Topics Summary for Final

1. Quantum Spin States
   a. State Vectors
      i. $\left|\psi\right\rangle = c_+\left|+z\right\rangle + c_-\left|-z\right\rangle = \left|+z\right\rangle\left\langle +z|\psi\right\rangle + \left|-z\right\rangle\left\langle -z|\psi\right\rangle$
      ii. $\left\langle\psi\right| = c_+^*\left\langle +z\right| + c_-^*\left\langle -z\right| = \left\langle\psi|+z\right\rangle\left\langle +z\right| + \left\langle\psi|-z\right\rangle\left\langle -z\right|$
   b. Inner Products:
      i. $\left\langle +z|+z\right\rangle = \left\langle -z|-z\right\rangle = 1$, normalized, $\left\langle +z|-z\right\rangle = \left\langle -z|+z\right\rangle = 0$. orthogonal
      ii. $\left\langle\psi|\psi\right\rangle = \left(c_+^*\left\langle +z\right| + c_-^*\left\langle -z\right|\right)\left(c_+\left|+z\right\rangle + c_-\left|-z\right\rangle\right) = \left|c_+\right|^2 + \left|c_-\right|^2$
      iii. $\left|\left\langle\varphi|\psi\right\rangle\right|^2 =$ Probability that a particle in state $\left|\psi\right\rangle$ can be found in state $\left|\varphi\right\rangle$.
   c. States in S_z-basis
      i. $\left|\pm x\right\rangle = \frac{1}{\sqrt{2}}\left|+z\right\rangle \pm \frac{1}{\sqrt{2}}\left|-z\right\rangle$, $\left\langle \pm x\right| = \frac{1}{\sqrt{2}}\left\langle +z\right| \pm \frac{1}{\sqrt{2}}\left\langle -z\right|$

         $\left|\pm y\right\rangle = \frac{1}{\sqrt{2}}\left|+z\right\rangle \pm \frac{i}{\sqrt{2}}\left|-z\right\rangle$, $\left\langle \pm y\right| = \frac{1}{\sqrt{2}}\left\langle +z\right| \mp \frac{i}{\sqrt{2}}\left\langle -z\right|$
      ii. General States
         1. $\left|\psi\right\rangle = \sum_n c_n\left|a_n\right\rangle$, $\left\langle\psi\right| = \sum_n c_n^*\left\langle a_n\right|$
         2. $\left\langle\psi|\psi\right\rangle = \sum_{i,j} c_i^* c_j\left\langle a_i|a_j\right\rangle = \sum_j \left|c_j\right|^2$
      iii. Expectation value: $\left\langle A\right\rangle = \sum_n P(a_n)a_n = \sum_n \left|c_n\right|^2 a_n$
      iv. Uncertainty: $\Delta A = \sqrt{\left\langle (A-\left\langle A\right\rangle)^2\right\rangle} = \sqrt{\left\langle A^2\right\rangle - \left\langle A\right\rangle^2}$
   d. Matrix Representations in S_z-basis
      i. $\left|\psi\right\rangle = c_+\left|+z\right\rangle + c_-\left|-z\right\rangle \rightarrow \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} \left\langle +z|\psi\right\rangle \\ \left\langle -z|\psi\right\rangle \end{pmatrix}$
      ii. $\left\langle\psi\right| = c_+^*\left\langle +z\right| + c_-^*\left\langle -z\right| \rightarrow \left(c_+^*, c_-^*\right) = \left(\left\langle\psi|+z\right\rangle, \left\langle\psi|-z\right\rangle\right)$
2. Operators
   a. Rotations
      i. $\hat{R}(d\phi\,\mathbf{k}) = 1 - \frac{i}{\hbar}\hat{J}_z\,d\phi$, $\hat{J}_z\left|\pm z\right\rangle = \pm\frac{\hbar}{2}\left|\pm z\right\rangle$
      ii. $\hat{R}(\phi\,\mathbf{k}) = e^{-i\hat{J}_z\phi/\hbar}$
      iii. $\left|+x\right\rangle = \hat{R}(\frac{\pi}{2}\mathbf{j})\left|+z\right\rangle$, $\left\langle +x\right| = \left\langle +z\right|\hat{R}^\dagger(\frac{\pi}{2}\mathbf{j})$
   b. Projections (Spin ½)
      i. $\hat{P}_+ = \left|+z\right\rangle\left\langle +z\right|$, $\hat{P}_- = \left|-z\right\rangle\left\langle -z\right|$, $\left|+z\right\rangle\left\langle +z\right| + \left|-z\right\rangle\left\langle -z\right| = I$

ii. $\hat{P}_{\pm}^{2}=\hat{P}_{\pm}, \hat{P}_{+}\hat{P}_{-}=\hat{P}_{-}\hat{P}_{+}=0$

c. Matrix Representations of Operators, $A_{ij}=\langle i|\hat{A}|j\rangle$

d. Commutators

i. $[\hat{J}_x,\hat{J}_y]=i\hbar\hat{J}_z,\ [\hat{J}_y,\hat{J}_z]=i\hbar\hat{J}_x,\ [\hat{J}_z,\hat{J}_x]=i\hbar\hat{J}_y$

ii. Commuting operators have simultaneous eigenstates.

