PHY 444 Final Exam **Name ________________________**

Instructions:

- Place your name on all of the pages.
- Do all of your work in this booklet. Do not tear off any sheets.
- Show all of your steps in the problems for full credit.
- Be clear and neat in your work. Any illegible work, or scribbling in the margins, will not be graded.
- Put a box around your answers when appropriate..
- If you need more space, you may use the back of a page or the extra sheet on the last page and write *On back of page #* in the problem. **No other scratch paper is allowed.**

Try to answer as many problems as possible. Provide as much information as possible. Show sufficient work or rationale for full credit. Remember that some problems may require less work than brute force methods. **Pace yourself!**

If you are stuck, or running out of time, indicate as completely as possible, the physics and steps you would take to tackle the problem. Also, indicate any relevant information that you would use.
Pay attention to the point distribution. Not all problems have the same weight.

Page	Pts	Score
1	20	
2	15	
3	16	
4	15	
5	14	
6	15	
7	14	
8	6	
Total	105	

$$R_H = 1.097\times10^{7}\,\mathrm{m}^{-1},\ h = 6.626\times10^{-34}\,\mathrm{Js},\quad \hbar = 1.055\times10^{-34}\,\mathrm{Js},\ 1\mathrm{eV} = 1.6\times10^{-19}\,\mathrm{J},$$
$$m_e = 9.11\times10^{-31}\,\mathrm{kg},\ m_p = 1.67\times10^{-27}\,\mathrm{kg},\ a_0 = 0.5292\times10^{-10}\,m$$

Spin-1 Matrices: $\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\quad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},\quad \hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

Eigenvectors of S_x: $\frac{1}{2}\begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Eigenvectors of S_y: $\frac{1}{2}\begin{pmatrix} \mp 1 \\ \sqrt{2}i \\ \pm 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Have a good break!

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1. (14 pts) Answer the following:
   a. If $[\hat{A},\hat{B}] = i\hat{C}$, what is the uncertainty relation?

   b. What does it mean for $\hat{A}$ to be a Hermitian operator?

   c. What does it mean for $\hat{U}$ to be a unitary operator?

   d. How many eigenstates have energy E_n in a hydrogen atom?

   e. Let $|\psi(0)\rangle$. Express $|\psi(t)\rangle$ in terms of its time evolution using $\hat{H}$.

   f. Write the time-independent Schrödinger equation for $|\psi(t)\rangle$.

   g. Give the Pauli matrix $\sigma_x =$

2. (6 pts) Evaluate and simplify the following
   a. $[\hat{J}_-,\hat{J}_z]^\dagger =$

   b. $S_+\left|\frac{3}{2},\frac{1}{2}\right\rangle =$

   c. ${}_z\left\langle\frac{1}{2},+\frac{1}{2}\middle|\frac{1}{2},-\frac{1}{2}\right\rangle_x =$

3. (4 pts) Determine the field gradient (in T/m) needed to deflect a beam of silver atoms 2.00 mm while moving 50.0 cm through a magnetic field at an average horizontal speed of 100.0 m/s. [$\mu_B = \dfrac{e\hbar}{2m_e c} = 9.274\times10^{-24}$ J/T, $M_{Ag} = 1.791\times10^{-25}$ kg]

4. (5 pts) Fill in exactly the missing parts of the following:

   a. $\dfrac{d}{dt}\langle A\rangle = \dfrac{i}{\hbar}\langle\_\_\_\_\_\_\_\rangle + \langle\_\_\_\_\_\rangle.$

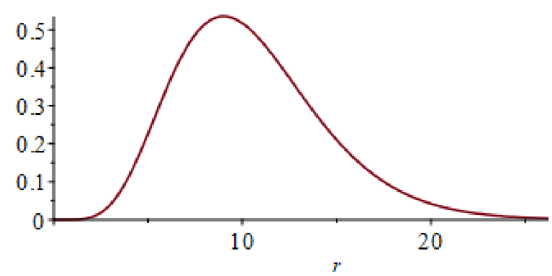
   b. $\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 +$ ________

   c. $\langle p|x\rangle =$

   d. $\Delta L_x \Delta L_y \geq$ ________.

