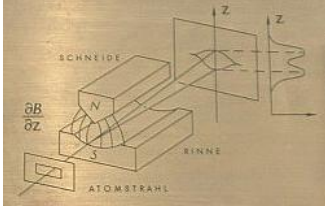


## Spin Eigenstates - Review

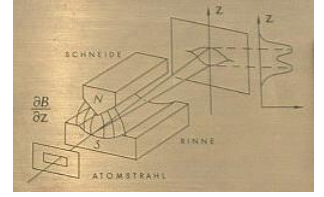
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## SG Devices Measure Spin



- Orient device in direction  $\mathbf{n}$
- The representation of  $|\psi\rangle$  in the  $S_n$ -basis for spin  $\frac{1}{2}$ :  
 $|\psi\rangle_n = I_n|\psi\rangle$ , where  
 $I_n = |+\mathbf{n}\rangle\langle+\mathbf{n}| + |-\mathbf{n}\rangle\langle-\mathbf{n}|$

$$\begin{aligned} |\psi\rangle_n &= |+\mathbf{n}\rangle\langle+\mathbf{n}|\psi\rangle + |-\mathbf{n}\rangle\langle-\mathbf{n}|\psi\rangle \\ &= a_+|+\mathbf{n}\rangle + a_-|-\mathbf{n}\rangle \\ &\rightarrow \begin{pmatrix} \langle+\mathbf{n}|\psi\rangle \\ \langle-\mathbf{n}|\psi\rangle \end{pmatrix} \end{aligned}$$

- Prob( $|+\mathbf{n}\rangle$ ) =  $|\langle+\mathbf{n}|\psi\rangle|^2$

## Representation of Operators

Matrix Representation of  $\hat{A}$  in  $S_n$ -basis

$$\hat{A} \rightarrow A_n = \begin{pmatrix} \langle+\mathbf{n}|\hat{A}|+\mathbf{n}\rangle & \langle+\mathbf{n}|\hat{A}|-\mathbf{n}\rangle \\ \langle-\mathbf{n}|\hat{A}|+\mathbf{n}\rangle & \langle-\mathbf{n}|\hat{A}|-\mathbf{n}\rangle \end{pmatrix}$$

Matrix Representations

$$\hat{A} \rightarrow A_n = S^\dagger A_z S,$$

where

$$S = \begin{pmatrix} \langle+z|+\mathbf{n}\rangle & \langle+z|-\mathbf{n}\rangle \\ \langle-z|+\mathbf{n}\rangle & \langle-z|-\mathbf{n}\rangle \end{pmatrix}$$

and

$$A_z = \begin{pmatrix} \langle+z|\hat{A}|+z\rangle & \langle+z|\hat{A}|-z\rangle \\ \langle-z|\hat{A}|+z\rangle & \langle-z|\hat{A}|-z\rangle \end{pmatrix}$$

## Change of Basis (z to n) - States

Transform Kets

$$\begin{aligned} |\psi\rangle_z &= |+\mathbf{z}\rangle\langle+\mathbf{z}|\psi\rangle + |-\mathbf{z}\rangle\langle-\mathbf{z}|\psi\rangle \\ &= (|+\mathbf{n}\rangle\langle+\mathbf{n}| + |-\mathbf{n}\rangle\langle-\mathbf{n}|)|+\mathbf{z}\rangle\langle+\mathbf{z}|\psi\rangle \\ &\quad + (|+\mathbf{n}\rangle\langle+\mathbf{n}| + |-\mathbf{n}\rangle\langle-\mathbf{n}|)|-\mathbf{z}\rangle\langle-\mathbf{z}|\psi\rangle \\ &= [ \langle+\mathbf{n}|+\mathbf{z}\rangle\langle+\mathbf{z}|\psi\rangle + \langle+\mathbf{n}|-\mathbf{z}\rangle\langle-\mathbf{z}|\psi\rangle ] |+\mathbf{n}\rangle \\ &\quad + [ \langle-\mathbf{n}|+\mathbf{z}\rangle\langle+\mathbf{z}|\psi\rangle + \langle-\mathbf{n}|-\mathbf{z}\rangle\langle-\mathbf{z}|\psi\rangle ] |-\mathbf{n}\rangle \end{aligned}$$

