PHY 444 Final Exam **Name ________________________**

Instructions:

- Place your name on all of the pages.
- Do all of your work in this booklet. Do not tear off any sheets.
- Show all of your steps in the problems for full credit.
- Be clear and neat in your work. Any illegible work, or scribbling in the margins, will not be graded.
- Put a box around your answers when appropriate.
- If you need more space, you may use the back of a page or the extra sheet on the last page and write *On back of page #* in the problem. **No other scratch paper is allowed.**

Try to answer as many problems as possible. Provide as much information as possible. Show sufficient work or rationale for full credit. Remember that some problems may require less work than brute force methods. **Pace yourself!**

If you are stuck, or running out of time, indicate as completely as possible, the physics and steps you would take to tackle the problem. Also, indicate any relevant information that you would use.

Pay attention to the point distribution. Not all problems have the same weight.

Page	Pts	Score
1	23	
2	17	
3	14	
4	12	
5	14	
6	11	
7	14	
Total	105	

$$R_H = 1.097\times10^{7}\,\text{m}^{-1},\; h = 6.626\times10^{-34}\,\text{Js},\quad \hbar = 1.055\times10^{-34}\,\text{Js},\; 1\text{eV} = 1.6\times10^{-19}\,\text{J},$$

$$m_e = 9.11\times10^{-31}\,\text{kg},\; m_p = 1.67\times10^{-27}\,\text{kg},\; a_0 = 0.5292\times10^{-10}\,m$$

Spin-1 Matrices: $\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$, $\hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$, $\hat{S}_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$.

Eigenvectors of S_x: $\frac{1}{2}\begin{pmatrix}1\\\pm\sqrt{2}\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$. Eigenvectors of S_y: $\frac{1}{2}\begin{pmatrix}\mp1\\\sqrt{2}i\\\pm1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$.

Have a great summer!

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1. (14 pts) **Two Point Short Answers:** Answer the following
   a. Write $\hat{S}_y$ in terms of $\hat{S}_+$ and $\hat{S}_-$.

   b. $\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 +$ _____________
   c. Uncertainty Principle: $\Delta A \Delta B \geq$ _________. [in terms of A and B.]

   d. A spin-½ particle with charge e and mass m is placed in a uniform magnetic field $\mathbf{B} = B_0\mathbf{j}$. Write the Hamiltonian, $\hat{H} = -\boldsymbol{\mu}\cdot\hat{\mathbf{S}}$, in terms of m, e, B_0, and a spin operator.

   e. How many possible electron states fill the $n = 3$ energy level? ________

   f. Eigenfunctions of the lowering operator, $\hat{a}$, are called _______________.

   g. Write the time-dependent Schrödinger equation in terms of H.

   h. Give the Pauli matrix $\sigma_y =$

2. (9 pts) **Three Point Short Answers:** Answer the following
   a. What wavelength photon is needed to cause a hydrogen electron to transitions from state $|\psi_{210}\rangle$ to $|\psi_{410}\rangle$. What type of radiation is absorbed?

   b. List the quantum numbers $n, \ell, j, s, \ldots,$ for the electronic configuration $2p_{\frac{3}{2}}$.

   c. A spin up hydrogen electron is in the state $|3, 2, -1\rangle$. Measurements of $\hat{L}^2$ and $\hat{L}_z$ yield what values?

3. (6 pts) Evaluate the following
   a. $[\hat{J}_-, \hat{J}_+]^\dagger =$

   b. $\hat{S}_- \left| \frac{5}{2}, -\frac{1}{2} \right\rangle =$

   c. ${}_y\left\langle \frac{1}{2}, -\frac{1}{2} \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle_z =$

4. (2 pts) What is the explicit relation between $\psi(x) = \langle x | \psi \rangle$ and $\varphi(p) = \langle p | \psi \rangle$?

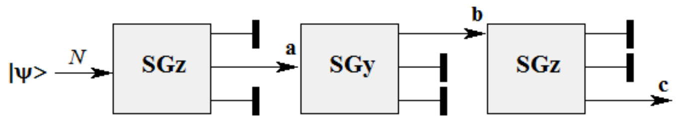
5. (6 pts) $\psi(\theta, \varphi) = C\left(3Y_{32} + 2Y_{52} + Y_{50}\right)$ is the angular part of a hydrogen wavefunction, where $Y_{\ell m} = Y_{\ell m}(\theta, \varphi)$ are normalized spherical harmonics.
   a. What is the normalization constant, *C*?

   b. What is the probability of finding an atom in the state with $m = 2$?

   c. What are the expectation values of $\hat{L}^2$ and $\hat{L}_z$?

