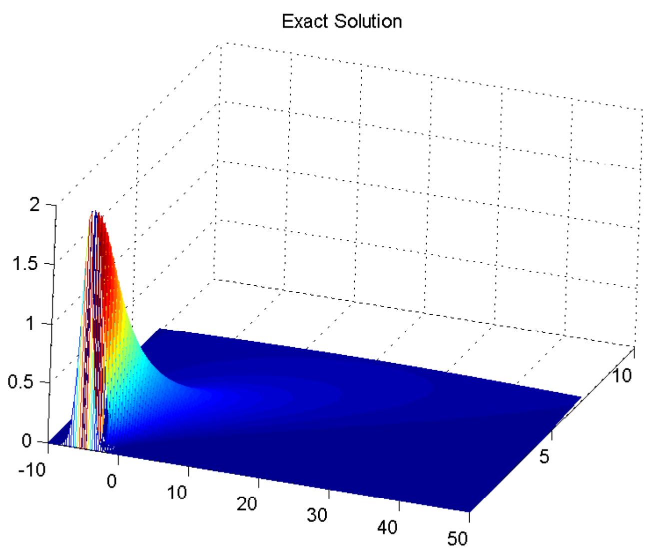

Figure 4: The exact solution using Equation (26).

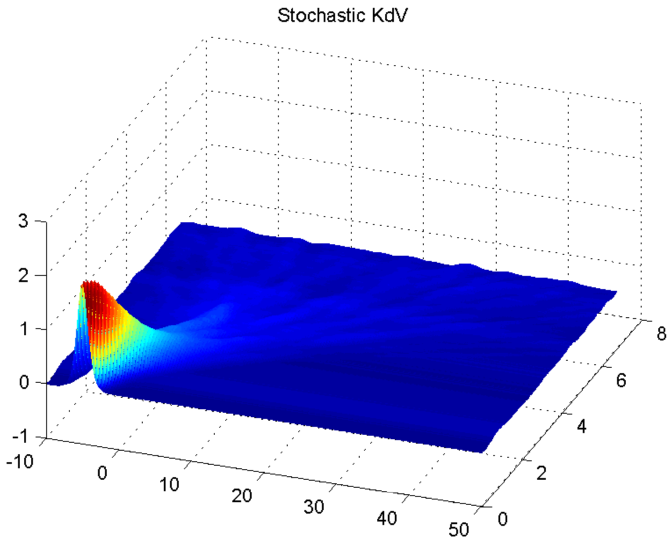

Figure 5: The solution generated by doing a simulation of the stochastic KdV equation.