Matrix Representation  $|\psi\rangle_n = \hat{S}^\dagger |\psi\rangle_z$

$$\begin{aligned} \begin{pmatrix} \langle+\mathbf{n}|\psi\rangle \\ \langle-\mathbf{n}|\psi\rangle \end{pmatrix} &= \underbrace{\begin{pmatrix} \langle+\mathbf{n}|+\mathbf{z}\rangle & \langle+\mathbf{n}|-\mathbf{z}\rangle \\ \langle-\mathbf{n}|+\mathbf{z}\rangle & \langle-\mathbf{n}|-\mathbf{z}\rangle \end{pmatrix}}_{\text{Components of z states}} \begin{pmatrix} \langle+\mathbf{z}|\psi\rangle \\ \langle-\mathbf{z}|\psi\rangle \end{pmatrix} \\ &\equiv S^\dagger \begin{pmatrix} \langle+\mathbf{z}|\psi\rangle \\ \langle-\mathbf{z}|\psi\rangle \end{pmatrix} \end{aligned}$$

## Change of Basis (z to n) - Operators

Begin with States

$$|\psi\rangle_n = \hat{S}^\dagger |\psi\rangle_z, \quad {}_n\langle\psi| = {}_z\langle\psi| \hat{S},$$

where

$$S^\dagger = \begin{pmatrix} \langle+\mathbf{n}|+\mathbf{z}\rangle & \langle+\mathbf{n}|-\mathbf{z}\rangle \\ \langle-\mathbf{n}|+\mathbf{z}\rangle & \langle-\mathbf{n}|-\mathbf{z}\rangle \end{pmatrix}$$

Relate  $\langle\psi|\hat{A}|\psi\rangle$  in z-basis to value in n-basis, using  $\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = I$ .

$$\begin{aligned} {}_z\langle\psi|\hat{A}_z|\psi\rangle &= {}_z\langle\psi|\hat{S}\hat{S}^\dagger A_z \hat{S}\hat{S}^\dagger|\psi\rangle_z \\ &= {}_n\langle\psi|\hat{S}^\dagger A_z \hat{S}|\psi\rangle_n \\ &= {}_n\langle\psi|\hat{A}_n|\psi\rangle_n \end{aligned} \quad (1)$$

If we define  $\hat{A}_{\text{new}} = \hat{A}_n$  and  $\hat{A}_{\text{old}} = \hat{A}_z$ , then

$$\hat{A}_{\text{new}} = \hat{S}^\dagger \hat{A}_{\text{old}} \hat{S}$$

## Angular Momentum Operators

Rotations and Generators ( $\hat{J}_n$  is Hermitian)

$$\hat{R}(\phi\mathbf{n}) = e^{-i\hat{J}_n\phi/\hbar}$$

Commutation Relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x, \quad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y.$$

Operators

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \quad \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

Eigenstates  $(2j+1)$  for  $-j, -j+1, \dots, j-1, j$

$$\begin{aligned} \hat{J}^2|j, m\rangle &= j(j+1)\hbar^2|j, m\rangle \\ \hat{J}_z|j, m\rangle &= m\hbar|j, m\rangle \\ \hat{J}_\pm|j, m\rangle &= \sqrt{j(j+1) - m(m \pm 1)}\hbar|j, m \pm 1\rangle \end{aligned}$$

### Spin 1/2 Representations

- ▶ Operators  $\hat{S}^2, \hat{S}_z, \hat{S}_\pm$
- ▶ States  $|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle,$
- ▶ Representations  $\hat{S} \rightarrow \frac{\hbar}{2}\sigma$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

▶ Expectation Values

$$\langle S_x \rangle = \langle \psi | \hat{S}_x | \psi \rangle = \left( \langle +z | \psi \rangle^* \quad \langle -z | \psi \rangle^* \right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle +z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix}$$

### Spin 1 Representations

- ▶ Operators  $\hat{S}^2, \hat{S}_z, \hat{S}_\pm$
- ▶ States  $|1, 1\rangle, |1, 0\rangle, |1, -1\rangle,$
- ▶ Representations

$$\hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{S}_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{S}_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}$$

▶ Representation of  $\hat{A}$ :

$$\hat{A} \rightarrow \begin{pmatrix} \langle 1, 1 | \hat{A} | 1, 1 \rangle & \langle 1, 1 | \hat{A} | 1, 0 \rangle & \langle 1, 1 | \hat{A} | 1, -1 \rangle \\ \langle 1, 0 | \hat{A} | 1, 1 \rangle & \langle 1, 0 | \hat{A} | 1, 0 \rangle & \langle 1, 0 | \hat{A} | 1, -1 \rangle \\ \langle 1, -1 | \hat{A} | 1, 1 \rangle & \langle 1, -1 | \hat{A} | 1, 0 \rangle & \langle 1, -1 | \hat{A} | 1, -1 \rangle \end{pmatrix}$$