6. (3 pts) Use $\hat{J}_+ = \hat{L}_+ + \hat{S}_+$ on the state $\left| j, m_j \right\rangle = \left| \frac{3}{2}, -\frac{3}{2} \right\rangle = |1, -1\rangle |-z\rangle$ to find $\left| \frac{3}{2}, -\frac{1}{2} \right\rangle = c_1 |1, m_1\rangle |s_1\rangle + c_2 |1, m_2\rangle |s_2\rangle$ by determining c_i, $|s_i\rangle$, and m_i, $i = 1, 2$. Here $\hat{J} = \hat{L} + \hat{S}$ is the total angular momentum.

7. (4 pts) A beam of spin-1 particles enters the Stern-Gerlach devices as depicted below with two of the three channels blocked after each transition. Find the probability that particles follow the given channel. [Hint: First write the appropriate probability amplitudes, such as ${}_z\langle 1,1|1,1\rangle_x$.]



   a. Probability that particles pass from **a to b**: ______

   b. Probability that particles pass from **a to c**: ______

8. (10 pts) Using the state vector $|\psi\rangle = N\left(12|+y\rangle - 5|-y\rangle\right)$,
   a. Normalize this state vector by determining N.

   b. What is the probability that a beam of particles in this state
      i. Entering an SGy device will measure $S_y = -\frac{\hbar}{2}$, or

      ii. Entering an SGz device will measure $S_z = -\frac{\hbar}{2}$?

   c. Calculate $\langle S_y \rangle$.

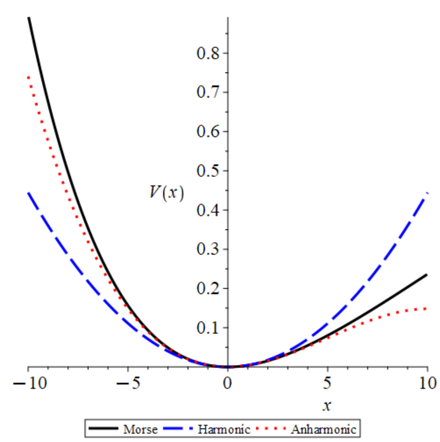
   d. Find the uncertainty, ΔS_y.

9. (5 pts) Using the normalized state $Y_{1,1}(\theta,\phi)$, do the following:
   a. Evaluate: $\hat{L}^2 Y_{1,1} =$ __________ and $\hat{L}_z Y_{1,1} =$ __________.
   b. Find the $Y_{1,1}(\theta,\phi)$ using the raising operator, $\hat{L}_+ = \frac{\hbar}{i} e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi}\right)$ and the state $Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$. [Recall $\hat{L}_+ |1,0\rangle$.]

10. (3 pts) Find the energy of the bound state of the potential $V(x) = -\alpha\,\delta(x)$ for $\alpha > 0$.. [Recall the condition $\left(\frac{d\psi}{dx}\right)_{0^+} - \left(\frac{d\psi}{dx}\right)_{0^-} = -\frac{2m\alpha}{\hbar^2}\psi(0)$.]

11. (4 pts) The Morse potential models the vibrational spectra of molecules and is given by $V(x) = V_0\left(1 - e^{-\mu x}\right)^2$. For small x, $V(x) = \mu^2 x^2 - \mu^3 x^3 + \frac{7}{12}\mu^4 x^4 + O(x^5)$. Keeping the first or first two terms for $\mu = 1/15$ is shown in comparison to the exact potential.

Obtain an approximation to the vibrational energies in terms of μ for the harmonic approximation.



Applying first order perturbation theory using $H_1 = -\mu^3 x^3$ we obtain $E_n^{(1)} = 0$. What is the next step to find nonzero corrections to the energy?

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12. (12 pts) Let $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$ be a mixture of states of the harmonic oscillator where $\hat{H}_0 = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hbar\omega$. Using this state, determine the following:

a. $\langle E \rangle$

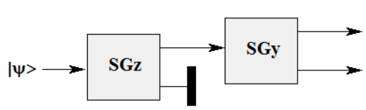
b. $\langle p \rangle$, where $\hat{p} = \beta(\hat{a} - \hat{a}^\dagger)$ for $\beta = \sqrt{\frac{m\omega\hbar}{2}}$.

