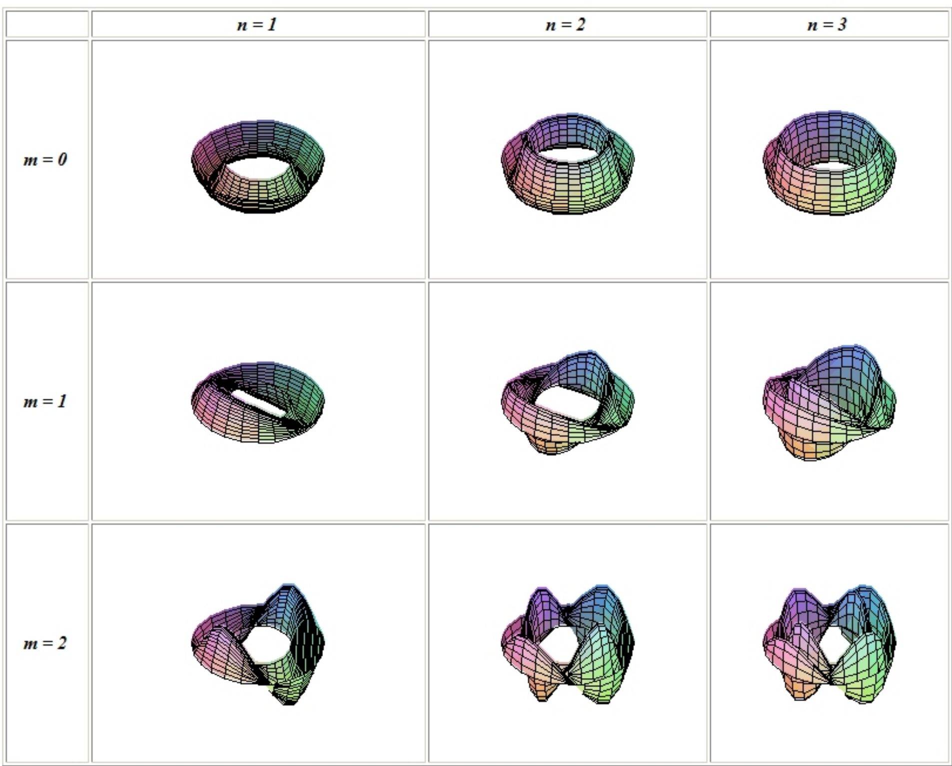
RUSSELL L. HERMAN

AN INTRODUCTION TO MATHEMATICAL PHYSICS VIA OSCILLATIONS



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Dedicated to those students who have endured previous editions of AN INTRODUCTION TO MATHEMATICAL PHYSICS VIA OSCILLATIONS *and to those about to embark on the journey.*

0 Prologue

"All science is either physics or stamp collecting." Ernest Rutherford (1871-1937)

0.1 Introduction

This is a set of notes on mathematical physics for undergraduate students who have completed a year long introductory course in physics. The intent of the course is to introduce students to many of the mathematical techniques useful in their undergraduate physics education long before they are exposed to more focused topics in physics.

This is an introduction to topics in mathematical physics, introduced using the physics of oscillations and waves. It is based upon a one semester junior level course in mathematics physics taught at the University of North Carolina Wilmington and originally set to book form in 2005. The notes were later modified and used in 2006, 2011, and 2012.

Most texts on mathematical physics are encyclopedic works which can never be covered in one semester and are often presented as a list of the seemingly unrelated topics with some examples from physics inserted to highlight the connection of the particular topic to the real world. The point of these excursions is to introduce the student to a variety of topics and not to delve into the rigor that one would find in some mathematics courses. Most of these topics have equivalent semester long courses which go into the details and proofs of the main conjectures in that topic. Students may decide to later enroll in such courses during their undergraduate, or graduate, study. Often the relevance to physics must be found in more advanced courses in physics when the particular methods are used for specialized applications.

So, why not teach the methods in the physics courses as they are needed? Part of the reason is that going into the details can take away from the global view of the course. Students often get lost in the mathematical details, as the proverbial tree can be lost in a forest of trees. Many of the mathematical techniques used in one course can be found in other courses. Collecting these techniques in one place, such as a course in mathematical physics, can provide a uniform background for students entering later courses in specialized topics in physics. Repeated exposure to standard methods also helps to ingrain these methods. Furthermore, in such a course as this, the student first sees both the physical and mathematical connections between different fields. Instructors can use this course as an opportunity to show students how the physics curriculum ties together what otherwise might appear to be a group of seemingly different courses.

The typical topics covered in a course on mathematical physics are vector analysis, vector spaces, linear algebra, complex variables, power series, ordinary and partial differential equations, Fourier series, Laplace and Fourier transforms, Sturm-Liouville theory, special functions and possibly other more advanced topics, such as tensors, group theory, the calculus of variations, or approximation techniques. We will cover many of these topics, but will do so in the guise of exploring specific physical problems. In particular, we will introduce these topics in the context of the physics of oscillations and waves.

0.2 *What is Mathematical Physics?*

WHAT DO YOU THINK when you hear the phrase "mathematical physics"? If one does a search on Google, one finds in Wikipedia the following:

> "Mathematical physics is an interdisciplinary field of academic study in between mathematics and physics, aimed at studying and solving problems inspired by physics within a mathematically rigorous framework. Although mathematical physics and theoretical physics are related, these two notions are often distinguished. Mathematical physics emphasizes the mathematical rigor of the same type as found in mathematics while theoretical physics emphasizes the links to actual observations and experimental physics which often requires the theoretical physicists to use heuristic, intuitive, and approximate arguments. Arguably, mathematical physics is closer to mathematics, and theoretical physics is closer to physics.
>
> Because of the required rigor, mathematical physicists often deal with questions that theoretical physicists have considered to be solved for decades. However, the mathematical physicists can sometimes (but neither commonly nor easily) show that the previous solution was incorrect.
>
> Quantum mechanics cannot be understood without a good knowledge of mathematics. It is not surprising, then, that its developed version under the name of quantum field theory is one of the most abstract, mathematically-based areas of physical sciences, being backward-influential to mathematics. Other subjects researched by mathematical physicists include operator algebras, geometric algebra, noncommutative geometry, string theory, group theory, statistical mechanics, random fields etc."

However, we will not adhere to the rigor suggested by this definition of mathematical physics, but will aim more towards the theoretical physics approach. Thus, this course could be called "A Course in Mathematical Methods in Physics." With this approach in mind, the course will be designed as a study of physical topics leading to the use of standard mathematical techniques. However, we should keep in mind Freeman Dyson's (b. 1923) words,

> "For a physicist mathematics is not just a tool by means of which phenomena can be calculated, it is the main source of concepts and principles by means of which new theories can be created." from *Mathematics in the Physical Sciences Mathematics in the Physical Sciences,* **Scientific American**, 211(3), September 1964, pp. 129-146.

It has not always been the case that we had to think about the differences between mathematics and physics. Until about a century ago people did not view physics and mathematics as separate disciplines. The Greeks did not separate the subjects, but developed an understanding of the natural sciences as part of their philosophical systems. Later, many of the big name physicists and mathematicians actually worked in both areas only to be placed in these categories through historical hindsight. People like Newton and Maxwell made just as many contributions to mathematics as they had to physics while trying to investigate the workings of the physical universe. Mathematicians such as Gauss, Leibniz and Euler had their share of contributions to physics.

Mathematics and physics are intimately related.

In the 1800's the climate changed. The study of symmetry lead to group theory, problems of convergence of the trigonometric series used by Fourier and others lead to the need for rigor in analysis, the appearance of non-Euclidean geometries challenged the millennia old Euclidean geometry, and the foundations of logic were challenged shortly after the turn of the century. This lead to a whole population of mathematicians interested in abstracting mathematics and putting it on a firmer foundation without much attention to applications in the real world. This split is summarized by Freeman Dyson:

> "I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce." from Missed Opportunities, 1972. (Gibbs Lecture)

In the meantime, many mathematicians have been interested in applying and extending their methods to other fields, such as physics, chemistry, biology and economics. These applied mathematicians have helped to mediate the divorce. Likewise, over the past century a number physicists with a strong bent towards mathematics have emerged as mathematical physicists. So, Dyson's report of a divorce might be premature.

Some of the most important fields at the forefront of physics are steeped in mathematics. Einstein's general theory of relativity, a theory of gravitation, involves a good dose of differential geometry. String theory is also highly mathematical, delving well beyond the topics in this book. While we will not get into these areas in this course, I would hope that students reading this book at least get a feel for the need to maintain the needed balance between mathematics and physics.

It is difficult to list all of the topics needed to study a subject like string theory. However, it is safe to say that a good grasp of the topics in this and more advance books on mathematical physics would help. A solid background in complex analysis, differential geometry, Lie groups and algebras, and variational principles should provide a good start.

0.3 An Overview of the Course

One of the problems with courses in mathematical physics and some of the courses taught in mathematics departments is that students do not always see the tie with physics. In this class we hope to enable students to see the mathematical techniques needed to enhance their future studies in physics. We will try not provide the mathematical topics devoid of physical motivation. We will instead introduce the methods studied in this course while studying one underlying theme from physics. We will tie the class mainly to the idea of oscillation in physics. Even though this theme is not the only possible collection of applications seen in physics, it is one of the most pervasive and has proven to be at the center of the revolutions of twentieth century physics.

In this section we provide an overview of the course in terms of the theme of oscillations even though at this writing there might be other topics introduced as the course is developed. There are many topics that could/might be included in the class depending upon the time that we have set aside. The current chapters/topics and their contents are:

1. **Introduction**

   In this chapter we review some of the key computational tools that you have seen in your first two courses in calculus and recall some of the basic formulae for elementary functions. Then we provide a short overview of your basic physics background, which will be useful in this course. It is meant to be a reference and additional topics may be added as we get further into the course.

   As the aim of this course is to introduce techniques useful in exploring the basic physics concepts in more detail through computation, we will also provide an overview of how one can use mathematical tables and computer algebra systems to help with the tedious tasks often encountered in solving physics problems.

   We will end with an example of how simple estimates in physics can lead to "back of the envelope" computations using dimensional analysis. While such computations do not require (at face value) the complex machinery seen in this course, it does use something that can be explained using the more abstract techniques of similarity analysis.

(a) What Do I Need to Know From Calculus?

(b) What I Need From My Intro Physics Class?

(c) Using Technology and Tables

(d) Dimensional Analysis

2. **Free Fall and Harmonic Oscillators**

A major theme throughout this book is that of oscillations, starting with simple vibrations and ending with the vibrations of membranes, electromagnetic fields, and the electron wave function. We will begin the first chapter by studying the simplest type of oscillation, simple harmonic motion. We will look at examples of a mass on a spring, LRC circuits, and oscillating pendula. These examples lead to constant coefficient differential equations, whose solutions we study along the way. We further consider the effects of damping and forcing in such systems, which are key ingredients to understanding the qualitative behavior of oscillating systems. Another important topic in physics is that of a nonlinear system. We will touch upon such systems in a few places in the text.

Even before introducing differential equations for solving problems involving simple harmonic motion, we will first look at differential equations for simpler examples, beginning with a discussion of free fall and terminal velocity. As you have been exposed to simple differential equations in your calculus class, we need only review some of the basic methods needed to solve standard applications in physics.

More complicated physical problems involve coupled systems. In fact, the problems in this chapter can be formulated as linear systems of differential equations. Such systems can be posed using matrices and the solutions are then obtained by solving eigenvalue problems, which is treated in the next chapter.

Other techniques for studying such problems described by differential equations involve power series methods and Laplace and other integral transforms. These ideas will be explored later in the book when we move on to exploring partial differential equations in higher dimensions. We will also touch on numerical solutions of differential equations, as not all problems can be solved analytically.

(a) Free Fall and Terminal Velocity; First Order ODEs

(b) The Simple Harmonic Oscillator; Second Order ODEs

(c) LRC Circuits

(d) Damped and Forced Oscillations; Nonhomogeneous ODEs

(e) Coupled Oscillators; Planar Systems

(f) The Nonlinear Pendulum

3. **Linear Algebra**

One of the most important mathematical topics in physics is linear algebra. Nowadays, the linear algebra course in most mathematics departments has evolved into a hybrid course covering matrix manipulations and some basics from vector spaces. However, it is seldom the case that applications, especially from physics, are covered. In this chapter we will introduce vector spaces, linear transformations and view matrices as representations of linear transformations. The main theorem of linear algebra is the spectral theorem, which means studying eigenvalue problems. Essentially, when can a given operator (or, matrix representation) be diagonalized? As operators act between vector spaces, it is useful to understand the concepts of both finite and infinite dimensional vector spaces and linear transformations on them.

The mathematical basis of much of physics relies on an understanding of both finite and infinite dimensional vector spaces. Linear algebra is important in the study of ordinary and partial differential equations, Fourier analysis, quantum mechanics and general relativity. We will return to this idea throughout the text. In this chapter we will introduce the concepts of vector spaces, linear transformations, and eigenvalue problems. We will also show how techniques from linear algebra are useful in solving coupled linear systems of differential equations. Later we shall see how much of what we do in physics involves linear transformations on vector spaces and eigenvalue problems.

(a) Finite Dimensional Vector Spaces

(b) Linear Transformations

(c) Matrices

(d) Eigenvalue Problems

(e) More Coupled Systems

(f) Diagonalization or The Spectral Theorem

4. **The Harmonics of Vibrating Strings**

The next type of oscillations which we will study are solutions of the one dimensional wave equation. A key example is provided by the finite length vibrating string. We will study traveling wave solutions and look at techniques for solving the wave equation. The standard technique is to use separation of variables, turning the solution of a partial differential equation into the solution of several ordinary differential equations. The resulting general solution will be written as an infinite series of sinusoidal functions, leading us to the study of Fourier series, which in turn provides the basis for studying the spectral content of complex signals.

In the meantime, we will also introduce the heat, or diffusion, equation as another example of a generic one dimensional partial differential equation exploiting the methods of this chapter. These problems begin our study of initial-boundary value problems, which pervade upper level physics courses, especially in electromagnetic theory and quantum mechanics.

(a) The Wave Equation in 1D

(b) Harmonics and Vibrations

(c) Fourier Trigonometric Series

(d) The Heat Equation in 1D

(e) Finite Length Strings

5. **Special Functions and the Space in Which They Live**

In our studies of systems in higher dimensions we encounter a variety of new solutions of boundary value problems. These collectively are referred to as Special Functions and have been known for a long time. They appear later in the undergraduate curriculum and we will cover several important examples. At the same time, we will see that these special functions may provide bases for infinite dimensional function spaces. Understanding these functions spaces goes a long way to understanding generalized Fourier theory, differential equations, and applications in electrodynamics and quantum theory.

In order to fully appreciate the special functions typically encountered in solving problem in higher dimensions, we will develop the Sturm-Liouville theory with some further excursion into the theory of infinite dimensional vector spaces.

(a) Infinite Dimensional Function Spaces

(b) Classical Orthogonal Polynomials

(c) Legendre Polynomials

(d) Gamma Function

(e) Bessel Functions

(f) Sturm-Liouville Eigenvalue Problems

6. **Complex Representations of The Real World**

   Another simple example, useful later for studying electromagnetic waves, is the infinite one-dimensional string. We begin with the solution of the finite length string, which consists of an infinite sum over a discrete set of frequencies, or a Fourier series. Allowing for the string length to get large will turn the infinite sum into a sum over a continuous set of frequencies. Such a sum is now an integration and the resulting integrals are defined as Fourier Transforms. Fourier transforms are useful in representing analog signals and localized waves in electromagnetism and quantum mechanics. Such integral transforms will be explored in the next chapter. However, useful results can only be obtained after first introducing complex variable techniques.

   So, we will spend some time exploring complex variable techniques and introducing the calculus of complex functions. In particular, we will become comfortable manipulating complex expressions and learn how to use contour methods to aid in the computation of integrals. We can apply these techniques to solving some special problems. We will first introduce a problem in fluid flow in two dimensions, which involve's solving Laplace's equation. We will explore dispersion relations, relations between frequency and wave number for wave propagation, and the computation of complicated integrals such as those encountered in computing induced current using Faraday's Law.

   (a) Complex Representations of Waves
   (b) Complex Numbers
   (c) Complex Functions and Their Derivatives
   (d) Harmonic Functions and Laplace's Equation
   (e) Complex Series Representations
   (f) Complex Integration
   (g) Applications to 2D Fluid Flow and AC Circuits

7. **Transforms of the Wave and Heat Equations**

   For problems defined on an infinite interval, solutions are no longer given in terms of infinite series. They can be represented in terms of integrals, which are associated with integral transforms. We will explore Fourier and Laplace transform methods for solving both ordinary and partial differential equations. By transforming our equations, we are lead to simpler equations in transform space. We will apply these methods to ordinary differential equations modeling forced oscillations and to the heat and wave equations.

(a) Transform Theory

(b) Exponential Fourier Transform

(c) The Dirac Delta Function

(d) The Laplace Transform and Its Applications

(e) Solution of Initial Value Problems; Circuits Problems

(f) The Inverse Laplace Transform

(g) Green's Functions and the Heat Equation

8. Electromagnetic Waves

One of the major theories is that of electromagnetism. In this chapter we will recall Maxwell's equations and use vector identities and vector theorems to derive the wave equation for electromagnetic waves. This will require us to recall some vector calculus from Calculus III. In particular, we will review vector products, gradients, divergence, curl, and standard vector identities useful in physics. In the next chapter we will solve the resulting wave equation for some physically interesting systems.

In preparation for solving problems in higher dimensions, we will pause to look at generalized coordinate systems and the transformation of gradients and other differential operators in these new systems. This will be useful in the next chapter for solving problems in other geometries.

(a) Maxwell's Equations

(b) Vector Analysis

(c) Electromagnetic Waves

(d) Curvilinear Coordinates

9. Problems in Higher Dimensions

Having studied one dimensional oscillations, we will now be prepared to move on to higher dimensional applications. These will involve heat flow and vibrations in different geometries primarily using the method of separation of variables. We will apply these methods to the solution of the wave and heat equations in higher dimensions.

Another major equation of interest that you will encounter in upper level physics is the Schrödinger equation. We will introduce this equation and explore solution techniques obtaining the relevant special functions involved in describing the wavefunction for a hydrogenic electron.

(a) Vibrations of a Rectangular Membrane

(b) Vibrations of a Kettle Drum

(c) Laplace's Equation in 3D

(d) Heat Equation in 3D

(e) Spherical Harmonics

(f) The Hydrogen Atom

0.4 *Tips for Students*

GENERALLY, SOME TOPICS in the course might seem difficult the first time through, especially not having had the upper level physics at the time the topics are introduced. However, like all topics in physics, you will understand many of the topics at deeper levels as you progress through your studies. It will become clear that the more adept one becomes in the mathematical background, the better your understanding of the physics.

You should read through this set of notes and then listen to the lectures. As you read the notes, be prepared to fill in the gaps in derivations and calculations. This is not a spectator sport, but a participatory adventure. Discuss the difficult points with others and your instructor. Work on problems as soon as possible. These are not problems that you can do the night before they are due. This is true of all physics classes. Feel free to go back and reread your old calculus and physics texts.

0.5 *Acknowledgments*

Most, if not all, of the ideas and examples are not my own. These notes are a compendium of topics and examples that I have used in teaching not only mathematical physics, but also in teaching numerous courses in physics and applied mathematics. Some of the notions even extend back to when I first learned them in courses I had taken. Some references to specific topics are included within the book, while other useful references [1], [2], [3], [4], [5] are provided in the bibliography for further study.

I would also like to express my gratitude to the many students who have found typos, or suggested sections needing more clarity. This not only applies to this set of notes, but also to my other two sets of notes, *An Introduction to Fourier and Complex Analysis with Application to the Spectral Analysis of Signals* and *A Second Course in Ordinary Differential Equations: Dynamical Systems and Boundary Value Problems* with which this text has some significant overlap.

[1] Mary L. Boas. *Mathematical Methods in the Physical Sciences*. John Wiley & Sons, Inc, third edition, 2006

[2] George Arfken. *Mathematical Methods for Physicists*. Academic Press, second edition, 1970

[3] Sadri Hassani. *Foundations of Mathematical Physics*. Allyn and Bacon, 1991

[4] Abdul J. Jerri. *Integral and Discrete Transforms with Applications and Error Analysis*. Marcal Dekker, Inc, 1992

[5] Susan M. Lea. *Mathematics for Physicists*. Brooks/Cole, 2004

1
Introduction

"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say." Bertrand Russell (1872-1970)

Before we begin our study of mathematical physics, perhaps we should review some things from your past classes. You definitely need to know something before taking this class. It is assumed that you have taken Calculus and are comfortable with differentiation and integration. You should also have taken some introductory physics class, preferably the calculus based course. Of course, you are not expected to know every detail from these courses. However, there are some topics and methods that will come up and it would be useful to have a handy reference to what it is you should know, especially when it comes to exams.

Most importantly, you should still have your introductory physics and calculus texts to which you can refer throughout the course. Looking back on that old material, you will find that it appears easier than when you first encountered the material. That is the nature of learning mathematics and physics. Your understanding is continually evolving as you explore topics more in depth. It does not always sink in the first time you see it.

In this chapter we will give a quick review of these topics. We will also mention a few new things that might be interesting. This review is meant to make sure that everyone is at the same level.

1.1 What Do I Need To Know From Calculus?

1.1.1 Introduction

There are two main topics in calculus: derivatives and integrals . You learned that derivatives are useful in providing rates of change in either time or space. Integrals provide areas under curves, but also are useful in providing other types of sums over continuous bodies, such as lengths, areas, volumes, moments of inertia, or flux integrals. In physics, one can look at graphs of position versus time and the slope (derivative) of such a function gives the velocity. By plotting velocity versus time you can either look at the derivative to obtain acceleration, or you could look at the area under the curve and get the displacement:

$$x = \int_{t_0}^{t} v\, dt. \tag{1.1}$$

Of course, you need to know how to differentiate and integrate given functions. Even before getting into differentiation and integration, you need to have a bag of functions useful in physics. Common functions are the polynomial and rational functions. You should be fairly familiar with these. Polynomial functions take the general form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{1.2}$$

where $a_n \neq 0$. This is the form of a polynomial of degree n. Rational functions, $f(x) = \frac{g(x)}{h(x)}$, consist of ratios of polynomials. Their graphs can exhibit vertical and horizontal asymptotes.

Next are the exponential and logarithmic functions. The most common are the natural exponential and the natural logarithm. The natural exponential is given by $f(x) = e^x$, where $e \approx 2.718281828\ldots$. The natural logarithm is the inverse to the exponential, denoted by $\ln x$. (One needs to be careful, because some mathematics and physics books use log to mean natural exponential, whereas many of us were first trained to use it to mean the common logarithm, which is the 'log base 10'. Here we will use $\ln x$ for the natural logarithm.)

The properties of the exponential function follow from the basic properties for exponents. Namely, we have:

Exponential properties.

$$\boxed{\begin{aligned} e^0 &= 1, && (1.3)\\ e^{-a} &= \frac{1}{e^a} && (1.4)\\ e^a e^b &= e^{a+b}, && (1.5)\\ (e^a)^b &= e^{ab}. && (1.6) \end{aligned}}$$

The relation between the natural logarithm and natural exponential is given by

$$y = e^x \Leftrightarrow x = \ln y. \tag{1.7}$$

Some common logarithmic properties are

Logarithmic properties.

$$
\begin{aligned}
\ln 1 &= 0, && (1.8)\\
\ln \frac{1}{a} &= -\ln a, && (1.9)\\
\ln(ab) &= \ln a + \ln b, && (1.10)\\
\ln \frac{a}{b} &= \ln a - \ln b, && (1.11)\\
\ln \frac{1}{b} &= -\ln b. && (1.12)
\end{aligned}
$$

We will see further applications of these relations as we progress through the course.

1.1.2 *Trigonometric Functions*

Another set of useful functions are the trigonometric functions. These functions have probably plagued you since high school. They have their origins as far back as the building of the pyramids. Typical applications in your introductory math classes probably have included finding the heights of trees, flag poles, or buildings. It was recognized a long time ago that similar right triangles have fixed ratios of any pair of sides of the two similar triangles. These ratios only change when the non-right angles change.

Thus, the ratio of two sides of a right triangle only depends upon the angle. Since there are six possible ratios (think about it!), then there are six possible functions. These are designated as sine, cosine, tangent and their reciprocals (cosecant, secant and cotangent). In your introductory physics class, you really only needed the first three. You also learned that they are represented as the ratios of the opposite to hypotenuse, adjacent to hypotenuse, etc. Hopefully, you have this down by now.

You should also know the exact values of these basic trigonometric functions for the special angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$, and their corresponding angles in the second, third and fourth quadrants. This becomes internalized after much use, but we provide these values in Table 1.1 just in case you need a reminder.

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{2}$	0	1	undefined

Table 1.1: Table of Trigonometric Values

The problems students often have using trigonometric functions in later courses stem from using, or recalling, identities. We will have many an occasion to do so in this class as well. What is an identity? It is a relation that holds true all of the time. For example, the most common identity for trigonometric functions is the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1. \tag{1.13}$$

This hold true for every angle θ! An even simpler identity is

$$\tan\theta = \frac{\sin\theta}{\cos\theta}. \tag{1.14}$$

Other simple identities can be derived from the Pythagorean identity. Dividing the identity by $\cos^2\theta$, or $\sin^2\theta$, yields

$$\begin{aligned} \tan^2\theta + 1 &= \sec^2\theta, & (1.15)\\ 1 + \cot^2\theta &= \csc^2\theta. & (1.16) \end{aligned}$$

Several other useful identities stem from the use of the sine and cosine of the sum and difference of two angles. Namely, we have that

Sum and difference identities.

$$\boxed{\begin{aligned} \sin(A \pm B) &= \sin A\cos B \pm \sin B\cos A, & (1.17)\\ \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B. & (1.18) \end{aligned}}$$

Note that the upper (lower) signs are taken together.

The double angle formulae are found by setting $A = B$:

Double angle formulae.

$$\boxed{\begin{aligned} \sin(2A) &= 2\sin A\cos B, & (1.19)\\ \cos(2A) &= \cos^2 A - \sin^2 A. & (1.20) \end{aligned}}$$

Using Equation (1.13), we can rewrite (1.20) as

$$\begin{aligned} \cos(2A) &= 2\cos^2 A - 1, & (1.21)\\ &= 1 - 2\sin^2 A. & (1.22) \end{aligned}$$

These, in turn, lead to the half angle formulae. Solving for $\cos^2 A$ and $\sin^2 A$, we find that

Half angle formulae.

$$\boxed{\begin{aligned} \sin^2 A &= \frac{1-\cos 2A}{2}, & (1.23)\\ \cos^2 A &= \frac{1+\cos 2A}{2}. & (1.24) \end{aligned}}$$

Finally, another useful set of identities are the product identities.

