Perturbative and WKB Expansions

1 Introduction

Solutions to the quantum harmonic oscillator problem,

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x), \tag{1}$$

are well known with associated energies

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

It is common to rescale the problem to obtain

$$-\frac{1}{2}\psi''(x) + \frac{1}{2}x^2\psi(x) = E\psi(x), \qquad E_n = n + \frac{1}{2}. \tag{2}$$

The first three numerically computed energy levels, E_n, and solutions, $\psi_n(x)$ are shown in Figure 1.

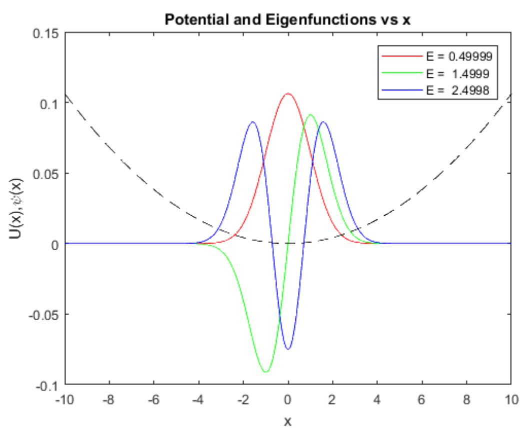

Figure 1: Solutions and energies.

Now we perturb the potential with a quartic term. Then, we have

$$-\frac{1}{2}\psi''(x) + \frac{1}{2}x^2\psi(x) + gx^4\psi(x) = E\psi(x), \tag{3}$$

where g is a coupling constant. Examples displaying the potential, solutions, and energies are shown in Figures 2-3.

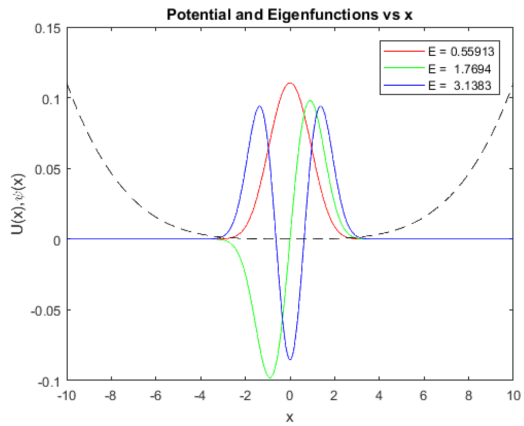

Figure 2: Solutions and energies for the problem with quartic perturbation and $g = 0.1$.

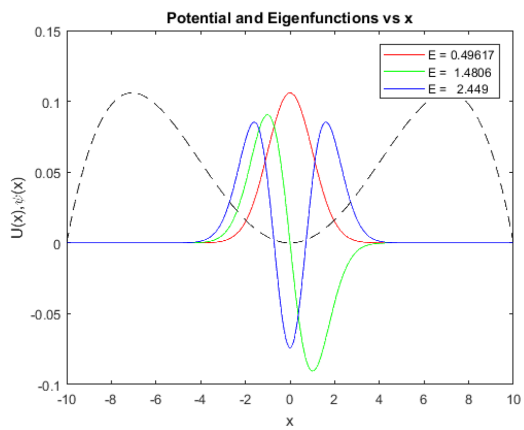

Figure 3: Solutions and energies for the problem with quartic perturbation and $g = -0.005$.

For small coupling, one can assume that the energies are close to those of the harmonic oscillator. The numerical results confirm this. How does one analytically obtain approximations to the energies? The usual methods for doing so are either Rayleigh-Schrödinger perturbation theory or a WKB analysis.

Bender and Wu[1] [1] had found for the ground state energy that corrections to the ground state energy in powers of the coupling parameter are given as

$$E_0 = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4 + \frac{916731}{256}g^5 + \mathcal{O}(g^6). \tag{4}$$

[1]Actually, they solved the problem

$$\left[-\frac{d^2}{dx^2} + \frac{1}{4}x^2 + \frac{1}{4}\lambda x^4\right]\psi(x) = E\psi(x).$$

Letting $x = \sqrt{2}q$, we obtain Equation (3) and get the same energies.