3. Matrix/Operator Types

a. Adjoint $A^{\dagger}=\left(A^{t}\right)^{*}$, Unitary $A^{\dagger}A=I$, Hermitian $A^{\dagger}=A$

4. Operator Relations

a. $\hat{J}^2=\hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2$

b. $\hat{J}_{\pm}=\hat{J}_x\pm i\hat{J}_y$

c. $\hat{J}_{-}\hat{J}_{+}=\hat{J}^2-\hat{J}_z^2-\hbar\hat{J}_z,\ \hat{J}_{+}\hat{J}_{-}=\hat{J}^2-\hat{J}_z^2+\hbar\hat{J}_z$

5. Eigenvalues and Eigenstates

a. $\hat{J}^2|j,m\rangle=j(j+1)\hbar^2|j,m\rangle,\ \hat{J}_z|j,m\rangle=m\hbar|j,m\rangle$

b. $m=-j,-j+1,\ldots,j-1,j$

6. Raising and Lowering Operators

a. $\hat{J}_{\pm}|j,m\rangle=\sqrt{j(j+1)-m(m\pm1)}\hbar|j,m\pm1\rangle$

7. Matrix Representations (Note similarity between *J*'s and *S*'s)

a. $\hat{A}|\psi\rangle=|\varphi\rangle\rightarrow\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\langle1|\psi\rangle\\ \langle2|\psi\rangle\end{pmatrix}=\begin{pmatrix}\langle1|\varphi\rangle\\ \langle2|\varphi\rangle\end{pmatrix}$

b. $\hat{P}_{+}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ \hat{P}_{-}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

c. $\hat{R}(\phi\mathbf{k})=\begin{pmatrix}e^{-i\phi/2} & 0\\ 0 & e^{i\phi/2}\end{pmatrix}$

8. Change of Basis

a. $|\psi'\rangle\xrightarrow[S_z\ Basis]{}\begin{pmatrix}\langle+z|\psi'\rangle\\ \langle-z|\psi'\rangle\end{pmatrix}=\begin{pmatrix}\langle+z|\hat{R}^{\dagger}|\psi\rangle\\ \langle-z|\hat{R}^{\dagger}|\psi\rangle\end{pmatrix}=\begin{pmatrix}\langle+x|\psi\rangle\\ \langle-x|\psi\rangle\end{pmatrix}\xleftarrow[S_x\ Basis]{}|\psi\rangle$

b. $\hat{A}\xrightarrow[S_z\text{ basis}]{}A_z=\begin{pmatrix}\langle+z|\hat{A}|+z\rangle & \langle+z|\hat{A}|-z\rangle\\ \langle-z|\hat{A}|+z\rangle & \langle-z|\hat{A}|-z\rangle\end{pmatrix}$

c. $\hat{A}\rightarrow A_x=S^{\dagger}A_zS$, where $S=\begin{pmatrix}\langle+z|+x\rangle & \langle+z|-x\rangle\\ \langle-z|+x\rangle & \langle-z|-x\rangle\end{pmatrix}$

d. Spin ½ - $\hat{\mathbf{S}}\rightarrow\frac{\hbar}{2}\boldsymbol{\sigma}$ in terms of Pauli spin matrices.

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

e. Matrix Elements: $(A_{mm'}) = \left(\langle j,m'|\hat{A}|j,m\rangle\right)$ where m is the row, m' is the column and one proceeds in the order $m = j, j-1,\ldots,-j+1,-j$.

f. 3D Rotation matrices

g. $$\hat{S}(\phi\mathbf{i}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \hat{S}(\phi\mathbf{j}) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \hat{S}(\phi\mathbf{k}) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

h. Spin 1

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_+ = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, S_- = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, S^2 = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i. Spin-1 Representation: $$A \to \begin{pmatrix} \langle 1,1|A|1,1\rangle & \langle 1,1|A|1,0\rangle & \langle 1,1|A|1,-1\rangle \\ \langle 1,0|A|1,1\rangle & \langle 1,0|A|1,0\rangle & \langle 1,0|A|1,-1\rangle \\ \langle 1,-1|A|1,1\rangle & \langle 1,-1|A|1,0\rangle & \langle 1,-1|A|1,-1\rangle \end{pmatrix}$$