Product Identities

For example, if we add the identities for $\sin(A+B)$ and $\sin(A-B)$, the second terms cancel and we have

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B.$$

Thus, we have that

$$\boxed{\sin A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B)). \qquad (1.25)}$$

Similarly, we have

$$\boxed{\cos A\cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B)). \qquad (1.26)}$$

and

$$\boxed{\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B)). \qquad (1.27)}$$

Know the above boxed identities!

These boxed equations are the most common trigonometric identities. They appear often and should just roll off of your tongue.

We will also need to understand the behaviors of trigonometric functions. In particular, we know that the sine and cosine functions are periodic. They are not the only periodic functions, as we shall see. [Just visualize the teeth on a carpenter's saw.] However, they are the most common periodic functions.

A periodic function $f(x)$ satisfies the relation

$$f(x+p) = f(x), \quad \text{for all } x$$

for some constant p. If p is the smallest such number, then p is called the period. Both the sine and cosine functions have period 2π. This means that the graph repeats its form every 2π units. Similarly, $\sin bx$ and $\cos bx$ have the common period $p = \frac{2\pi}{b}$. We will make use of this fact in later chapters.

Related to these are the inverse trigonometric functions. For example, $f(x) = \sin^{-1} x$, or $f(x) = \arcsin x$. Inverse functions give back angles, so you should think

In Feynman's *Surely You're Joking Mr. Feynman!*, Richard Feynman (1918-1988) talks about his invention of his own notation for both trigonometric and inverse trigonometric functions as the standard notation did not make sense to him.

$$\theta = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin\theta. \qquad (1.28)$$

Also, you should recall that $y = \sin^{-1} x = \arcsin x$ is only a function if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Similar relations exist for $y = \cos^{-1} x = \arccos x$ and $\tan^{-1} x = \arctan x$.

Once you think about these functions as providing angles, then you can make sense out of more complicated looking expressions, like $\tan(\sin^{-1} x)$. Such expressions often pop up in evaluations of integrals. We can untangle this in order to produce a simpler form by referring to expression (1.28). $\theta = \sin^{-1} x$ is simple an angle whose sine is x. Knowing the sine is the opposite side of a right triangle divided by its hypotenuse, then one just draws a triangle in this proportion. Namely, the side opposite the angle has length x and the hypotenuse has length 1. Using the Pythagorean Theorem, the missing side (adjacent to the angle) is simply $\sqrt{1-x^2}$. Having obtained the lengths for all three sides, we can now produce the tangent of the angle as

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}.$$

1.1.3 *Hyperbolic Functions*

So, ARE THERE ANY other functions that are useful in physics? Actually, there are many more. However, you have probably not see many of them to date. We will see by the end of the semester that there are many important functions that arise as solutions of some fairly generic, but important, physics problems. In your calculus classes you have also seen that some relations are represented in parametric form. However, there is at least one other set of elementary functions, which you should already know about. These are the hyperbolic functions. Such functions are useful in representing hanging cables, unbounded orbits, and special traveling waves called solitons. They also play a role in special and general relativity.

Solitons are special solutions to some generic nonlinear wave equations. They typically experience elastic collisions and play special roles in a variety of fields in physics, such as hydrodynamics and optics. A simple soliton solution is of the form $u(x,t) = 2\eta^2 \operatorname{sech}^2 \eta(x - 4\eta^2 t)$.

Hyperbolic functions are actually related to the trigonometric functions, as we shall see after a little bit of complex function theory. For now, we just want to recall a few definitions and an identity. Just as all of the trigonometric functions can be built from the sine and the cosine, the hyperbolic functions can be defined in terms of the hyperbolic sine and hyperbolic cosine:

Hyperbolic functions; We will see later the connection between the hyperbolic and trigonometric functions.

$$\sinh x = \frac{e^x - e^{-x}}{2}, \tag{1.29}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{1.30}$$

There are four other hyperbolic functions. These are defined in terms of the above functions similar to the relations between the trigonometric functions. We have

Hyperbolic identities

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{1.31}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{1.32}$$
$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \tag{1.33}$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \qquad (1.34)$$

There are also a whole set of identities, similar to those for the trigonometric functions. For example, the Pythagorean identity for trigonometric functions, $\sin^2\theta + \cos^2\theta = 1$, is replaced by the identity

$$\cosh^2 x - \sinh^2 x = 1.$$

This is easily shown by simply using the definitions of these functions. This identity is also useful for providing a parametric set of equations describing hyperbolae. Letting $x = a\cosh t$ and $y = b\sinh t$, one has

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cosh^2 t - \sinh^2 t = 1.$$

A list of commonly needed hyperbolic function identities are given by the following:

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1, & (1.35)\\
\tanh^2 x + \operatorname{sech}^2 x &= 1, & (1.36)\\
\cosh(A \pm B) &= \cosh A\cosh B \pm \sinh A\sinh B, & (1.37)\\
\sinh(A \pm B) &= \sinh A\cosh B \pm \sinh B\cosh A, & (1.38)\\
\cosh 2x &= \cosh^2 x + \sinh^2 x, & (1.39)\\
\sinh 2x &= 2\sinh x\cosh x, & (1.40)\\
\cosh^2 x &= \frac{1}{2}\left(1 + \cosh 2x\right), & (1.41)\\
\sinh^2 x &= \frac{1}{2}\left(\cosh 2x - 1\right). & (1.42)
\end{aligned}$$

Note the similarity with the trigonometric identities. Other identities can be derived from these.

There also exist inverse hyperbolic functions and these can be written in terms of logarithms. As with the inverse trigonometric functions, we begin with the definition

$$y = \sinh^{-1} x \quad \Leftrightarrow \quad x = \sinh y. \qquad (1.43)$$

The aim is to write y in terms of x without using the inverse function. First, we note that

$$x = \frac{1}{2}\left(e^y - e^{-y}\right). \qquad (1.44)$$

Now, we solve for e^y. This is done by noting that $e^{-y} = \frac{1}{e^y}$ and rewriting the previous equation as

$$0 = (e^y)^2 - 2xe^y - 1. \qquad (1.45)$$

Inverse Hyperbolic Functions:

$$\sinh^{-1} x = \ln\left(x + \sqrt{1 + x^2}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1 + x}{1 - x}$$

This equation is in quadratic form which we can solve as

$$e^y = x + \sqrt{1+x^2}.$$

(There is only one root as we expect the exponential to be positive.) The final step is to solve for y,

$$y = \ln\left(x + \sqrt{1+x^2}\right). \tag{1.46}$$

1.1.4 Derivatives

NOW THAT WE KNOW some elementary functions, we seek their derivatives. We will not spend time exploring the appropriate limits in any rigorous way. We are only interested in the results. We provide these in Table 1.2. We expect that you know the meaning of the derivative and all of the usual rules, such as the product and quotient rules.

Function	Derivative
a	0
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
$\csc ax$	$-a\csc ax\cot ax$
$\sec ax$	$a\sec ax\tan ax$
$\cot ax$	$-a\csc^2 ax$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\,\mathrm{sech}^2 ax$
$\mathrm{csch}\, ax$	$-a\,\mathrm{csch}\, ax\coth ax$
$\mathrm{sech}\, ax$	$-a\,\mathrm{sech}\, ax\tanh ax$
$\coth ax$	$-a\,\mathrm{csch}^2 ax$

Table 1.2: Table of Derivatives (a is a constant.)

Also, you should be familiar with the Chain Rule. Recall that this rule tells us that if we have a composition of functions, such as the elementary functions above, then we can compute the derivative of the composite function. Namely, if $h(x) = f(g(x))$, then

$$\frac{dh}{dx} = \frac{d}{dx}\left(f(g(x))\right) = \left.\frac{df}{dg}\right|_{g(x)}\frac{dg}{dx} = f'(g(x))g'(x). \tag{1.47}$$

For example, let $H(x) = 5\cos\left(\pi\tanh 2x^2\right)$. This is a composition of three functions, $H(x) = f(g(h(x)))$, where $f(x) = 5\cos x$, $g(x) =$

$\pi \tanh x$, and $h(x) = 2x^2$. Then the derivative becomes

$$\begin{aligned} H'(x) &= 5\left(-\sin\left(\pi \tanh 2x^2\right)\right) \frac{d}{dx}\left(\left(\pi \tanh 2x^2\right)\right) \\ &= -5\pi \sin\left(\pi \tanh 2x^2\right) \operatorname{sech}^2 2x^2 \frac{d}{dx}\left(2x^2\right) \\ &= -20\pi x \sin\left(\pi \tanh 2x^2\right) \operatorname{sech}^2 2x^2. \end{aligned} \tag{1.48}$$

1.1.5 *Integrals*

INTEGRATION IS TYPICALLY a bit harder. Imagine being given the last result result in (1.48) and having to figure out what I differentiated in order to get that function. As you may recall from the Fundamental Theorem of Calculus, the integral is the inverse operation to differentiation:

$$\int \frac{df}{dx}\, dx = f(x) + C. \tag{1.49}$$

It is not always easy to evaluate a given integral. In fact some integrals are not even doable! However, you learned in calculus that there are some methods that could yield an answer. While you might be happier using a computer algebra system, such as Maple or WolframAlpha.com, or a fancy calculator, you should know a few basic integrals and know how to use tables for some of the more complicated ones. In fact, it can be exhilarating when you can do a given integral without reference to a computer or a Table of Integrals. However, you should be prepared to do some integrals using what you have been taught in calculus. We will review a few of these methods and some of the standard integrals in this section.

First of all, there are some integrals you are expected to know without doing any work. These integrals appear often and are just an application of the Fundamental Theorem of Calculus to the previous Table 1.2. The basic integrals that students should know off the top of their heads are given in Table 1.3.

These are not the only integrals you should be able to do. However, we can expand the list by recalling a few of the techniques that you learned in calculus. There are just a few: The Method of Substitution, Integration by Parts, Integration Using Partial Fraction Decomposition, and Trigonometric Integrals.

Example 1.1. *When confronted with an integral, you should first ask if a simple substitution would reduce the integral to one you know how to do. So, as an example, consider the following integral*

$$\int \frac{x}{\sqrt{x^2+1}}\, dx.$$

Table 1.3: Table of Integrals

Function	Indefinite Integral
a	ax
x^n	$\frac{x^{n+1}}{n+1}$
e^{ax}	$\frac{1}{a}e^{ax}$
$\frac{1}{x}$	$\ln x$
$\sin ax$	$-\frac{1}{a}\cos ax$
$\cos ax$	$\frac{1}{a}\sin ax$
$\sec^2 ax$	$\frac{1}{a}\tan ax$
$\sinh ax$	$\frac{1}{a}\cosh ax$
$\cosh ax$	$\frac{1}{a}\sinh ax$
$\operatorname{sech}^2 ax$	$\frac{1}{a}\tanh ax$
$\sec x$	$\ln \lvert\sec x + \tan x\rvert$
$\frac{1}{a+bx}$	$\frac{1}{b}\ln(a+bx)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1} ax$
$\frac{1}{\sqrt{a^2-x^2}}$	$\frac{1}{a}\sin^{-1} ax$
$\frac{1}{\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1} ax$

The ugly part of this integral is the x^2+1 under the square root. So, we let $u = x^2+1$. Noting that when $u = f(x)$, we have $du = f'(x)\,dx$. For our example, $du = 2x\,dx$. Looking at the integral, part of the integrand can be written as $x\,dx = \frac{1}{2}u\,du$. Then, the integral becomes

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}}.$$

The substitution has converted our integral into an integral over u. Also, this integral is doable! It is one of the integrals we should know. Namely, we can write it as

$$\frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du.$$

This is now easily finished after integrating and using our substitution variable to give

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} + C = \sqrt{x^2+1} + C.$$

Note that we have added the required integration constant and that the derivative of the result easily gives the original integrand (after employing the Chain Rule).

Often we are faced with definite integrals, in which we integrate between two limits. There are several ways to use these limits. However, students often forget that a change of variables generally means that the limits have to change.

Example 1.2. *Consider the above example with limits added.*

$$\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx.$$

We proceed as before. We let $u = x^2 + 1$. As x goes from 0 to 2, u takes values from 1 to 5. So, this substitution gives

$$\int_0^2 \frac{x}{\sqrt{x^2+1}}\, dx = \frac{1}{2}\int_1^5 \frac{du}{\sqrt{u}} = \sqrt{u}\Big|_1^5 = \sqrt{5} - 1.$$

When the Method of Substitution fails, there are other methods you can try. One of the most used is the Method of Integration by Parts. Recall the Integration by Parts Formula:

Integration by Parts Formula.

$$\boxed{\int u\, dv = uv - \int v\, du. \qquad (1.50)}$$

The idea is that you are given the integral on the left and you can relate it to an integral on the right. Hopefully, the new integral is one you can do, or at least it is an easier integral than the one you are trying to evaluate.

However, you are not usually given the functions u and v. You have to determine them. The integral form that you really have is a function of another variable, say x. Another form of the formula can be given as

$$\boxed{\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx. \qquad (1.51)}$$

This form is a bit more complicated in appearance, though it is clearer what is happening. The derivative has been moved from one function to the other. Recall that this formula was derived by integrating the product rule for differentiation.

Note: Often in physics one needs to move a derivative between functions inside an integrand. The key - use integration by parts to move the derivative from one function to the other under an integral.

The two formulae are related by using the differential relations

$$\begin{aligned} u = f(x) &\quad \to \quad du = f'(x)\, dx, \\ v = g(x) &\quad \to \quad dv = g'(x)\, dx. \end{aligned} \tag{1.52}$$

This also gives a method for applying the Integration by Parts Formula.

Example 1.3. *Consider the integral $\int x \sin 2x\, dx$. We choose $u = x$ and $dv = \sin 2x\, dx$. This gives the correct left side of the Integration by Parts Formula. We next determine v and du:*

$$du = \frac{du}{dx} dx = dx,$$

$$v = \int dv = \int \sin 2x\, dx = -\frac{1}{2}\cos 2x.$$

We note that one usually does not need the integration constant. Inserting these expressions into the Integration by Parts Formula, we have

$$\int x \sin 2x\, dx = -\frac{1}{2} x \cos 2x + \frac{1}{2}\int \cos 2x\, dx.$$

We see that the new integral is easier to do than the original integral. Had we picked $u = \sin 2x$ *and* $dv = x\,dx$*, then the formula still works, but the resulting integral is not easier.*

For completeness, we finish the integration. The result is

$$\int x \sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

As always, you can check your answer by differentiating the result, a step students often forget to do. Namely,

$$\begin{aligned}
\frac{d}{dx}\left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C\right) &= -\frac{1}{2}\cos 2x + x\sin 2x + \frac{1}{4}(2\cos 2x) \\
&= x\sin 2x. \qquad (1.53)
\end{aligned}$$

So, we do get back the integrand in the original integral.

We can also perform integration by parts on definite integrals. The general formula is written as

Integration by Parts for Definite Integrals.

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx. \qquad (1.54)}$$

Example 1.4. *Consider the integral*

$$\int_0^\pi x^2\cos x\,dx.$$

This will require two integrations by parts. First, we let $u = x^2$ *and* $dv = \cos x$*. Then,*

$$du = 2x\,dx. \quad v = \sin x.$$

Inserting into the Integration by Parts Formula, we have

$$\begin{aligned}
\int_0^\pi x^2\cos x\,dx &= x^2\sin x\Big|_0^\pi - 2\int_0^\pi x\sin x\,dx \\
&= -2\int_0^\pi x\sin x\,dx. \qquad (1.55)
\end{aligned}$$

We note that the resulting integral is easier that the given integral, but we still cannot do the integral off the top of our head (unless we look at Example 3!). So, we need to integrate by parts again. (Note: In your calculus class you may recall that there is a tabular method for carrying out multiple applications of the formula. We will show this method in the next example.)

We apply integration by parts by letting $U = x$ *and* $dV = \sin x\,dx$*. This gives* $dU = dx$ *and* $V = -\cos x$*. Therefore, we have*

$$\begin{aligned}
\int_0^\pi x\sin x\,dx &= -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx \\
&= \pi + \sin x\Big|_0^\pi \\
&= \pi. \qquad (1.56)
\end{aligned}$$

The final result is

$$\int_0^{\pi} x^2 \cos x \, dx = -2\pi.$$

There are other ways to compute integrals of this type. First of all, there is the Tabular Method to perform integration by parts. A second method is to use differentiation of parameters under the integral. We will demonstrate this using examples.

Using the Tabular Method.

Example 1.5. *Compute the integral $\int_0^{\pi} x^2 \cos x \, dx$ using the Tabular Method.*

First we identify the two functions under the integral, x^2 and $\cos x$. We then write the two functions and list the derivatives and integrals of each, respectively. This is shown in Table 1.5. Note that we stopped when we reached 0 in the left column.

Table 1.4: Tabular Method

D		I
x^2		$\cos x$
	$\searrow +$	
$2x$		$\sin x$
	$\searrow -$	
2		$-\cos x$
	$\searrow +$	
0		$-\sin x$

Next, one draws diagonal arrows, as indicated, with alternating signs attached, starting with $+$. The indefinite integral is then obtained by summing the products of the functions at the ends of the arrows along with the signs on each arrow:

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

To find the definite integral, one evaluates the antiderivative at the given limits.

$$\begin{aligned} \int_0^{\pi} x^2 \cos x \, dx &= \left[x^2 \sin x + 2x \cos x - 2 \sin x \right]_0^{\pi} \\ &= (\pi^2 \sin \pi + 2\pi \cos \pi - 2 \sin \pi) - 0 \\ &= -2\pi. \end{aligned} \tag{1.57}$$

Actually, the Tabular Method works even if a 0 does not appear on the left side. One can go as far as possible, and if a 0 does not appear, then one needs only integrate, if possible, the product of the functions in the last row, adding the next sign in the alternating sign progression. The next example shows how this works.

Example 1.6. *Use the Tabular Method to compute $\int e^{2x} \sin 3x \, dx$.*

As before, we first set up the table.

D		I
$\sin 3x$		e^{2x}
	$\searrow +$	
$3\cos 3x$		$\frac{1}{2}e^{2x}$
	$\searrow -$	
$-9\sin 3x$		$\frac{1}{4}e^{2x}$

Table 1.5: Tabular Method - Non-terminating Example.

Putting together the pieces, noting that the derivatives in the left column will never vanish, we have

$$\int e^{2x}\sin 3x\,dx = (\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x)e^{2x} + \int (-9\sin 3x)\left(\frac{1}{4}e^{2x}\right)\,dx.$$

The integral on the right is a multiple of the one on the left, so we can combine them,

$$\frac{13}{4}\int e^{2x}\sin 3x\,dx = (\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x)e^{2x},$$

or

$$\int e^{2x}\sin 3x\,dx = (\frac{2}{13}\sin 3x - \frac{3}{13}\cos 3x)e^{2x}.$$

Another method that one can use to evaluate this integral is to differentiate under the integral sign. This is mentioned in the Richard Feynman's memoir *Surely You're Joking, Mr. Feynman!*. In the book Feynman recounts using this "trick" to be able to do integrals that his MIT classmates could not do. This is based on a theorem in Advanced Calculus.

Differentiation Under the Integral Sign and Feynman's trick.

Theorem 1.1. *Let the functions $f(x,t)$ and $\frac{\partial f(x,t)}{\partial x}$ be continuous in both t, and x, in the region of the (t,x) plane which includes $a(x) \le t \le b(x)$, $x_0 \le x \le x_1$, where the functions $a(x)$ and $b(x)$ are continuous and have continuous derivatives for $x_0 \le x \le x_1$. Defining*

$$F(x) \equiv \int_{a(x)}^{b(x)} f(x,t)\,dt,$$

then

$$\begin{aligned} \frac{dF(x)}{dx} &= \left(\frac{\partial F}{\partial b}\right)\frac{db}{dx} + \left(\frac{\partial F}{\partial a}\right)\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt \\ &= f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt. \end{aligned} \tag{1.58}$$

for $x_0 \le x \le x_1$. This is a generalized version of the Fundamental Theorem of Calculus.

In the next examples we show how we can use this theorem to bypass integration by parts.

Example 1.7. *Use differentiation under the integral sign to evaluate* $\int xe^x\,dx$. *First, consider the integral*

$$I(x,a) = \int e^{ax}\,dx.$$

Then

$$\frac{\partial I(x,a)}{\partial a} = \int xe^{ax}\,dx.$$

So,

$$\begin{aligned}
\int xe^{ax}\,dx &= \frac{\partial I(x,a)}{\partial a} \\
&= \frac{\partial}{\partial a}\left(\int e^{ax}\,dx\right) \\
&= \frac{\partial}{\partial a}\left(\frac{e^{ax}}{a}\right) \\
&= \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \qquad (1.59)
\end{aligned}$$

Evaluating this result at $a = 1$, *we have*

$$\int xe^x\,dx = (x-1)e^x.$$

Example 1.8. *We will do the integral* $\int_0^\pi x^2 \cos x\,dx$ *once more. First, consider the integral*

$$\begin{aligned}
I(a) &\equiv \int_0^\pi \cos ax\,dx \\
&= \left.\frac{\sin ax}{a}\right|_0^\pi \\
&= \frac{\sin a\pi}{a}. \qquad (1.60)
\end{aligned}$$

Differentiating the integral with respect to a twice gives

$$\frac{d^2 I(a)}{da^2} = -\int_0^\pi x^2 \cos ax\,dx. \qquad (1.61)$$

Evaluation of this result at $a = 1$ *leads to the desired result. Thus,*

$$\begin{aligned}
\int_0^\pi x^2 \cos x\,dx &= -\left.\frac{d^2 I(a)}{da^2}\right|_{a=1} \\
&= -\frac{d^2}{da^2}\left(\frac{\sin a\pi}{a}\right)\Big|_{a=1} \\
&= -\frac{d}{da}\left(\frac{a\pi\cos a\pi - \sin a\pi}{a^2}\right)\Big|_{a=1} \\
&= -\left(\frac{a^2\pi^2 \sin a\pi + 2a\pi\cos a\pi - 2\sin a\pi}{a^3}\right)\Big|_{a=1} \\
&= -2\pi. \qquad (1.62)
\end{aligned}$$

Other types of integrals that you will see often are trigonometric integrals. In particular, integrals involving powers of sines and cosines. For odd powers, a simple substitution will turn the integrals into simple powers.

Integration of odd powers of sine and cosine.

Example 1.9. *For example, consider*

$$\int \cos^3 x\, dx.$$

This can be rewritten as

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx.$$

Let $u = \sin x$. *Then* $du = \cos x\, dx$. *Since* $\cos^2 x = 1 - \sin^2 x$, *we have*

$$\begin{aligned} \int \cos^3 x\, dx &= \int \cos^2 x \cos x\, dx \\ &= \int (1 - u^2)\, du \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C. \end{aligned} \tag{1.63}$$

A quick check confirms the answer:

$$\frac{d}{dx}\left(\sin x - \frac{1}{3}\sin^3 x + C\right) = \cos x - \sin^2 x \cos x = \cos x(1 - \sin^2 x) = \cos^3 x.$$

Even powers of sines and cosines are a little more complicated, but doable. In these cases we need the half angle formulae:

Integration of even powers of sine and cosine.

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}, \tag{1.64}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}. \tag{1.65}$$

Example 1.10. *As an example, we will compute*

$$\int_0^{2\pi} \cos^2 x\, dx.$$

Substituting the half angle formula for $\cos^2 x$, *we have*

$$\begin{aligned} \int_0^{2\pi} \cos^2 x\, dx &= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2x)\, dx \\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)_0^{2\pi} \\ &= \pi. \end{aligned} \tag{1.66}$$

We note that this result appears often in physics. When looking at root mean square averages of sinusoidal waves, one needs the average of the square of sines and cosines. Recall that the average of a function on interval $[a, b]$ is given as

$$f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.67}$$

So, the average of $\cos^2 x$ over one period is

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\,dx = \frac{1}{2}. \tag{1.68}$$

The root mean square is then $\frac{1}{\sqrt{2}}$.

1.1.6 Geometric Series

Geometric series are fairly common and will be used throughout the book. You should learn to recognize them and work with them.

Infinite series occur often in mathematics and physics. Two series which occur often are the geometric series and the binomial series. we will discuss these in the next two sections.

A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots. \tag{1.69}$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is r.

Example 1.11. *For example,*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

is an example of a geometric series. We can write this using summation notation,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} 1\left(\frac{1}{2}\right)^n.$$

Thus, $a = 1$ is the first term and $r = \frac{1}{2}$ is the common ratio of successive terms. Next, we seek the sum of this infinite series, if it exists.

The sum of a geometric series, when it converges, can easily be determined. We consider the nth partial sum:

$$s_n = a + ar + \ldots + ar^{n-2} + ar^{n-1}. \tag{1.70}$$

Now, multiply this equation by r.

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \tag{1.71}$$

Subtracting these two equations, while noting the many cancelations, we have

$$\begin{aligned}(1-r)s_n &= (a + ar + \ldots + ar^{n-2} + ar^{n-1}) \\ &\quad -(ar + ar^2 + \ldots + ar^{n-1} + ar^n) \\ &= a - ar^n \\ &= a(1-r^n).\end{aligned} \tag{1.72}$$

Thus, the nth partial sums can be written in the compact form

$$s_n = \frac{a(1-r^n)}{1-r}. \tag{1.73}$$

Recall that the sum, if it exists, is given by $S = \lim_{n\to\infty} s_n$. Letting n get large in the partial sum (1.73), we need only evaluate $\lim_{n\to\infty} r^n$. From our special limits we know that this limit is zero for $|r| < 1$. Thus, we have

Geometric Series

The sum of the geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad |r| < 1. \tag{1.74}$$

The reader should verify that the geometric series diverges for all other values of r. Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example 1.12. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is

$$S = \frac{1}{1-\frac{1}{2}} = 2.$$

This agrees with the plot of the partial sums in Figure A.6.

Example 1.13. $\sum_{k=2}^{\infty} \frac{4}{3^k}$

In this example we note that the first term occurs for $k = 2$. So, $a = \frac{4}{9}$. Also, $r = \frac{1}{3}$. So,

$$S = \frac{\frac{4}{9}}{1-\frac{1}{3}} = \frac{2}{3}.$$

Example 1.14. $\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right)$

Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed [1]. Thus, we have

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right) = \sum_{n=1}^{\infty} \frac{3}{2^n} - \sum_{n=1}^{\infty} \frac{2}{5^n}.$$

Now we can add both geometric series:

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n}\right) = \frac{\frac{3}{2}}{1-\frac{1}{2}} - \frac{\frac{2}{5}}{1-\frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

[1] A rearrangement of terms in an infinite series is allowed when the series is absolutely convergent.