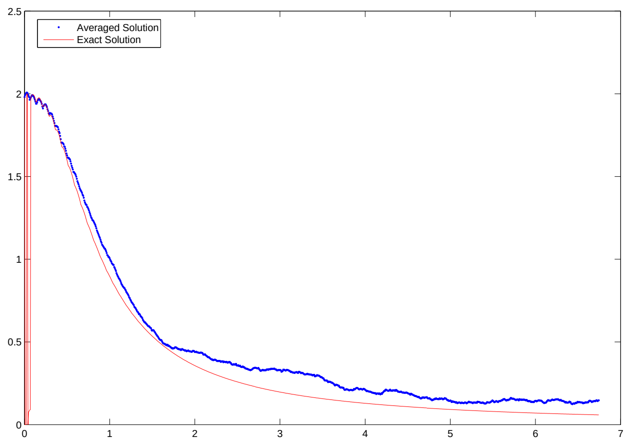

Figure 6: Comparison of the amplitudes from the exact solution and a simulation as provided in the code below.

```
% Soliton Parameters
eta=1.0; x0=-5.0; epsilon=0.1; mu=sqrt(2*epsilon);

% Mesh Parameters
N=200; Tsteps=1000; a=-10; b=50; x=linspace(a,b,N+1); dx=(b-a)/N;
dt=dx^3/4; t=(0:Tsteps-1)*dt;

% Stochastic Run
M=200; uu=zeros(Tsteps,N+1);

for k=1:M
    k
    r=randn(Tsteps,1);
    w=zeros(Tsteps,1);
    Iw=zeros(Tsteps,1);
    w(1)=0;
    Iw(1)=w(1)/2*dt;
    for i=2:Tsteps
        w(i)=w(i-1)+mu*sqrt(dt)*r(i);
        Iw(i)=Iw(i-1)+(w(i-1)+w(i))/2*dt;
    end

    u=zeros(Tsteps,N+1);
    for i=1:Tsteps;
        for j=1:N+1;
            u(i,j)=w(i)+2*eta^2*(sech(eta*(x(j)-4*eta^2*t(i)-x0-6*Iw(i)))).^2;
        end
    end
uu=uu+u; end uu=uu/M;

% Exact Solution
for j=1:N+1
    u(1,j)=2*eta^2*(sech(eta*(x(j)-x0)))^2;
end for i=2:Tsteps;
    i
    B=48*eta^2*epsilon*t(i)^3;
    for j=1:N+1;
        A=2*(eta*(x(j)-x0)-4*eta^3*t(i));
        F = @(s)(sech(s/2)).^2.*exp(-(A-s).^2/4/B);
        u(i,j) =eta^2/sqrt(pi*B)*quad(F,-20,20);
    end
end

% Results
figure(1) mesh(x,t,u) title('Exact Solution') figure(2)
mesh(x,t,uu) title('Stochastic KdV') figure(3)
```

```
plot(t,max(uu'),'b.') hold on plot(t,max(u'),'r-')
legend('Averaged Solution','Exact Solution',2) hold off
```

3 Problems

There are several problems with Wadati's derivation. These also appear elsewhere in the literature references to Wadati's paper.

First, we note that the series expansion for the $\operatorname{sech}^2 z$ is not quite right. We should instead have derived it as follows (for $z \neq 0$):

$$\begin{aligned}
\operatorname{sech}^2 z &= \frac{4}{(e^z + e^{-z})^2} \\
&= \frac{4e^{-2|z|}}{(1 + e^{-2|z|})^2} \\
&= 2\operatorname{sgn}(z) \frac{d}{dz} \frac{1}{1 + e^{-2|z|}} \\
&= 2\operatorname{sgn}(z) \frac{d}{dz} \left(\sum_{n=0}^{\infty} (-e^{-2|z|})^n \right) \\
&= 2 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-2n|z|}, \qquad (29)
\end{aligned}$$

This accounts for the convergence of the geometric series used in the derivation. Namely, in the original derivation, one should have noted that $|e^{2z}| < 1$, or $z < 0$. This new derivation accounts for the $z > 0$ case. Konotop and Vázquez [2] appear to have used this in their review of Wadati's derivation. They present the infinite series result as

$$< u(x,t) >= 8\eta^2 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n|a| + n^2 b}. \qquad (30)$$

There also appears to be a problem with the derivation of the average. Wadati should actually have computed

$$< u(x,t) >= 8\eta^2 \sum_{n=1}^{\infty} (-1)^{n+1} n \left\langle \exp\left[-2n\eta \left| x - 4\eta^2 t - x_0 - 6 \int_0^t W(t')\, dt' \right| \right] \right\rangle.$$

One could get around this problem by computing the average for space-time regions where $x - 4\eta^2 t - x_0 - 6\int_0^t W(t')\, dt'$ is definitely of one sign. Another approach would instead be to directly expand

$$u(x,t) = 2\eta^2 \operatorname{sech}^2(\eta(x - x_0 - 4\eta^2 t) - 6\eta \int_0^t W(t')\, dt') \equiv 2\eta^2 \operatorname{sech}^2(\theta + \sigma)$$

about $\sigma = 0$ for

$$\sigma \equiv -6\eta \int_0^t W(t')\, dt'.$$

We have that

$$2\eta^2 \operatorname{sech}^2(\theta+\sigma) = 2\eta^2 \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^n}{\partial\theta^n} \operatorname{sech}^2\theta. \tag{31}$$

The average can now be computed as

$$< u(x,t) >= 2\eta^2 \sum_{n=0}^{\infty} \frac{<\sigma^n>}{n!} \frac{\partial^n}{\partial\theta^n} \operatorname{sech}^2\theta \tag{32}$$

provided that we can compute

$$< \sigma^n >= \left\langle \left(-6\eta \int_0^t W(t')\,dt' \right)^n \right\rangle. \tag{33}$$

Herman [1] shows that such averages can be computed based upon the nature of the Gaussian noise as

$$< \sigma^n >= \begin{cases} 0, & n \text{ odd}, \\ (2\ell-1)!! < \sigma^2 >^\ell, & n = 2\ell, \text{ even}. \end{cases} \tag{34}$$

Thus, we just need to compute $< \sigma^2 >$.

