Angular Momentum

1. Generators: $\hat{R}(\phi\mathbf{n}) = e^{-i\mathbf{J}\cdot\mathbf{n}\phi/\hbar}$
2. Commutators
   a. $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$, $[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$, $[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$
   b. Commuting operators have simultaneous eigenstates.
3. Operators Relations
   a. $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$
   b. $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$
   c. $\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$, $\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$
4. Eigenvalues and Eigenstates
   a. $\hat{J}^2 \left| j,m \right\rangle = j(j+1)\hbar \left| j,m \right\rangle$, $\hat{J}_z \left| j,m \right\rangle = m\hbar \left| j,m \right\rangle$
   b. $\hat{S}^2 \left| s,m \right\rangle = s(s+1)\hbar \left| s,m \right\rangle$, $\hat{S}_z \left| s,m \right\rangle = m\hbar \left| s,m \right\rangle$
   c. $m = -j, -j+1, \ldots, j-1, j$
5. Raising and Lowering Operators
   a. $\hat{J}_\pm \left| j,m \right\rangle = \sqrt{j(j+1) - m(m\pm1)}\hbar \left| j,m\pm1 \right\rangle$
   b. $\hat{S}_\pm \left| s,m \right\rangle = \sqrt{s(s+1) - m(m\pm1)}\hbar \left| s,m\pm1 \right\rangle$
6. Matrix Representations (Note similarity between *J*'s and *S*'s)
   a. Rotation – 3D

$$\hat{S}(\phi\mathbf{i}) \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \hat{S}(\phi\mathbf{j}) \to \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \hat{S}(\phi\mathbf{k}) \to \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   b. Spin ½ - $\hat{\mathbf{S}} \to \frac{\hbar}{2}\boldsymbol{\sigma}$ in terms of Pauli spin matrices.

$$\hat{S}_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}_y \to \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{S}_z \to \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   c. Spin 1

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_+ = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, S_- = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, S^2 = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7. Matrix Elements: $\left(A_{mm'}\right)=\left(\langle j,m'|\hat{A}|j,m\rangle\right)$ where m is the row, m' is the column and one proceeds in the order $m=j,j-1,\ldots,-j+1,-j$.
   a. Two state example: $\hat{A}\xrightarrow[S_z\text{ basis}]{}\begin{pmatrix}\langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle\end{pmatrix}$
   b. Spin-1 $A\to\begin{pmatrix}\langle 1,1|A|1,1\rangle & \langle 1,1|A|1,0\rangle & \langle 1,1|A|1,-1\rangle \\ \langle 1,0|A|1,1\rangle & \langle 1,0|A|1,0\rangle & \langle 1,0|A|1,-1\rangle \\ \langle 1,-1|A|1,1\rangle & \langle 1,-1|A|1,0\rangle & \langle 1,-1|A|1,-1\rangle\end{pmatrix}$
8. Uncertainty $\Delta A\Delta B\geq\frac{|\langle C\rangle|}{2}$ for $[A,B]=iC$, and A,B,C Hermitian. $\Delta J_x\Delta J_y\geq\frac{\hbar}{2}\left|\langle J_z\rangle\right|$
9. Send particles with spin s through Stern-Gerlach device oriented in different positions. Determine the number of beam channels and the probability a given particle will be found in a given channel;.
10. Time Evolution:
    a. $|\psi(t)\rangle=\hat{U}(t)|\psi(0)\rangle=e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$
    b. $i\hbar\frac{d}{dt}|\psi(t)\rangle=\hat{H}(t)|\psi(t)\rangle$, Schrödinger Equation
    c. $\frac{d}{dt}\langle A\rangle=\langle\psi(t)|\left(\frac{i}{\hbar}[\hat{H},\hat{A}]+\frac{\partial\hat{A}}{\partial t}\right)|\psi(t)\rangle$
11. Energy eigenstates: $\hat{H}|E_n\rangle=E_n|E_n\rangle$, $|\psi\rangle=\sum_n\langle E_n|\psi\rangle|E_n\rangle$
12. Precession of Spin: $\langle S_z\rangle=0,\langle S_x\rangle=\frac{\hbar}{2}\cos\omega_0 t,\langle S_y\rangle=\frac{\hbar}{2}\sin\omega_0 t$, $\omega_0=\frac{ge}{2mc}B_0$
13. Energy-time Uncertainty, $\Delta E\Delta t\geq\frac{\hbar}{2}$
14. Two Spin ½ Particle Eigenstates
    a. $|+z,+z\rangle=|+z\rangle_1|+z\rangle_2$,
15. Fine vs hyperfine splitting
16. Addition of Angular Momenta, Eigenstates of Total Angular Momenta – two particles
    a. $R(d\theta\mathbf{n})=1-\frac{i}{\hbar}\hat{\mathbf{S}}\cdot\mathbf{n}\,d\theta=\left(1-\frac{i}{\hbar}\hat{\mathbf{S}}_1\cdot\mathbf{n}\,d\theta\right)\otimes\left(1-\frac{i}{\hbar}\hat{\mathbf{S}}_2\cdot\mathbf{n}\,d\theta\right)$
    b. $\hat{\mathbf{S}}=\hat{\mathbf{S}}_1\otimes 1+1\otimes\hat{\mathbf{S}}_2$
    c. $\hat{\mathbf{S}}^2=\left(\hat{\mathbf{S}}_1+\hat{\mathbf{S}}_2\right)^2=\hat{\mathbf{S}}_1^2+\hat{\mathbf{S}}_2^2+2\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2$, $\hat{S}_z=\hat{S}_{1z}+\hat{S}_{2z}$
17. What are entangled states?
18. Hamiltonians: $\hat{H}=-\hat{\mu}\cdot\mathbf{B}=\omega_0\hat{S}_z,\hat{H}=\frac{2A}{\hbar^2}\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2$