Geometric series are important because they are easily recognized and summed. Other series, which can be summed, are special cases of Taylor series, as we will see later. Another type of series that can be summed is a *telescoping series* as seen in the next example.

Example 1.15. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *The first few terms of this series are*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots.$$

It does not appear that we can sum this infinite series. However, if we used the partial fraction expansion

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

then we find the partial sums can be written as

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{k} - \frac{1}{k+1} \right). \end{aligned} \tag{1.75}$$

We see that there are many cancelations of neighboring terms, leading to the series collapsing (like a telescope) to something manageable:

$$s_k = 1 - \frac{1}{k+1}.$$

Taking the limit as $k \to \infty$, we find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Example 1.16. *The Partition Function*

A common occurrence of geometric series is a series of exponentials. An example of this occurs in statistical mechanics. Statistical mechanics is the branch of physics which explores the thermodynamic behavior of systems containing a large number of particles. An important tool is the partition function, Z. This function is the sum of terms, $e^{-\epsilon_n/kT}$, over all possible quantum states of the system. Here ϵ_n is the energy of the nth state, T the temperature, and k is Boltzmann's constant. Given Z, one can compute macroscopic quantities, such as the average energy,

$$< E >= -\frac{\partial \ln Z}{\partial \beta},$$

where $\beta = 1/kT$.

For the case of the quantum harmonic oscillator, the energy states are given by $\epsilon_n = \left(n + \frac{1}{2}\right) \hbar\omega$. The partition function is then

$$Z = \sum_{n=0}^{\infty} e^{-\beta \epsilon_n}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega} \\ &= e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega}. \end{aligned} \tag{1.76}$$

The terms in the last sum are really powers of an exponential,

$$e^{-\beta n\hbar\omega} = \left(e^{-\beta\hbar\omega}\right)^n.$$

So,

$$Z = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n.$$

This is a geometric series, which can be summed as long as $e^{-\beta\hbar\omega} < 1$. *Thus,*

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}.$$

Multiplying the numerator and denominator by $e^{\beta\hbar\omega/2}$, *we have*

$$Z = \frac{1}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} = (2\sinh\beta\hbar\omega/2)^{-1}.$$

1.1.7 The Binomial Expansion

The binomial expansion is a special series expansion used to approximate expressions of the form $(a+b)^p$ for $b \ll a$, or $(1+x)^p$ for $|x| \ll 1$.

ONE SERIES EXPANSION which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$ in powers of a and b. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p. While the binomial expansion can be obtained using Taylor series, we will provide a more interesting derivation here to show that

$$(a+b)^p = \sum^{\infty} C_p^r a^{n-r} b^r, \tag{1.77}$$

where the C_p^r are called the *binomial coefficients*.

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned} (a+b)^0 &= 1 \\ (a+b)^1 &= a+b \\ (a+b)^2 &= a^2+2ab+b^2 \end{aligned}$$

$$\begin{aligned}(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\&\cdots\end{aligned} \tag{1.78}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $(k+1)$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not yet know the numerical coefficient in the expansion.

Let's list the coefficients for the above expansions.

$$\begin{array}{lccccccccc} n=0: & & & & & 1 & & & & \\ n=1: & & & & 1 & & 1 & & & \\ n=2: & & & 1 & & 2 & & 1 & & \\ n=3: & & 1 & & 3 & & 3 & & 1 & \\ n=4: & 1 & & 4 & & 6 & & 4 & & 1 \end{array} \tag{1.79}$$

This pattern is the famous Pascal's triangle.[2] There are many interesting features of this triangle. But we will first ask how each row can be generated.

[2] Pascal's triangle is named after Blaise Pascal (1623-1662). While such configurations of number were known earlier in history, Pascal published them and applied them to probability theory.

Pascal's triangle has many unusual properties and a variety of uses:

- Horizontal rows add to powers of 2.
- The horizontal rows are powers of 11 (1, 11, 121, 1331, etc.).
- Adding any two successive numbers in the diagonal 1-3-6-10-15-21-28... results in a perfect square.
- When the first number to the right of the 1 in any row is a prime number, all numbers in that row are divisible by that prime number.
- Sums along certain diagonals leads to the Fibonacci sequence.

We see that each row begins and ends with a one. The second term and next to last term have a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have for rows $n=2$ and $n=3$ that $1+2=3$ and $2+1=3$:

$$\begin{array}{lccccccccc} n=2: & & 1 & & & 2 & & & 1 & \\ & & & \searrow & \swarrow & & \searrow & \swarrow & & \\ n=3: & 1 & & 3 & & & & 3 & & 1 \end{array} \tag{1.80}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{lccccccccccccc} n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\ n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \tag{1.81}$$

So, we use the numbers in row $n=4$ to generate entries in row $n=5$: $1+4=5$, $4+6=10$. We then use row $n=5$ to get row $n=6$, etc.

Of course, it would take a while to compute each row up to the desired n. Fortunately, there is a simple expression for computing a specific coefficient. Consider the kth term in the expansion of $(a+b)^n$.

Let $r = k - 1$. Then this term is of the form $C_r^n a^{n-r} b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the binomial coefficients have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \left(\begin{array}{c} n \\ r \end{array} \right).$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count the number of ways that we can arrange r products of b with $n - r$ products of a. There are n slots to place the b's. For example, the $r = 2$ case for $n = 4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*. Thus, it is natural to use this notation.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) a^{n-r} b^r. \tag{1.82}$$

Now consider the geometric series $1 + x + x^2 + \ldots$. We have seen that such a series converges for $|x| < 1$, giving

$$1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$.

This is again a binomial to a power, but the power is not an integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (A.35).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$(1+x)^p = \sum_{r=0}^{\infty} \left(\begin{array}{c} p \\ r \end{array} \right) x^r. \tag{1.83}$$

However, we quickly run into problems with this form. Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\left(\begin{array}{c} -1 \\ 1 \end{array} \right) = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned}\begin{pmatrix} p \\ r \end{pmatrix} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \tag{1.84}$$

With this in mind we now state the theorem:

General Binomial Expansion

The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (A.35). For p real, we have the following binomial series:

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r, \quad |x| < 1. \tag{1.85}$$

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3). \tag{1.86}$$

Example 1.17. *Approximate* $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ *for* $v \ll c$. *This can be rewritten as*

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}.$$

The factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is important in special relativity. Namely, this is the factor relating differences in time and length measurements by observers moving relative inertial frames. For celestial speeds, this is an appropriate approximation.

Using the binomial expansion for $p = -1/2$, *we have*

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) = 1 + \frac{v^2}{2c^2}.$$

Example 1.18. *Small differences in large numbers.*

As an example, we could compute $f(R,h) = \sqrt{R^2+h^2} - R$ *for* $R = 6378.164$ *km and* $h = 1.0$ *m. Inserting these values into a scientific calculator, one finds that*

$$f(6378164, 1) = \sqrt{6378164^2 + 1} - 6378164 = 1 \times 10^{-7}\ m.$$

In some calculators one might obtain 0, in other calculators, or computer algebra systems like Maple, one might obtain other answers. What answer do you get and how accurate is your answer?

The problem with this computation is that $R \gg h$. *Therefore, the computation of* $f(R,h)$ *depends on how many digits the computing device can handle.*

The best way to get an answer is to use the binomial approximation. Writing $x = \frac{h}{R}$, *we have*

$$\begin{aligned} f(R,h) &= \sqrt{R^2+h^2} - R \\ &= R\sqrt{1+x^2} - R \\ &\simeq R\left[1+\frac{1}{2}x^2\right] - R \\ &= \frac{1}{2}Rx^2 \\ &= \frac{1}{2}\frac{h}{R^2} = 7.83926 \times 10^{-8}\ m. \end{aligned} \tag{1.87}$$

Of course, you should verify how many digits should be kept in reporting the result.

In the next examples, we show how computations taking a more general form can be handled. Such general computations appear in proofs involving general expansions without specific numerical values given.

Example 1.19. *Obtain an approximation to* $(a+b)^p$ *when* a *is much larger than* b, *denoted by* $a \gg b$.

If we neglect b *then* $(a+b)^p \simeq a^p$. *How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion. Namely, what is the power of* b/a *of the first neglected term in this expansion?*

In order to do this we first divide out a *as*

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p.$$

Now we have a small parameter, $\frac{b}{a}$. *According to what we have seen earlier, we can use the binomial expansion to write*

$$\left(1+\frac{b}{a}\right)^n = \sum_{r=0}^{\infty}\left(\begin{array}{c} p \\ r \end{array}\right)\left(\frac{b}{a}\right)^r. \tag{1.88}$$

Thus, we have a sum of terms involving powers of $\frac{b}{a}$. *Since* $a \gg b$, *most of these terms can be neglected. So, we can write*

$$\left(1+\frac{b}{a}\right)^p = 1 + p\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

Here we used $O()$, *big-Oh notation, to indicate the size of the first neglected term. (This notation is formally defined in another section.)*

Summarizing, this then gives

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p$$

$$\begin{aligned} &= a^p \left(1 + p\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right)\right) \\ &= a^p + pa^p\frac{b}{a} + a^pO\left(\left(\frac{b}{a}\right)^2\right). \end{aligned} \tag{1.89}$$

Therefore, we can approximate $(a+b)^p \simeq a^p + pba^{p-1}$*, with an error on the order of* b^2a^{p-2}*. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that* $(a+b)^p \simeq a^p$*, but it is not typically good enough in applications because the error in this case is of the order* ba^{p-1}*.*

Example 1.20. *Approximate* $f(x) = (a+x)^p - a^p$ *for* $x \ll a$*.*

In an earlier example we computed $f(R,h) = \sqrt{R^2+h^2} - R$ *for* $R = 6378.164$ *km and* $h = 1.0$ *m. We can make use of the binomial expansion to determine the behavior of similar functions in the form* $f(x) = (a+x)^p - a^p$*. Inserting the binomial expression into* $f(x)$*, we have as* $\frac{x}{a} \to 0$ *that*

$$\begin{aligned} f(x) &= (a+x)^p - a^p \\ &= a^p\left[\left(1+\frac{x}{a}\right)^p - 1\right] \\ &= a^p\left[\frac{px}{a} + O\left(\left(\frac{x}{a}\right)^2\right)\right] \\ &= O\left(\frac{x}{a}\right) \quad \text{as } \frac{x}{a} \to 0. \end{aligned} \tag{1.90}$$

This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as

$$(a+x)^p - a^p = a^{p-1}px + O\left(\left(\frac{x}{a}\right)^2\right) \quad \text{as } \frac{x}{a} \to 0.$$

We could use this approximation to answer the original question by letting $a = R^2$*,* $x = 1$ *and* $p = \frac{1}{2}$*. Then, our approximation would be of order*

$$O\left(\left(\frac{x}{a}\right)^2\right) = O\left(\left(\frac{1}{6378164^2}\right)^2\right) \sim 2.4 \times 10^{-14}.$$

Thus, we have

$$\sqrt{6378164^2+1} - 6378164 \approx a^{p-1}px$$

where

$$a^{p-1}px = (6378164^2)^{-1/2}(0.5)1 = 7.83926 \times 10^{-8}.$$

This is the same result we had obtained before.

So far, this is enough to get started in the course. We will recall other topics as we need them. For example, we will discuss the method of partial fraction decomposition when we discuss terminal velocity in the next chapter and when we cover applications of the Laplace transform later in the book.

1.2 *What I Need From My Intro Physics Class?*

So, WHAT DO WE NEED to know about physics? You should be comfortable with common terms from mechanics and electromagnetism. In some cases, we will review specific topics. However, it would be helpful to review some topics from your introductory and modern physics texts.

As you may recall, your study of physics began with the simplest systems. You first studied motion for point masses. You were then introduced to the concepts of position, displacement, velocity and acceleration. You studied motion first in one dimension and even then can only do problems in which the acceleration is constant, or piecewise constant. You looked at horizontal motion and then vertical motion, in terms of free fall. Finally, you moved into two dimensions and considered projectile motion. Some calculus was introduced and you learned how to represent vector quantities.

You then asked, "What causes a change in the state of motion of a body?" We are lead to a discussion of forces. The types of forces encountered are the weight, the normal force, tension, the force of gravity and then centripetal forces. You might have also seen spring forces, which we will see shortly, lead to oscillatory motion - the underlying theme of this book.

Next, you found out that there are well known conservation principles for energy and momentum. In these cases you were lead to the concepts of work, kinetic energy and potential energy. You found out that even when mechanical energy is not conserved, you could account for the missing energy as the work done by nonconservative forces. Momentum becomes important in collision problems or when looking at impulses.

With these basic ideas under your belt, you proceeded to study more complicated systems. Looking at extended bodies, most notably rigid bodies, led to the study of rotational motion. you found out that there are analogues to all of the previously discussed concepts for point masses. For example, there are the natural analogues of rotational velocity and acceleration. The cause of rotational acceleration is the torque. The analogue to mass is the moment of inertia.

The next level of complication, which sometimes is not covered, are bulk systems. One can study fluids, solids and gases. These can be investigated by looking at things like mass density, pressure, volume and temperature. This leads to the study of thermodynamics in which one studies the transfer of energy between a system and its surroundings. This involves the relationship between the work done on the system, the heat energy added to a systems and its change in internal energy.

Bulk systems can also suffer deformations when a force per area is applied. This can lead to the idea that small deformations can lead to the propagation of energy throughout the system in the form of waves. We will later explore this wave motion in several systems.

The second course in physics is spent on electricity and magnetism, leading to electromagnetic waves. You first learned about charges and charge distributions, electric fields, electric potentials. Then you found out that moving charges produce magnetic fields and are affected by external magnetic fields. Furthermore, changing magnetic fields produce currents. This can all be summarized by Maxwell's equations, which we will recall later in the course. These equations, in turn, predict the existence of electromagnetic waves.

Depending how far you delved into the book, you may have seen excursions into optics and the impact that trying to understand the existence of electromagnetic waves has had on the development of so-called "modern physics". For example, in trying to understand what medium electromagnetic waves might propagate through, Einstein proposed an answer that completely changed the way we understand the nature of space and time. In trying to understand how ionized gases radiate and interact with matter, Einstein and others were lead down a path that has lead to quantum mechanics and further challenges to our understanding of reality.

So, that is the introductory physics course in a nutshell. In fact, that is most of physics. The rest is detail, which you will explore in your other courses as you progress toward a degree in physics.

1.3 *Technology and Tables*

As we progress through the course, you will often have to compute integrals and derivatives by hand. However, many of you know that some of the tedium can be alleviated by using computers, or even looking up what you need in tables. In some cases you might even find applets online that can quickly give you the answers you seek.

However, you also need to be comfortable in doing many computations by hand. This is necessary, especially in your early studies, for several reasons. For example, you should try to evaluate integrals by hand when asked to do them. This reinforces the techniques, as outlined earlier. It exercises your brain in much the same way that you might jog daily to exercise your body. Who knows, keeping your brain active this way might even postpone Alzheimer's. The more comfortable you are with derivations and evaluations, the easier it is to follow future lectures without getting bogged down by the details, wondering how your professor got from step A to step D. You can always use a computer algebra system, or a Table of Integrals, to check on your work.

Problems can arise when depending purely on the output of computers, or other "black boxes". Once you have a firm grasp on the techniques and a feeling as to what answers should look like, then you can feel comfortable with what the computer gives you. Sometimes, Computer Algebra Systems (CAS) like Maple can give you strange looking answers, and sometimes even wrong answers. Also, these programs cannot do every integral, or solve every differential equation, that you ask them to do. Even some of the simplest looking expressions can cause computer algebra systems problems. Other times you might even provide wrong input, leading to erroneous results.

Another source of indefinite integrals, derivatives, series expansions, etc, is a Table of Mathematical Formulae. There are several good books that have been printed. Even some of these have typos in them, so you need to be careful. However, it may be worth the investment to have such a book in your personal library. Go to the library, or the bookstore, and look at some of these tables to see how useful they might be.

There are plenty of online resources as well. For example, there is the Wolfram Integrator at http://integrals.wolfram.com/ as well as the recent http://www.wolframalpha.com/. There is also a wealth of information at the following sites: http://www.sosmath.com/, http://www.math2.org/, http://mathworld.wolfram.com/, and http://functions.wolfram.com/.

1.4 Appendix: Dimensional Analysis

In the first chapter in your introductory physics text you were introduced to dimensional analysis. Dimensional analysis is useful for recalling particular relationships between variables by looking at the units involved, independent of the system of units employed. Though most of the time you have used SI, or MKS, units in most of your physics problems.

There are certain basic units - length, mass and time. By the second course, you found out that you could add charge to the list. We can represent these as [L], [M], [T] and [C]. Other quantities typically have units that can be expressed in terms of the basic units. These are called derived units. So, we have that the units of acceleration are $[L]/[T]^2$ and units of mass density are $[M]/[L]^3$. Slightly more complicated units arise for force. Since $F = ma$, the units of force are

$$[F] = [m][a] = [M]\frac{[L]}{[T]^2}.$$

Similarly, units of magnetic field can be found, though with a little more effort. Recall that $F = qvB\sin\theta$ for a charge q moving with speed v through a magnetic field B at an angle of θ. $\sin\theta$ has no units. So,

$$\begin{aligned}[B] &= \frac{[F]}{[q][v]} \\ &= \frac{\frac{[\mathrm{M}][\mathrm{L}]}{[\mathrm{T}]^2}}{[\mathrm{C}]\frac{[\mathrm{L}]}{[\mathrm{T}]}} \\ &= \frac{[\mathrm{M}]}{[\mathrm{C}][\mathrm{T}]}. \end{aligned} \tag{1.91}$$

Now, assume that you do not know how B depended on F, q and v, but you knew the units of all of the quantities. Can you figure out the relationship between them? We could write

$$[B] = [F]^\alpha [q]^\beta [v]^\gamma$$

and solve for the exponents by inserting the dimensions. Thus, we have

$$[\mathrm{M}][\mathrm{C}]^{-1}[\mathrm{T}]^{-1} = \left([\mathrm{M}][\mathrm{L}][\mathrm{T}]^{-2}\right)^\alpha [\mathrm{C}]^\beta \left([\mathrm{L}][\mathrm{T}]^{-1}\right)^\gamma.$$

Right away we can see that $\alpha = 1$ and $\beta = -1$ by looking at the powers of [M] and [C], respectively. Thus,

$$[\mathrm{M}][\mathrm{C}]^{-1}[\mathrm{T}]^{-1} = [\mathrm{M}][\mathrm{L}][\mathrm{T}]^{-2}[\mathrm{C}]^{-1}\left([\mathrm{L}][\mathrm{T}]^{-1}\right)^\gamma = [\mathrm{M}][\mathrm{C}]^{-1}[\mathrm{L}]^{1+\gamma}[\mathrm{T}]^{-2-\gamma}.$$

We see that picking $\gamma = -1$ balances the exponents and gives the correct relation

$$[B] = [F][q]^{-1}[v]^{-1}.$$

An important theorem at the heart of dimensional analysis is the Buckingham Π Theorem. In essence, this theorem tells us that physically meaningful equations in n variables can be written as an equation involving $n - m$ dimensionless quantities, where m is the number of dimensions used. The importance of this theorem is that one can actually compute useful quantities without even knowing the exact form of the equation!

The Buckingham Π Theorem.

The Buckingham Π Theorem was introduced by Edgar Buckingham (1867-1940) in 1914. Let q_i be n physical variables that are related by

$$f(q_1, q_2, \ldots, q_n) = 0. \tag{1.92}$$

Assuming that m dimensions are involved, we let π_i be $k = n - m$ dimensionless variables. Then the equation (1.92) can be rewritten as a function of these dimensionless variables as

$$F(\pi_1, \pi_2, \ldots, \pi_k) = 0, \tag{1.93}$$

where the π_i's can be written in terms of the physical variables as

$$\pi_i = q_1^{k_1} q_2^{k_2} \cdots q_n^{k_n}, \quad i = 1, \ldots, k. \tag{1.94}$$

Well, this is our first really new concept (apart from some mathematical tricks) and it is probably a mystery as to its importance. It also seems a bit abstract. However, this is the basis for some of the proverbial "back of the envelope calculations" which you might have heard about. So, let's see how it can be used.

Example 1.21. *Using dimensional analysis to obtain the period of a simple pendulum.*

Formula for the period of a pendulum.

Let's consider the period of a simple pendulum; e.g., a point mass hanging on a massless string. The period, T, of the pendulum's swing could depend upon the the string length, ℓ, the mass of the "pendulum bob", m, and gravity in the form of the acceleration due to gravity, g. These are the q_i's in the theorem. We have four physical variables. The only units involved are length, mass and time. So, $m = 3$. This means that there are $k = n - m = 1$ dimensionless variables, call it π. So, there must be an equation of the form

$$F(\pi) = 0$$

in terms of the dimensionless variable

$$\pi = \ell^{k_1} m^{k_2} T^{k_3} g^{k_4}.$$

We just need to find the k_i's. This could be done by inspection, or we could write out the dimensions of each factor and determine how π can be dimensionless. Thus,

$$\begin{aligned} [\pi] &= [\ell]^{k_1}[m]^{k_2}[T]^{k_3}[g]^{k_4} \\ &= [L]^{k_1}[M]^{k_2}[T]^{k_3}\left(\frac{[L]}{[T]^2}\right)^{k_4} \\ &= [L]^{k_1+k_4}[M]^{k_2}[T]^{k_3-2k_4}. \end{aligned} \tag{1.95}$$

π will be dimensionless when

$$\begin{aligned} k_1 + k_4 &= 0, \\ k_2 &= 0, \\ k_3 - 2k_4 &= 0. \end{aligned} \tag{1.96}$$

This is a linear homogeneous system of three equations and four unknowns. We can satisfy these equations by setting $k_1 = -k_4$, $k_2 = 0$, and $k_3 = 2k_4$. Therefore, we have

$$\pi = \ell^{-k_4} T^{2k_4} g^{k_4} = \left(\ell^{-1} T^2 g\right)^{k_4}.$$

k_4 is arbitrary, so we can pick the simplest value, $k_4 = 1$. Then,

$$F\left(\frac{T^2 g}{\ell}\right) = 0.$$

Assuming that this equation has one zero, z, which has to be verified by other means, we have that

$$\frac{gT^2}{\ell} = z = const.$$

Thus, we have determined that the period is independent of the mass and proportional to the square root of the length. The constant can be determined by experiment as $z = 4\pi^2$. Thus,

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

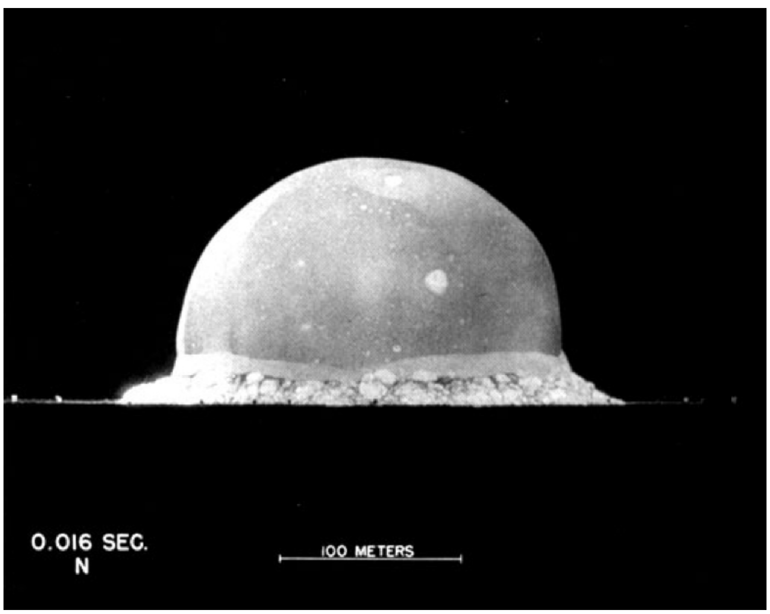


Figure 1.1: A photograph of the first atomic bomb test. This image was found at http://www.atomicarchive.com.

Example 1.22. *Estimating the energy of an atomic bomb.*

A more interesting example was provided by Sir Geoffrey Taylor in 1941 for determining the energy release of an atomic bomb. Let's assume that the energy is released in all directions from a single point. Possible physical variables are the time since the blast, t, the energy, E, the distance from the blast, r, the atmospheric density ρ and the atmospheric pressure, p. We have five physical variables and only three units. So, there should be two dimensionless quantities. Let's determine these.

Energy release in the first atomic bomb.

We set

$$\pi = E^{k_1} t^{k_2} r^{k_3} p^{k_4} \rho^{k_5}.$$

Inserting the respective units, we find that

$$\begin{aligned}
[\pi] &= [E]^{k_1}[t]^{k_2}[r]^{k_3}[p]^{k_4}[\rho]^{k_5} \\
&= \left([M][L]^2[T]^{-2}\right)^{k_1}[T]^{k_2}[L]^{k_3}\left([M][L]^{-1}[T]^{-2}\right)^{k_4}\left([M][L]^{-3}\right)^{k_5} \\
&= [M]^{k_1+k_4+k_5}[L]^{2k_1+k_3-k_4-3k_5}[T]^{-2k_1+k_2-2k_4}. \qquad (1.97)
\end{aligned}$$

Note: You should verify the units used. For example, the units of force can be found using $F = ma$ and work (energy) is force times distance. Similarly, you need to know that pressure is force per area.

For π to be dimensionless, we have to solve the system:

$$\begin{aligned}
k_1 + k_4 + k_5 &= 0, \\
2k_1 + k_3 - k_4 - 3k_5 &= 0, \\
-2k_1 + k_2 - 2k_4 &= 0. \qquad (1.98)
\end{aligned}$$

This is a set of three equations and five unknowns. The only way to solve this system is to solve for three unknowns in term of the remaining two. (In linear algebra one learns how to solve this using matrix methods.) Let's solve for k_1, k_2, and k_5 in terms of k_3 and k_4. The system can be written as

$$\begin{aligned}
k_1 + k_5 &= -k_4, \\
2k_1 - 3k_5 &= k_4 - k_3, \\
2k_1 - k_2 &= -2k_4. \qquad (1.99)
\end{aligned}$$

These can be solved by solving for k_1 and k_4 using the first two equations and then finding k_2 from the last one. Solving this system yields:

$$k_1 = -\frac{1}{5}(2k_4 + k_3) \quad k_2 = \frac{2}{5}(3k_4 - k_3) \quad k_5 = \frac{1}{5}(k_3 - 3k_4).$$

We have the freedom to pick values for k_3 and k_4. Two independent sets of simple values are obtained by picking one variable as zero and the other as one. This will give the following two cases:

Case I. $k_3 = 1$ *and* $k_4 = 0$.