### Representation of $\hat{S}_x$ - Spin 1 Case

Noting that  $\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$ ,  $A_{ij} = 0$ ,  $i = 1, 2, 3$ , and

$$\begin{aligned} A_{12} &= \langle 1, 1 | \hat{S}_x | 1, 0 \rangle = A_{21}^* \\ &= \frac{1}{2} \langle 1, 1 | \hat{S}_+ + \hat{S}_- | 1, 0 \rangle \\ &= \frac{1}{2} (\sqrt{2} \langle 1, 1 | 1, 1 \rangle + \sqrt{2} \langle 1, 1 | 1, -1 \rangle) = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} A_{13} &= \langle 1, 1 | \hat{S}_x | 1, -1 \rangle = A_{31}^* \\ &= \frac{1}{2} \langle 1, 1 | \hat{S}_+ + \hat{S}_- | 1, -1 \rangle = 0 \end{aligned}$$

$$\begin{aligned} A_{23} &= \langle 1, 0 | \hat{S}_x | 1, -1 \rangle = A_{32}^* \\ &= \frac{1}{2} \langle 1, 0 | \hat{S}_+ + \hat{S}_- | 1, -1 \rangle = \frac{1}{\sqrt{2}} \end{aligned}$$

The final representation is ...

### Eigenstates of $\hat{S}_x$ - Spin 1 Case

Find the eigenstates of  $S_x$  in  $S_z$ -basis

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Eigenvalues  $[\lambda = -1, 0, 1]$

$$\begin{vmatrix} -\lambda & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -\lambda & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda \left( \lambda^2 - \frac{1}{2} \right) + \frac{1}{2} \lambda = 0$$

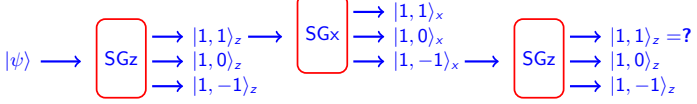
Eigenvectors for  $\lambda = -1$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = - \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$b = -\sqrt{2}a = -\sqrt{2}c, \quad a + c = -\sqrt{2}b \Rightarrow |1, -1\rangle_x \xrightarrow{S_z} \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

### Spin 1 Particles - SG Devices

Send spin 1 particles through 3 Stern Gerlach devices.



Probability [to find  $|\psi\rangle$  in state  $|\phi\rangle$ ] =  $|\langle \phi | \psi \rangle|^2$ ,

$$|z(1, 1 | \psi\rangle|^2, \quad |x(1, -1 | 1, 1\rangle_z|^2, \quad |z(1, 1 | 1, -1\rangle_x|^2,$$

**Example:** Evaluate  $|x(1, -1 | 1, 1\rangle_z|^2$ .

Recall that  $|1, -1\rangle_x$  is an eigenstate of  $S_x$  :

$$\text{Therefore, } |1, -1\rangle_x \xrightarrow{S_z} \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow |x(1, -1 | 1, 1\rangle_z|^2 = \frac{1}{4}.$$

### Ch. 4: Time Evolution

The Schrödinger Equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

For  $\hat{H}$  time-independent,

$$|\psi(t)\rangle = \hat{U} |\psi(0)\rangle, \quad \hat{U} = e^{-i\hat{H}t/\hbar}$$

Energy Eigenstates  $\hat{H}|E\rangle = E|E\rangle,$

Time Evolution: Initial State  $|\psi(0)\rangle = \sum_n |E_n\rangle \langle E_n | \psi(0)\rangle$

$$\Rightarrow |\psi(t)\rangle = \sum_n e^{-iE_n t/\hbar} |E_n\rangle \langle E_n | \psi(0)\rangle$$

Expectation Values:  $i\hbar \frac{d}{dt} \langle A \rangle = \langle \psi(t) | [\hat{H}, \hat{A}] + \frac{\partial \hat{A}}{\partial t} | \psi(t)\rangle$

## Precession in a Magnetic Field

Constant Magnetic Field  $\mathbf{B} = B_0 \mathbf{k}$

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} = \frac{ge}{2mc} B_0 S_z \equiv \omega_0 S_z$$

Evolution of States  $\hat{H}|\pm z\rangle = \pm \frac{\hbar\omega_0}{2}|\pm z\rangle$

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}|\mathbf{x}\rangle \\ &= \frac{1}{\sqrt{2}} \left( e^{-iE_+t/\hbar}|\mathbf{x}\rangle + e^{-iE_-t/\hbar}|\mathbf{-z}\rangle \right) \\ &= \frac{1}{\sqrt{2}} \left( e^{-i\omega_0 t/2}|\mathbf{x}\rangle + e^{i\omega_0 t/2}|\mathbf{-z}\rangle \right) \end{aligned}$$