5. (6 pts) A plot of $r^2R_{32}^2(r)$ is shown below, where with radial wave function $R_{32}(r) = \dfrac{2\sqrt{30}}{1215}a_0^{-7/2}r^2e^{-r/3a_0}$ and a_0 is the Bohr radius.

   a. Calculate the most probable position.



   b. Determine the average position, $\langle r\rangle$?

6. (3 pts) An eigenstate of S_x with eigenvalue 0 is represented by

$$|1,0\rangle_x \xrightarrow[S_x]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Applying a rotation of $\frac{\pi}{2}$ about the z-axis to $|1,0\rangle_x$ results in what state?

7. (4 pts) A Hydrogen electron absorbs a photon and its state changes from $|\psi_{320}\rangle$ to $|\psi_{520}\rangle$. What are the wavelength and (unperturbed) energy of the photon?

8. (9 pts) Let $\psi(x) = \langle x|\psi\rangle = \begin{cases} Ne^{-2x}, & x > 0, \\ Ne^{2x}, & x < 0, \end{cases}$ where $a > 0$.

   a. Find the normalization constant.

   b. Find Δx. [Hint: Tap your inner Feynman.]

   c. Find the momentum wavefunction, $\varphi(p) = \langle p|\psi\rangle$.

9. (8 pts) Consider spin-1/2 particles in the state $|\psi\rangle = \frac{4}{5}|+z\rangle - \frac{3}{5}i|-z\rangle$.

   a. What is the probability that a beam of particles in this state

      i. Entering an SGz device will measure $S_z = -\frac{\hbar}{2}$?

      ii. Entering an SGx device will measure $S_x = +\frac{\hbar}{2}$?

   b. Find the expectation value $\langle S_z \rangle$.

   c. Determine the uncertainty in measuring S_z.

10. (4 pts) Let $\psi(x) = Ae^{-b^2x^2/2}$ be the ground state of a one-dimensional harmonic oscillator. Find $\langle x^2 \rangle$ knowing that $\int_{-\infty}^{\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}$.

11. (3 pts) Let $\psi(x) = \left(\frac{\beta^2}{\pi}\right)^{1/4} \frac{1}{\sqrt{48}}\left(8\beta^3x^3 - 12\beta x\right)e^{-\beta^2x^2/2}$, where $\beta = \sqrt{\frac{m\omega}{\hbar}}$, be an energy eigenstate of the quantum harmonic oscillator.

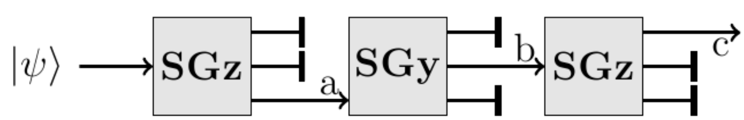
    a. Without doing much work, determine the energy of this state: __________

    b. Explain how you obtained your answer.

12. (4 pts) An external field is applied to a hydrogen atom. One finds the matrix representation of the perturbed Hamiltonian is $H_1 = \begin{pmatrix} 0 & -4\gamma \\ -\gamma & 0 \end{pmatrix}$. Use degenerate perturbation theory to determine the first order energy shifts and form of the mixed (eigen-) states.

13. (4 pts) Find the correctly normalized state $Y_2^1(\theta,\phi)$ using the lowering operator, $\hat{L}_- = \hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)$ and the state $Y_2^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi}$.

14. (6 pts) A beam of spin-1 particles in state $|\psi\rangle = \frac{2}{\sqrt{14}}|1,1\rangle + \frac{i}{\sqrt{14}}|1,0\rangle - \frac{3}{\sqrt{14}}|1,-1\rangle$ enters the sequence of Stern-Gerlach devices below. Find the probabilities of emerging from each device as indicated.



i) $|\psi\rangle$ to a ____________

ii) From a to b. ____________

iii) From b to c. ____________

15. (7 pts) We have seen that total angular momentum ($\hat{J} = \hat{L} + \hat{S}$) states are mixed states in the form $\left|j, m_j\right\rangle = c_1\left|\ell, m, +z\right\rangle + c_2\left|\ell, m+1, -z\right\rangle$.

   a. For an electron in the *p*-orbital, what are the possible *j*-values?