In this case we then have $k_1 = -\frac{1}{5}$, $k_2 = -\frac{2}{5}$, and $k_5 = \frac{1}{5}$. This gives

$$\pi_1 = E^{-1/5}t^{-2/5}r\rho^{1/5} = r\left(\frac{\rho}{Et^2}\right)^{1/5}.$$

Case II. $k_3 = 0$ *and* $k_4 = 1$.

In this case we then have $k_1 = -\frac{2}{5}$, $k_2 = \frac{6}{5}$, and $k_5 = -\frac{3}{5}$.

$$\pi_2 = E^{-2/5}t^{6/5}p\rho^{-3/5} = p\left(\frac{t^6}{\rho^3 E^2}\right)^{1/5}.$$

Thus, we have that the relation between the energy and the other variables is of the form

$$F\left(r\left(\frac{\rho}{Et^2}\right)^{1/5}, p\left(\frac{t^6}{\rho^3 E^2}\right)^{1/5}\right) = 0.$$

Of course, this is not enough to determine the explicit equation. However, Taylor was able to use this information to get an energy estimate.

Note that π_1 is dimensionless. It can be represented as a function of the dimensionless variable π_2. So, assuming that $\pi_1 = h(\pi_2)$, we have that

$$h(\pi_2) = r\left(\frac{\rho}{Et^2}\right)^{1/5}.$$

Note that for $t = 1$ second, the energy is expected to be huge, so $\pi_2 \approx 0$. Thus,

$$r\left(\frac{\rho}{Et^2}\right)^{1/5} \approx h(0).$$

Simple experiments suggest that $h(0)$ is of order one, so

$$r \approx \left(\frac{Et^2}{\rho}\right)^{1/5}.$$

In 1947 Taylor applied his earlier analysis to movies of the first atomic bomb test in 1945 and his results were close to the actual values. How can one do this? You can find pictures of the first atomic bomb test with a superimposed length scale online.

We can rewrite the above result to get the energy estimate:

$$E \approx \frac{r^5 \rho}{t^2}.$$

As an exercise, you can estimate the radius of the explosion at the given time and determine the energy of the blast in so many tons of TNT.

Problems

1. Prove the following identities using only the definitions of the trigonometric functions, the Pythagorean identity, or the identities for sines and cosines of sums of angles.

a. $\cos 2x = 2\cos^2 x - 1$.

b. $\sin 3x = A \sin^3 x + B \sin x$, for what values of A and B?

2. Do the following.

a. Write $(\cosh x - \sinh x)^6$ in terms of exponentials.

b. Prove $\cosh 2x = \cosh^2 x + \sinh^2 x$.

c. If $\cosh x = \frac{13}{12}$ and $x < 0$, find $\sinh x$ and $\tanh x$.

d. Find the exact value of $\sinh(\operatorname{arccosh} 3)$

3. Compute the following integrals

a. $\int xe^{2x^2}\,dx$.
b. $\int_0^3 \frac{5x}{\sqrt{x^2+16}}dx$.
c. $\int x^3 \sin 3x\,dx$. (Do this using integration by parts, the Tabular Method, and differentiation under the integral sign.)
d. $\int \cos^4 3x\,dx$.
e. $\int_0^{\pi/2} \sec^3 x\,dx$.
f. $\int \sqrt{9-x^2}\,dx$
g. $\int \frac{dx}{(4-x^2)^2}$, using the substitution $x = 2\tanh u$.
h. $\int \frac{dx}{(x+4)^{3/2}}$, using the substitutions
- $x = 2\tan u$ and
- $x = 2\sinh u$.

4. Find the sum for each of the series:

a. $\sum_{n=0}^{\infty} \frac{(-1)^n 3}{4^n}$.
b. $\sum_{n=2}^{\infty} \frac{2}{5^n}$.
c. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$.
d. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$.

5. Evaluate the following expressions at the given point. Use your calculator or your computer (such as Maple). Then use series expansions to find an approximation to the value of the expression to as many places as you trust.

a. $f(x) = \frac{1}{\sqrt{1+2x^2}} - 1 + x^2$ at $x = 5.00 \times 10^{-3}$.
b. $f(R,h) = R - \sqrt{R^2+h^2}$ for $R = 1.374 \times 10^3$ km and $h = 1.00$ m.
c. $f(x) = 1 - \frac{1}{\sqrt{1-x}}$ for $x = 2.5 \times 10^{-13}$.

6. Use dimensional analysis to derive a possible expression for the drag force F_D on a soccer ball of diameter D moving at speed v through air of density ρ and viscosity μ. [Hint: Assuming viscosity has units $\frac{[M]}{[L][T]}$, there are two possible dimensionless combinations: $\pi_1 = \mu D^\alpha \rho^\beta v^\gamma$ and $\pi_2 = F_D D^\alpha \rho^\beta v^\gamma$. Determine α, β, and γ for each case and interpret your results.]

7. Challenge: Read the section on dimensional analysis. In particular, look at the results of Example 1.22. Using measurements in/on Figure 1.1, obtain an estimate of the energy of the blast in tons of TNT. Explain your work. Does your answer make sense? Why?

2

Free Fall and Harmonic Oscillators

"Mathematics began to seem too much like puzzle solving. Physics is puzzle solving, too, but of puzzles created by nature, not by the mind of man." Maria Goeppert-Mayer (1906-1972)

2.1 Free Fall and Terminal Velocity

In this chapter we will study some common differential equations that appear in physics. We will begin with the simplest types of equations and standard techniques for solving them We will end this part of the discussion by returning to the problem of free fall with air resistance. We will then turn to the study of oscillations, which are modeled by second order differential equations.

Free fall example.

Let us begin with a simple example from introductory physics. Recall that free fall is the vertical motion of an object solely under the force of gravity. It has been experimentally determined that an object at near the surface of the Earth falls at a constant acceleration in the absence of other forces, such as air resistance. This constant acceleration is denoted by $-g$, where g is called the acceleration due to gravity. The negative sign is an indication that we have chosen a coordinate system in which up is positive.

We are interested in determining the position, $y(t)$, of the falling body as a function of time. From the definition of free fall, we have

$$\ddot{y}(t) = -g. \tag{2.1}$$

Differentiation with respect to time is often denoted by dots instead of primes.

Note that we will occasionally use a dot to indicate time differentiation. This notation is standard in physics and we will begin to introduce you to this notation, though at times we might use the more familiar prime notation to indicate spatial differentiation, or general differentiation.

In Equation (2.1) we know g. It is a constant. Near the Earth's surface it is about 9.81 $\mathrm{m/s^2}$ or 32.2 $\mathrm{ft/s^2}$. What we do not know is $y(t)$. This is our first differential equation. In fact it is natural to see differential equations appear in physics ofteh through Newton's Second Law, $F = ma$, as it plays an important role in classical physics. We will return to this point later.

So, how does one solve the differential equation in (2.1)? We can do so by using what we know about calculus. It might be easier to see when we put in a particular number instead of g. You might still be getting used to the fact that some letters are used to represent constants. We will come back to the more general form after we see how to solve the differential equation.

Consider

$$\ddot{y}(t) = 5. \tag{2.2}$$

Recalling that the second derivative is just the derivative of a derivative, we can rewrite this equation as

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = 5. \tag{2.3}$$

This tells us that the derivative of dy/dt is 5. Can you think of a function whose derivative is 5? (Do not forget that the independent variable is t.) Yes, the derivative of $5t$ with respect to t is 5. Is this the only function whose derivative is 5? No! You can also differentiate $5t + 1$, $5t + \pi$, $5t - 6$, etc. In general, the derivative of $5t + C$ is 5.

So, the equation can be reduced to

$$\frac{dy}{dt} = 5t + C. \tag{2.4}$$

Now we ask if you know a function whose derivative is $5t + C$. Well, you might be able to do this one in your head, but we just need to recall the Fundamental Theorem of Calculus, which relates integrals and derivatives. Thus, we have

$$y(t) = \frac{5}{2}t^2 + Ct + D,$$

where D is a second integration constant.

This is a solution to the original equation. That means the solution is a function that when placed into the differential equation makes both sides of the equal sign the same. You can always check your answer by showing that the solution satisfies the equation. In this case we have

$$\ddot{y}(t) = \frac{d^2}{dt^2}\left(\frac{5}{2}t^2 + Ct + D\right) = \frac{d}{dt}(5t + C) = 5.$$

So, it is a solution.

We also see that there are two arbitrary constants, C and D. Picking any values for these gives a whole family of solutions. As we will see, the equation $\ddot{y}(t) = 5$ is a linear second order ordinary differential equation. The general solution of such an equation always has two arbitrary constants.

Let's return to the free fall problem. We solve it the same way. The only difference is that we can replace the constant 5 with the constant $-g$. So, we find that

$$\frac{dy}{dt} = -gt + C, \tag{2.5}$$

and

$$y(t) = -\frac{1}{2}gt^2 + Ct + D. \tag{2.6}$$

Once you get down the process, it only takes a line or two to solve.

There seems to be a problem. Imagine dropping a ball that then undergoes free fall. We just determined that there are an infinite number of solutions to where the ball is at any time! Well, that is not possible. Experience tells us that if you drop a ball you expect it to behave the same way every time. Or does it? Actually, you could drop the ball from anywhere. You could also toss it up or throw it down. So, there are many ways you can release the ball before it is in free fall. That is where the constants come in. They have physical meanings.

If you set $t = 0$ in the equation, then you have that $y(0) = D$. Thus, D gives the initial position of the ball. Typically, we denote initial values with a subscript. So, we will write $y(0) = y_0$. Thus, $D = y_0$.

That leaves us to determine C. It appears at first in Equation (2.5). Recall that $\frac{dy}{dt}$, the derivative of the position, is the vertical velocity, $v(t)$. It is positive when the ball moves upward. We will denote the initial velocity $v(0) = v_0$. Inserting $t = 0$ in Equation (2.5), we find that $\dot{y}(0) = C$. This implies that $C = v(0) = v_0$.

Putting this all together, we have the physical form of the solution for free fall as

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0. \tag{2.7}$$

Doesn't this equation look familiar? Now we see that the infinite family of solutions consists of free fall resulting from initially dropping a ball at position y_0 with initial velocity v_0. The conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$ are called the initial conditions. A solution of a differential equation satisfying a set of initial conditions is often called a particular solution.

So, we have solved the free fall equation. Along the way we have begun to see some of the features that will appear in the solutions of other problems that are modeled with differential equation. Throughout the book we will see several applications of differential equations. We will extend our analysis to higher dimensions, in which we case will be faced with so-called partial differential equations, which involve the partial derivatives of functions of more that one variable.

But are we done with free fall? Not at all! We can relax some of the conditions that we have imposed. We can add air resistance. We will visit this problem later in this chapter after introducing some more techniques.

Finally, we should also note that free fall at constant g only takes place near the surface of the Earth. What if a tile falls off the shuttle far from the surface? It will also fall to the Earth. Actually, it may undergo projectile motion, which you may recall is a combination of horizontal motion and free fall.

To look at this problem we need to go to the origins of the acceleration due to gravity. This comes out of Newton's Law of Gravitation. Consider a mass m at some distance $h(t)$ from the surface of the (spherical) Earth. Letting M and R be the Earth's mass and radius, respectively, Newton's Law of Gravitation states that Newton's Law of Gravitation

$$\begin{aligned} ma &= F \\ m\frac{d^2h(t)}{dt^2} &= G\frac{mM}{(R+h(t))^2}. \end{aligned} \tag{2.8}$$

Thus, we arrive at a differential equation

$$\frac{d^2h(t)}{dt^2} = \frac{GM}{(R+h(t))^2}. \tag{2.9}$$

This equation is not as easy to solve. We will leave it as a homework exercise for the reader.

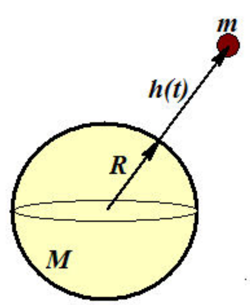


Figure 2.1: Free fall far from the Earth from a height $h(t)$ from the surface.

2.2 *First Order Differential Equations*

BEFORE MOVING ON, we first define an *n-th order ordinary differential equation* is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F(y^{(n)}(x), y^{(n-1)}(x), \dots, y'(x), y(x), x) = 0. \tag{2.10}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$.

An *initial value problem* consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{2.11}$$

A *linear nth order differential equation* takes the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x)) = f(x). \tag{2.12}$$

If $f(x) \equiv 0$, then the equation is said to be *homogeneous*, otherwise it is *nonhomogeneous*.

Typically, the first differential equations encountered are first order equations. A *first order differential equation* takes the form

$$F(y', y, x) = 0. \tag{2.13}$$

There are two general forms for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

2.2.1 Separable Equations

A FIRST ORDER EQUATION IS SEPARABLE if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{2.14}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be *autonomous*.

The *general solution* to equation (2.14) is obtained in terms of two integrals:

Separable equations.

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)\, dx + C,} \tag{2.15}$$

where C is an integration constant. This yields a *1-parameter family of solutions* to the differential equation corresponding to different values of C. If one can solve (2.15) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a *particular solution*.

Example 2.1. $y' = 2xy$, $y(0) = 2$.

Applying (2.15), one has

$$\int \frac{dy}{y} = \int 2x\, dx + C.$$

Integrating yields

$$\ln |y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. *Since C is an arbitrary constant, A is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure 2.2.*

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, *one finds that* $A = 2$. *So, the particular solution satisfying the initial condition is* $y(x) = 2e^{x^2}$.

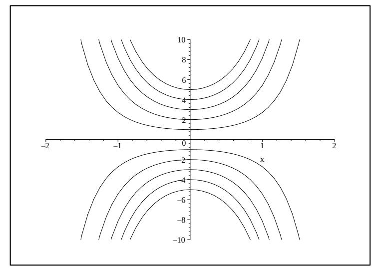


Figure 2.2: Plots of solutions from the 1-parameter family of solutions of Example 2.1 for several initial conditions.

Example 2.2. $yy' = -x$.

Following the same procedure as in the last example, one obtains:

$$\int y\, dy = -\int x\, dx + C \Rightarrow y^2 = -x^2 + A, \quad \textit{where} \quad A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, *we see that this is a family of circles for* $A > 0$ *and the origin for* $A = 0$. *Plots of some solutions in this family are shown in Figure 2.3.*

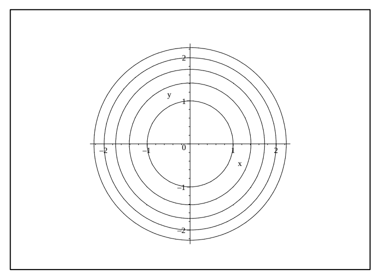


Figure 2.3: Plots of solutions of Example 2.2 for several initial conditions.

2.2.2 *Linear First Order Equations*

Linear first order differential equations.

The second type of first order equation encountered is the *linear first order differential equation* in the standard form

$$y'(x) + p(x)y(x) = q(x). \tag{2.16}$$

In this case one seeks an *integrating factor*, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu(x)q(x). \tag{2.17}$$

The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi)\, d\xi)$. One can derive $\mu(x)$ by expanding the derivative in Equation (2.17),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \tag{2.18}$$

and comparing this equation to the one obtained from multiplying (2.16) by $\mu(x)$:

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \tag{2.19}$$

Note that these last two equations would be the same if

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

This is a separable first order equation whose solution is the above given form for the integrating factor,

Integrating factor.

$$\boxed{\mu(x) = \exp\left(\int^x p(\xi)\, d\xi\right). \tag{2.20}}$$

Equation (2.17) is now easily integrated to obtain the solution

$$\boxed{y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\, d\xi + C\right]. \tag{2.21}}$$

Example 2.3. $xy' + y = x, \quad x > 0, y(1) = 0.$

One first notes that this is a linear first order differential equation. Solving for y', one can see that the equation is not separable. Furthermore, it is not in the standard form (2.16). So, we first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{2.22}$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (2.22) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, the differential equation becomes

$$(xy)' = x.$$

Integrating, one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is

$$y(x) = \frac{1}{2}(x - \frac{1}{x}).$$

Example 2.4. $(\sin x)y' + (\cos x)y = x^2$.

Actually, this problem is easy if you realize that

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, rewrite the original differential equation in standard form:

$$y' + (\cot x)y = x^2 \csc x.$$

Then, compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot \xi \, d\xi\right) = e^{\ln(\sin x)} = \sin x.$$

Using the integrating factor, the original equation becomes

$$\frac{d}{dx}\left((\sin x)y\right) = x^2.$$

Integrating, we have

$$y \sin x = \frac{1}{3}x^3 + C.$$

So, the solution is

$$y = \left(\frac{1}{3}x^3 + C\right)\csc x.$$

There are other first order equations that one can solve for closed form solutions. However, many equations are not solvable, or one is simply interested in the behavior of solutions. In such cases one turns to direction fields. We will return to a discussion of the qualitative behavior of differential equations later.

2.2.3 *Terminal Velocity*

NOW LET'S RETURN to free fall. What if there is air resistance? We first need to model the air resistance. As an object falls faster and faster, the drag force becomes greater. So, this resistive force is a function of the velocity. There are a couple of standard models that people use to test this. The idea is to write $F = ma$ in the form

$$m\ddot{y} = -mg + f(v), \tag{2.23}$$

where $f(v)$ gives the resistive force and mg is the weight. Recall that this applies to free fall near the Earth's surface. Also, for it to be resistive, $f(v)$ should oppose the motion. If the body is falling, then $f(v)$ should be positive. If it is rising, then $f(v)$ would have to be negative to indicate the opposition to the motion.

One common determination derives from the drag force on an object moving through a fluid. This force is given by

$$f(v) = \frac{1}{2}CA\rho v^2, \tag{2.24}$$

where C is the drag coefficient, A is the cross sectional area and ρ is the fluid density. For laminar flow the drag coefficient is constant.

Unless you are into aerodynamics, you do not need to get into the details of the constants. So, it is best to absorb all of the constants into one to simplify the computation. So, we will write $f(v) = bv^2$. The differential equation including drag can then be rewritten as

$$\dot{v} = kv^2 - g, \tag{2.25}$$

where $k = b/m$. Note that this is a first order equation for $v(t)$. It is separable too!

Formally, we can separate the variables and integrate over time to obtain

$$t + K = \int^v \frac{dz}{kz^2 - g}. \tag{2.26}$$

(Note: We used an integration constant of K since C is the drag coefficient in this problem.) If we can do the integral, then we have a solution for v. In fact, we can do this integral. You need to recall another common method of integration, which we have not reviewed yet. Do you remember Partial Fraction Decomposition? It involves factoring the denominator in the integral. Of course, this is ugly because the constants are represented by letters and are not specific numbers. Letting $\alpha^2 = g/k$, we can write the integrand as

This is the first use of Partial Fraction Decomposition. We will explore this method further in the section on Laplace Transforms.

$$\frac{1}{kz^2 - g} = \frac{1}{k}\frac{1}{z^2 - \alpha^2} = \frac{1}{2\alpha k}\left[\frac{1}{z - \alpha} - \frac{1}{z + \alpha}\right]. \tag{2.27}$$

Now, the integrand can be easily integrated giving

$$t + K = \frac{1}{2\alpha k}\ln\left|\frac{v - \alpha}{v + \alpha}\right|. \tag{2.28}$$

Solving for v, we have

$$v(t) = \frac{1 - Be^{2\alpha kt}}{1 + Be^{2\alpha kt}}\alpha, \tag{2.29}$$

where $B \equiv e^K$. B can be determined using the initial velocity.

There are other forms for the solution in terms of a tanh function, which the reader can determine as an exercise. One important conclusion is that for large times, the ratio in the solution approaches -1. Thus, $v \to -\alpha = -\sqrt{\frac{g}{k}}$. This means that the falling object will reach a terminal velocity.

As a simple computation, we can determine the terminal velocity. We will take an 80 kg skydiver with a cross sectional area of about 0.093 m^2. (The skydiver is falling head first.) Assume that the air density is a constant 1.2 kg/m^3 and the drag coefficient is $C = 2.0$. We first note that

$$v_{\text{terminal}} = -\sqrt{\frac{g}{k}} = -\sqrt{\frac{2mg}{CA\rho}}.$$

So,

$$v_{\text{terminal}} = -\sqrt{\frac{2(70)(9.8)}{(2.0)(0.093)(1.2)}} = 78m/s.$$

This is about 175 mph, which is slightly higher than the actual terminal velocity of a sky diver. One would need a more accurate determination of C for a more realistic answer.

2.3 *The Simple Harmonic Oscillator*

THE NEXT PHYSICAL PROBLEM of interest is that of simple harmonic motion. Such motion comes up in many places in physics and provides a generic first approximation to models of oscillatory motion. This is the beginning of a major thread running throughout this course. You have seen simple harmonic motion in your introductory physics class. We will review SHM (or SHO in some texts) by looking at springs and pendula (the plural of pendulum). We will use this as our jumping board into second order differential equations and later see how such oscillatory motion occurs in AC circuits.

2.3.1 *Mass-Spring Systems*

WE BEGIN with the case of a single block on a spring as shown in Figure 2.4. The net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant. Here x is the elongation, or displacement of the spring from equilibrium. When the displacement is positive, the spring force is negative and when the displacement is negative the spring force is positive. We have depicted a horizontal system sitting on a frictionless surface. A similar model can be provided for vertically oriented springs. However, you need to account for gravity to determine the location of equilibrium. Otherwise, the oscillatory motion about equilibrium is modeled the same.

From Newton's Second Law, $F = m\ddot{x}$, we obtain the equation for the motion of the mass on the spring:

$$m\ddot{x} + kx = 0.$$

We will later derive solutions of such equations in a methodical way. For now we note that two solutions of this equation are given by

$$x(t) = A\cos\omega t,$$
$$x(t) = A\sin\omega t, \tag{2.30}$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

is the angular frequency, measured in rad/s. It is related to the frequency by

$$\omega = 2\pi f,$$

where f is measured in cycles per second, or Hertz. Furthermore, this is related to the period of oscillation, the time it takes the mass to go through one cycle:

$$T = 1/f.$$

Finally, A is called the amplitude of the oscillation.

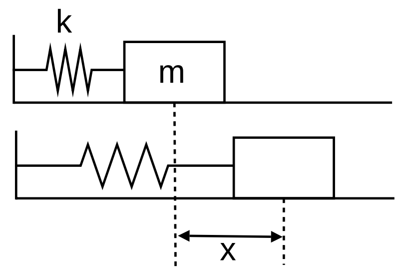


Figure 2.4: Spring-Mass system.

2.3.2 *The Simple Pendulum*

THE SIMPLE PENDULUM consists of a point mass m hanging on a string of length L from some support. [See Figure 2.5.] One pulls the mass back to some stating angle, θ_0, and releases it. The goal is to find the angular position as a function of time.

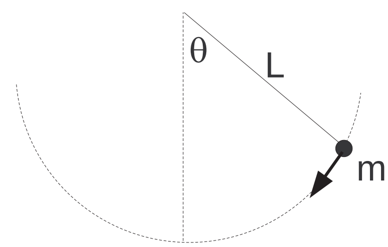


Figure 2.5: A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

There are a couple of possible derivations. We could either use Newton's Second Law of Motion, $F = ma$, or its rotational analogue in terms of torque, $\tau = I\alpha$. We will use the former only to limit the amount of physics background needed.

There are two forces acting on the point mass. The first is gravity. This points downward and has a magnitude of mg, where g is the standard symbol for the acceleration due to gravity. The other force is the tension in the string. In Figure 2.6 these forces and their sum are shown. The magnitude of the sum is easily found as $F = mg\sin\theta$ using the addition of these two vectors.

Now, Newton's Second Law of Motion tells us that the net force is the mass times the acceleration. So, we can write

$$m\ddot{x} = -mg\sin\theta.$$

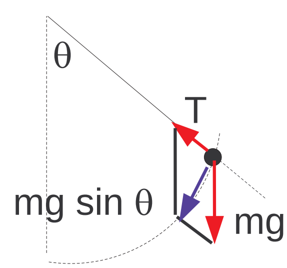


Figure 2.6: There are two forces acting on the mass, the weight mg and the tension T. The net force is found to be $F = mg\sin\theta$.

Next, we need to relate x and θ. x is the distance traveled, which is the length of the arc traced out by the point mass. The arclength is related to the angle, provided the angle is measure in radians. Namely, $x = r\theta$ for $r = L$. Thus, we can write

$$mL\ddot{\theta} = -mg\sin\theta.$$

Canceling the masses, this then gives us the nonlinear pendulum equation

Nonlinear pendulum equation.

$$L\ddot{\theta} + g\sin\theta = 0. \tag{2.31}$$

There are several variations of Equation (2.31) which will be used in this text. The first one is the linear pendulum. This is obtained by making a small angle approximation. For small angles we know that $\sin\theta \approx \theta$. Under this approximation (2.31) becomes

Linear pendulum equation.

$$L\ddot{\theta} + g\theta = 0. \tag{2.32}$$

The equation for a compound pendulum takes a similar form. We start with the rotational form of Newton's second law $\tau = I\alpha$. Noting that the torque due to gravity acts at the center of mass position ℓ, the torque is given by $\tau = -mg\ell\sin\theta$. Since $\alpha = \ddot{\theta}$, we have $I\ddot{\theta} = -mg\ell\sin\theta$. Then, for small angles $\ddot{\theta} + \omega^2\theta = 0$, where $\omega = \frac{mg\ell}{I}$. for a point mass, $\ell = L$ and $I = mL^2$, leading to the result in the text.

We note that this equation is of the same form as the mass-spring system. We define $\omega = \sqrt{g/L}$ and obtain the equation for simple harmonic motion,

$$\ddot{\theta} + \omega^2\theta = 0.$$

2.4 *Second Order Linear Differential Equations*

In the last section we saw how second order differential equations naturally appear in the derivations for simple oscillating systems. In this section we will look at more general second order linear differential equations.

Second order differential equations are typically harder than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.33}$$

One can rewrite this equation using operator terminology. Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (2.33) becomes

$$Ly = f. \tag{2.34}$$

The solutions of linear differential equations are found by making use of the linearity of L. Namely, we consider the *vector space*[1] consisting of real-valued functions over some domain. Let f and g be vectors in this function space. L is a *linear operator* if for two vectors f and g and scalar a, we have that

[1] We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

a. $L(f+g) = Lf + Lg$

b. $L(af) = aLf$.