Expectation values

$$\langle S_z \rangle = 0, \quad \langle S_x \rangle = \frac{\hbar}{2} \cos \omega_0 t, \quad \langle S_y \rangle = \frac{\hbar}{2} \sin \omega_0 t$$

## Uncertainty

For Hermitian matrices  $\hat{A}, \hat{B}, [\hat{A}, \hat{B}] = i\hat{C}$ ,

$$\Delta A \Delta B \geq \frac{|\langle C \rangle|}{2}$$

Example:  $\Delta J_x \Delta J_y \geq \frac{\hbar}{2} |\langle J_z \rangle|$

Recall

$$\Delta A^2 = \langle A^2 \rangle - \langle A \rangle^2$$

and

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

## Ch. 5: Two Spin Systems

Basis States for Spin- $\frac{1}{2}$  Particles

$$\begin{aligned} |z, z\rangle &= \left| \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle \\ |x, z\rangle &= \frac{1}{\sqrt{2}} |z, z\rangle + \frac{1}{\sqrt{2}} |z, -z\rangle \end{aligned}$$

Hyperfine Splitting  $\hat{H} = \frac{2A}{\hbar^2} \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$

$$\begin{aligned} \hat{H} &= \frac{A}{\hbar^2} (\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S}_{1z}\hat{S}_{2z}) \\ &\rightarrow \begin{pmatrix} \frac{A}{2} & 0 & 0 & 0 \\ 0 & -\frac{A}{2} & A & 0 \\ 0 & A & -\frac{A}{2} & 0 \\ 0 & 0 & 0 & \frac{A}{2} \end{pmatrix} \quad (2) \end{aligned}$$

## Eigenvalue Problem

Seek Eigenvalues and Energy Eigenstates  $\hat{H}|E\rangle = E|E\rangle$ .

$$\begin{pmatrix} \frac{A}{2} & 0 & 0 & 0 \\ 0 & -\frac{A}{2} & A & 0 \\ 0 & A & -\frac{A}{2} & 0 \\ 0 & 0 & 0 & \frac{A}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = E \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Eigenvalue Equation

$$\begin{aligned} 0 &= \begin{vmatrix} \frac{A}{2} - E & 0 & 0 & 0 \\ 0 & -\frac{A}{2} - E & A & 0 \\ 0 & A & -\frac{A}{2} - E & 0 \\ 0 & 0 & 0 & \frac{A}{2} - E \end{vmatrix} \\ &= \left( \frac{A}{2} - E \right)^2 \left[ \left( E + \frac{A}{2} \right)^2 - A^2 \right] \end{aligned}$$

## Energy Eigenstates

For eigenvalues  $E = \frac{A}{2}$ , we get the triplet

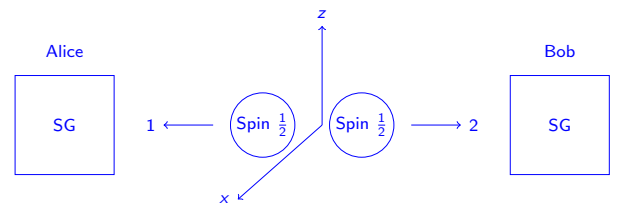
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{or}$$

$$\begin{aligned} |1, 1\rangle &= |z, z\rangle, \quad |1, 0\rangle = \frac{1}{\sqrt{2}} |z, -z\rangle + \frac{1}{\sqrt{2}} |z, +z\rangle, \\ |1, -1\rangle &= |z, -z\rangle. \end{aligned}$$

For eigenvalues  $E = -\frac{3A}{2}$ , we get the singlet

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{or } |0, 0\rangle = \frac{1}{\sqrt{2}} |z, -z\rangle - \frac{1}{\sqrt{2}} |z, +z\rangle.$$

## EPR Paradox - $|0, 0\rangle$ Decay



A spin-0 particle decays into two spin- $\frac{1}{2}$  particles.

$$\begin{aligned} |0, 0\rangle &= \frac{1}{\sqrt{2}} |z, -z\rangle - \frac{1}{\sqrt{2}} |z, +z\rangle \\ &= \frac{1}{\sqrt{2}} |z\rangle_1 |z\rangle_2 - \frac{1}{\sqrt{2}} |z\rangle_1 |z\rangle_2. \end{aligned}$$

What do Alice and Bob measure?