Writing

$$E_0 = \frac{1}{2} + \sum_{n=1}^{\infty} A_n g^n,$$

they found that for large n,[2]

$$\begin{aligned} A_n &\sim (-1)^{n+1}\sqrt{\frac{6}{\pi^3}}3^n\Gamma(n+\frac{1}{2}) \\ &= -\sqrt{\frac{6}{\pi^2}}\left(-\frac{3}{2}\right)^n (2n-1)!!. \end{aligned} \tag{5}$$

Thus, the series is divergent with factorial growth.

2 Riccati Equation

Following [3], we begin with the Schrödinger equation in the form

$$\hat{H}\psi = E\psi, \qquad \hat{H} = -\frac{g^2}{2}\frac{d^2}{dq^2} + V(g,q), \tag{6}$$

where g is a coupling constant. One can think of $g = \hbar$.

Letting $S(q) = -g\psi'/\psi$, we obtain the Riccati equation

$$gS' - S^2 + 2(V - E) = 0. \tag{7}$$

We next introduce two independent solutions satisfying the boundary conditions

$$\lim_{q\to+\infty} \psi_2(q) = 0, \quad \lim_{q\to-\infty} \psi_1(q) = 0.$$

Leading order WKB for large q and $V(q) > gE$ gives

$$\begin{aligned} \psi_1(q) &\sim \exp\left(-\frac{1}{g}\int dq\sqrt{2(V(q)-E)}\right) \\ \psi_2(q) &\sim \exp\left(\frac{1}{g}\int dq\sqrt{2(V(q)-E)}\right) \end{aligned} \tag{8}$$

Defining

$$S_\pm(q) = \frac{1}{2}\left(\frac{\psi_1'}{\psi_1} + \frac{\psi_2'}{\psi_2}\right),$$

we have

$$\begin{aligned} gS_-' - S_+^2 + 2V(q) - 2gE = 0, \\ gS_+' - 2S_+S_- = 0. \end{aligned} \tag{9}$$

[2] Recall that $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$.

The Bohr-Sommerfeld condition,

$$B(E,g) \equiv -\frac{1}{2\pi i g}\oint_C dz\, S_+(z,g,E) = N + \frac{1}{2}, \tag{10}$$

is used to find the energy, E, as

$$E = \sum_{k=0}^{\infty} g^k E_k.$$

The contour C is chosen to be a positively oriented simple closed contour enclosing all real poles of $S_+(z,g,E)$.

3 Perturbative Expansion

Let

$$S(q) = \sum_{k=0}^{\infty} g^k s_k(q)$$

and define $U(q) = \sqrt{2V(q)}$. From the Riccati equation, one finds

$$\begin{aligned}
s_0(q) &= U(q), \\
s_1(q) &= \frac{1}{2U(q)}\left(U'(q) - 2E\right), \\
s_2(q) &= \frac{U''}{4U^2} - \frac{3U'^2}{8U^3} + \frac{EU'}{2U^3} - \frac{E^2}{2U^3}, \\
s_k(q) &= \frac{1}{2U(q)}\left(s'_{k-1}(q) - \sum_{\ell=1}^{k-1} s_{k-\ell}(q)s_\ell(q)\right), \quad k = 2,3,\ldots.
\end{aligned} \tag{11}$$

Assuming reflection symmetry, $B(-E,-g) = -B(E,g)$, we have

$$\begin{aligned}
S_+(z,g,E) &= \frac{1}{2}(S(z,g,E) + S(z,-g,-E)) \\
&= U - g\frac{E}{U} + g^2\left(\frac{1}{4}\frac{U''}{U^2} - \frac{3}{8}\frac{U'^2}{U^3} - \frac{1}{2}\frac{E^2}{U^3}\right) + \mathcal{O}(g^3).
\end{aligned} \tag{12}$$

4 WKB Expansion

In this case we expand $S(q)$ in powers of g with gE fixed,

$$S(q) = \sum_{k=0}^{\infty} g^k S_k(q).$$

Letting $W(q) = \sqrt{2(V(q) - gE)}$, gives

$$S_0(q) = W(q),$$

$$
\begin{aligned}
S_1(q) &= \frac{1}{2}\frac{S_0'(q)}{S_0(q)}, \\
S_2(q) &= \frac{S_1'(q) - S_1^2(q)}{2S_0(q)} \\
&= \frac{S_0^{''}(q)}{4S_0^2(q)} - \frac{3}{8}\frac{{S_0'}^2(q)}{S_0^3(q)}, \\
S_k(q) &= \frac{1}{2S_0(q)}\left(S_{k-1}'(q) - \sum_{\ell=1}^{k-1} S_{k-\ell}(q)S_\ell(q)\right), \quad k = 2, 3, \ldots. \qquad (13)
\end{aligned}
$$