One typically solves (2.33) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (2.33) is simply given as $y = y_h + y_p$. This is true because of the linearity of L. Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{2.35}$$

There are methods for finding a particular solution of a differential equation. These range from pure guessing to the Method of Undetermined Coefficients, or by making use of the Method of Variation of Parameters. We will review some of these methods later.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, others have studied a variety of second order linear equations and have saved us the trouble for some of the differential equations that often appear in applications.

Again, linearity is useful in producing the general solution of a homogeneous linear differential equation. If y_1 and y_2 are solutions of the homogeneous equation, then the *linear combination* $y = c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation. In fact, if y_1 and y_2 are *linearly independent*,[2] then $y = c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem. As you may recall, linear independence is established if the Wronskian of the solutions in not zero. In this case, we have

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0. \tag{2.36}$$

[2] Recall, a set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1y(1(x) + \ldots + c_ny_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \ldots, n$.

2.4.1 Constant Coefficient Equations

THE SIMPLEST AND MOST SEEN second order differential equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{2.37}$$

where a, b, and c are constants.

Solutions to (2.37) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (2.37) leads to the *characteristic equation*

$$ar^2 + br + c = 0. \tag{2.38}$$

The characteristic equation for $ay'' + by' + cy = 0$ is $ar^2 + br + c = 0$. Solutions of this quadratic equation lead to solutions of the differential equation.

Namely, we compute the derivatives of $y(x) = e^{rx}$, to get $y(x) = re^{rx}$, and $y(x) = r^2e^{rx}$. Inserting into (2.37), we have

$$0 = ay''(x) + by'(x) + cy(x) = (ar^2 + br + c)e^{rx}.$$

Since the exponential is never zero, we find that $ar^2 + br + c = 0$.

Two real, distinct roots, r_1 and r_2, give solutions of the form $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$.

The roots of this equation, r_1, r_2, in turn lead to three types of solution depending upon the nature of the roots. In general, we have two linearly independent solutions, $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$, and the general solution is given by a linear combination of these solutions,

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}.$$

For two real distinct roots, we are done. However, when the roots are real, but equal, or complex conjugate roots, we need to do a little more work to obtain usable solutions.

Repeated roots, $r_1 = r_2 = r$, give solutions of the form

$$y(x) = (c_1 + c_2x)e^{rx}.$$

In the case when there is a repeated real root, one has only one independent solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second solution? Since the solutions are independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. For constant coefficient second order equations, we can write the equation as

$$(D - r)^2y = 0,$$

where $D = \frac{d}{dx}$.

We now insert $y_2(x)$ into this equation. First we compute

$$(D - r)ve^{rx} = v'e^{rx}.$$

Then,

$$(D - r)^2ve^{rx} = (D - r)v'e^{rx} = v''e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, $(D - r)^2y_2 = 0$, then $v''(x)e^{rx} = 0$ for all x. So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

Complex roots, $r = \alpha \pm i\beta$, give solutions of the form

$$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x).$$

So,

$$y_2(x) = (ax + b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$.

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula

$$e^{i\beta x} = \cos \beta x + i \sin \beta x. \tag{2.39}$$

Then the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right] \\ &= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] \\ &\equiv e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x). \end{aligned} \tag{2.40}$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$.

The three cases are summarized below followed by several examples.

Classification of Roots of the Characteristic Equation for Second Order Constant Coefficient ODEs

1. **Real, distinct roots** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$. [This is covered in the appendix to this chapter.]

3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$. [This is covered in the appendix to this chapter.]

Example 2.5. $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0$.

The characteristic equation for this problem is $r^2 - r - 6 = 0$. *The roots of this equation are found as* $r = -2, 3$. *Therefore, the general solution can be* *quickly written down:*

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1 e^{-2x} + 3c_2 e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{2.41}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Example 2.6. $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. Again, the solution is easily obtained as $y(x) = (c_1 + c_2 x)e^{-3x}$.

Example 2.7. $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$ and the general solution consists purely of sinusoidal functions: $y(x) = c_1 \cos(2x) + c_2 \sin(2x)$.

Example 2.8. $y'' + 2y' + 4y = 0$.

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as $y(x) = \left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right] e^{-x}$.

Example 2.9. $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example 2.7. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A \sin x$ and determine what A needs to be. Recall, this is the Method of Undetermined Coefficients which we review in later in the chapter. Inserting our guess in the equation gives $(-A + 4A) \sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \sin x$.

As we have seen, one of the most important applications of such equations is in the study of oscillations. Typical systems are a mass on a spring, or a simple pendulum. For a mass m on a spring with spring constant $k > 0$, one has from Hooke's law that the position as a function of time, $x(t)$, satisfies the equation

$$m\ddot{x} + kx = 0.$$

This constant coefficient equation has pure imaginary roots ($\alpha = 0$) and the solutions are pure sines and cosines. This is called simple harmonic motion. Adding a damping term and periodic forcing complicates the dynamics, but is nonetheless solvable. We will return to damped oscillations later and also investigate nonlinear oscillations.

2.5 *LRC Circuits*

ANOTHER TYPICAL PROBLEM often encountered in a first year physics class is that of an LRC series circuit. This circuit is pictured in Figure 2.7. The resistor is a circuit element satisfying Ohm's Law. The capacitor is a device that stores electrical energy and an inductor, or coil, store magnetic energy.

The physics for this problem stems from Kirchoff's Rules for circuits. Namely, the sum of the drops in electric potential are set equal to the rises in electric potential. The potential drops across each circuit element are given by

1. Resistor: $V = IR$.
2. Capacitor: $V = \frac{q}{C}$.
3. Inductor: $V = L\frac{dI}{dt}$.

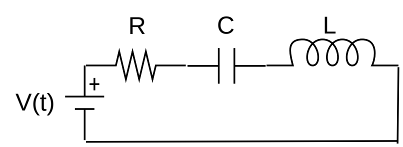


Figure 2.7: Series LRC Circuit.

Furthermore, we need to define the current as $I = \frac{dq}{dt}$. where q is the charge in the circuit. Adding these potential drops, we set them equal to the voltage supplied by the voltage source, $V(t)$. Thus, we obtain

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t).$$

Since both q and I are unknown, we can replace the current by its expression in terms of the charge to obtain

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t).$$

This is a second order equation for $q(t)$.

More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors.

This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 2.8. This is not a problem that can be covered in the first year physics course. One can set up a system of second order equations and proceed to solve them. We will see how to solve such problems later in the text.

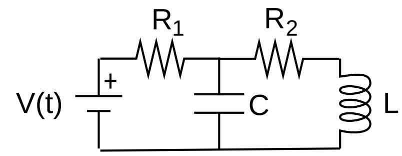


Figure 2.8: Parallel LRC Circuit.

2.5.1 *Special Cases*

IN THIS SECTION we will look at special cases that arise for the series LRC circuit equation. These include RC circuits, solvable by first order methods and LC circuits, leading to oscillatory behavior.

Case I. RC Circuits

We first consider the case of an RC circuit in which there is no inductor. Also, we will consider what happens when one charges a capacitor with a DC battery ($V(t) = V_0$) and when one discharges a charged capacitor ($V(t) = 0$).

Charging a capacitor.

For charging a capacitor, we have the initial value problem

$$R\frac{dq}{dt} + \frac{q}{C} = V_0, \quad q(0) = 0. \tag{2.42}$$

This equation is an example of a linear first order equation for $q(t)$. However, we can also rewrite it and solve it as a separable equation, since V_0 is a constant. We will do the former only as another example of finding the integrating factor.

We first write the equation in standard form:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}. \tag{2.43}$$

The integrating factor is then

$$\mu(t) = e^{\int \frac{dt}{RC}} = e^{t/RC}.$$

Thus,

$$\frac{d}{dt}\left(qe^{t/RC}\right) = \frac{V_0}{R}e^{t/RC}. \tag{2.44}$$

Integrating, we have

$$qe^{t/RC} = \frac{V_0}{R}\int e^{t/RC}\,dt = CV_0e^{t/RC} + K. \tag{2.45}$$

Note that we introduced the integration constant, K. Now divide out the exponential to get the general solution:

$$q = CV_0 + Ke^{-t/RC}. \tag{2.46}$$

(If we had forgotten the K, we would not have gotten a correct solution for the differential equation.)

Next, we use the initial condition to get the particular solution. Namely, setting $t = 0$, we have that

$$0 = q(0) = CV_0 + K.$$

So, $K = -CV_0$. Inserting this into the solution, we have

$$q(t) = CV_0(1 - e^{-t/RC}). \tag{2.47}$$

Now we can study the behavior of this solution. For large times the second term goes to zero. Thus, the capacitor charges up, asymptotically, to the final value of $q_0 = CV_0$. This is what we expect, because the current is no longer flowing over R and this just gives the relation between the potential difference across the capacitor plates when a charge of q_0 is established on the plates.

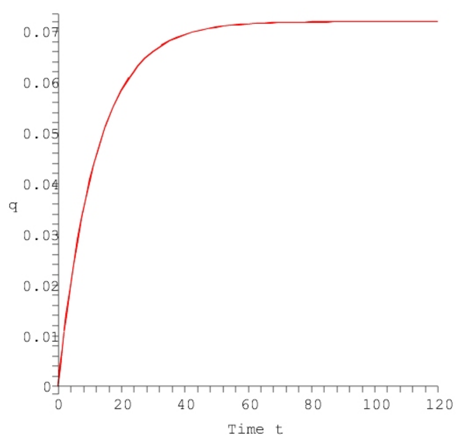


Figure 2.9: The charge as a function of time for a charging capacitor with $R = 2.00$ kΩ, $C = 6.00$ mF, and $V_0 = 12$ V.

Let's put in some values for the parameters. We let $R = 2.00$ kΩ, $C = 6.00$ mF, and $V_0 = 12$ V. A plot of the solution is given in Figure 2.9. We see that the charge builds up to the value of $CV_0 = 0.072$ C. If we use a smaller resistance, $R = 200\ \Omega$, we see in Figure 2.10 that the capacitor charges to the same value, but much faster.

The rate at which a capacitor charges, or discharges, is governed by the time constant, $\tau = RC$. This is the constant factor in the exponential. The larger it is, the slower the exponential term decays. If we set $t = \tau$, we find that

$$q(\tau) = CV_0(1 - e^{-1}) = (1 - 0.3678794412\ldots)q_0 \approx 0.63q_0.$$

Thus, at time $t = \tau$, the capacitor has almost charged to two thirds of its final value. For the first set of parameters, $\tau = 12$s. For the second set, $\tau = 1.2$s.

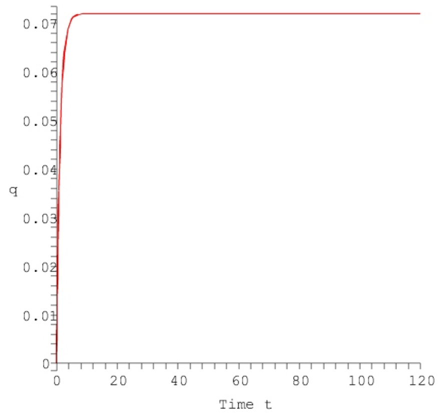


Figure 2.10: The charge as a function of time for a charging capacitor with $R = 200\ \Omega$, $C = 6.00$ mF, and $V_0 = 12$ V.

Discharging a capacitor.

Now, let's assume the capacitor is charged with charge $\pm q_0$ on its plates. If we disconnect the battery and reconnect the wires to complete the circuit, the charge will then move off the plates, discharging the capacitor. The relevant form of the initial value problem becomes

$$R\frac{dq}{dt} + \frac{q}{C} = 0, \quad q(0) = q_0. \tag{2.48}$$

This equation is simpler to solve. Rearranging, we have

$$\frac{dq}{dt} = -\frac{q}{RC}. \tag{2.49}$$

This is a simple exponential decay problem, which you can solve using separation of variables. However, by now you should know how to immediately write down the solution to such problems of the form $y' = ky$. The solution is

$$q(t) = q_0 e^{-t/\tau}, \quad \tau = RC.$$

We see that the charge decays exponentially. In principle, the capacitor never fully discharges. That is why you are often instructed to place a shunt across a discharged capacitor to fully discharge it.

In Figure 2.11 we show the discharging of the two previous RC circuits. Once again, $\tau = RC$ determines the behavior. At $t = \tau$ we have

$$q(\tau) = q_0 e^{-1} = (0.3678794412\ldots)q_0 \approx 0.37 q_0.$$

So, at this time the capacitor only has about a third of its original value.

Case II. LC Circuits

LC Oscillators.

Another simple result comes from studying LC circuits. We will now connect a charged capacitor to an inductor. In this case, we consider the initial value problem

$$L\ddot{q} + \frac{1}{C}q = 0, \quad q(0) = q_0, \dot{q}(0) = I(0) = 0. \tag{2.50}$$

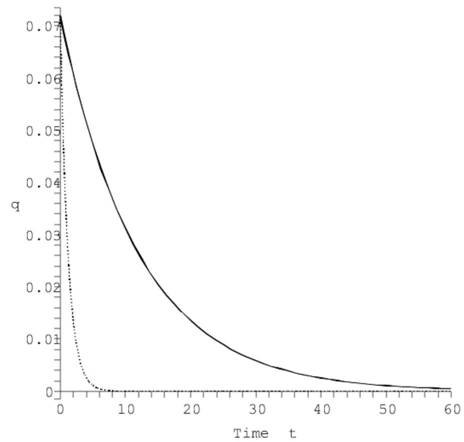


Figure 2.11: The charge as a function of time for a discharging capacitor with $R = 2.00$ kΩ or $R = 200\ \Omega$, and $C = 6.00$ mF, and $q_0 = 0.072$ C.

Dividing out the inductance, we have

$$\ddot{q} + \frac{1}{LC} q = 0. \tag{2.51}$$

This equation is a second order, constant coefficient equation. It is of the same form as the ones for simple harmonic motion of a mass on a spring or the linear pendulum. So, we expect oscillatory behavior. The characteristic equation is

$$r^2 + \frac{1}{LC} = 0.$$

The solutions are

$$r_{1,2} = \pm \frac{i}{\sqrt{LC}}.$$

Thus, the solution of (2.51) is of the form

$$q(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad \omega = (LC)^{-1/2}. \tag{2.52}$$

Inserting the initial conditions yields

$$q(t) = q_0 \cos(\omega t). \tag{2.53}$$

The oscillations that result are understandable. As the charge leaves the plates, the changing current induces a changing magnetic field in the inductor. The stored electrical energy in the capacitor changes to stored magnetic energy in the inductor. However, the process continues until the plates are charged with opposite polarity and then the process begins in reverse. The charged capacitor then discharges and the capacitor eventually returns to its original state and the whole system repeats this over and over.

The frequency of this simple harmonic motion is easily found. It is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \frac{1}{\sqrt{LC}}. \tag{2.54}$$

This is called the tuning frequency because of its role in tuning circuits.

Of course, this is an ideal situation. There is always resistance in the circuit, even if only a small amount from the wires. So, we really need to account for resistance, or even add a resistor. This leads to a slightly more complicated system in which damping will be present.

2.6 *Damped Oscillations*

AS WE HAVE INDICATED, simple harmonic motion is an ideal situation. In real systems we often have to contend with some energy loss in the system. This leads to the damping of the oscillations. This energy loss could be in the spring, in the way a pendulum is attached to its support, or in the resistance to the flow of current in an LC circuit. The simplest models of resistance are the addition of a term in first derivative of the dependent variable. Thus, our three main examples with damping added look like:

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{2.55}$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \tag{2.56}$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0. \tag{2.57}$$

These are all examples of the general constant coefficient equation

$$ay''(x) + by'(x) + cy(x) = 0. \tag{2.58}$$

We have seen that solutions are obtained by looking at the characteristic equation $ar^2 + br + c = 0$. This leads to three different behaviors depending on the discriminant in the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.59}$$

We will consider the example of the damped spring. Then we have

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}. \tag{2.60}$$

For $b > 0$, there are three types of damping.

Damped oscillator cases.

I. Overdamped, $b^2 > 4mk$

In this case we obtain two real root. Since this is Case I for constant coefficient equations, we have that

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

We note that $b^2 - 4mk < b^2$. Thus, the roots are both negative. So, both terms in the solution exponentially decay. The damping is so strong that there is no oscillation in the system.

II. Critically Damped, $b^2 = 4mk$

In this case we obtain one real root. This is Case II for constant coefficient equations and the solution is given by

$$x(t) = (c_1 + c_2 t)e^{rt},$$

where $r = -b/2m$. Once again, the solution decays exponentially. The damping is just strong enough to hinder any oscillation. If it were any weaker the discriminant would be negative and we would need the third case.

III. Underdamped, $b^2 < 4mk$

In this case we have complex conjugate roots. We can write $\alpha = -b/2m$ and $\beta = \sqrt{4mk - b^2}/2m$. Then the solution is

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

These solutions exhibit oscillations due to the trigonometric functions, but we see that the amplitude may decay in time due the the overall factor of $e^{\alpha t}$ when $\alpha < 0$. Consider the case that the initial conditions give $c_1 = A$ and $c_2 = 0$. (When is this?) Then, the solution, $x(t) = Ae^{\alpha t} \cos \beta t$, looks like the plot in Figure 2.12.

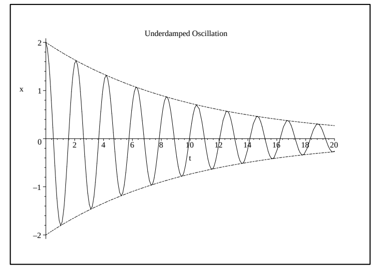


Figure 2.12: A plot of underdamped oscillation given by $x(t) = 2e^{0.1t} \cos 3t$. The dashed lines are given by $x(t) = \pm 2e^{0.1t}$, indicating the bounds on the amplitude of the motion.

2.7 *Forced Systems*

All of the systems presented at the beginning of the last section exhibit the same general behavior when a damping term is present. An additional term can be added that can cause even more complicated behavior. In the case of LRC circuits, we have seen that the voltage source makes the system nonhomogeneous. It provides what is called a source term. Such terms can also arise in the mass-spring and pendulum systems. One can drive such systems by periodically pushing the mass, or having the entire system moved, or impacted by an outside force. Such systems are called forced, or driven.

Typical systems in physics can be modeled by nonhomogenous second order equations. Thus, we want to find solutions of equations of the form

$$Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.61}$$

Recall, that one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (2.33) is simply given as $y = y_h + y_p$.

To date, we only know how to solve constant coefficient, homogeneous equations. So, by adding a nonhomogeneous to such equations we need to figure out what to do with the extra term. In other words, how does one find the particular solution?

You could guess a solution, but that is not usually possible without a little bit of experience. So we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically developed the particular solution. We will come back to this method the Method of Variation of Parameters, later in this section.

2.7.1 *Method of Undetermined Coefficients*

LET'S SOLVE a simple differential equation highlighting how we can handle nonhomogeneous equations.

Example 2.10. *Consider the equation*

$$y'' + 2y' - 3y = 4. \tag{2.62}$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{2.63}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. *The roots are* $r = 1, -3$. *So, we can immediately write the solution*

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (2.62). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x, then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ *does not work. But, we could try an arbitrary constant,* $y = A$.

Let's see. Inserting $y = A$ *into (2.62), we obtain*

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ *and this works. So, we have a particular solution,* $y_p(x) = -\frac{4}{3}$. *This step is done.*

Combining the two solutions, we have the general solution to the original nonhomogeneous equation (2.62). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine the arbitrary constants.

Example 2.11. *What if we had a different source term? Consider the equation*

$$y'' + 2y' - 3y = 4x. \tag{2.64}$$

The only thing that would change is the particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (2.64), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (2.64)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{2.65}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{2.66}$$

So, the particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 2.7.1.

More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have the solution. This solution is then added to the general solution of the homogeneous differential equation.

$f(x)$	Guess
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

Example 2.12. *As a final example, let's consider the equation*

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{2.67}$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^x and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If the driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$. We compute the derivative of our guess, $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$. Inserting these into the equation, we obtain

$$[(9x-6) + 2(1-3x) - 3x]Ae^{-3x} = 2e^{-3x},$$

or

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$.

Modified Method of Undetermined Coefficients

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k, where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

2.7.2 Forced Oscillations

As an example of a simple forced system, we can consider forced linear oscillations. For example one can force the mass-spring system. In general, such as system would satisfy the equation

$$m\ddot{x} + b\dot{(x)} + kx = F(t), \tag{2.68}$$

where m is the mass, b is the damping constant, k is the spring constant, and $F(t)$ is the driving force. If $F(t)$ is of simple form, then we can employ the Method of Undetermined Coefficients. As the damping term only complicates the solution we will assume that $b = 0$. Furthermore, we will introduce a sinusoidal driving force, $F(t) = F_0 \cos \omega t$. Then, the simple driven system becomes

$$m\ddot{x} + kx = F_0 \cos \omega t. \tag{2.69}$$

As we have seen, one first determines the solution to the homogeneous problem,

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. In order to apply the Method of Undetermined Coefficients, one has to make a guess which is not a solution of the homogeneous solution. The first guess would be to use $x_p = A \cos \omega t$. This is fine if $\omega \neq \omega_0$. Otherwise, one would need to use the Modified Method of Undetermined Coefficients as described in the last section. The details will be left to the reader.

The general solution to the problem is thus

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \begin{cases} \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t, & \omega \neq \omega_0, \\ \frac{F_0}{2m\omega_0} t \sin \omega_0 t, & \omega = \omega_0. \end{cases} \tag{2.70}$$

The case of resonance.

Special cases of these solutions provide interesting physics, which can be explored by the reader in the homework. In the case that $\omega = \omega_0$, we see that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency. As the system is moving to the left, one pushes it to the left. If it is moving to the right, one is adding energy in that direction. This forces the amplitude of oscillation to continue to grow until the system breaks.

In the case that $\omega \neq \omega_0$, one can rewrite the solution in a simple form. Let's choose the initial conditions as $x(0) = 0$, $\dot{x}(0) = 0$. Then one has (see Problem 13)

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}. \tag{2.71}$$

For values of ω near ω_0, one finds the solution consists of a rapid oscillation, due to the $\sin\frac{(\omega_0+\omega)t}{2}$ factor, with a slowly varying amplitude, $\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}$. The reader can investigate this solution. This leads to what are called beats.

2.7.3 Cauchy-Euler Equations

ANOTHER CLASS OF SOLVABLE linear differential equations that is of interest are the Cauchy-Euler type of equations. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{2.72}$$

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

The solutions of Cauchy-Euler equations can be found using the characteristic equation $ar(r-1)+br+c=0$.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \qquad y''(x) = r(r-1)x^{r-2},$$

into Equation (2.72), we have

$$[ar(r-1)+br+c]\,x^r = 0.$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

$$ar(r-1)+br+c = 0. \tag{2.73}$$

For two real, distinct roots, the general solution takes the form

$$y(x) = c_1x^{r_1} + c_2x^{r_2}.$$

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. If there are two real, distinct roots, then the general solution takes the form $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

For one root, $r_1 = r_2 = r$, the general solution is of the form

$$y(x) = (c_1 + c_2\ln|x|)x^r.$$

Deriving the solution for Case 2 for the Cauchy-Euler equations works in the same way as the second for constant coefficient equations, but it is a bit messier. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r-r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r-1)+br+c = 0.$$

Rewriting this, we have

$$r^2 + (\frac{b}{a}-1)r + \frac{c}{a} = 0.$$

Comparing equations, we find

$$\frac{b}{a} = 1 - 2r_1, \qquad \frac{c}{a} = r_1^2.$$

So, the general Cauchy-Euler equation in this case takes the form

$$x^2 y'' + (1 - 2r_1)xy' + r_1^2 y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1 v)x^{r_1 - 1}, \\ y_2''(x) &= (x^2 v'' + 2r_1 xv' + r_1(r_1 - 1)v)x^{r_1 - 2}. \end{aligned} \tag{2.74}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2 y'' + (1 - 2r_1)xy' + r_1^2 y \\ &= (xv'' + v')x^{r_1 + 1}. \end{aligned} \tag{2.75}$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln |v'| = -\ln |x| + C.$$

Exponentiating, we have one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A \ln |x| + k.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1} \ln |x|.$$

We now turn to the case of complex conjugate roots, $r = \alpha \pm i\beta$. When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha + i\beta}$. The key to obtaining real solutions is to first recall that

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

For complex conjugate roots, $r = \alpha \pm i\beta$, the general solution takes the form

$$y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|)).$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha+i\beta} = x^{\alpha} e^{i\beta \ln x}, \quad x > 0.$$

We can now find two real, linearly independent solutions, $x^{\alpha}\cos(\beta \ln |x|)$ and $x^{\alpha}\sin(\beta \ln |x|)$ following the same steps as earlier for the constant coefficient case.

The results are summarized in the table below followed by examples.

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations

1. Real, distinct roots r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.

2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln |x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln |x|)x^r$.

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^{\alpha}\cos(\beta \ln |x|)$ and $x^{\alpha}\sin(\beta \ln |x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^{\alpha}(c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|))$.

Example 2.13. $x^2 y'' + 5xy' + 12y = 0$

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned} \tag{2.76}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = \left[c_1 \cos(2\sqrt{2}\ln |x|) + c_2 \sin(2\sqrt{2}\ln |x|)\right] x^{-2}$

Example 2.14. $t^2 y'' + 3ty' + y = 0, \quad y(1) = 0, y'(1) = 1.$

For this example the characteristic equation takes the form

$$r(r-1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$*. Therefore, the general solution is*

$$y(t) = (c_1 + c_2 \ln|t|)t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ *we know the values of* y *and* y'*. Using the general solution, we first have that*

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln|t|t^{-1}$*. Now, using the second condition and*

$$y'(t) = c_2(1 - \ln|t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln|t|t^{-1}$*.*

Nonhomogeneous Cauchy-Euler Equations We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients. We will demonstrate this with a couple of examples.

Example 2.15. *Find the solution of* $x^2y'' - xy' - 3y = 2x^2$*.*

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$*. So, the roots are* $r = -1, 3$ *and the solution is* $y_h(x) = c_1x^{-1} + c_2x^3$*.*

We next need a particular solution. Let's guess $y_p(x) = Ax^2$*. Inserting the guess into the nonhomogeneous differential equation, we have*

$$\begin{aligned} 2x^2 &= x^2y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{2.77}$$

So, $A = -2/3$*. Therefore, the general solution of the problem is*

$$y(x) = c_1x^{-1} + c_2x^3 - \frac{2}{3}x^2.$$

Example 2.16. *Find the solution of* $x^2y'' - xy' - 3y = 2x^3$*.*

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2y'' + bxy' + cy = dx^r,$$

where r *is a solution of* $ar(r-1) + br + c = 0$*. Let's guess a solution of the form* $y = Ax^r \ln x$*. Then one finds that the differential equation reduces to* $Ax^r(2ar - a + b) = dx^r$*. [You should verify this for yourself.]*

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. *Inserting into the equation, we obtain* $4Ax^3 = 2x^3$, *or* $A = 1/2$. *The general solution of the problem can now be written as*

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x.$$

2.7.4 *Method of Variation of Parameters*

A MORE SYSTEMATIC way to find particular solutions is through the use of the Method of Variation of Parameters. The derivation is a little detailed and the solution is sometimes messy, but the application of the method is straight forward if you can do the required integrals. We will first derive the needed equations and then do some examples.