To see how this is done, we have

$$\begin{aligned} < \sigma^2 > &= \left\langle 36\eta^2 \int_0^t W(t_1)\,dt_1 \int_0^t W(t_2)\,dt_2 \right\rangle \\ &= 36\eta^2 \int_0^t \int_0^t < W(t_1)W(t_2) > \,dt_1 dt_2 \\ &= 72\epsilon\eta^2 \int_0^t \int_0^t \min(t_1,t_2)\,dt_1 dt_2 \\ &= 72\epsilon\eta^2 \int_0^t \left(\int_0^{t_2} \min(t_1,t_2)\,dt_1 + \int_{t_2}^t \min(t_1,t_2)\,dt_1 \right) dt_2 \\ &= 72\epsilon\eta^2 \int_0^t \left(\int_0^{t_2} t_1\,dt_1 + \int_{t_2}^t t_2\,dt_1 \right) dt_2 \\ &= 72\epsilon\eta^2 \int_0^t \left(\frac{t_2^2}{2} + t_2(t-t_2) \right) dt_2 \\ &= 24\epsilon\eta^2 t^3. \end{aligned} \tag{35}$$

Inserting these results in Equation (32) yields

$$\begin{aligned} < u(x,t) > &= 2\eta^2 \sum_{n=0}^{\infty} \frac{<\sigma^n>}{n!} \frac{\partial^n}{\partial\theta^n} \operatorname{sech}^2\theta \\ &= 2\eta^2 \sum_{\ell=0}^{\infty} \frac{<\sigma^2>^\ell}{(2\ell)!} (2\ell-1)!! \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} \operatorname{sech}^2\theta \end{aligned}$$

$$
\begin{aligned}
&= \quad 2\eta^2 \sum_{\ell=0}^{\infty} \frac{<\sigma^2>^\ell}{2^\ell \ell!} \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} \operatorname{sech}^2\theta \\
&= \quad 2\eta^2 \sum_{\ell=0}^{\infty} \frac{(12\epsilon\eta^2 t^3)^\ell}{\ell!} \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} \operatorname{sech}^2\theta. \qquad (36)
\end{aligned}
$$

In order to see the agreement with Wadati's result for small $b = 48\epsilon\eta^2 t^3$, we need to set $\theta = a/2$. Noting that $\frac{\partial^{2\ell}}{\partial\theta^{2\ell}} = 2^{2\ell}\frac{\partial^{2\ell}}{\partial a^{2\ell}}$, we obtain

$$
< u(x,t) >= 2\eta^2 \sum_{\ell=0}^{\infty} \frac{(48\epsilon\eta^2 t^3)^\ell}{\ell!} \frac{\partial^{2\ell}}{\partial a^{2\ell}} \operatorname{sech}^2\frac{a}{2} = 2\eta^2 \sum_{\ell=0}^{\infty} \frac{b^\ell}{\ell!} \frac{\partial^{2\ell}}{\partial a^{2\ell}} \operatorname{sech}^2\frac{a}{2}.
$$

We further note that this solution again satisfies the heat equation and that for $b = 0$ this solution reduces to the soliton initial condition. Thus, we have seemingly bypassed any problem with the computing the average with an absolute value. However, this series is divergent for $b > 1$. This divergence problem still needs to be addressed.

References

[1] Herman R L 1990 The Stochastic, Damped KdV Equation *J. Phys. A* **23** 1063-1084.

[2] Konotop V V and Vázquez L 1994 *Nonlinear Random Waves* (World Scientific).

[3] Scalerandi M, Romano A and Condat C A 1998 Korteweg-de Vries Solitons Under Additive Stochastic Perturbations *Phys. Rev. E* **58** 4166-4173.

[4] Wadati M 1983 Stochastic Korteweg-de Vries Equation *J. Phys. Soc. Jpn.* **52** 2642-2648.