Spin Eigenstates - Review

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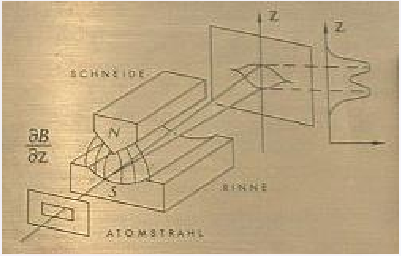
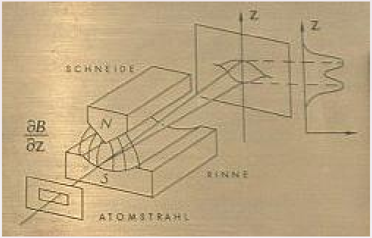


SG Devices Measure Spin



- Orient device in direction $\mathbf{n}$
- The representation of $|\psi\rangle$ in the S_n-basis for spin $\frac{1}{2}$:
  $|\psi\rangle_n = I_n|\psi\rangle$, where
  $I_n = |+n\rangle\langle+n| + |-n\rangle\langle-n|$

$$
\begin{aligned}
|\psi\rangle_n &= |+n\rangle\langle+n|\psi\rangle + |-n\rangle\langle-n|\psi\rangle \\
&= a_+|+n\rangle + a_-|-n\rangle \\
&\rightarrow \begin{pmatrix} \langle+n|\psi\rangle \\ \langle-n|\psi\rangle \end{pmatrix}
\end{aligned}
$$

- $\text{Prob}(|+n\rangle) = |\langle+n|\psi\rangle|^2$

Representation of Operators

Matrix Representation of $\hat{A}$ in S_n-basis

$$\hat{A} \quad \rightarrow \quad A_n = \begin{pmatrix} \langle +n|\hat{A}|+n\rangle & \langle +n|\hat{A}|-n\rangle \\ \langle -n|\hat{A}|+n\rangle & \langle -n|\hat{A}|-n\rangle \end{pmatrix}$$

Matrix Representations

$$\hat{A} \rightarrow A_n = \mathbb{S}^\dagger A_z \mathbb{S},$$

where

$$\mathbb{S} = \begin{pmatrix} \langle +z|+n\rangle & \langle +z|-n\rangle \\ \langle -z|+n\rangle & \langle -z|-n\rangle \end{pmatrix}$$

and

$$A_z = \begin{pmatrix} \langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix}$$

Change of Basis (z to n) - States

Transform Kets

$$
\begin{aligned}
|\psi\rangle_z &= |{+z}\rangle\langle{+z}|\psi\rangle + |{-z}\rangle\langle{-z}|\psi\rangle \\
&= \left(|{+n}\rangle\langle{+n}| + |{-n}\rangle\langle{-n}|\right)|{+z}\rangle\langle{+z}|\psi\rangle \\
&\quad + \left(|{+n}\rangle\langle{+n}| + |{-n}\rangle\langle{-n}|\right)|{-z}\rangle\langle{-z}|\psi\rangle \\
&= \left[\langle{+n}|{+z}\rangle\langle{+z}|\psi\rangle + \langle{+n}|{-z}\rangle\langle{-z}|\psi\rangle\right]|{+n}\rangle \\
&\quad + \left[\langle{-n}|{+z}\rangle\langle{+z}|\psi\rangle + \langle{-n}|{-z}\rangle\langle{-z}|\psi\rangle\right]|{-n}\rangle
\end{aligned}
$$

Matrix Representation $|\psi\rangle_n = \hat{S}^\dagger|\psi\rangle_z$

$$
\begin{aligned}
\begin{pmatrix} \langle{+n}|\psi\rangle \\ \langle{-n}|\psi\rangle \end{pmatrix} &= \underbrace{\begin{pmatrix} \langle{+n}|{+z}\rangle & \langle{+n}|{-z}\rangle \\ \langle{-n}|{+z}\rangle & \langle{-n}|{-z}\rangle \end{pmatrix}}_{\text{Components of } z \text{ states}} \begin{pmatrix} \langle{+z}|\psi\rangle \\ \langle{-z}|\psi\rangle \end{pmatrix} \\
&\equiv S^\dagger \begin{pmatrix} \langle{+z}|\psi\rangle \\ \langle{-z}|\psi\rangle \end{pmatrix}
\end{aligned}
$$