9. Uncertainty $\Delta A \Delta B \ge \frac{|\langle C\rangle|}{2}$ for $[A,B] = iC$, and A,B,C Hermitian
10. Send particles with spin s through Stern-Gerlach device oriented in different positions. Determine the number of beam channels and the probability a given particle will be found in a given channel.
11. Time Evolution:
    a. $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$
    b. $i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$, Schrödinger Equation
    c. $\frac{d}{dt}\langle A\rangle = \langle\psi(t)|\left(\frac{i}{\hbar}[\hat{H},\hat{A}] + \frac{\partial\hat{A}}{\partial t}\right)|\psi(t)\rangle$
12. Hamiltonians: $\hat{H} = -\hat{\mu}\cdot\mathbf{B} = \omega_0\hat{S}_z, \hat{H} = \frac{2A}{\hbar^2}\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2$
13. Precession of Spin: $\langle S_z\rangle = 0, \langle S_x\rangle = \frac{\hbar}{2}\cos\omega_0 t, \langle S_y\rangle = \frac{\hbar}{2}\sin\omega_0 t$, $\omega_0 = \frac{ge}{2mc}B_0$

14. Addition of Angular Momenta, Eigenstates of Total Angular Momenta – two particles
    a. $R(d\theta\mathbf{n}) = 1 - \frac{i}{\hbar}\hat{\mathbf{S}}\cdot\mathbf{n}\,d\theta = \left(1 - \frac{i}{\hbar}\hat{\mathbf{S}}_1\cdot\mathbf{n}\,d\theta\right)\otimes\left(1 - \frac{i}{\hbar}\hat{\mathbf{S}}_2\cdot\mathbf{n}\,d\theta\right)$
    b. $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 \otimes 1 + 1 \otimes \hat{\mathbf{S}}_2$
    c. $\hat{\mathbf{S}}^2 = \left(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2\right)^2 = \hat{\mathbf{S}}_1^2 + \hat{\mathbf{S}}_2^2 + 2\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2$, $\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$
15. Continuous States
    a. $|\psi\rangle = \int da\,|a\rangle\langle a|\psi\rangle, \langle a'|a\rangle = \delta(a'-a)$
    b. $\hat{x}|x\rangle = x|x\rangle$, $\hat{p}_x|p\rangle = p|p\rangle$.
    c. $\hat{T}(a) = e^{-i\hat{p}_x a/\hbar}, \hat{T}(a)|x\rangle = |x+a\rangle$
    d. $\langle x|\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$
    e. $\langle p|\psi\rangle = \int dx\langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\langle x|\psi\rangle dx$
16. Schrödinger's Equation
    a. $\langle x|\hat{H}|\psi\rangle = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\langle x|\psi\rangle = i\hbar\frac{\partial}{\partial t}\langle x|\psi\rangle$
    b. 1D Time Independent: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$
    c. 3D Time Independent: $-\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = E\psi$
17. Energy eigenstates: $\hat{H}|E_n\rangle = E_n|E_n\rangle$, $|\psi\rangle = \sum_n \langle E_n|\psi\rangle|E_n\rangle$
18. One Dimensional Problems
    a. Ehrenfest's Theorem $\frac{d\langle x\rangle}{dt} = \frac{\langle p_x\rangle}{m}, \frac{d\langle p_x\rangle}{dt} = \left\langle -\frac{dV}{dx}\right\rangle \Rightarrow m\frac{d^2\langle x\rangle}{dt^2} = \left\langle -\frac{dV}{dx}\right\rangle$
    b. Free particle – definition and time evolution
    c. Gaussian wave packet - $\int e^{-ipx/\hbar-\alpha x^2}dx$ - Computing Gaussian integrals $\int_{-\infty}^{\infty} x^n e^{-\beta x^2}dx$,

       $\langle x\rangle, \langle x^2\rangle, \Delta x, \langle p_x\rangle, \langle p_x^2\rangle, \Delta p_x$, and uncertainty $\Delta x\Delta p_x$
    d. Particle in Box, Infinite Square Well, $0 \le x \le a$.
        i. $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}, E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2}$
    e. Finite Square Well
    f. Bound states and scattering states
    g. Reflection, Transmission probabilities

h. Tunneling

i. Harmonic Oscillator

i. $\hat{H} = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)\hbar\omega, [\hat{a}, \hat{a}^\dagger] = 1$

ii. $\hat{a}^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle, \hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

iii. $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \hat{p}_x = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^\dagger\right)$

iv. $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi$

v. $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

vi. $\langle x|n\rangle = \frac{1}{\sqrt{n!}}\langle x|\left(\hat{a}^\dagger\right)^n|0\rangle, \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-m\omega x^2/2\hbar}$