The difference in this case is that when computing $B(E,g)$ one encounters branch points and takes the branch cut along the positive real axis.

$$
\begin{aligned}
S_0 &= \left(U^2 - gE\right)^{\frac{1}{2}} \\
&= U\left(1 - gEU^{-2}\right)^{\frac{1}{2}} \\
&= U\sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+1)\Gamma(-\frac{1}{2})}\left(-gEU^{-2}\right)^n \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
S_{2+} &= \frac{1}{8}\frac{{S_0'}^2(q)}{S_0^3(q)} \\
&= \frac{U'^2U^2}{8S_0^5} \\
&= \frac{1}{8}\sum_{n=-1}^{\infty} \frac{\Gamma(n+\frac{1}{5})}{\Gamma(n+1)\Gamma(\frac{5}{2})}\left(2gE\right)^n U'^2U^{-2n-3} \qquad (15)
\end{aligned}
$$

5 Example

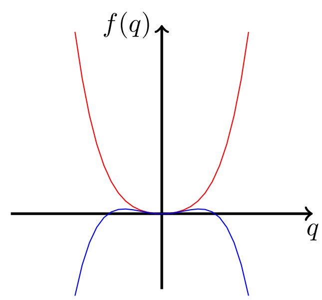

Figure 4: Plots of $f(q) = \frac{1}{2}q^2 + q^4$ (red) and $f(q) = \frac{1}{2}q^2 - q^4$ (blue).

$$
\begin{aligned}
B &= E + \left(-3/2\,E^2 - 3/8\right) G + \left(\frac{35\,E^3}{4} + \frac{85\,E}{16}\right) G^2 + \left(-\frac{1155\,E^4}{16} - \frac{2625\,E^2}{32} - \frac{1995}{256}\right) G^3 \\
&\quad + \left(\frac{400785\,E}{1024} + \frac{165165\,E^3}{128} + \frac{45045\,E^5}{64}\right) G^4 + \ldots . \qquad (16)
\end{aligned}
$$

Appendix - WKB Approximation

The WKB Approximation, named after Wentzel–Kramers–Brillouin, is a method to approximate solutions to linear differential equations with spatially varying coefficients. According to [2], this method was discussed by several researchers such as George Green (1793-1841), Lord Rayleigh (1842-1909), Richard Gans (1880-1954), Harold Jeffreys (1891-1989), only to have the names of Gregor Wentzel (1898-1978), Henrich A. Kramers (1884-1952), and Leon Brillouin (1889-1969) stick as the “WKB” approximation (or, method). This is probably because of the application to quantum mechanics. Sometimes Jeffreys is added giving the WKBJ or JWKB approximation.

We begin with the time-independent Schödinger equation,

$$
-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x), \qquad (17)
$$

or

$$
\psi''(x) + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x) = 0, \qquad (18)
$$

and consider the scattering problem shown in Figure 5. A particle moving from left to right with energy E encounters potential $V(x)$ and is reflected at the turning point x_r. We seek to find the solutions on either side of the turning point and match them at the turning point. We will also provide the results for particles incident on a potential from the right with a turning point at $x = x_\ell$.

In general, this is not exactly solvable, so we seek approximate solutions valid when the wavelength is much smaller than some characteristic length, $\lambda \ll L$. Recalling that $\lambda = h/p$, this is a condition on $p(x)$, $p(x) \gg \frac{h}{L}$, where

$$
p(x) = \sqrt{2m\left[E - V(x)\right]}.
$$

Therefore, there is a restriction on the kinetic energy:

$$
E - V(x) = \frac{p^2(x)}{2m} \gg \frac{h^2}{2mL^2}.
$$

We make the WKB ansätz,

$$
\begin{aligned}
\psi(x) &= A(x)e^{\frac{i}{\hbar}S(x)} \\
&= e^{\frac{i}{\hbar}S(x) + \ln A} \\
&\equiv e^{\frac{i}{\hbar}W(x)}. \qquad (19)
\end{aligned}
$$