   b. For the state $\left|j, m_j\right\rangle = \left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(\left|1, -1, +z\right\rangle + 2\left|1, 0, -z\right\rangle\right)$ find

      i. $\hat{J}_+\left|\frac{3}{2}, -\frac{1}{2}\right\rangle =$

      ii. $\left(\hat{L}_+ + \hat{S}_+\right)\left(\frac{1}{\sqrt{3}}\left(\left|1, -1, +z\right\rangle + 2\left|1, 0, -z\right\rangle\right)\right) =$

   c. Use these results to obtain $\left|\frac{3}{2}, \frac{1}{2}\right\rangle = c_1\left|1, m_1\right\rangle\left|s_1\right\rangle + c_2\left|1, m_2\right\rangle\left|s_2\right\rangle$ for coefficients, c_i, spins, $\left|s_i\right\rangle$, and m_i, $i = 1, 2$.

16. (8 pts) A collection of hydrogen atoms are in state

$$\psi(r,\theta,\phi) = \frac{1}{\sqrt{5}}\psi_{211} - \frac{1}{\sqrt{5}}\psi_{321} + \sqrt{\frac{3}{5}}\psi_{311}.$$

   a. What are the possible energies (in eV) that could be measured?

   b. What are the probabilities of measuring each of these energies?

   c. What is the probability that a measurement of $\hat{L}^2$ yields $6\hbar^2$?

17. (8 pts) Consider a particle in the infinite square well $V_0(x)=\begin{cases}0, & 0\le x\le 2,\\ \infty, & \text{elsewhere.}\end{cases}$

a. What are $\psi_3(x)$ and E_3?

b. If the particle is in state $\psi_3(x)$, what is the probability of finding the particle in the interval $0.5\le x\le 1$?

c. Let $V(x)=V_0(x)+V_1(x)$, where $V_1(x)=\begin{cases}\lambda x(1-x), & 0\le x\le 1,\\ 0, & 1\le x\le 2,\\ \infty, & \text{elsewhere.}\end{cases}$

Assuming λ is small, what is the first order energy shift in E_3?

18. (6 pts) A harmonic oscillator, with Hamiltonian, $\hat{H}=\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\hbar\omega$ and energy eigenstates $|n\rangle$, is in the initial state $|\psi(0)\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle+i|1\rangle\right)$. Letting $\hat{x}=\beta(\hat{a}+\hat{a}^\dagger)$,

a. What is $\langle x\rangle$ for the initial state?

b. Determine the state $|\psi(t)\rangle$.

Find $\langle x\rangle$ as a function of t and simplify.

19. (6 pts) Consider the potential barrier $V(x) = V_0\,\delta(x)$ for $V_0 = \dfrac{\alpha\hbar^2}{2m}$, where $\alpha > 0$.

a. Find the transmission coefficient.

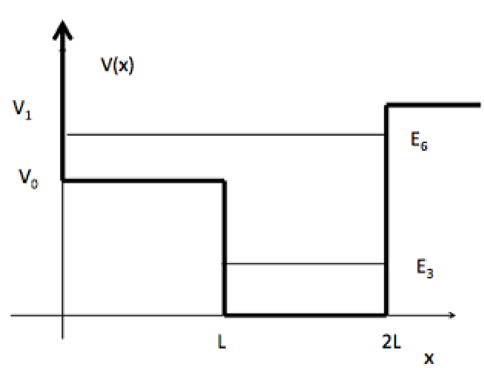
[Recall the condition $\left(\dfrac{d\psi}{dx}\right)_{x>0} - \left(\dfrac{d\psi}{dx}\right)_{x<0} = \alpha\psi(0).$]

b. Place this barrier in an infinite square well, $V(x) = \begin{cases} V_0\delta(x), & |x| < a \\ \infty, & |x| > a \end{cases}$.

Find an equation for the bound state energies for even wavefunctions.

Bonus

A particle moves in the potential well below. Sketch the wavefunctions for the third and sixth energy levels shown in the figure. Annotate your sketches indicating classically allowed and forbidden regions, discussing important features such as where there is exponential or oscillatory behavior, where the wavefunction is zero, etc.



Extra Space