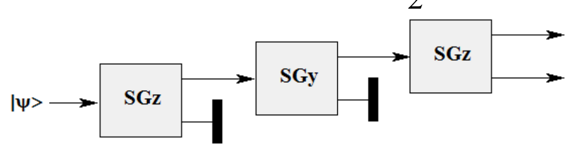
c. $|\psi(t)\rangle$

d. Find $\langle p(t) \rangle$ as a function of t and simplify.

e. For the Hamiltonian $\hat{H}_0 + \alpha\hat{H}_1$, where $\widehat{H}_1 = \frac{1}{2}\hbar\omega\left(\hat{a} + \hat{a}^\dagger\right)^2$, find the first order (in α) energy correction $E_n^{(1)}$.

13. (2 pts) Spin-$\frac{1}{2}$ particles in state $|\psi\rangle = \frac{1}{\sqrt{5}}|+z\rangle + \frac{2}{\sqrt{5}}|-z\rangle$ enter each device. Record the percentage of particles emerging from the end device with $S = \frac{\hbar}{2}$.





a. ______________________

b. ______________________

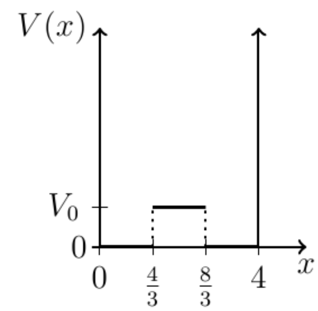
14. (8 pts) Consider a particle in the infinite square well $V(x) = \begin{cases} 0, & 0 \le x \le 4, \\ \infty, & \text{elsewhere.} \end{cases}$

a. What are $\psi_3(x)$ and E_3?

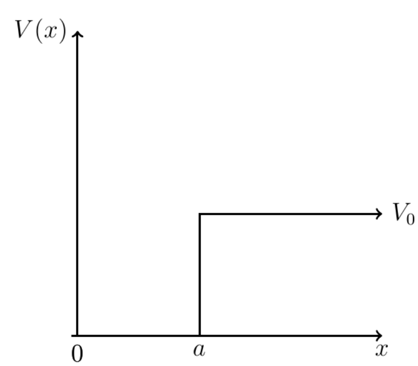
b. If the particle is in state $\psi_3(x)$, what is the probability of finding the particle in the interval $0 \le x \le 1$?

c. Consider the following perturbation to the potential well with $V_0 = \dfrac{\pi^2\hbar^2}{32m}$.

What is the first order energy shift for E_3?



15. (3 pts) For the semi-infinite square well potential with energies $0 < E < V_0$, show that the energy must satisfy the transcendental equation, $\cot \dfrac{a\sqrt{2mE}}{\hbar} = -\sqrt{\dfrac{V_0 - E}{E}}$.



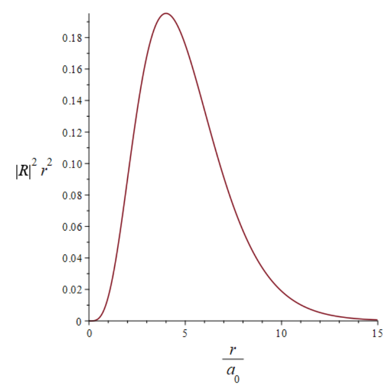
16. (5 pts) $\psi(x) = Ae^{-b^2x^2}$ is the ground state of a one-dimensional harmonic oscillator.

  a. Assuming A is real, normalize the wavefunction.

  b. Find $\langle x^2 \rangle$.

17. (4 pts) For $n=2$ and $\ell=1$, the hydrogen radial wavefunction is given by $R_{2,1}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{3/2}\frac{1}{a_0}re^{-r/2a_0}$, where a_0 is the Bohr radius. The radial probability is found using $dP = |R_{2,1}(r)|^2\, r^2\, dr$.

  a. Determine the most probable position?



  b. Determine the average position, $\langle r \rangle$?

18. (5 pts) Consider the Hamiltonian $H = H_0 + \lambda H_1 = \begin{pmatrix} 1.0 & 0 \\ 0 & 1.0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0.4 \\ 0.1 & 0 \end{pmatrix}$. .

  a. Find the energies and eigenvectors of the unperturbed Hamiltonian.

  b. Use degenerate perturbation theory to determine the first order energy corrections.

Extra Space