Here we have defined

$$
W(x) = S(x) - i\ln A\hbar.
$$

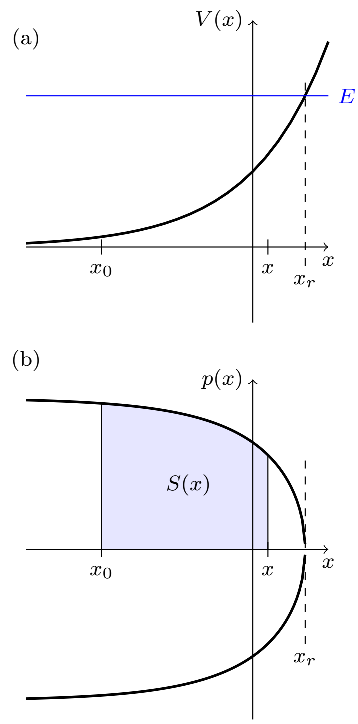

Figure 5: (a) Potential plot showing turning point at $x = x_r$ associated with scattering energy E. (b) Phase space plot of curves $p(x) = \pm\sqrt{2m\,[E - V(x)]}$ and the shaded area, $S(x)$.

Since $\hbar$ is small, it looks like we have the beginning of a power series expansion in powers of $\hbar$,

$$W(x) = W_0(x) + W_1(x)\hbar + W_2(x)\hbar^2 + \dots.$$

Here we have

$$W_0(x) = S(x), \quad W_1(x) = -i \ln A.$$

The typical introduction to WKB stops at these two terms. The question as to how well two terms approximates the exact solution was answered in 1954 by F. W. J. Olver [2].

Inserting the ansätz into the Schrödinger equation, one obtains a differential equation for $W(x)$:

$$\frac{1}{2m} W'^2 - \frac{i\hbar}{2m} W'' + V = E, \tag{20}$$

or

$$W'^2 - i\hbar W'' + 2m[V - E] = 0. \tag{21}$$

Note that this is a Riccati equation[3] for $y(x) = W'(x)$.

Inserting the expansion for $W(x)$ and collecting like powers of $\hbar$, we obtain

$$\frac{1}{2m} W_0'^2 + V(x) = E,$$

[3] A general nonlinear Riccati equation takes the form $y' = q_0(x) + q_1(x)y + q_2(x)y^2$

$$\frac{1}{m} W_0' W_1' - \frac{i}{2m} W_0'' = 0. \tag{22}$$

Since $W_0(x) = S(x)$ and $W_1(x) = -i \ln A$, these equations give

$$\frac{1}{2m} S'^2 + V(x) = E, \tag{23}$$

$$0 = \frac{A'}{A} S' + 12 S'' = \frac{d}{dx} \left(A^2 S' \right). \tag{24}$$

Equation (23) is in the form of the Hamilton-Jacobi equation, so $S(x)$ is called the action. Equation (24) is an amplitude transport equation.

Solving for S', we have

$$S'(x) = \sqrt{2m[E - V(x)]} = p(x).$$

Integrating, we have

$$S(x) = \int_{x_0}^{x} p(\xi)\, d\xi = \pm \int_{x_0}^{x} \sqrt{2m \left[E - V(\xi)\right]}\, d\xi.$$

Therefore, we see that $S(x)$ is the area in the phase plane as shown in Figure 5.

Integrating Equation (24), we find

$$A(x) = \frac{\text{const.}}{\sqrt{p(x)}}.$$

Combining these solutions, we have found the general solution

$$\psi(x) = \frac{c_r}{\sqrt{p(x)}} e^{\frac{i}{\hbar} S(x)} + \frac{c_\ell}{\sqrt{p(x)}} e^{-\frac{i}{\hbar} S(x)}, \quad E > V(x)$$

in the classically allowed region.

In the classically forbidden region, $E < V(x)$. Therefore, the general solution takes on the form

$$\psi(x) = \frac{c_g}{\sqrt{|p(x)|}} e^{K(x)/\hbar} + \frac{c_d}{\sqrt{|p(x)|}} e^{-K(x)/\hbar}, \quad E < V(x),$$

where

$$K(x) = \int_{x_0}^{x} |p(\xi)|\, d\xi = \int_{x_0}^{x} \sqrt{2m \left[V(\xi) - E\right]}\, d\xi.$$

A similar analysis can be provided for particles incident from the right as shown on the left side of Figure 5.