Change of Basis (z to n) - Operators

Begin with States

$$|\psi\rangle_n = \hat{S}^\dagger |\psi\rangle_z, \qquad {}_n\langle\psi| =_z \langle|\psi|\hat{S},$$

where

$$S^\dagger = \begin{pmatrix} \langle +n|+z\rangle & \langle +n|-z\rangle \\ \langle -n|+z\rangle & \langle -n|-z\rangle \end{pmatrix}$$

Relate $\langle\psi|\hat{A}|\psi\rangle$ in z-basis to value in n-basis, using $\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = I$.

$$\begin{aligned} {}_z\langle\psi|\hat{A}_z|\psi\rangle_z &= {}_z\langle\psi|\hat{S}\hat{S}^\dagger A_z \hat{S}\hat{S}^\dagger|\psi\rangle_z \\ &= {}_n\langle\psi|\hat{S}^\dagger A_z \hat{S}\hat{|}\psi\rangle_n \\ &= {}_n\langle\psi|\hat{A}_n|\psi\rangle_n \end{aligned} \tag{1}$$

If we define $\hat{A}_{\text{new}} = \hat{A}_{\text{n}}$ and $\hat{A}_{\text{old}} = \hat{A}_{\text{z}}$, then

$$\hat{A}_{\text{new}} = \hat{S}^\dagger \hat{A}_{\text{old}} \hat{S}$$

Angular Momentum Operators

Rotations and Generators ($\hat{J}_n$ is Hermitian)

$$\hat{R}\left(\phi\mathbf{n}\right) = e^{-i\hat{J}_n\phi/\hbar}$$

Commutation Relations

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x, \quad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y.$$

Operators

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \qquad \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

Eigenstates $(2j+1)$ for $-j, -j+1, \ldots, j-1, j$

$$\begin{aligned}
\hat{J}^2|j,m\rangle &= j(j+1)\hbar^2|j,m\rangle \\
\hat{J}_z|j,m\rangle &= m\hbar|j,m\rangle \\
\hat{J}_\pm|j,m\rangle &= \sqrt{j(j+1) - m(m\pm 1)}\hbar|j,m\pm 1\rangle
\end{aligned}$$

Spin $\frac{1}{2}$ Representations

- Operators $\hat{S}^2, \hat{S}_z, \hat{S}_\pm$
- States $|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle,$
- Representations $\hat{\boldsymbol{S}} \to \frac{\hbar}{2}\boldsymbol{\sigma}$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Expectation Values

$$\langle S_x \rangle = \langle \psi | \hat{S}_x | \psi \rangle = \begin{pmatrix} \langle +z|\psi\rangle^* & \langle -z|\psi\rangle^* \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$$

Spin 1 Representations

- Operators $\hat{S}^2, \hat{S}_z, \hat{S}_\pm$
- States $|1,1\rangle, |1,0\rangle, |1,-1\rangle$,
- Representations

$$\hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \hat{S}_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \hat{S}_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}$$

- Representation of $\hat{A}$:

$$\hat{A} \rightarrow \begin{pmatrix} \langle 1,1|\hat{A}|1,1\rangle & \langle 1,1|\hat{A}|1,0\rangle & \langle 1,1|\hat{A}|1,-1\rangle \\ \langle 1,0|\hat{A}|1,1\rangle & \langle 1,0|\hat{A}|1,0\rangle & \langle 1,0|\hat{A}|1,-1\rangle \\ \langle 1,-1|\hat{A}|1,1\rangle & \langle 1,-1|\hat{A}|1,0\rangle & \langle 1,-1|\hat{A}|1,-1\rangle \end{pmatrix}$$

Representation of $\hat{S}_x$ - Spin 1 Case

Noting that $\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$, $A_{ii} = 0$, $i = 1, 2, 3$, and

$$\begin{aligned}
A_{12} &= \langle 1, 1|\hat{S}_x|1, 0\rangle = A_{21}^* \\
&= \frac{1}{2}\langle 1, 1|\hat{S}_+ + \hat{S}_-|1, 0\rangle \\
&= \frac{1}{2}\left(\sqrt{2}\langle 1, 1|1, 1\rangle + \sqrt{2}\langle 1, 1||1, -1\rangle\right) = \frac{1}{\sqrt{2}}. \\
A_{13} &= \langle 1, 1|\hat{S}_x|1, -1\rangle = A_{31}^* \\
&= \frac{1}{2}\langle 1, 1|\hat{S}_+ + \hat{S}_-|1, -1\rangle = 0. \\
A_{23} &= \langle 1, 0|\hat{S}_x|1, -1\rangle = A_{32}^* \\
&= \frac{1}{2}\langle 1, 0|\hat{S}_+ + \hat{S}_-|1, -1\rangle = \frac{1}{\sqrt{2}}.
\end{aligned}$$

The final representation is ...

