Summary of State Vectors and Operators

1. Quantum State Vectors
    a. Kets - $|+z\rangle, |-z\rangle, |\psi\rangle$
    b. Bras - $\langle +z|, \langle -z|, \langle \psi|$
2. Superposition
    a. $$\begin{aligned}|\psi\rangle &= c_+|+z\rangle + c_-|-z\rangle \\ &= |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle\end{aligned}$$
    b. $$\begin{aligned}\langle\psi| &= c_+^*\langle +z| + c_-^*\langle -z| \\ &= \langle\psi|+z\rangle\langle +z| + \langle\psi|-z\rangle\langle -z|\end{aligned}$$
3. Inner Products:
    a. $\langle +z|+z\rangle = \langle -z|-z\rangle = 1$, normalized
    b. $\langle +z|-z\rangle = \langle -z|+z\rangle = 0$. orthogonal
    c. $$\begin{aligned}\langle\psi|\psi\rangle &= \left(c_+^*\langle +z| + c_-^*\langle -z|\right)\left(c_+|+z\rangle + c_-|-z\rangle\right) \\ &= |c_+|^2 + |c_-|^2\end{aligned}$$
    d. $\langle\varphi|\psi\rangle =$ Probability amplitude that a particle in state $|\psi\rangle$ can be found in state $|\varphi\rangle$.
    e. $|\langle\varphi|\psi\rangle|^2 =$ Probability that a particle in state $|\psi\rangle$ can be found in state $|\varphi\rangle$.
4. States in S_z-basis
    a. $|\pm x\rangle = \frac{1}{\sqrt{2}}|+z\rangle \pm \frac{1}{\sqrt{2}}|-z\rangle$, $\langle \pm x| = \frac{1}{\sqrt{2}}\langle +z| \pm \frac{1}{\sqrt{2}}\langle -z|$
    b. $|\pm y\rangle = \frac{1}{\sqrt{2}}|+z\rangle \pm \frac{i}{\sqrt{2}}|-z\rangle$, $\langle \pm y| = \frac{1}{\sqrt{2}}\langle +z| \mp \frac{i}{\sqrt{2}}\langle -z|$
5. General States
    a. $|\psi\rangle = \sum_n c_n|a_n\rangle$, $\langle\psi| = \sum_n c_n^*\langle a_n|$
    b. $\langle\psi|\psi\rangle = \sum_{i,j} c_i^* c_j \langle a_i|a_j\rangle = \sum_j |c_j|^2$
    c. Expectation value: $\langle A\rangle = \sum_n P(a_n)a_n = \sum_n |c_n|^2 a_n$
    d. Uncertainty: $\Delta A = \sqrt{\langle (A - \langle A\rangle)^2\rangle} = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$
6. Matrix Representations of States in S_z-basis
    a. $|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle \rightarrow \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$
    b. $|+z\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-z\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

c. $\langle\psi| = c_+^* \langle +z| + c_-^* \langle -z| \to \left(c_+^*, c_-^*\right) = \left(\langle\psi|+z\rangle, \langle\psi|-z\rangle\right)$

d. $\langle +z| \to (1,0)$, $\langle -z| \to (0,1)$

e. $|\pm x\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$, $\langle \pm x| \to \frac{1}{\sqrt{2}}(1,\pm 1)$

f. $|\pm y\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$, $\langle \pm y| \to \frac{1}{\sqrt{2}}(1,\mp i)$

7. Rotations
   a. $\hat{R}(d\phi\,\mathbf{k}) = 1 - \frac{i}{\hbar}\hat{J}_z\, d\phi$
   b. $\hat{R}(\phi\,\mathbf{k}) = e^{-i\hat{J}_z\phi/\hbar}$
   c. $|+x\rangle = \hat{R}(\frac{\pi}{2}\mathbf{j})|+z\rangle$, $\langle +x| = \langle +z|\hat{R}^\dagger(\frac{\pi}{2}\mathbf{j})$
8. Projections
   a. $\hat{P}_+ = |+z\rangle\langle +z|$, $\hat{P}_- = |-z\rangle\langle -z|$,
   b. $|+z\rangle\langle +z| + |-z\rangle\langle -z| = I$, Completeness
   c. $\hat{P}_\pm^2 = \hat{P}_\pm, \hat{P}_+\hat{P}_- = \hat{P}_-\hat{P}_+ = 0$
9. Eigenvalues
   a. $\hat{J}_z|\pm z\rangle = \pm\frac{\hbar}{2}|\pm z\rangle$
10. Matrix Representations of Operators, $A_{ij} = \langle i|\hat{A}|j\rangle$
    a. $\hat{A}|\psi\rangle = |\varphi\rangle \to \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle 1|\varphi\rangle \\ \langle 2|\varphi\rangle \end{pmatrix}$
    b. $\hat{A} \xrightarrow[S_z \text{ basis}]{} A_z = \begin{pmatrix} \langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix}$
    c. $\hat{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\hat{P}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
    d. $\hat{R}(\phi\mathbf{k}) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$
11. Change of Basis
    a. $|\psi'\rangle \xrightarrow[S_z \text{ Basis}]{} \begin{pmatrix} \langle +z|\psi'\rangle \\ \langle -z|\psi'\rangle \end{pmatrix} = \begin{pmatrix} \langle +z|\hat{R}^\dagger|\psi\rangle \\ \langle -z|\hat{R}^\dagger|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix} \xleftarrow[S_x \text{ Basis}]{} |\psi\rangle$
    b. $\hat{A} \to A_x = S^\dagger A_z S$, where $S = \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix}$

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12. Matrix/Operator Types
    a. Adjoint $A^{\dagger} = \left(A^{t}\right)^{*}$,
    b. Unitary $A^{\dagger}A = I$,
    c. Hermitian $A^{\dagger} = A$

Other Topics for Exam

1. Stern-Gerlach Devices – Behaviors of SGz, SGx, SGy, modified devices
2. Normalization of bras and kets
3. Probabilities, expectation values, uncertainties
4. Use of completeness relation for identity operator
5. Representation of states and operators in different bases
6. Definition/recognition of different operators/matrices – unitary, Hermitian, etc
7. Rotation, generator, projection operators
8. Composition of operators, products of matrices
9. Change of basis for ket and operator matrix representations