We first consider the case of particles incident from the left as shown on the right in Figure 6. There are two regions of interest. These regions correspond to the regions on the left ($x < x_r$) and right ($x > x_r$) sides of the turning point. However, near the turning point these solutions are invalid since $\lambda = 2\pi\hbar/p$ becomes unbounded as $x \to x_r$. Therefore, we need to find the solution behavior near x_r. In Figure 6 this region is designated as Region II.

In Region II we expand the potential about $x = x_r$, or use the linear approximation

$$V(x) \approx V(x_r) + V'(x_r)(x - x_r).$$

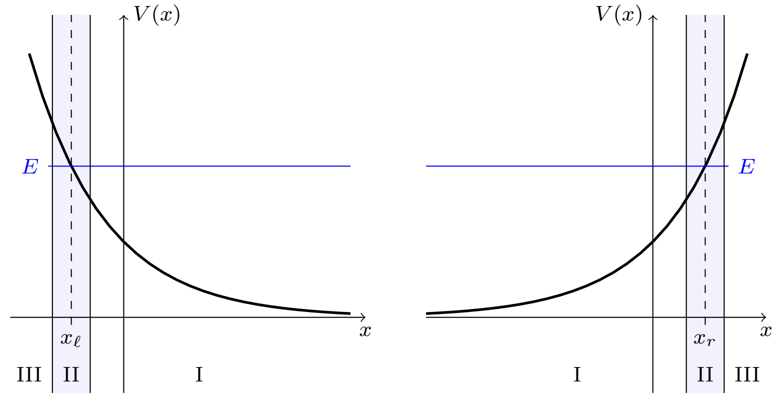

Figure 6: Potential plot showing three regions used in the WKB analysis. On the left is for an incident particle from the right and on the right is for an incident particle from the left.

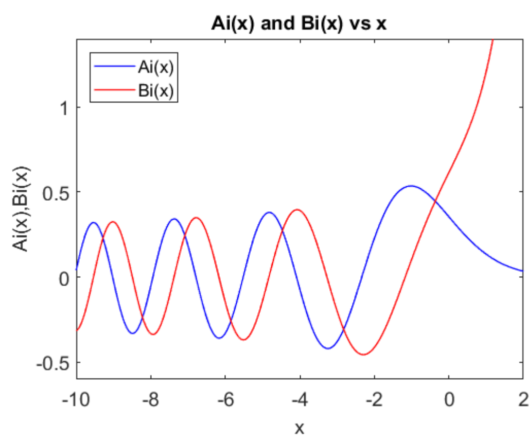

Figure 7: Plots of the Airy functions of the first and second kind.

Inserting this in the Schrödinger equation and defining $x - x_r = az$, $\psi(x) = \phi(z)$, we find

$$\phi''(z) - z\psi(z) = 0, \qquad a = \left(\frac{\hbar^2}{2mV'(x_r)}\right)^{1/3}.$$

This is the Airy equation with solutions given by the Airy function of the first kind, $\mathrm{Ai}(x)$, and the Airy function of the second kind, $\mathrm{Bi}(x)$. Therefore, the general solution is in the form

$$\phi(z) = c_a \mathrm{Ai}(z) + c_b \mathrm{Bi}(z).$$

The behavior of these functions is shown in Figures 7-8 with the asymptotic behavior listed in Table 1.

The general forms for the wavefunctions in the three regions are

$$\psi_I(x) \quad = \quad \frac{c_r e^{i\pi/4}}{\sqrt{p(x)}} e^{\frac{i}{\hbar}S(x,x_r)} + \frac{c_\ell e^{-i\pi/4}}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}S(x,x_r)}, \tag{25}$$

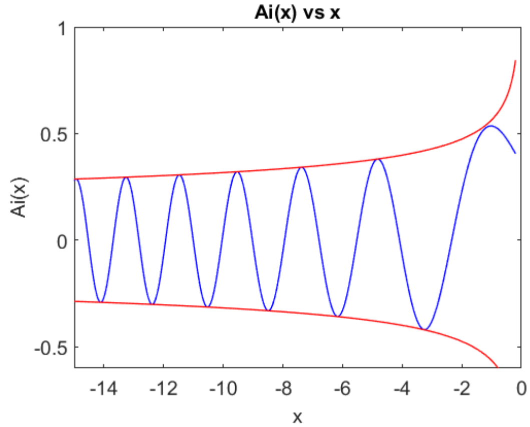

Figure 8: Plots of the $\mathrm{Ai}(x)$ and its envelope, $y = \frac{1}{\sqrt{\pi\sqrt{-x}}}$ for $x \leq 0$.