We begin with the nonhomogeneous equation. Let's assume it is of the standard form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.78}$$

We know that the solution of the homogeneous equation can be written in terms of two linearly independent solutions, which we will call $y_1(x)$ and $y_2(x)$:

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Replacing the constants with functions, then we now longer have a solution to the homogeneous equation. Is it possible that we could stumble across the right functions with which to replace the constants and somehow end up with $f(x)$ when inserted into the left side of the differential equation? It turns out that we can.

So, let's assume that the constants are replaced with two unknown functions, which we will call $c_1(x)$ and $c_2(x)$. This change of the parameters is where the name of the method derives. Thus, we are assuming that a particular solution takes the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{2.79}$$

We assume the nonhomogeneous equation has a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

If this is to be a solution, then insertion into the differential equation should make it true. To do this we will first need to compute some derivatives.

The first derivative is given by

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{2.80}$$

Next we will need the second derivative. But, this will give use eight terms. So, we will first make an assumption. Let's assume that the last two terms add to zero:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{2.81}$$

It turns out that we will get the same results in the end if we did not assume this. The important thing is that it works!

So, we now have the first derivative as

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x). \tag{2.82}$$

The second derivative is then only four terms:

$$y_p'(x) = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x). \tag{2.83}$$

Now that we have the derivatives, we can insert the guess into the differential equation. Thus, we have

$$\begin{aligned} f(x) \quad = \quad & a(x)(c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)) \\ & + b(x)(c_1(x)y_1'(x) + c_2(x)y_2'(x)) \\ & + c(x)(c_1(x)y_1(x) + c_2(x)y_2(x)). \end{aligned} \tag{2.84}$$

Regrouping the terms, we obtain

$$\begin{aligned} f(x) \quad = \quad & c_1(x)(a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x)) \\ & c_2(x)(a(x)y_2''(x) + b(x)y_2'(x) + c(x)y_2(x)) \\ & + a(x)(c_1'(x)y_1'(x) + c_2'(x)y_2'(x)). \end{aligned} \tag{2.85}$$

Note that the first two rows vanish since y_1 and y_2 are solutions of the homogeneous problem. This leaves the equation

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}. \tag{2.86}$$

In summary, we have assumed a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

This is only possible if the unknown functions $c_1(x)$ and $c_2(x)$ satisfy the system of equations

To solve the differential equation $Ly = f$, we assume $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, for $Ly_{1,2} = 0$. Then, one need only solve a simple system of equations.

$$\boxed{\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{a(x)}. \end{aligned} \qquad (2.87)}$$

It is standard to solve this system for the derivatives of the unknown functions and then present the integrated forms. However, one could just start from here.

System (2.87) can be solved as

$$c_1'(x) = -\frac{fy_2}{aW(y_1, y_2)},$$

$$c_1'(x) = \frac{fy_1}{aW(y_1, y_2)},$$

where $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the Wronskian.

Example 2.17. *Consider the problem: $y'' - y = e^{2x}$. We want the general solution of this nonhomogeneous problem.*

The general solution to the homogeneous problem $y_h'' - y_h = 0$ is

$$y_h(x) = c_1 e^x + c_2 e^{-x}.$$

In order to use the Method of Variation of Parameters, we seek a solution of the form

$$y_p(x) = c_1(x)e^x + c_2(x)e^{-x}.$$

We find the unknown functions by solving the system in (2.87), which in this case becomes

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= e^{2x}. \end{aligned} \tag{2.88}$$

Adding these equations we find that

$$2c_1'e^x = e^{2x} \to c_1' = \frac{1}{2}e^x.$$

Solving for $c_1(x)$ we find

$$c_1(x) = \frac{1}{2}\int e^x\,dx = \frac{1}{2}e^x.$$

Subtracting the equations in the system yields

$$2c_2'e^{-x} = -e^{2x} \to c_2' = -\frac{1}{2}e^{3x}.$$

Thus,

$$c_2(x) = -\frac{1}{2}\int e^{3x}\,dx = -\frac{1}{6}e^{3x}.$$

The particular solution is found by inserting these results into y_p:

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (\frac{1}{2}e^x)e^x + (-\frac{1}{6}e^{3x})e^{-x} \\ &= \frac{1}{3}e^{2x}. \end{aligned} \tag{2.89}$$

Thus, we have the general solution of the nonhomogeneous problem as

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{3}e^{2x}.$$

Example 2.18. *Now consider the problem: $y'' + 4y = \sin x$.*

The solution to the homogeneous problem is

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x. \tag{2.90}$$

We now seek a particular solution of the form

$$y_h(x) = c_1(x)\cos 2x + c_2(x)\sin 2x.$$

We let $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, $a(x) = 1$, $f(x) = \sin x$ in system (2.87):

$$\begin{aligned} c_1'(x)\cos 2x + c_2'(x)\sin 2x &= 0 \\ -2c_1'(x)\sin 2x + 2c_2'(x)\cos 2x &= \sin x. \end{aligned} \tag{2.91}$$

Now, use your favorite method for solving a system of two equations and two unknowns. In this case, we can multiply the first equation by $2\sin 2x$ and the second equation by $\cos 2x$. Adding the resulting equations will eliminate the c_1' terms. Thus, we have

$$c_2'(x) = \frac{1}{2}\sin x \cos 2x = \frac{1}{2}(2\cos^2 x - 1)\sin x.$$

Inserting this into the first equation of the system, we have

$$c_1'(x) = -c_2'(x)\frac{\sin 2x}{\cos 2x} = -\frac{1}{2}\sin x \sin 2x = -\sin^2 x \cos x.$$

These can easily be solved:

$$c_2(x) = \frac{1}{2}\int (2\cos^2 x - 1)\sin x\, dx = \frac{1}{2}(\cos x - \frac{2}{3}\cos^3 x).$$

$$c_1(x) = -\int \sin^x \cos x\, dx = -\frac{1}{3}\sin^3 x.$$

The final step in getting the particular solution is to insert these functions into $y_p(x)$. This gives

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (-\frac{1}{3}\sin^3 x)\cos 2x + (\frac{1}{2}\cos x - \frac{1}{3}\cos^3 x)\sin x \\ &= \frac{1}{3}\sin x. \end{aligned} \tag{2.92}$$

So, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\sin x. \tag{2.93}$$

2.8 Numerical Solutions of ODEs

So FAR WE HAVE SEEN some of the standard methods for solving first and second order differential equations. However, we have had to restrict ourselves to very special cases in order to get nice analytical solutions to our initial value problems. While these are not the only equations for which we can get exact results (see Section 2.7.3 for another common class of second order differential equations), there are many cases in which exact solutions are not possible. In such cases we have to rely on approximation techniques, including the numerical solution of the equation at hand.

The use of numerical methods to obtain approximate solutions of differential equations and systems of differential equations has been known for some time. However, with the advent of powerful computers and desktop computers, we can now solve many of these problems with relative ease. The simple ideas used to solve first order differential equations can be extended to the solutions of more complicated systems of partial differential equations, such as the large scale problems of modeling ocean dynamics, weather systems and even cosmological problems stemming from general relativity.

In this section we will look at the simplest method for solving first order equations, Euler's Method. While it is not the most efficient method, it does provide us with a picture of how one proceeds and can be improved by introducing better techniques, which are typically covered in a numerical analysis text.

Let's consider the class of first order initial value problems of the form

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{2.94}$$

We are interested in finding the solution $y(x)$ of this equation which passes through the initial point (x_0, y_0) in the xy-plane for values of x in the interval $[a,b]$, where $a = x_0$. We will seek approximations of the solution at N points, labeled x_n for $n = 1, \ldots, N$. For equally spaced points we have $\Delta x = x_1 - x_0 = x_2 - x_1$, etc. Then, $x_n = x_0 + n\Delta x$. In Figure 2.13 we show three such points on the x-axis.

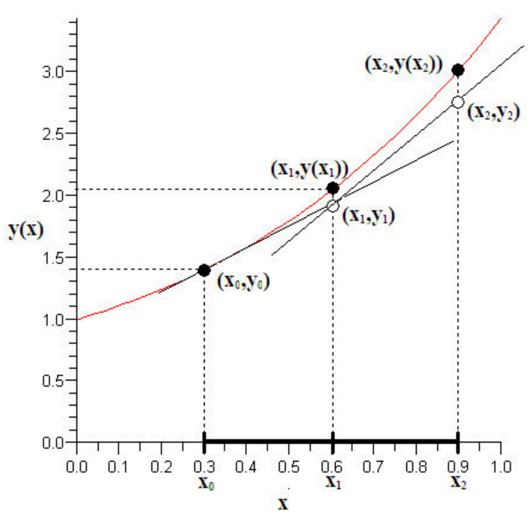


Figure 2.13: The basics of Euler's Method are shown. An interval of the x axis is broken into N subintervals. The approximations to the solutions are found using the slope of the tangent to the solution, given by $f(x,y)$. Knowing previous approximations at (x_{n-1}, y_{n-1}), one can determine the next approximation, y_n.

We will develop a simple numerical method, called Euler's Method. We rely on Figure 2.13 to do this. As already noted, we first break the interval of interest into N subintervals with $N+1$ points x_n. We already know a point on the solution $(x_0, y(x_0)) = (x_0, y_0)$. How do we find the solution for other x values?

We first note that the differential equation gives us the slope of the tangent line at $(x, y(x))$ of the solution $y(x)$. The slope is $f(x, y(x))$. Referring to Figure 2.13, we see the tangent line drawn at (x_0, y_0). We look now at $x = x_1$. A vertical line intersects both the solution curve and the tangent line. While we do not know the solution, we can determine the tangent line and find the intersection point. As seen in the figure, this intersection point is in theory close to the point on the solution curve. So, we will designate y_1 as the approximation of the solution $y(x_1)$. We just need to determine y_1.

The idea is simple. We approximate the derivative in the differential equation by its difference quotient:

$$\frac{dy}{dx} \approx \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{\Delta x}. \tag{2.95}$$

But, we have by the differential equation that the slope of the tangent to the curve at (x_0, y_0) is

$$y'(x_0) = f(x_0, y_0).$$

Thus,

$$\frac{y_1 - y_0}{\Delta x} \approx f(x_0, y_0). \tag{2.96}$$

So, we can solve this equation for y_1 to obtain

$$y_1 = y_0 + \Delta x f(x_0, y_0). \tag{2.97}$$

This give y_1 in terms of quantities that we know.

We now proceed to approximate $y(x_2)$. Referring to Figure 2.13, we see that this can be done by using the slope of the solution curve at (x_1, y_1). The corresponding tangent line is shown passing though (x_1, y_1) and we can then get the value of y_2. Following the previous argument, we find that

$$y_2 = y_1 + \Delta x f(x_1, y_1). \tag{2.98}$$

Continuing this procedure for all x_n, we arrive at the following numerical scheme for determining a numerical solution to Euler's equation:

$$\begin{aligned} y_0 &= y(x_0), \\ y_n &= y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}), \quad n = 1, \ldots, N. \end{aligned} \tag{2.99}$$

Example 2.19. *We will consider a standard example for which we know the exact solution. This way we can compare our results. The problem is given that*

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \tag{2.100}$$

find an approximation for $y(1)$.

First, we will do this by hand. We will break up the interval $[0,1]$, since we want the solution at $x=1$ and the initial value is at $x=0$. Let $\Delta x = 0.50$. Then, $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1.0$. Note that $N = \frac{b-a}{\Delta x} = 2$.

We can carry out Euler's Method systematically. We set up a table for the needed values. Such a table is shown in Table 2.1.

n	x_n	$y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1} = 0.5x_{n-1} + 1.5y_{n-1}$
0	0	1
1	0.5	$0.5(0) + 1.5(1.0) = 1.5$
2	1.0	$0.5(0.5) + 1.5(1.5) = 2.5$

Table 2.1: Application of Euler's Method for $y' = x + y$, $y(0) = 1$ and $\Delta x = 0.5$.

Note how the table is set up. There is a column for each x_n and y_n. The first row is the initial condition. We also made use of the function $f(x,y)$ in computing the y_n's. This sometimes makes the computation easier. As a result, we find that the desired approximation is given as $y_2 = 2.5$.

Is this a good result? Well, we could make the spatial increments smaller. Let's repeat the procedure for $\Delta x = 0.2$, or $N = 5$. The results are in Table 2.2.

n	x_n	$y_n = 0.2x_{n-1} + 1.2y_{n-1}$
0	0	1
1	0.2	$0.2(0) + 1.2(1.0) = 1.2$
2	0.4	$0.2(0.2) + 1.2(1.2) = 1.48$
3	0.6	$0.2(0.4) + 1.2(1.48) = 1.856$
4	0.8	$0.2(0.6) + 1.2(1.856) = 2.3472$
5	1.0	$0.2(0.8) + 1.2(2.3472) = 2.97664$

Table 2.2: Application of Euler's Method for $y' = x + y$, $y(0) = 1$ and $\Delta x = 0.2$.

Now we see that our approximation is $y_1 = 2.97664$. So, it looks like the value is near 3, but we cannot say much more. Decreasing Δx more shows that we are beginning to converge to a solution. We see this in Table 2.3.

Δx	$y_N \approx y(1)$
0.5	2.5
0.2	2.97664
0.1	3.187484920
0.01	3.409627659
0.001	3.433847864
0.0001	3.436291854

Table 2.3: Results of Euler's Method for $y' = x + y$, $y(0) = 1$ and varying Δx

Of course, these values were not done by hand. The last computation would have taken 1000 lines in the table, or at least 40 pages! One could use a computer to do this. A simple code in Maple would look like the following:

```
> restart:
> f:=(x,y)->y+x;
> a:=0:  b:=1:  N:=100:  h:=(b-a)/N;
> x[0]:=0: y[0]:=1:
  for i from 1 to N do
  y[i]:=y[i-1]+h*f(x[i-1],y[i-1]):
  x[i]:=x[0]+h*(i):
  od:
  evalf(y[N]);
```

In this case we could simply use the exact solution. The exact solution is easily found as

$$y(x) = 2e^x - x - 1.$$

(The reader can verify this.) So, the value we are seeking is

$$y(1) = 2e - 2 = 3.4365636\ldots.$$

Thus, even the last numerical solution was off by about 0.00027.

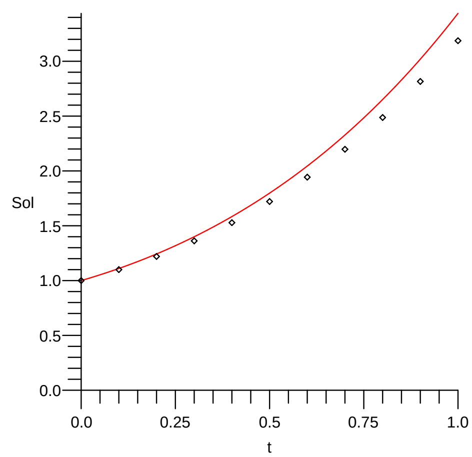


Figure 2.14: A comparison of the results Euler's Method to the exact solution for $y' = x + y$, $y(0) = 1$ and $N = 10$.

Adding a few extra lines for plotting, we can visually see how well the approximations compare to the exact solution. The Maple code for doing such a plot is given below.

```
> with(plots):
> Data:=[seq([x[i],y[i]],i=0..N)]:
> P1:=pointplot(Data,symbol=DIAMOND):
> Sol:=t->-t-1+2*exp(t);
> P2:=plot(Sol(t),t=a..b,Sol=0..Sol(b)):
> display({P1,P2});
```

We show in Figures 2.14-2.15 the results for $N = 10$ and $N = 100$. In Figure 2.14 we can see how quickly the numerical solution diverges from the exact solution. In Figure 2.15 we can see that visually the solutions agree, but we note that from Table 2.3 that for $\Delta x = 0.01$, the solution is still off in the second decimal place with a relative error of about 0.8%.

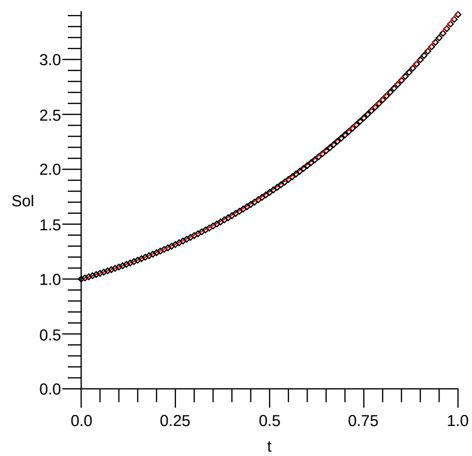


Figure 2.15: A comparison of the results Euler's Method to the exact solution for $y' = x + y$, $y(0) = 1$ and $N = 100$.

Why would we use a numerical method when we have the exact solution? Exact solutions can serve as test cases for our methods. We can make sure our code works before applying them to problems whose solution is not known.

There are many other methods for solving first order equations. One commonly used method is the fourth order Runge-Kutta method. This method has smaller errors at each step as compared to Euler's Method. It is well suited for programming and comes built-in in many packages like Maple and MATLAB. Typically, it is set up to handle systems of first order equations.

In fact, it is well known that nth order equations can be written as a system of n first order equations. Consider the simple second order equation

$$y'' = f(x, y).$$

This is a larger class of equations than the second order constant coefficient equation. We can turn this into a system of two first order differential equations by letting $u = y$ and $v = y' = u'$. Then, $v' = y'' = f(x, u)$. So, we have the first order system

$$\begin{aligned} u' &= v, \\ v' &= f(x, u). \end{aligned} \tag{2.101}$$

We will not go further into the Runge-Kutta Method here. You can find more about it in a numerical analysis text. However, we will see that systems of differential equations do arise naturally in physics. Such systems are often coupled equations and lead to interesting behaviors.

2.9 *Linear Systems*

2.9.1 *Coupled Oscillators*

IN THE LAST SECTION we saw that the numerical solution of second order equations, or higher, can be cast into systems of first order equations. Such systems are typically coupled in the sense that the solution of at least one of the equations in the system depends on knowing one of the other solutions in the system. In many physical systems this coupling takes place naturally. We will introduce a simple model in this section to illustrate the coupling of simple oscillators. However, we will reserve solving the coupled system of oscillators until the next chapter after exploring the needed mathematics.

There are many problems in physics that result in systems of equations. This is because the most basic law of physics is given by Newton's Second Law, which states that if a body experiences a net force, it will accelerate. Thus,

$$\sum \mathbf{F} = m\mathbf{a}.$$

Since $\mathbf{a} = \ddot{\mathbf{x}}$ we have a system of second order differential equations in general for three dimensional problems, or one second order differential equation for one dimensional problems.

We have already seen the simple problem of a mass on a spring as shown in Figure 2.4. Recall that the net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant and x is the elongation of the spring. When it is positive, the spring force is negative and when it is negative the spring force is positive. The equation for simple harmonic motion for the mass-spring system was found to be given by

$$m\ddot{x} + kx = 0.$$

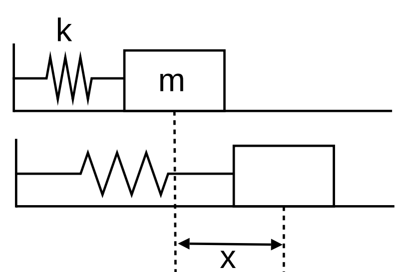


Figure 2.16: Spring-Mass system.

This second order equation can be written as a system of two first order equations in terms of the unknown position and velocity. We first set $y = \dot{x}$ and then rewrite the second order equation in terms of x and y. Thus, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{2.102}$$

The coefficient matrix for this system is $\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$, where $\omega^2 = \frac{k}{m}$.

One can look at more complicated spring-mass systems. Consider two blocks attached with two springs as in Figure 2.17. In this case we apply Newton's second law for each block. We will designate the elongations of each spring from equilibrium as x_1 and x_2. These are shown in Figure 2.17.

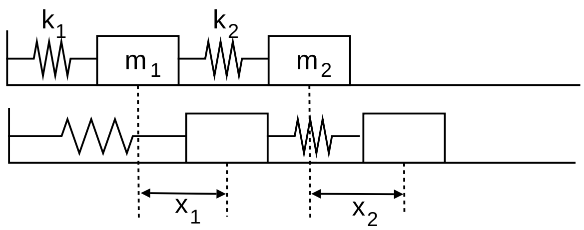


Figure 2.17: Spring-Mass system.

For mass m_1, the forces acting on it are due to each spring. The first spring with spring constant k_1 provides a force on m_1 of $-k_1x_1$. The second spring is stretched, or compressed, based upon the relative locations of the two masses. So, it will exert a force on m_1 of $k_2(x_2 - x_1)$.

Similarly, the only force acting directly on mass m_2 is provided by the restoring force from spring 2. So, that force is given by $-k_2(x_2 - x_1)$. The reader should think about the signs in each case.

Putting this all together, we apply Newton's Second Law to both masses. We obtain the two equations

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \tag{2.103}$$

Thus, we see that we have a coupled system of two second order differential equations.

One can rewrite this system of two second order equations as a system of four first order equations by letting $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$. This leads to the system

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) \\ \dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1). \end{aligned} \tag{2.104}$$

As we will see, this system can be written more compactly in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.105}$$

However, before we can solve this system of first order equations, we need to recall a few things from linear algebra. This will be done in the next chapter. For now, we will return to simpler systems and explore the behavior of typical solutions in these planar systems.

2.9.2 *Planar Systems*

We now consider examples of solving a coupled system of first order differential equations in the plane. We will focus on the theory of linear systems with constant coefficients. Understanding these simple systems helps in future studies of nonlinear systems, which contain much more interesting behaviors, such as the onset of chaos. In the next chapter we will return to these systems and describe a matrix approach to obtaining the solutions.

A general form for first order systems in the plane is given by a system of two equations for unknowns $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= P(x,y,t) \\ y'(t) &= Q(x,y,t). \end{aligned} \tag{2.106}$$

An *autonomous* system is one in which there is no explicit time dependence:

$$\begin{aligned} x'(t) &= P(x,y) \\ y'(t) &= Q(x,y). \end{aligned} \tag{2.107}$$

Otherwise the system is called *nonautonomous*.

A *linear system* takes the form

$$\begin{aligned} x' &= a(t)x + b(t)y + e(t) \\ y' &= c(t)x + d(t)y + f(t). \end{aligned} \tag{2.108}$$

A *homogeneous* linear system results when $e(t) = 0$ and $f(t) = 0$.

A *linear, constant coefficient system* of first order differential equations is given by

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f. \end{aligned} \tag{2.109}$$

We will focus on linear, homogeneous systems of constant coefficient first order differential equations:

$$\boxed{\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned}} \tag{2.110}$$

As we will see later, such systems can result by a simple translation of the unknown functions. These equations are said to be coupled if either $b \neq 0$ or $c \neq 0$.

We begin by noting that the system (2.110) can be rewritten as a second order constant coefficient linear differential equation, which we already know how to solve. We differentiate the first equation in system system (2.110) and systematically replace occurrences of y and y', since we also know from the first equation that $y = \frac{1}{b}(x' - ax)$. Thus, we have

$$\begin{aligned} x'' &= ax' + by' \\ &= ax' + b(cx + dy) \\ &= ax' + bcx + d(x' - ax). \end{aligned} \tag{2.111}$$

Rewriting the last line, we have

$$\boxed{x'' - (a + d)x' + (ad - bc)x = 0.} \tag{2.112}$$

This is a linear, homogeneous, constant coefficient ordinary differential equation. We know that we can solve this by first looking at the roots of the characteristic equation

$$r^2 - (a + d)r + ad - bc = 0 \tag{2.113}$$

and writing down the appropriate general solution for $x(t)$. Then we can find $y(t)$ using Equation (2.110):

$$y = \frac{1}{b}(x' - ax).$$

We now demonstrate this for a specific example.

Example 2.20. *Consider the system of differential equations*

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{2.114}$$

Carrying out the above outlined steps, we have that $x'' + 3x' - 4x = 0$. This can be shown as follows:

$$\begin{aligned} x'' &= -x' + 6y' \\ &= -x' + 6(x - 2y) \\ &= -x' + 6x - 12\left(\frac{x' + x}{6}\right) \\ &= -3x' + 4x \end{aligned} \tag{2.115}$$

The resulting differential equation has a characteristic equation of $r^2 + 3r - 4 = 0$. *The roots of this equation are* $r = 1, -4$. *Therefore,* $x(t) = c_1e^t + c_2e^{-4t}$. *But, we still need* $y(t)$. *From the first equation of the system we have*

$$y(t) = \frac{1}{6}(x' + x) = \frac{1}{6}(2c_1e^t - 3c_2e^{-4t}).$$

Thus, the solution to the system is

$$\begin{aligned} x(t) &= c_1e^t + c_2e^{-4t}, \\ y(t) &= \tfrac{1}{3}c_1e^t - \tfrac{1}{2}c_2e^{-4t}. \end{aligned} \tag{2.116}$$

Sometimes one needs initial conditions. For these systems we would specify conditions like $x(0) = x_0$ and $y(0) = y_0$. These would allow the determination of the arbitrary constants as before.

Example 2.21. *Solve*

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \tag{2.117}$$

given $x(0) = 2$, $y(0) = 0$.

We already have the general solution of this system in (2.116). Inserting the initial conditions, we have

$$\begin{aligned} 2 &= c_1 + c_2, \\ 0 &= \tfrac{1}{3}c_1 - \tfrac{1}{2}c_2. \end{aligned} \tag{2.118}$$

Solving for c_1 *and* c_2 *gives* $c_1 = 6/5$ *and* $c_2 = 4/5$. *Therefore, the solution of the initial value problem is*

$$\begin{aligned} x(t) &= \tfrac{2}{5}\left(3e^t + 2e^{-4t}\right), \\ y(t) &= \tfrac{2}{5}\left(e^t - e^{-4t}\right). \end{aligned} \tag{2.119}$$

2.9.3 Equilibrium Solutions and Nearby Behaviors

In studying systems of differential equations, it is often useful to study the behavior of solutions without obtaining an algebraic form for the solution. This is done by exploring equilibrium solutions and solutions nearby equilibrium solutions. Such techniques will be seen to be useful later in studying nonlinear systems.

