# L-Squared

## 1 Classical Identity

In these notes we discuss the computation of $\mathbf{L}^2 = (\mathbf{r}\times\mathbf{p})\cdot(\mathbf{r}\times\mathbf{p})$ and it's quantum operator version. First we compute the classical version.

$$\begin{aligned} \mathbf{L}^2 &= (\mathbf{r}\times\mathbf{p})\cdot(\mathbf{r}\times\mathbf{p}) \\ &= \mathbf{L}\cdot(\mathbf{r}\times\mathbf{p}) \\ &= \mathbf{r}\cdot(\mathbf{p}\times\mathbf{L}). \end{aligned} \tag{1}$$

Note that

$$\begin{aligned} \mathbf{p}\times\mathbf{L} &= \mathbf{p}\times(\mathbf{r}\times\mathbf{p}) \\ &= \mathbf{r}(\mathbf{p}\cdot\mathbf{p}) - \mathbf{p}(\mathbf{p}\cdot\mathbf{r}). \end{aligned} \tag{2}$$

using the BAC-CAB rule,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}).$$

This gives the identity.

$$\mathbf{L}^2 = r^2p^2 - (\mathbf{r}\cdot\mathbf{p})^2. \tag{3}$$

## 2 Quantum Operator Identity

Now we will turn to $\hat{\mathbf{L}}^2 = (\hat{\mathbf{r}}\times\mathbf{p})\cdot(\hat{\mathbf{r}}\times\hat{\mathbf{p}})$. Here we need to be careful with the order of operators. Also, we will make use of the Levi-Civita permutation symbol. We first note that

$$\mathbf{r}\times\mathbf{p} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} x_j p_k \hat{\mathbf{e}}_i,$$

where

$$\epsilon_{ijk} = \begin{cases} 1, & ijk \text{ an even permutation of } 123, \\ -1, & ijk \text{ an odd permutation of } 123, \\ 0, & i=j, j=k, \text{ or } k=i. \end{cases}$$

Furthermore, we will need the identities

$$\sum_{k=1}^{3} \delta_{kk} = 3, \tag{4}$$

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{\ell k}, \tag{5}$$

$$\hat{p}_k \hat{x}_j = \hat{x}_j \hat{p}_k - i\hbar \delta_{jk}. \tag{6}$$

Now we compute $\hat{\mathbf{L}}^2$ :

$$\begin{aligned}
\hat{\mathbf{L}}^2 &= (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \\
&= \sum_i (\hat{\mathbf{r}} \times \hat{\mathbf{p}})_i (\hat{\mathbf{r}} \times \hat{\mathbf{p}})_i \\
&= \sum_i \left( \sum_{j,k} \epsilon_{ijk} \hat{x}_j \hat{p}_k \right) \left( \sum_{\ell,m} \epsilon_{i\ell m} \hat{x}_\ell \hat{p}_m \right) \\
&= \sum_{j,k} \sum_{\ell,m} \sum_i \epsilon_{ijk} \epsilon_{i\ell m} \hat{x}_j \hat{p}_k \hat{x}_\ell \hat{p}_m \\
&= \sum_{j,k} \sum_{\ell,m} \left[ \delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{\ell k} \right] \hat{x}_j \hat{p}_k \hat{x}_\ell \hat{p}_m \\
&= \sum_{j,k} \left[ \hat{x}_j \hat{p}_k \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_k \hat{x}_k \hat{p}_j \right].
\end{aligned}$$

Looking at the last line, we see see the possibility of some dot products, like $\sum_k A_k B_k = \mathbf{A} \cdot \mathbf{B}$. However, we need to be careful when switching the order of some operators. So, we need to insert expressions like (6).

$$\begin{aligned}
\hat{\mathbf{L}}^2 &= \sum_{j,k} \left[ \hat{x}_j \hat{p}_k \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_k \hat{x}_k \hat{p}_j \right]. \\
&= \sum_{j,k} \left[ \hat{x}_j \left( \hat{x}_j \hat{p}_k - i\hbar \delta_{jk} \right) \hat{p}_k - \hat{x}_j \hat{p}_k \left( \hat{p}_j \hat{x}_k + i\hbar \delta_{jk} \right) \right]. \\
&= \sum_{j,k} \left[ \hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - i\hbar \delta_{jk} \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_j \hat{p}_k \hat{x}_k - i\hbar \delta_{jk} \hat{x}_j \hat{p}_k \right]. \\
&= \sum_{j,k} \left[ \hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - i\hbar \delta_{jk} \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_j \left( \hat{x}_k \hat{p}_k - i\hbar \delta_{kk} \right) - i\hbar \delta_{jk} \hat{x}_j \hat{p}_k \right]. \\
&= \sum_{j} \left[ \hat{x}_j \hat{x}_j \hat{p}^2 - i\hbar \hat{x}_j \hat{p}_j - \hat{x}_j \hat{p}_j \left( \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} - 3i\hbar \right) - i\hbar \hat{x}_j \hat{p}_j \right]. \\
&= \hat{r}^2 \hat{p}^2 - i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} - (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})^2 + 3i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} - i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} \\
&= \hat{r}^2 \hat{p}^2 - (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})^2 + i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}}. \qquad (7)
\end{aligned}$$

# 3 Rewriting the Hamiltonian

We begin with the state $\hat{\mathbf{L}}^2 |\psi\rangle$. Projecting onto the position basis,

$$\begin{aligned}
\langle \mathbf{r} | \hat{\mathbf{L}}^2 | \psi \rangle &= \langle \mathbf{r} | [\hat{r}^2 \hat{p}^2 + i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} - (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})^2] | \psi \rangle \\
&= r^2 \langle \mathbf{r} | \hat{p}^2 | \psi \rangle + i\hbar \mathbf{r} \cdot \frac{\hbar}{i} \nabla \langle \mathbf{r} | \psi \rangle + \hbar^2 (\mathbf{r} \cdot \nabla)^2 \langle \mathbf{r} | \psi \rangle \\
r^2 \langle \mathbf{r} | \hat{p}^2 | \psi \rangle &= \langle \mathbf{r} | \hat{\mathbf{L}}^2 | \psi \rangle - \hbar^2 r \frac{\partial}{\partial r} \langle \mathbf{r} | \psi \rangle - \hbar^2 r \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) \langle \mathbf{r} | \psi \rangle
\end{aligned}$$

$$\langle \mathbf{r}|\hat{H}|\psi\rangle = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)\langle \mathbf{r}|\psi\rangle + \frac{\langle \mathbf{r}|\hat{\mathbf{L}}^2|\psi\rangle}{2\mu r^2} + V(r)\langle \mathbf{r}|\psi\rangle = E\langle \mathbf{r}|\psi\rangle \tag{8}$$

Now, we make use of the eigenvalue problem for $\hat{\mathbf{L}}^2$. Namely,

$$\hat{\mathbf{L}}^2|E,\ell,m\rangle = \ell(\ell+1)\hbar^2|E,\ell,m\rangle.$$

So, with $|\psi\rangle = |E,\ell,m\rangle$, we have

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}+V(r)\right]\langle \mathbf{r}|E,\ell,m\rangle = E\langle \mathbf{r}|E,\ell,m\rangle$$

Letting $\langle \mathbf{r}|E,\ell,m\rangle = R(r)Y(\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$,

$$\begin{aligned} -\frac{\hbar^2}{2\mu}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}R+V(r)R &= ER, \\ \hat{\mathbf{L}}^2 Y &= \ell(\ell+1)\hbar^2 Y \\ \hat{L}_z Y &= m\hbar Y. \end{aligned} \tag{9}$$