	$z \ll 0$	$z \gg 0$
$\mathrm{Ai}(z)$	$\frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos\alpha(z)$	$\frac{1}{2\sqrt{\pi}z^{1/4}} \exp(-\beta(z))$
$\mathrm{Bi}(z)$	$\frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin\alpha(z)$	$\frac{1}{\sqrt{\pi}z^{1/4}} \exp(\beta(z))$

Table 1: The asymptotic behavior of Airy functions which can also be seen in Figure 7. Here we have defined $\alpha(z) = -\frac{2}{3}(-z)^{2/3} + \frac{\pi}{4}$ and $\beta(z) = \frac{2}{3}z^{3/2}$.

$$\psi_{II}(x) = c_a \mathrm{Ai}(z) + c_b \mathrm{Bi}(z), \tag{26}$$

$$\psi_{III}(x) = \frac{c_g}{\sqrt{|p(x)|}} e^{K(x,x_r)/\hbar} + \frac{c_d}{\sqrt{|p(x)|}} e^{-K(x,x_r)/\hbar}, \tag{27}$$

where

$$S(x, x_r) = \int_{x_r}^{x} \sqrt{2m\,[E - V(\xi)]}\, d\xi,$$

$$K(x, x_r) = \int_{x_r}^{x} \sqrt{2m\,[V(\xi) - E]}\, d\xi,$$

$$az = x - x_r, \qquad a = \left(\frac{\hbar^2}{2mV'(x_r)}\right)^{1/3},$$

and phase shifts of $\pm\frac{i\pi}{4}$ were introduced to make matching with the asymptotics of the Airy functions simpler.

In the forbidden region we consider the asymptotics of the Airy function for $z \gg 0$. Then, matching solutions in Regions II and III requires finding the asymptotic form for $K(x, x_r)$.

$$\begin{aligned} K(x, x_r) &= \int_{x_r}^{x} \sqrt{2m\,[V(\xi) - E]}\, d\xi \\ &\approx \sqrt{\frac{2mV'(x_r)}{\hbar^2}} \int_{x_r}^{x} \sqrt{\xi - x_r}\, d\xi \text{ for } x \approx x_r, \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}a^{-3/2}(x-x_r)^{3/2} \\
&= \frac{2}{3}z^{3/2} = \beta(z).
\end{aligned} \tag{28}
$$

Furthermore, we note that near the turning point

$$
\begin{aligned}
|p(z)|^2 &= 2m(E-V(x)) \\
&\approx 2m(x-x_r)V'(x_r) \text{ for } x \approx x_r, \\
&= 2mV'(x_r)az \\
&= \frac{\hbar^2}{a^2}z.
\end{aligned} \tag{29}
$$

Therefore, we have found that

$$
\begin{aligned}
\psi_{III}(x) &= \frac{c_g}{\sqrt{|p(x)|}}e^{K(x,x_r)/\hbar} + \frac{c_d}{\sqrt{|p(x)|}}e^{-K(x,x_r)/\hbar} \\
&\sim \frac{c_g}{z^{1/4}}\sqrt{\frac{a}{\hbar}}e^{\beta(z)} + \frac{c_d}{z^{1/4}}\sqrt{\frac{a}{\hbar}}e^{-\beta(z)}.
\end{aligned} \tag{30}
$$

We compare this with the asymptotic solution for the Airy functions in Region II,

$$
\begin{aligned}
\psi_{II}(x) &= c_a\mathrm{Ai}(z) + c_b\mathrm{Bi}(z) \\
&\sim \frac{c_a}{2\sqrt{\pi}z^{1/4}}e^{-\beta(z)} + \frac{c_b}{\sqrt{\pi}z^{1/4}}e^{\beta(z)}.
\end{aligned} \tag{31}
$$

We see that

$$
c_d = \frac{1}{2}\sqrt{\frac{\hbar}{\pi a}}c_a, \quad c_g = \frac{1}{2}\sqrt{\frac{\hbar}{\pi a}}c_b.
$$