We begin this section by studying *equilibrium solutions* of system (2.109). For equilibrium solutions the system does not change in time. Therefore, equilibrium solutions satisfy the equations $x' = 0$ and $y' = 0$. Of course, this can only happen for constant solutions. Let x_0 and y_0

be the (constant) equilibrium solutions. Then, x_0 and y_0 must satisfy the system

$$\begin{aligned} 0 &= ax_0 + by_0 + e, \\ 0 &= cx_0 + dy_0 + f. \end{aligned} \tag{2.120}$$

This is a linear system of nonhomogeneous algebraic equations. One only has a unique solution when the determinant of the system is not zero, i.e., $ad - bc \neq 0$. Using Cramer's (determinant) Rule for solving such systems, we have

$$\boxed{x_0 = -\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y_0 = -\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \qquad (2.121)}$$

If the system is homogeneous, $e = f = 0$, then we have that the origin is the equilibrium solution; i.e., $(x_0, y_0) = (0, 0)$. Often we will have this case since one can always make a change of coordinates from (x, y) to (u, v) by $u = x - x_0$ and $v = y - y_0$. Then, $u_0 = v_0 = 0$.

Next we are interested in the behavior of solutions near the equilibrium solutions. Later this behavior will be useful in analyzing more complicated nonlinear systems. We will look at some simple systems that are readily solved.

Example 2.22. Stable Node (sink)

Consider the system

$$\begin{aligned} x' &= -2x \\ y' &= -y. \end{aligned} \tag{2.122}$$

This is a simple uncoupled system. Each equation is simply solved to give

$$x(t) = c_1 e^{-2t} \text{ and } y(t) = c_2 e^{-t}.$$

In this case we see that all solutions tend towards the equilibrium point, $(0, 0)$. *This will be called a stable node, or a sink.*

Before looking at other types of solutions, we will explore the stable node in the above example. There are several methods of looking at the behavior of solutions. We can look at solution plots of the dependent versus the independent variables, or we can look in the xy-plane at the parametric curves $(x(t), y(t))$.

Solution Plots: One can plot each solution as a function of t given a set of initial conditions. Examples are are shown in Figure 2.18 for several initial conditions. Note that the solutions decay for large t. Special cases result for various initial conditions. Note that for $t = 0$,

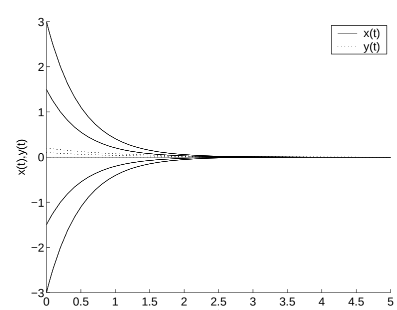


Figure 2.18: Plots of solutions of Example 2.22 for several initial conditions.

$x(0) = c_1$ and $y(0) = c_2$. (Of course, one can provide initial conditions at any $t = t_0$. It is generally easier to pick $t = 0$ in our general explanations.) If we pick an initial condition with c_1=0, then $x(t) = 0$ for all t. One obtains similar results when setting $y(0) = 0$.

Phase Portrait: There are other types of plots which can provide additional information about the solutions even if we cannot find the exact solutions as we can for these simple examples. In particular, one can consider the solutions $x(t)$ and $y(t)$ as the coordinates along a parameterized path, or curve, in the plane: $\mathbf{r} = (x(t), y(t))$ Such curves are called *trajectories* or *orbits*. The xy-plane is called the *phase plane* and a collection of such orbits gives a *phase portrait* for the family of solutions of the given system.

One method for determining the equations of the orbits in the phase plane is to eliminate the parameter t between the known solutions to get a relationship between x and y. In the above example we can do this, since the solutions are known. In particular, we have

$$x = c_1 e^{-2t} = c_1 \left(\frac{y}{c_2}\right)^2 \equiv Ay^2.$$

Another way to obtain information about the orbits comes from noting that the slopes of the orbits in the xy-plane are given by dy/dx. For autonomous systems, we can write this slope just in terms of x and y. This leads to a first order differential equation, which possibly could be solved analytically, solved numerically, or just used to produce a *direction field*. We will see that direction fields are useful in determining qualitative behaviors of the solutions without actually finding explicit solutions.

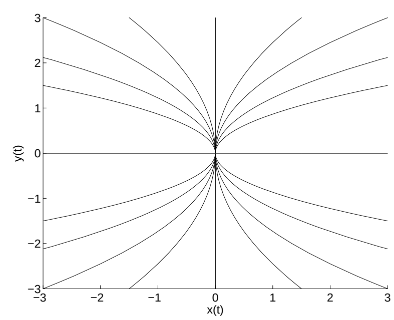


Figure 2.19: Orbits for Example 2.22.

First we will obtain the orbits for Example 2.22 by solving the corresponding slope equation. First, recall that for trajectories defined parametrically by $x = x(t)$ and $y = y(t)$, we have from the Chain Rule for $y = y(x(t))$ that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Therefore,

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.} \tag{2.123}$$

For the system in (2.122) we use Equation (2.123) to obtain the equation for the slope at a point on the orbit:

$$\frac{dy}{dx} = \frac{y}{2x}.$$

The general solution of this first order differential equation is found using separation of variables as $x = Ay^2$ for A an arbitrary constant.

Plots of these solutions in the phase plane are given in Figure 2.19. [Note that this is the same form for the orbits that we had obtained above by eliminating t from the solution of the system.]

Once one has solutions to differential equations, we often are interested in the long time behavior of the solutions. Given a particular initial condition (x_0, y_0), how does the solution behave as time increases? For orbits near an equilibrium solution, do the solutions tend towards, or away from, the equilibrium point? The answer is obvious when one has the exact solutions $x(t)$ and $y(t)$. However, this is not always the case.

Let's consider the above example for initial conditions in the first quadrant of the phase plane. For a point in the first quadrant we have that

$$dx/dt = -2x < 0,$$

meaning that as $t \to \infty$, $x(t)$ get more negative. Similarly,

$$dy/dt = -y < 0,$$

indicates that $y(t)$ is also getting smaller for this problem. Thus, these orbits tend towards the origin as $t \to \infty$. This qualitative information was obtained without relying on the known solutions to the problem.

Direction Fields: Another way to determine the behavior of our system is to draw the direction field. Recall that a direction field is a vector field in which one plots arrows in the direction of tangents to the orbits. This is done because the slopes of the tangent lines are given by dy/dx. For the system (2.110), the slope is

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

In general, for nonautonomous systems, we obtain a first order differential equation of the form

$$\frac{dy}{dx} = F(x, y).$$

This particular equation can be solved by the reader.

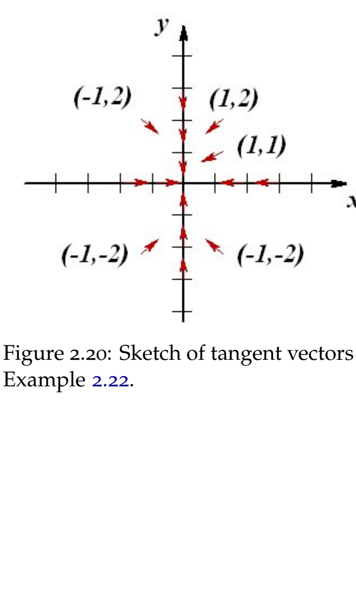


Figure 2.20: Sketch of tangent vectors for Example 2.22.

Example 2.23. *Draw the direction field for Example 2.22.*

We can use software to draw direction fields. However, one can sketch these fields by hand. we have that the slope of the tangent at this point is given by

$$\frac{dy}{dx} = \frac{-y}{-2x} = \frac{y}{2x}.$$

For each point in the plane one draws a piece of tangent line with this slope. In Figure 2.20 we show a few of these. For $(x, y) = (1, 1)$ the slope is $dy/dx = 1/2$. So, we draw an arrow with slope $1/2$ at this point. From system (2.122), *we have that x' and y' are both negative at this point. Therefore, the vector points down and to the left.*

We can do this for several points, as shown in Figure 2.20. Sometimes one can quickly sketch vectors with the same slope. For this example, when $y = 0$, the slope is zero and when $x = 0$ the slope is infinite. So, several vectors can be provided. Such vectors are tangent to curves known as isoclines in which $\frac{dy}{dx}$ =constant.

It is often difficult to provide an accurate sketch of a direction field. Computer software can be used to provide a better rendition. For Example 2.22 the direction field is shown in Figure 2.21. Looking at this direction field, one can begin to "see" the orbits by following the tangent vectors.

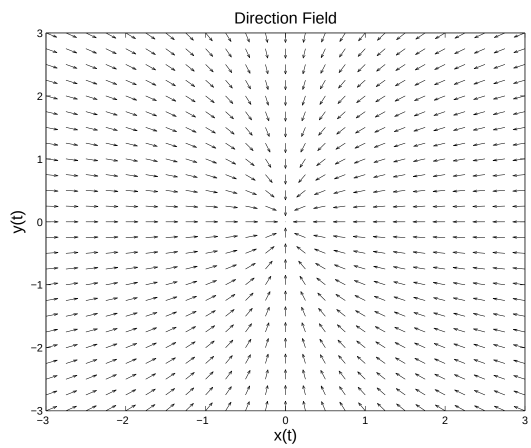


Figure 2.21: Direction field for Example 2.22.

Of course, one can superimpose the orbits on the direction field. This is shown in Figure 2.22. Are these the patterns you saw in Figure 2.21?

In this example we see all orbits "flow" towards the origin, or equilibrium point. Again, this is an example of what is called a stable node or a sink. (Imagine what happens to the water in a sink when the drain is unplugged.)

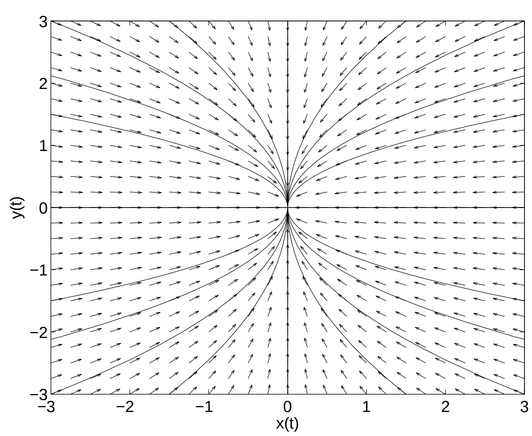


Figure 2.22: Phase portrait for Example 2.22.

Example 2.24. Saddle

Consider the system

$$\begin{aligned} x' &= -x \\ y' &= y. \end{aligned} \tag{2.124}$$

This is another uncoupled system. The solutions are again simply gotten by integration. We have that $x(t) = c_1 e^{-t}$ and $y(t) = c_2 e^t$. Here we have that x decays as t gets large and y increases as t gets large. In particular, if one picks initial conditions with $c_2 = 0$, then orbits follow the x-axis towards the origin. For initial points with $c_1 = 0$, orbits originating on the y-axis will flow away from the origin. Of course, in these cases the origin is an equilibrium point and once at equilibrium, one remains there.

In fact, there is only one line on which to pick initial conditions such that the orbit leads towards the equilibrium point. No matter how small c_2 is, sooner, or later, the exponential growth term will dominate the solution. One can see this behavior in Figure 2.23.

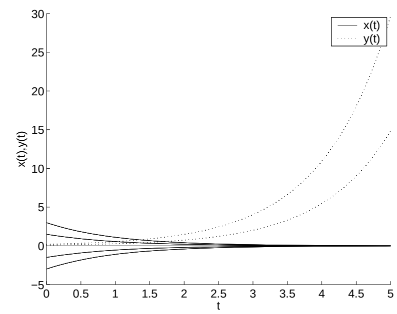


Figure 2.23: Plots of solutions of Example 2.24 for several initial conditions.

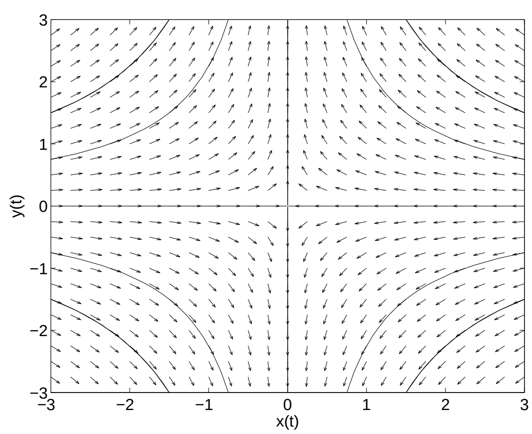


Figure 2.24: Phase portrait for Example 2.24, a saddle.

Similar to the first example, we can look at a variety of plots. These are given by Figures 2.23-2.24. The orbits can be obtained from the system as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{x}.$$

The solution is $y = \frac{A}{x}$. For different values of $A \neq 0$ we obtain a family of hyperbolae. These are the same curves one might obtain for the level curves of a surface known as a saddle surface, $z = xy$. Thus, this type of equilibrium point is classified as a *saddle* point. From the phase portrait we can verify that there are many orbits that lead away from the origin (equilibrium point), but there is one line of initial conditions that leads to the origin and that is the x-axis. In this case, the line of initial conditions is given by the x-axis.

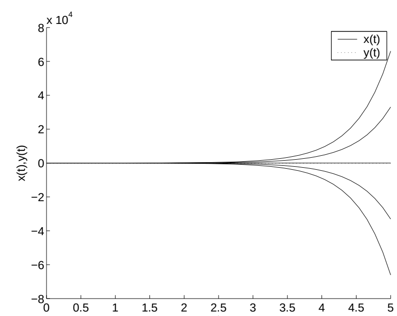


Figure 2.25: Plots of solutions of Example 2.25 for several initial conditions.

Example 2.25. Unstable Node (source)

$$\begin{aligned} x' &= 2x \\ y' &= y. \end{aligned} \tag{2.125}$$

This example is similar to Example 2.22. The solutions are obtained by replacing t with $-t$. The solutions, orbits and direction fields can be seen in Figures 2.25-2.26. This is once again a node, but all orbits lead away from the equilibrium point. It is called an unstable node or a source.

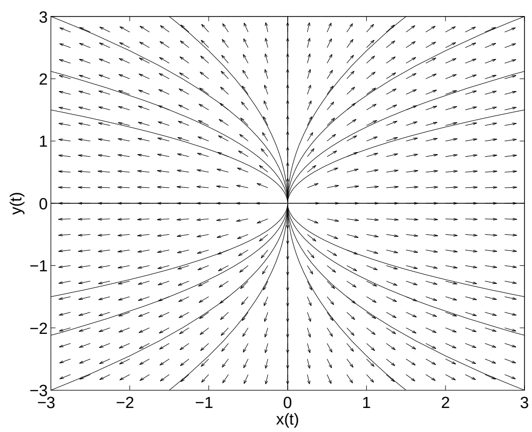


Figure 2.26: Phase portrait for Example 2.25, an unstable node or source.

Example 2.26. Center

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{2.126}$$

This system is a simple, coupled system. Neither equation can be solved without some information about the other unknown function. However, we can differentiate the first equation and use the second equation to obtain

$$x'' + x = 0.$$

We recognize this equation from the last chapter as one that appears in the study of simple harmonic motion. The solutions are pure sinusoidal oscillations:

$$x(t) = c_1 \cos t + c_2 \sin t, \quad y(t) = -c_1 \sin t + c_2 \cos t.$$

In the phase plane the trajectories can be determined either by looking at the direction field, or solving the first order equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

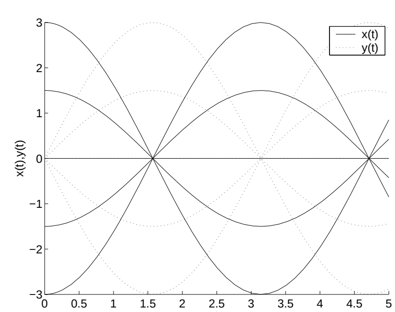


Figure 2.27: Plots of solutions of Example 2.26 for several initial conditions.

Performing a separation of variables and integrating, we find that

$$x^2 + y^2 = C.$$

Thus, we have a family of circles for $C > 0$. (Can you prove this using the general solution?) Looking at the results graphically in Figures 2.27-2.28 confirms this result. This type of point is called a *center*.

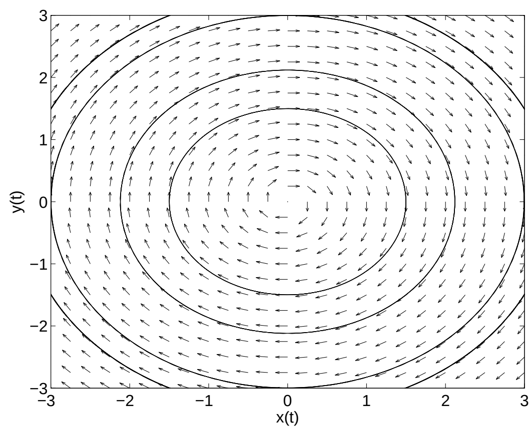


Figure 2.28: Phase portrait for Example 2.26, a center.

Example 2.27. Focus (spiral)

$$\begin{aligned} x' &= \alpha x + y \\ y' &= -x. \end{aligned} \tag{2.127}$$

In this example, we will see an additional set of behaviors of equilibrium points in planar systems. We have added one term, αx, to the system in Example 2.26. We will consider the effects for two specific values of the parameter: $\alpha = 0.1, -0.2$. The resulting behaviors are shown in the remaining graphs. We see orbits that look like spirals. These orbits are stable and unstable *spirals* (or *foci*, the plural of focus.)

We can understand these behaviors by once again relating the system of first order differential equations to a second order differential equation. Using the usual method for obtaining a second order equation form a system, we find that $x(t)$ satisfies the differential equation

$$x'' - \alpha x' + x = 0.$$

We recall from our first course that this is a form of *damped simple harmonic motion*. We will explore the different types of solutions that will result for various α's.

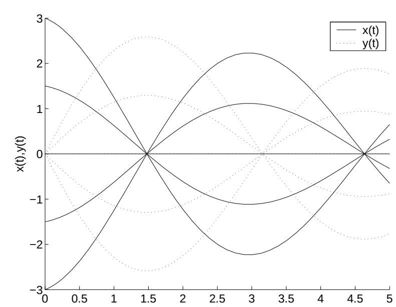


Figure 2.29: Plots of solutions of Example 2.27 for several initial conditions, $\alpha = -0.2$.

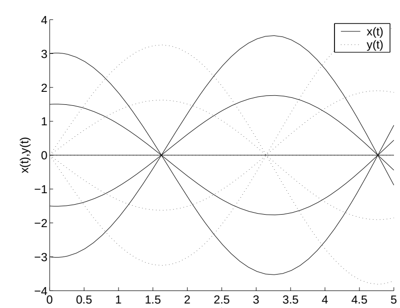


Figure 2.30: Plots of solutions of Example 2.27 for several initial conditions, $\alpha = 0.1$.

The characteristic equation is $r^2 - \alpha r + 1 = 0$. The solution of this quadratic equation is

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

There are five special cases to consider as shown below.

Classification of Solutions of $x'' - \alpha x' + x = 0$

1. $\alpha = -2$. There is one real solution. This case is called *critical damping* since the solution $r = -1$ leads to exponential decay. The solution is $x(t) = (c_1 + c_2 t)e^{-t}$.

2. $\alpha < -2$. There are two real, negative solutions, $r = -\mu, -\nu$, $\mu, \nu > 0$. The solution is $x(t) = c_1 e^{-\mu t} + c_2 e^{-\nu t}$. In this case we have what is called *overdamped* motion. There are no oscillations

3. $-2 < \alpha < 0$. There are two complex conjugate solutions $r = \alpha/2 \pm i\beta$ with real part less than zero and $\beta = \frac{\sqrt{4-\alpha^2}}{2}$. The solution is $x(t) = (c_1 \cos \beta t + c_2 \sin \beta t)e^{\alpha t/2}$. Since $\alpha < 0$, this consists of a decaying exponential times oscillations. This is often called an *underdamped oscillation*.

4. $\alpha = 0$. This leads to *simple harmonic motion*.

5. $0 < \alpha < 2$. This is similar to the underdamped case, except $\alpha > 0$. The solutions are growing oscillations.

6. $\alpha = 2$. There is one real solution. The solution is $x(t) = (c_1 + c_2 t)e^t$. It leads to unbounded growth in time.

7. For $\alpha > 2$. There are two real, positive solutions $r = \mu, \nu > 0$. The solution is $x(t) = c_1 e^{\mu t} + c_2 e^{\nu t}$, which grows in time.

For $\alpha < 0$ the solutions are losing energy, so the solutions can oscillate with a diminishing amplitude. (See Figure 2.29.) For $\alpha > 0$, there is a growth in the amplitude, which is not typical. (See Figure 2.30.) Of course, there can be overdamped motion if the magnitude of α is too large.

Example 2.28. Degenerate Node *For this example, we will write out the solutions. It is a coupled system for which only the second equation is coupled.*

$$\begin{aligned} x' &= -x \\ y' &= -2x - y. \end{aligned} \tag{2.128}$$

There are two possible approaches:

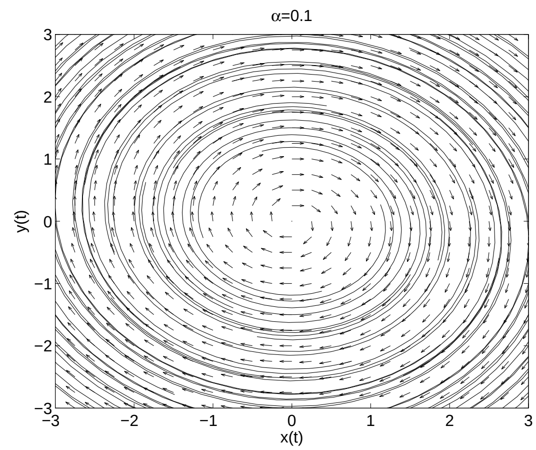


Figure 2.31: Phase portrait for Example 2.27 with $\alpha = 0.1$. This is an unstable focus, or spiral.

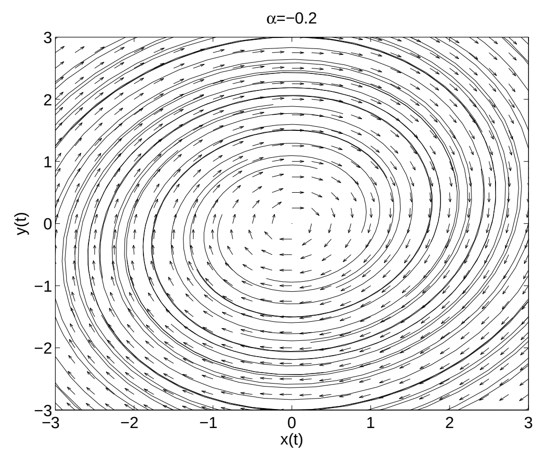


Figure 2.32: Phase portrait for Example 2.27 with $\alpha = -0.2$. This is a stable focus, or spiral.

a. We could solve the first equation to find $x(t) = c_1 e^{-t}$. Inserting this solution into the second equation, we have

$$y' + y = -2c_1 e^{-t}.$$

This is a relatively simple linear first order equation for $y = y(t)$. The integrating factor is $\mu = e^t$. The solution is found as $y(t) = (c_2 - 2c_1 t)e^{-t}$.

b. Another method would be to proceed to rewrite this as a second order equation. Computing x'' does not get us very far. So, we look at

$$\begin{aligned} y'' &= -2x' - y' \\ &= 2x - y' \\ &= -2y' - y. \end{aligned} \tag{2.129}$$

Therefore, y satisfies

$$y'' + 2y' + y = 0.$$

The characteristic equation has one real root, $r = -1$. So, we write

$$y(t) = (k_1 + k_2 t)e^{-t}.$$

This is a stable degenerate node. Combining this with the solution $x(t) = c_1e^{-t}$, *we can show that* $y(t) = (c_2 - 2c_1t)e^{-t}$ *as before.*

In Figure 2.33 we see several orbits in this system. It differs from the stable node show in Figure 2.19 in that there is only one direction along which the orbits approach the origin instead of two. If one picks $c_1 = 0$, *then* $x(t) = 0$ *and* $y(t) = c_2e^{-t}$. *This leads to orbits running along the y-axis as seen in the figure.*

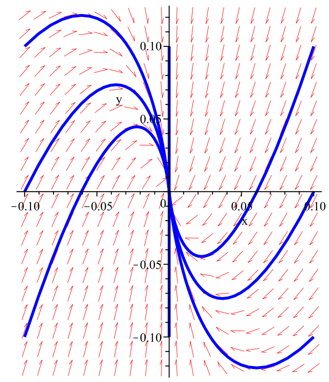


Figure 2.33: Plots of solutions of Example 2.28 for several initial conditions.

Example 2.29. A Line of Equilibria, Zero Root

$$\begin{aligned} x' &= 2x - y \\ y' &= -2x + y. \end{aligned} \tag{2.130}$$

In this last example, we have a coupled set of equations. We rewrite it as a second order differential equation:

$$\begin{aligned} x'' &= 2x' - y' \\ &= 2x' - (-2x + y) \\ &= 2x' + 2x + (x' - 2x) = 3x'. \end{aligned} \tag{2.131}$$

So, the second order equation is

$$x'' - 3x' = 0$$

and the characteristic equation is $0 = r(r - 3)$. *This gives the general solution as*

$$x(t) = c_1 + c_2e^3t$$

and thus

$$y = 2x - x' = 2(c_1 + c_2^3t) - (3c_2e^{3t}) = 2c_1 - c_2e^{3t}.$$

In Figure 2.34 we show the direction field. The constant slope field seen in this example is confirmed by a simple computation:

$$\frac{dy}{dx} = \frac{-2x + y}{2x - y} = -1.$$

Furthermore, looking at initial conditions with $y = 2x$, *we have at* $t = 0$,

$$2c_1 - c_2 = 2(c_1 + c_2) \quad \Rightarrow \quad c_2 = 0.$$

Therefore, points on this line remain on this line forever, $(x, y) = (c_1, 2c_1)$. *This line of fixed points is called a line of equilibria.*

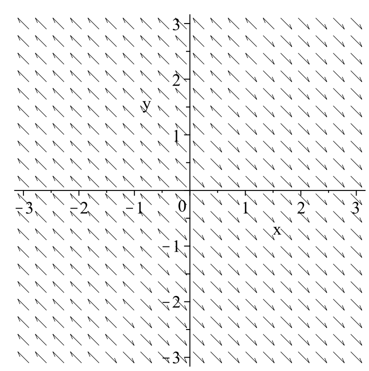


Figure 2.34: Plots of direction field of Example 2.29.

