Basics of the One-dimensional Schrödinger Equation

Time-Dependent Schrödinger Equation: $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

Time Evolution: $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$

Energy Eigenstates: $\hat{H}|E\rangle = E|E\rangle$

Wavefunctions:

Position space wavefunction: $\psi(x,t) = \langle x|\psi(t)\rangle = \int dp \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\varphi(p,t)$

Momentum space wavefunction: $\varphi(p,t) = \langle p|\psi(t)\rangle = \int dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\psi(x,t)$

Time-Independent Schrödinger Equation

Energy Eigenstates: $\psi_E(x,t) = \langle x|E\rangle e^{-iEt/\hbar}$

TISE:
$$\begin{cases} \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\langle x|E\rangle = E\langle x|E\rangle \\ -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \end{cases}$$

Stationary states: $\psi_n(x) = \langle x|E_n\rangle$, and eigenvalues: $E_n,\ n = 1, 2, \ldots$

Time Evolution (Ket vs wavefunction notation):

Initial states: $|\psi(0)\rangle = \sum_{n=1}^{\infty} c_n|E_n\rangle$, where $c_n = \langle E_n|\psi(0)\rangle$,

$$\psi(x,0) = \sum_{n=1}^{\infty} c_n\psi_n(x),\ \text{where } c_n = \langle \psi_n|\psi(0)\rangle = \int_{-\infty}^{\infty} \psi_n{}^*(x)\psi(x,0)\,dx$$

Time Evolution: $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-i\hat{H}t/\hbar}\sum_{n=1}^{\infty} c_n|E_n\rangle = \sum_{n=1}^{\infty} c_n e^{-i\hat{H}t/\hbar}|E_n\rangle = \sum_{n=1}^{\infty} c_n e^{-iE_n t/\hbar}|E_n\rangle$

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_n t/\hbar}$$

Measurements:

Probability to measure $E_n = |c_n|^2$

$P(\text{particle} \in [x, x+dx]) = |\langle x|\psi\rangle|^2\,dx = |\psi(x,t)|^2\,dx$

$P(\text{particle} \in [a,b]) = \int_a^b dx|\langle x|\psi\rangle|^2 = \int_a^b |\psi(x,t)|^2\,dx$

$P(\text{particle has energy } E) = |c_n|^2 = \int_{-\infty}^{\infty} \psi_n{}^*(x)\psi(x,t)\,dx$

Scattering States:

Probability density, $\rho(x,t) = |\psi(x,t)|^2$, Probability current, $j_x(x,t) = \frac{\hbar}{2mi}[\psi^*{}'\psi - \psi^*\psi']$

States – Far left: $\psi_L(x) = Ae^{ikx} + Be^{-ikx}$ Far Right: $\psi_R(x) = Fe^{ikx} + Ge^{-ikx}$

Scattering Matrix $\begin{pmatrix} B \\ F \end{pmatrix} = S\begin{pmatrix} A \\ G \end{pmatrix}$ gives (scattering from left, $G = 0$): $R = \frac{|B|^2}{|A|^2} = |S_{11}|^2$, $T = \frac{|F|^2}{|A|^2} = |S_{21}|^2$

Can use S to get bound states ($k \to i\kappa$ and seek singularities)