A similar analysis can be done relating regions I and II. Then, we find that the coefficients are related by

$$
c_r = \frac{1}{2}\sqrt{\frac{\hbar}{\pi a}}(c_a - ic_b), \quad c_\ell = \frac{1}{2}\sqrt{\frac{\hbar}{\pi a}}(c_a + ic_b).
$$

For a particle incident from the left, we need $c_b = c_g = 0$. Then these expressions simplify to

$$
c_r = \frac{1}{2}\sqrt{\frac{\hbar}{\pi a}}c_a = c_\ell
$$

and the wavefunctions take the form

$$
\psi_I(x) = \frac{c_r}{\sqrt{p(x)}}\cos\left(\frac{S(x,x_r)}{\hbar} + \frac{\pi}{4}\right), \tag{32}
$$

$$
\psi_{II}(x) = c_a\mathrm{Ai}(z), \tag{33}
$$

$$
\psi_{III}(x) = \frac{c_d}{\sqrt{|p(x)|}}e^{-K(x,x_r)/\hbar}. \tag{34}
$$

Setting $c_d = 1$, we have $c_a = 2\sqrt{\frac{\pi a}{\hbar}}$ and, therefore, $c_r = c_\ell = 1$.

In general, we have found the connection rules

$$\begin{pmatrix} c_r \\ c_\ell \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} & 1 \\ \frac{i}{2} & 1 \end{pmatrix} \begin{pmatrix} c_g \\ c_d \end{pmatrix}.$$

These are independent of a.

We next consider bound state solutions for the potential in Figure 9(a) with the associated phase plane plot in Figure 9(b). In this case there are two turning points at $x = x_\ell$ and $x = x_r$. Based on the previous analysis, we write the wavefunctions for each region as

$$\begin{aligned} \psi_I(x) &= \frac{a_g}{\sqrt{|p(x)|}} e^{K(x,x_\ell)/\hbar} + \frac{a_d}{\sqrt{|p(x)|}} e^{-K(x,x_\ell)/\hbar}, && (35) \\ \psi_{II}(x) &= \frac{b_r e^{-i\pi/4}}{\sqrt{p(x)}} e^{\frac{i}{\hbar}S(x,x_\ell)} + \frac{b_\ell e^{i\pi/4}}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}S(x,x_\ell)}, && (36) \\ &= \frac{b'_r e^{i\pi/4}}{\sqrt{p(x)}} e^{\frac{i}{\hbar}S(x,x_r)} + \frac{b'_\ell e^{-i\pi/4}}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}S(x,x_r)}, && (37) \\ \psi_{III}(x) &= \frac{c_g}{\sqrt{|p(x)|}} e^{K(x,x_r)/\hbar} + \frac{c_d}{\sqrt{|p(x)|}} e^{-K(x,x_r)/\hbar}. && (38) \end{aligned}$$

Considering the behavior in the classically forbidden regions, we have that $a_d = c_g = 0$. We also note that there are two equivalent expressions for Region II depending if we use the left or right turning point.

The coefficients are represented by the connection formulae found in the single turning point problem. Thus, we have for the right turning point,

$$\begin{pmatrix} b'_r \\ b'_\ell \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} & 1 \\ \frac{i}{2} & 1 \end{pmatrix} \begin{pmatrix} c_g \\ c_d \end{pmatrix}. \tag{39}$$

We have similar relations for the left turning point:

$$\begin{pmatrix} b_r \\ b_\ell \end{pmatrix} = \begin{pmatrix} 1 & \frac{i}{2} \\ 1 & -\frac{i}{2} \end{pmatrix} \begin{pmatrix} a_g \\ a_d \end{pmatrix}. \tag{40}$$

If $a_d = 0$, $a_g = 1$, then from Equation (39) we have $b_r = 1 = b_\ell$.

We need to relate b'_r, b'_ℓ to b_r and b_ℓ. Focusing on the first terms in each expression for $\psi_{II}(x)$, we have

$$\begin{aligned} S(x, x_\ell) &= \int_{x_\ell}^{x} p(x)\, dx \\ &= \int_{x_\ell}^{x_r} p(x)\, dx + \int_{x_r}^{x} p(x)\, dx \\ &= \pi I + S(x, x_r), && (41) \end{aligned}$$