19. Two-body Problem

a. $\hat{H}|E,\ell,m\rangle = E|E,\ell,m\rangle$

b. $\hat{L}^2|E,\ell,m\rangle = \ell(\ell+1)\hbar^2|E,\ell,m\rangle$

c. $\hat{L}_z|E,\ell,m\rangle = m\hbar|E,\ell,m\rangle$

d. $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, etc.

e. $\hat{L}_z \to \frac{\hbar}{i}\frac{\partial}{\partial\varphi}, \hat{L}_x \to \frac{\hbar}{i}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right), \hat{L}_y \to \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$

f. Raising and lowering operators: $\hat{L}_\pm \to \frac{\hbar}{i}e^{\pm i\varphi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right)$ - applied to $\langle\theta,\varphi|\ell,\ell\rangle = c_\ell e^{i\ell\varphi}\sin^\ell\theta,\ L_-|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m-1)}\hbar|\ell,m-1\rangle$. Here $c_\ell = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}$.

g. Spherical Harmonics $Y_{\ell m}(\theta,\varphi) = P_\ell^m(\cos\theta)e^{\pm im\varphi}$

h. Hydrogen Atom

i. $-\frac{\hbar^2}{2\mu}\nabla^2\psi + \left[\frac{\ell(\ell+1)\hbar^2}{2\mu r^2} - \frac{Zke^2}{r}\right]\psi = E\psi$

ii. $\psi(r,\theta,\varphi) = R(r)Y_\ell^m(\theta,\varphi)$

iii. $E = -\frac{\mu k^2 e^4}{2\hbar^2 n^2}Z^2 = -\frac{\mu c^2\alpha^2}{2n^2}Z^2 = -13.6\,\text{eV}\frac{Z}{n^2}$

iv. $a_0 = \frac{\hbar}{\mu c \alpha} = 0.529\,\mathring{\text{A}},\ \alpha = \frac{ke^2}{\hbar c} \approx \frac{1}{137.036}$

i. Deuteron Model, $R(r) = ru(r)$

i. $-\frac{\hbar^2}{2\mu}u'' - V_0 u = Eu,\ r < a \Rightarrow \tan k_0 a = -\frac{k_0}{q}$

j. Infinite Square Well, spherical Bessel functions, and Magic Numbers

20. Time Independent Perturbation Theory

a. $\hat{H} = \hat{H}_0 + \lambda \hat{H}_1,\ \hat{H}_0 \left|\phi_n^{(0)}\right\rangle = E_n^{(0)} \left|\phi_n^{(0)}\right\rangle$

b. Energy Shift: $E_n^{(1)} = \left\langle \phi_n^{(0)} \middle| \hat{H}_1 \middle| \phi_n^{(0)} \right\rangle$,

c. Wave function: $\left|\phi_n^{(1)}\right\rangle = \sum_{k \neq n} \left|\phi_n^{(0)}\right\rangle \frac{\left\langle \phi_k^{(0)} \middle| \hat{H}_1 \middle| \phi_n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}}$

d. Degenerate States, $(H_1)_{ji} = \left\langle \phi_{n,j}^{(0)} \middle| \hat{H}_1 \middle| \phi_{n,i}^{(0)} \right\rangle$

i. $\begin{pmatrix} (H_1)_{11} & (H_1)_{12} \\ (H_1)_{21} & (H_1)_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E_n^{(1)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

e. Stark Effect, $\hat{H}_1 = -\hat{\boldsymbol{\mu}}_e \cdot \mathbf{E} = e\mathbf{r} \cdot \mathrm{E}$

f. Spin-orbit Coupling $\hat{H}_1 = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} = \frac{Zke^2}{m^2 c^2 r^3} \mathbf{S} \cdot \mathbf{L}$

g. Darwin Term, Relativistic Correction

h. Fine and hyperfine structure, fine structure constant

i. Zeeman Effect, $\hat{H}_1 = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} = \left(\frac{e}{2mc}\mathbf{L} + \frac{e}{mc}\mathbf{S} \right) \cdot \mathbf{B}$

j. Perturbations of: particle in a box (1D, 2D, 3D), harmonic oscillators (1D, 2D, 3D), Spin systems (Spin ½, Spin 1, etc.)