Eigenstates of $\hat{S}_x$ - Spin 1 Case

Find the eigenstates of S_x in S_z-basis

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Eigenvalues $[\lambda = -1, 0, 1]$

$$\begin{vmatrix} -\lambda & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -\lambda & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda\left(\lambda^2 - \frac{1}{2}\right) + \frac{1}{2}\lambda = 0$$

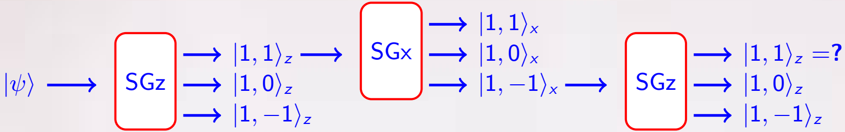
Eigenvectors for $\lambda = -1$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$b = -\sqrt{2}a = -\sqrt{2}c, a + c = -\sqrt{2}b, \Rightarrow \quad |1, -1\rangle_x \xrightarrow[S_z]{} \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Spin 1 Particles - SG Devices

Send spin 1 particles through 3 Stern Gerlach devices.



Probability [to find $|\psi\rangle$ in state $|\phi\rangle$] $= |\langle\phi|\psi\rangle|^2$,

$$|{}_z\langle 1,1|\psi\rangle|^2, \qquad |{}_x\langle 1,-1|1,1\rangle_z|^2, \qquad |{}_z\langle 1,1|1,-1\rangle_x|^2,$$

Example: Evaluate $|{}_x\langle 1,-1|1,1\rangle_z|^2$.

Recall that $|1,-1\rangle_x$ is an eigenstate of S_x :

$$\text{Therefore,} \quad |1,-1\rangle_x \xrightarrow[S_z]{} \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow |{}_x\langle 1,-1|1,1\rangle_z|^2 = \frac{1}{4}.$$

Ch. 4: Time Evolution

The Schrödinger Equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

For $\hat{H}$ time-independent,

$$|\psi(t)\rangle = \hat{U}|\psi(0)\rangle, \quad \hat{U} = e^{-i\hat{H}t/\hbar}$$

Energy Eigenstates $\hat{H}|E\rangle = E|E\rangle$,
Time Evolution: Initial State $|\psi(0)\rangle = \sum_n |E_n\rangle\langle E_n|\psi(0)\rangle$

$$\Rightarrow |\psi(t)\rangle = \sum_n e^{-i\hat{E}_n t/\hbar}|E_n\rangle\langle E_n|\psi(0)\rangle$$

Expectation Values: $i\hbar \frac{d}{dt}\langle A\rangle = \langle\psi(t)|[\hat{H}, \hat{A}] + \frac{\partial \hat{A}}{\partial t}|\psi(t)\rangle$

Precession in a Magnetic Field

Constant Magnetic Field $\mathbf{B} = B_0\mathbf{k}$

$$\hat{H} = -\hat{\mu}\cdot\mathbf{B} = \frac{ge}{2mc}B_0 S_z \equiv \omega_0 S_z$$

Evolution of States $\hat{H}|\pm\mathbf{z}\rangle = \pm\frac{\hbar\omega_0}{2}|\pm\mathbf{z}\rangle$

$$\begin{aligned}
|\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}|+\mathbf{x}\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-iE_+t/\hbar}|+\mathbf{z}\rangle + e^{-iE_-t/\hbar}|-\mathbf{z}\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|+\mathbf{z}\rangle + e^{i\omega_0 t/2}|-\mathbf{z}\rangle\right)
\end{aligned}$$

Expectation values

$$\langle S_z\rangle = 0, \quad \langle S_x\rangle = \frac{\hbar}{2}\cos\omega_0 t, \quad \langle S_y\rangle = \frac{\hbar}{2}\sin\omega_0 t$$