where I is the action integral,

$$I = \frac{1}{2\pi} \oint p(x)\, dx.$$

The integral in the definition of the action gives the total phase change,

$$\Delta\phi = \oint p(x)\, dx - 2\frac{\pi}{2} = 2n\pi.$$

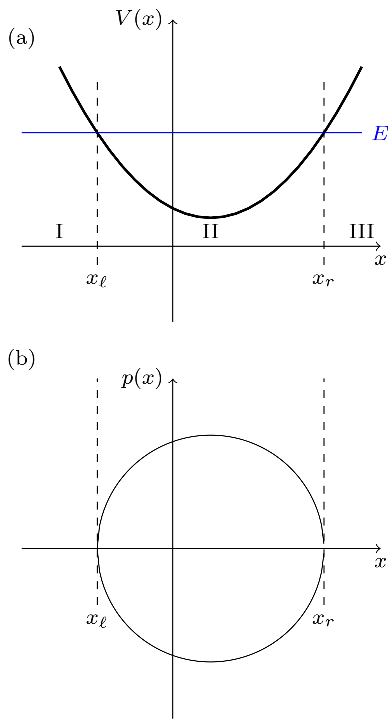

Figure 9: (a) Potential plot showing turning points at $x = x_\ell$ and $x = x_r$ associated with bound state energy E. (b) Phase space plot of curves $p(x) = \pm\sqrt{2m(E - V(x))}$.

There are two phase shifts at the turning points contributing shifts of $\pi/2$. This in tuen gives the Bohr-Sommerfeld quantization rule

$$\oint p(x)\,dx = \left(n + \frac{1}{2}\right) h.$$

Thus, the allowed orbits have the action integrals

$$I_n = \frac{1}{2\pi}\oint p(x)\,dx = \left(n + \frac{1}{2}\right) \hbar. \tag{42}$$

Now we can related the expressions for $\psi_{II}(x)$,

$$\begin{aligned}
\psi_{II}(x) &= \frac{b_r e^{-i\pi/4}}{\sqrt{p(x)}} e^{\frac{i}{\hbar}S(x,x_\ell)} + \frac{b_\ell e^{i\pi/4}}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}S(x,x_\ell)} \\
&= \frac{b_r e^{-i\pi/4}}{\sqrt{p(x)}} e^{\frac{i}{\hbar}S(x,x_r)+\frac{i}{\hbar}\pi I} + \frac{b_\ell e^{i\pi/4}}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}S(x,x_r)-\frac{i}{\hbar}\pi I} \\
&= \frac{b'_r e^{i\pi/4}}{\sqrt{p(x)}} e^{\frac{i}{\hbar}S(x,x_r)} + \frac{b'_\ell e^{-i\pi/4}}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}S(x,x_r)}.
\end{aligned} \tag{43}$$

Comparing the last two lines, we have

$$\begin{aligned}
b'_r &= e^{\frac{i\pi}{\hbar}I - i\pi/2} b_r \\
b'_\ell &= e^{-\frac{i\pi}{\hbar}I + i\pi/2} b_\ell.
\end{aligned} \tag{44}$$

,

From Equation (40), we have

$$\begin{pmatrix} c_g \\ c_d \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -i & i \end{pmatrix} \begin{pmatrix} b'_r \\ b'_\ell \end{pmatrix}. \tag{45}$$

Therefore, since $b_\ell = b_r = 1$,

$$c_g = ib'_r - ib'_\ell = -2\sin\left(\frac{\pi I}{\hbar} - \frac{\pi}{2}\right) = 2\cos\frac{\pi I}{\hbar} = 0.$$

This implies that $I = \left(n + \frac{1}{2}\right)\hbar$, which is the Bohr-Sommerfeld condition. Furthermore, we find that

$$c_d = \frac{1}{2}(b'_r + b'_\ell) = \cos\left(\frac{\pi I}{\hbar} - \frac{\pi}{2}\right) = \cos n\pi = (-1)^n.$$

This analysis gives the wavefunctions in the three regions, away from the turning points, as

$$\psi_I(x) = \frac{1}{\sqrt{|p(x)|}} e^{K(x,x_\ell)/\hbar}, \tag{46}$$

$$\psi_{II}(x) = \frac{2}{\sqrt{p(x)}} \cos\left(\frac{S(x,x_\ell)}{\hbar} - \frac{\pi}{4}\right), \tag{47}$$

$$\psi_{III}(x) = \frac{(-1)^n}{\sqrt{|p(x)|}} e^{-K(x,x_r)/\hbar}. \tag{48}$$

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