Uncertainty

For Hermitian matrices $\hat{A}$, $\hat{B}$, $[\hat{A}, \hat{B}] = i\hat{C}$,

$$\Delta A \Delta B \geq \frac{|\langle C \rangle|}{2}$$

Example: $\Delta J_x \Delta J_y \geq \frac{\hbar}{2} |\langle J_z \rangle|$

Recall

$$\Delta A^2 = \langle A^2 \rangle - \langle A \rangle^2$$

and

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

Ch. 5: Two Spin Systems

Basis States for Spin-$\frac{1}{2}$ Particles

$$\begin{aligned}
|+z,+z\rangle &= |\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}\rangle \\
|+x,+z\rangle &= \frac{1}{\sqrt{2}}|+z,+z\rangle + \frac{1}{\sqrt{2}}|-z,+z\rangle
\end{aligned}$$

Hyperfine Splitting $\hat{H} = \frac{2A}{\hbar^2}\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$

$$\begin{aligned}
\hat{H} &= \frac{A}{\hbar^2}(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+} + 2\hat{S}_{1z}\hat{S}_{2z}) \\
&\rightarrow \begin{pmatrix} \frac{A}{2} & 0 & 0 & 0 \\ 0 & -\frac{A}{2} & A & 0 \\ 0 & A & -\frac{A}{2} & 0 \\ 0 & 0 & 0 & \frac{A}{2} \end{pmatrix}
\end{aligned} \tag{2}$$

Eigenvalue Problem

Seek Eigenvalues and Energy Eigenstates $\hat{H}|E\rangle = E|E\rangle$.

$$\begin{pmatrix} \frac{A}{2} & 0 & 0 & 0 \\ 0 & -\frac{A}{2} & A & 0 \\ 0 & A & -\frac{A}{2} & 0 \\ 0 & 0 & 0 & \frac{A}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = E \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Eigenvalue Equation

$$\begin{aligned} 0 &= \begin{vmatrix} \frac{A}{2} - E & 0 & 0 & 0 \\ 0 & -\frac{A}{2} - E & A & 0 \\ 0 & A & -\frac{A}{2} - E & 0 \\ 0 & 0 & 0 & \frac{A}{2} - E \end{vmatrix} \\ &= \left(\frac{A}{2} - E\right)^2 \left[\left(E + \frac{A}{2}\right)^2 - A^2\right] \end{aligned}$$

Energy Eigenstates

For eigenvalues $E = \frac{A}{2}$, we get the triplet

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \text{ or}$$

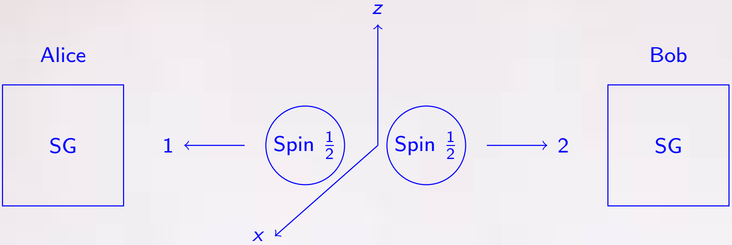
$$|1,1\rangle = |+z,+z\rangle, \quad |1,0\rangle = \frac{1}{\sqrt{2}}|+z,-z\rangle + \frac{1}{\sqrt{2}}|-z,+z\rangle,$$

$$|1,-1\rangle = |-z,-z\rangle.$$

For eigenvalues $E = -\frac{3A}{2}$, we get the singlet

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \text{ or } |0,0\rangle = \frac{1}{\sqrt{2}}|+z,-z\rangle - \frac{1}{\sqrt{2}}|-z,+z\rangle.$$

EPR Paradox - $|0,0\rangle$ Decay



A spin-0 particle decays into two spin-$\frac{1}{2}$ particles.

$$
\begin{aligned}
|0,0\rangle &= \frac{1}{\sqrt{2}}|+z,-z\rangle - \frac{1}{\sqrt{2}}|-z,+z\rangle \\
&= \frac{1}{\sqrt{2}}|+z\rangle_1|-z\rangle_2 - \frac{1}{\sqrt{2}}|-z\rangle_1|+z\rangle_2.
\end{aligned}
$$

What do Alice and Bob measure?