2.9.4 Polar Representation of Spirals

IN THE EXAMPLES with a center or a spiral, one might be able to write the solutions in polar coordinates. Recall that a point in the plane can be described by either Cartesian (x,y) or polar (r,θ) coordinates. Given the polar form, one can find the Cartesian components using

$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Given the Cartesian coordinates, one can find the polar coordinates using

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}. \tag{2.132}$$

Since x and y are functions of t, then naturally we can think of r and θ as functions of t. The equations that they satisfy are obtained by differentiating the above relations with respect to t.

Differentiating the first equation in (2.132) gives

$$rr' = xx' + yy'.$$

Inserting the expressions for x' and y' from system 2.110, we have

$$rr' = x(ax+by) + y(cx+dy).$$

In some cases this may be written entirely in terms of r's. Similarly, we have that

$$\theta' = \frac{xy' - yx'}{r^2},$$

which the reader can prove for homework.

In summary, when converting first order equations from rectangular to polar form, one needs the relations below.

Time Derivatives of Polar Variables

$$\begin{aligned} r' &= \frac{xx'+yy'}{r}, \\ \theta' &= \frac{xy'-yx'}{r^2}. \end{aligned} \tag{2.133}$$

Example 2.30. *Rewrite the following system in polar form and solve the resulting system.*

$$\begin{aligned} x' &= ax+by \\ y' &= -bx+ay. \end{aligned} \tag{2.134}$$

We first compute r' and θ':

$$rr' = xx' + yy' = x(ax+by) + y(-bx+ay) = ar^2.$$

$$r^2\theta' = xy' - yx' = x(-bx + ay) - y(ax + by) = -br^2.$$

This leads to simpler system

$$\begin{aligned} r' &= ar \\ \theta' &= -b. \end{aligned} \tag{2.135}$$

This system is uncoupled. The second equation in this system indicates that we traverse the orbit at a constant rate in the clockwise direction. Solving these equations, we have that $r(t) = r_0 e^{at}, \quad \theta(t) = \theta_0 - bt$. *Eliminating* t *between these solutions, we finally find the polar equation of the orbits:*

$$r = r_0 e^{-a(\theta-\theta_0)t/b}.$$

If you graph this for $a \neq 0$, *you will get stable or unstable spirals.*

Example 2.31. *Consider the specific system*

$$\begin{aligned} x' &= -y + x \\ y' &= x + y. \end{aligned} \tag{2.136}$$

In order to convert this system into polar form, we compute

$$rr' = xx' + yy' = x(-y + x) + y(x + y) = r^2.$$

$$r^2\theta' = xy' - yx' = x(x + y) - y(-y + x) = r^2.$$

This leads to simpler system

$$\begin{aligned} r' &= r \\ \theta' &= 1. \end{aligned} \tag{2.137}$$

Solving these equations yields

$$r(t) = r_0 e^t, \quad \theta(t) = t + \theta_0.$$

Eliminating t *from this solution gives the orbits in the phase plane,* $r(\theta) = r_0 e^{\theta-\theta_0}$.

A more complicated example arises for a nonlinear system of differential equations. Consider the following example.

Example 2.32.

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2). \end{aligned} \tag{2.138}$$

Transforming to polar coordinates, one can show that in order to convert this system into polar form, we compute

$$r' = r(1 - r^2), \quad \theta' = 1.$$

This uncoupled system can be solved and this is left to the reader.

2.10 Appendix: The Nonlinear Pendulum

We can also make the simple pendulum more realistic by adding damping. This could be due to energy loss in the way the string is attached to the support or due to the drag on the mass, etc. Assuming that the damping is proportional to the angular velocity, we have equations for the damped nonlinear and damped linear pendula:

$$L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0. \tag{2.139}$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \tag{2.140}$$

Finally, we can add forcing. Imagine that the support is attached to a device to make the system oscillate horizontally at some frequency. Then we could have equations such as

$$L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t. \tag{2.141}$$

We will look at these and other oscillation problems later in our discussion.

Before returning to studying the equilibrium solutions of the nonlinear pendulum, we will look at how far we can get at obtaining analytical solutions. First, we investigate the simple linear pendulum.

The linear pendulum equation (2.32) is a constant coefficient second order linear differential equation. The roots of the characteristic equations are $r = \pm\sqrt{\frac{g}{L}}i$. Thus, the general solution takes the form

$$\theta(t) = c_1\cos(\sqrt{\frac{g}{L}}t) + c_2\sin(\sqrt{\frac{g}{L}}t). \tag{2.142}$$

We note that this is usually simplified by introducing the angular frequency

$$\omega \equiv \sqrt{\frac{g}{L}}. \tag{2.143}$$

One consequence of this solution, which is used often in introductory physics, is an expression for the period of oscillation of a simple pendulum. The period is found to be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{g}{L}}. \tag{2.144}$$

As we have seen, this value for the period of a simple pendulum was derived assuming a small angle approximation. How good is this approximation? What is meant by a *small* angle? We could recall from calculus that the Taylor series approximation of $\sin\theta$ about $\theta = 0$:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots. \tag{2.145}$$

One can obtain a bound on the error when truncating this series to one term after taking a numerical analysis course. But we can just simply plot the relative error, which is defined as

$$\text{Relative Error } = \frac{\sin\theta - \theta}{\sin\theta}.$$

A plot of the relative error is given in Figure 2.35. Thus for $\theta \approx 0.4$ radians (or, degrees) we have that the relative error is about 4%.

We would like to do better than this. So, we now turn to the nonlinear pendulum. We first rewrite Equation (2.141) is the simpler form

$$\ddot{\theta} + \omega^2\theta = 0. \tag{2.146}$$

We next employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 + \int^{\theta(t)} F(\phi)\, d\phi\right] = (\ddot{\theta} + F(\theta))\dot{\theta}.$$

For our problem, we multiply Equation (2.146) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2\theta\dot{\theta} = 0$$

and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \tag{2.147}$$

Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \sqrt{2(c + \omega^2\cos\theta)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2\cos\theta)}}.$$

Of course, one needs to be able to do the integral. When one gets a solution in this implicit form, one says that the problem has been solved by quadratures. Namely, the solution is given in terms of some integral.

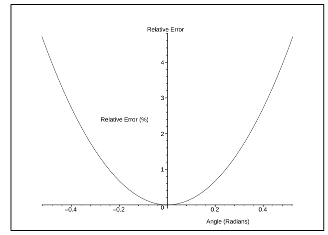


Figure 2.35: The relative error in percent when approximating $\sin\theta$ by θ.

In fact, the above integral can be transformed into what is know as an elliptic integral of the first kind. We will rewrite our result and then use it to obtain an approximation to the period of oscillation of the nonlinear pendulum, leading to corrections to the linear result found earlier.

We will first rewrite the constant found in (2.147). This requires a little physics. The swinging of a mass on a string, assuming no energy loss at the pivot point, is a conservative process. Namely, the total mechanical energy is conserved. Thus, the total of the kinetic and gravitational potential energies is a constant. The kinetic energy of the masses on the string is given as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

The potential energy is the gravitational potential energy. If we set the potential energy to zero at the bottom of the swing, then the potential energy is $U = mgh$, where h is the height that the mass is from the bottom of the swing. A little trigonometry gives that $h = L(1-\cos\theta)$. So,

$$U = mgL(1-\cos\theta).$$

So, the total mechanical energy is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1-\cos\theta). \tag{2.148}$$

We note that a little rearranging shows that we can relate this to Equation (2.147):

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \frac{1}{mL^2}E - \omega^2 = c.$$

We can use Equation (2.148) to get a value for the total energy. At the top of the swing the mass is not moving, if only for a moment. Thus, the kinetic energy is zero and the total energy is pure potential energy. Letting θ_0 denote the angle at the highest position, we have that

$$E = mgL(1-\cos\theta_0) = mL^2\omega^2(1-\cos\theta_0).$$

Therefore, we have found that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \omega^2(1-\cos\theta_0). \tag{2.149}$$

Using the half angle formula,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta),$$

we can rewrite Equation (2.149) as

$$\frac{1}{2}\dot{\theta}^2 = 2\omega^2\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right]. \tag{2.150}$$

Solving for θ', we have

$$\frac{d\theta}{dt} = 2\omega \left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right]^{1/2}. \tag{2.151}$$

One can now apply separation of variables and obtain an integral similar to the solution we had obtained previously. Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, we have that

$$T = \frac{2}{\omega}\int_0^{\theta_0} \frac{d\phi}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}. \tag{2.152}$$

This result is not much different than our previous result, but we can now easily transform the integral into an elliptic integral. [3] We define

$$z = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$

and

$$k = \sin\frac{\theta_0}{2}.$$

Then Equation (2.152) becomes

$$T = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{2.153}$$

This is done by noting that $dz = \frac{1}{2k}\cos\frac{\theta}{2}\,d\theta = \frac{1}{2k}(1-k^2z^2)^{1/2}\,d\theta$ and that $\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2} = k^2(1-z^2)$. The integral in this result is an elliptic integral of the first kind. In particular, the elliptic integral of the first kind is defined as

$$F(\phi, k) \equiv \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

In some contexts, this is known as the incomplete elliptic integral of the first kind and $K(k) = F(\frac{\pi}{2}, k)$ is called the complete integral of the first kind.

There are table of values for elliptic integrals and now one can use a computer algebra system to compute values of such integrals. For small angles, we have that k is small. So, we can develop a series expansion for the period, T, for small k. This is simply done by first expanding

$$(1-k^2z^2)^{-1/2} = 1 + \frac{1}{2}k^2z^2 + \frac{3}{8}k^2z^4 + O((kz)^6)$$

using the binomial expansion which we review later in the text. Inserting the expansion in the integrand and integrating term by term, one finds that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right]. \tag{2.154}$$

[3] Elliptic integrals were first studied by Leonhard Euler and Giulio Carlo de' Toschi di Fagnano (1682-1766) , who studied the lengths of curves such as the ellipse and the lemniscate, $(x^2+y^2)^2 = x^2 - y^2$.

This expression gives further corrections to the linear result, which only provides the first term. In Figure 2.36 we show the relative errors incurred when keeping the k^2 and k^4 terms versus not keeping them.

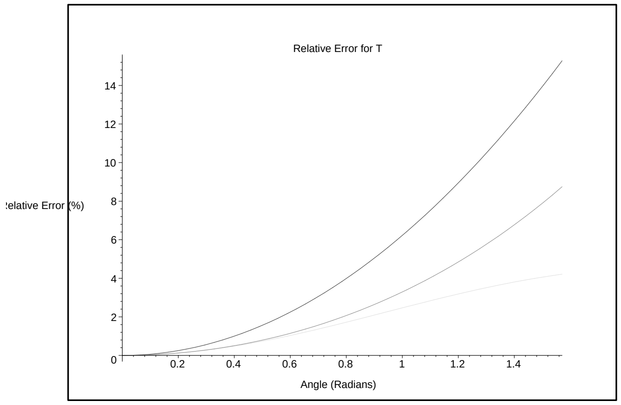


Figure 2.36: The relative error in percent when approximating the exact period of a nonlinear pendulum with one, two, or three terms in Equation (2.154).

Problems

1. Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $\frac{dy}{dx} = \frac{e^x}{2y}$.

b. $\frac{dy}{dt} = y^2(1+t^2)$, $y(0) = 1$.

c. $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{x}$.

d. $xy' = y(1-2y), \quad y(1) = 2$.

e. $y' - (\sin x)y = \sin x$.

f. $xy' - 2y = x^2, y(1) = 1$.

g. $\frac{ds}{dt} + 2s = st^2, \quad , s(0) = 1$.

h. $x' - 2x = te^{2t}$.

i. $\frac{dy}{dx} + y = \sin x$, $y(0) = 0$.

j. $\frac{dy}{dx} - \frac{3}{x}y = x^3$, $y(1) = 4$.

2. Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 9y' + 20y = 0$.

b. $y'' - 3y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

c. $x^2y'' + 5xy' + 4y = 0, \quad x > 0.$

d. $x^2y'' - 2xy' + 3y = 0, \quad x > 0.$

3. Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

a. Find the 1-parameter family of solutions (general solution) of this equation.

b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

4. The initial value problem

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad y(1) = 1$$

does not fall into the class of problems considered in our review. However, if one substitutes $y(x) = xz(x)$ into the differential equation, one obtains an equation for $z(x)$ which can be solved. Use this substitution to solve the initial value problem for $y(x)$.

5. Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

a. Find the general solution of the homogenous equation.

b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.

c. Use your answers in the previous parts to write down the general solution for this problem.

6. Find the general solution of the given equation by the method given.

a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.

b. $y'' + y' = 3x^2$. Variation of Parameters.

7. Find the general solution of each differential equation. When an initial condition is given, find the particular solution satisfying that condition.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

b. $y'' + y = 2\sin 3x.$

c. $y'' + y = 1 + 2\cos x.$

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad x > 0.$

8. Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0, \quad y_1(x) = x^4.$

b. $xy'' - y' + 4x^3y = 0, \quad y_1(x) = \sin(x^2).$

9. A ball is thrown upward with an initial velocity of 49 m/s from 539 m high. How high does the ball get and how long does in take before it hits the ground? [Use results from first problem done in class, free fall, $y'' = -g$.]

10. Consider the solution of a simple growth and decay problem,$y(t) = y_0e^{kt}$, to solve this typical radioactive decay problem: Forty percent of a radioactive substance disappears in 100 years.

a. What is the half-life of the substance?

b. After how many years will 90% be gone?

11. A spring fixed at its upper end is stretched six inches by a 10-pound weight attached at its lower end. The spring-mass system is suspended in a viscous medium so that the system is subjected to a damping force of $5\frac{dx}{dt}$ lbs. Describe the motion of the system if the weight is drawn down an additional 4 inches and released. What would happen if you changed the coefficient "5" to "4"? [You may need to consult your introductory physics text.]

12. Consider an LRC circuit with $L = 1.00$ H, $R = 1.00 \times 10^2$ Ω, $C = 1.00 \times 10^{-4}$ f, and $V = 1.00 \times 10^3$ V. Suppose that no charge is present and no current is flowing at time $t = 0$ when a battery of voltage V is inserted. Find the current and the charge on the capacitor as functions of time. Describe how the system behaves over time.

13. Consider the problem of forced oscillations as described in section 2.7.2.

a. Derive the general solution in Equation (2.70).

b. Use a CAS to plot the general solution in Equation (2.70) for the following cases:

c. Derive the form in Equation (2.71).

d. Use a CAS to plot the solution in Equation (2.71) for the following cases:

14. A certain model of the motion of a tossed whiffle ball is given by

$$mx'' + cx' + mg = 0, \quad x(0) = 0, \quad x'(0) = v_0.$$

Here m is the mass of the ball, g=9.8 m/s^2 is the acceleration due to gravity and c is a measure of the damping. Since there is no x term, we can write this as a first order equation for the velocity $v(t) = x'(t)$:

$$mv' + cv + mg = 0.$$

a. Find the general solution for the velocity $v(t)$ of the linear first order differential equation above.

b. Use the solution of part a to find the general solution for the position $x(t)$.

c. Find an expression to determine how long it takes for the ball to reach it's maximum height?

d. Assume that $c/m = 10\ \mathrm{s}^{-1}$. For $v_0 = 5, 10, 15, 20$ m/s, plot the solution, $x(t)$, versus the time.

e. From your plots and the expression in part c, determine the rise time. Do these answers agree?

f. What can you say about the time it takes for the ball to fall as compared to the rise time?

15. Consider the system

$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

a. Determine the second order differential equation satisfied by $x(t)$.

b. Solve the differential equation for $x(t)$.

c. Using this solution, find $y(t)$.

d. Verify your solutions for $x(t)$ and $y(t)$.

e. Find a particular solution to the system given the initial conditions $x(0) = 1$ and $y(0) = 0$.

16. Use the transformations relating polar and Cartesian coordinates to prove that

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right].$$

17. Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.)

a.

$$\begin{aligned} x' &= 3x \\ y' &= -2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= 2y \\ y' &= -3x. \end{aligned}$$

d.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

e.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

18. In example 2.32 a conversion to polar coordinates lead to the equation $r' = r(1 - r^2)$. Solve this equation for initial values of $r(0) = 0, 0.5, 1.0, 2.0$. Based upon these solutions, describe the behavior of all solutions to the original system in Cartesian coordinates.

3
Linear Algebra

"Physics is much too hard for physicists." David Hilbert (1862-1943)

As the reader is aware by now, calculus has its roots in physics and has become a very useful tool for modeling the physical world. Another very important area of mathematics is linear algebra. Physics students who have taken a course in linear algebra in a mathematics department might not come away with this perception. It is not until students take more advanced classes in physics that they begin to realize that a good grounding in linear algebra can lead to a better understanding of the behavior of physical systems.

Linear algebra is the backbone of most of applied mathematics and underlies many areas of physics, such as quantum mechanics.

In this chapter we will introduce some of the basics of linear algebra for finite dimensional vector spaces and we will reinforce these concepts through generalizations in later chapters to infinite dimensional vector spaces. In keeping with the theme of our text, we will apply some of these ideas to the coupled systems introduced in the last chapter. Such systems lead to linear and nonlinear oscillations in dynamical systems.

3.1 Vector Spaces

Much of the discussion and terminology that we will use comes from the theory of vector spaces . Up until now you may only have dealt with finite dimensional vector spaces. Even then, you might only be comfortable with two and three dimensions. We will review a little of what we know about finite dimensional spaces so that we can introduce more general function spaces.

The notion of a vector space is a generalization of three dimensional vectors and operations on them. In three dimensions, we have things called vectors[1] , which are arrows of a specific length and pointing in a given direction. To each vector, we can associate a point in a three dimensional Cartesian system. We just attach the tail of the vector $\mathbf{v}$ to

[1] In introductory physics one defines a vector as any quantity that has both magnitude and direction.

the origin and the head lands at (x, y, z).[2] We then use unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the coordinate axes to write

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

[2] In multivariate calculus one concentrates on the component form of vectors. These representations are easily generalized as we will see.

Having defined vectors, we then learned how to add vectors and multiply vectors by numbers, or scalars. Under these operations, we expected to get back new vectors. Then we learned that there were two types of multiplication of vectors. We could multiply them to get a scalar or a vector. This leads to dot products and cross products, respectively. The dot product is useful for determining the length of a vector, the angle between two vectors, or if the vectors were orthogonal. The cross product is used to produce orthogonal vectors, areas of parallelograms, and volumes of parallelepipeds.

In physics you first learned about vector products when you defined work, $W = \mathbf{F} \cdot \mathbf{r}$. Cross products were useful in describing things like torque, $\tau = \mathbf{r} \times \mathbf{F}$, or the force on a moving charge in a magnetic field, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. We will return to these more complicated vector operations later when reviewing Maxwell's equations of electrodynamics.

These notions are then generalized to spaces of more than three dimensions in linear algebra courses. The properties outlined roughly above need to be preserved. So, we have to start with a space of vectors and the operations between them. We also need a set of scalars, which generally come from some *field* . However, in our applications the field will either be the set of real numbers or the set of complex numbers.[3]

[3] A field is a set together with two operations, usually addition and multiplication, such that we have

- Closure under addition and multiplication
- Associativity of addition and multiplication
- Commutativity of addition and multiplication
- Additive and multiplicative identity
- Additive and multiplicative inverses
- Distributivity of multiplication over addition

A *vector space* V over a field F is a set that is closed under addition and scalar multiplication and satisfies the following conditions: For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. There exists a $\mathbf{0}$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$.
4. There exists an additive inverse, $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = 0$.

   There are several distributive properties:
5. $a(b\mathbf{v}) = (ab)\mathbf{v}$.
6. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.
7. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
8. There exists a multiplicative identity, 1, such that $1(\mathbf{v}) = \mathbf{v}$.

For now, we will restrict our examples to two and three dimensions and the field will consist of the real numbers.

In three dimensions the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ play an important role. Any vector in the three dimensional space can be written as a linear combination of these vectors,

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

In fact, given any three non-coplanar vectors, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, all vectors can be written as a linear combination of those vectors,

$$\mathbf{v} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3.$$

Such vectors are said to span the space and are called a basis for the space.

We can generalize these ideas. In an n-dimensional vector space any vector in the space can be represented as the sum over n linearly independent vectors (the equivalent of non-coplanar vectors). Such a *linearly independent* set of vectors $\{\mathbf{v}_j\}_{j=1}^n$ satisfies the condition

$$\sum_{j=1}^n c_j\mathbf{v}_j = \mathbf{0} \quad \Leftrightarrow \quad c_j = 0.$$

Note that we will often use summation notation instead of writing out all of the terms in the sum.

This leads to the idea of a basis set. The *standard basis* in an n-dimensional vector space is a generalization of the standard basis in three dimensions ($\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$). We define

The standard basis vectors, $\mathbf{e}_k$ are a natural generalization of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$$\mathbf{e}_k = (0, \ldots, 0, \underbrace{1}_{k\text{th space}}, 0, \ldots, 0), \quad k = 1, \ldots, n. \tag{3.1}$$

Then, we can expand any $\mathbf{v} \in V$ as

$$\mathbf{v} = \sum_{k=1}^n v_k\mathbf{e}_k, \tag{3.2}$$

where the v_k's are called the components of the vector in this basis. Sometimes we will write $\mathbf{v}$ as an n-tuple $(v_1, v_2, \ldots, v_n)$. This is similar to the ambiguous use of (x, y, z) to denote both vectors and points in the three dimensional space.

The only other thing we will need at this point is to generalize the dot product. Recall that there are two forms for the dot product in three dimensions. First, one has that

For more general vector spaces the term inner product is used to generalize the notions of dot and scalar products as we will see below.

$$\mathbf{u} \cdot \mathbf{v} = uv\cos\theta, \tag{3.3}$$

where u and v denote the length of the vectors. The other form is the component form:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{k=1}^3 u_kv_k. \tag{3.4}$$

Of course, this form is easier to generalize. So, we define the *scalar product* between two n-dimensional vectors as

$$< \mathbf{u}, \mathbf{v} >= \sum_{k=1}^{n} u_k v_k. \tag{3.5}$$

Actually, there are a number of notations that are used in other texts. One can write the scalar product as $(\mathbf{u}, \mathbf{v})$ or even in the Dirac bra-ket notation[4] $< \mathbf{u}|\mathbf{v} >$.

[4] The bra-ket notation was introduced by Paul Adrien Maurice Dirac (1902-1984) in order to facilitate computations of inner products in quantum mechanics. In the notation $< \mathbf{u}|\mathbf{v} >$, $< \mathbf{u}|$ is the bra and $|\mathbf{v} >$ is the ket. The kets live in a vector space and represented by column vectors with respect to a given basis. The bras live in the dual vector space and are represented by row vectors. The correspondence between bra and kets is $|\mathbf{v} >= \overline{|\mathbf{v} >}^T$. One can operate on kets, $A|\mathbf{v} >$, and make sense out of operations like$< \mathbf{u}|A|\mathbf{v} >$, which are used to obtain expectation values associated with the operator. Finally, the outer product, $|\mathbf{v} >< \mathbf{v}|$ is used to perform vector space projections.

We note that the (real) scalar product satisfies some simple properties. For vectors $\mathbf{v}$, $\mathbf{w}$ and real scalar α we have

1. $< \mathbf{v}, \mathbf{v} >\geq 0$ and $< \mathbf{v}, \mathbf{v} >= 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $< \mathbf{v}, \mathbf{w} >=< \mathbf{w}, \mathbf{v} >$.
3. $< \alpha\mathbf{v}, \mathbf{w} >= \alpha < \mathbf{v}, \mathbf{w} >$.

While it does not always make sense to talk about angles between general vectors in higher dimensional vector spaces, there is one concept that is useful. It is that of orthogonality, which in three dimensions is another way of saying the vectors are perpendicular to each other. So, we also say that vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if and only if $< \mathbf{u}, \mathbf{v} >= 0$. If $\{\mathbf{a}_k\}_{k=1}^n$, is a set of basis vectors such that

$$< \mathbf{a}_j, \mathbf{a}_k >= 0, \quad k \neq j,$$

then it is called an *orthogonal basis*.

Orthogonal basis vectors.

If in addition each basis vector is a unit vector, then one has an *orthonormal basis*. This generalization of the unit basis can be expressed more compactly. We will denote such a basis of unit vectors by $\mathbf{e}_j$ for $j = 1 \ldots n$. Then,

$$< \mathbf{e}_j, \mathbf{e}_k >= \delta_{jk}, \tag{3.6}$$

where we have introduced the Kronecker delta (named after Leopold Kronecker (1823-1891))

$$\delta_{jk} \equiv \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases} \tag{3.7}$$

The process of making vectors have unit length is called *normalization*. This is simply done by dividing by the *length* of the vector. Recall that the length of a vector, $\mathbf{v}$, is obtained as $v = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. So, if we want to find a unit vector in the direction of $\mathbf{v}$, then we simply normalize it as

Normalization of vectors.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{v}.$$

Notice that we used a hat to indicate that we have a unit vector. Furthermore, if $\{\mathbf{a}_j\}_{j=1}^n$, is a set of orthogonal basis vectors, then

$$\hat{\mathbf{a}}_j = \frac{\mathbf{a}_i}{\sqrt{< \mathbf{a}_j, \mathbf{a}_j >}}, \quad j = 1 \ldots n.$$

Example 3.1. *Find the angle between the vectors* $\mathbf{u} = (-2, 1, 3)$ *and* $\mathbf{v} = (1, 0, 2)$. *we need the lengths of each vector,*

$$u = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14},$$

$$v = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}.$$

We also need the scalar product of these vectors,

$$\mathbf{u} \cdot \mathbf{v} = -2 + 6 = 4.$$

This gives

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{uv} = \frac{4}{\sqrt{5}\sqrt{14}}.$$

So, $\theta = 61.4^\circ$.

Example 3.2. *Normalize the vector* $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

The length of the vector is $v = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$. *So, the unit vector in the direction of* $\mathbf{v}$ *is* $\hat{\mathbf{v}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$.

Let $\{\mathbf{a}_k\}_{k=1}^n$, be a set of orthogonal basis vectors for vector space V. We know that any vector $\mathbf{v}$ can be represented in terms of this basis, $\mathbf{v} = \sum_{k=1}^n v_k \mathbf{a}_k$. If we know the basis and vector, can we find the components, v_k? The answer is yes. We can use the scalar product of $\mathbf{v}$ with each basis element $\mathbf{a}_j$. Using the properties of the scalar product, we have for $j = 1, \ldots, n$

$$\begin{aligned} < \mathbf{a}_j, \mathbf{v} > &= < \mathbf{a}_j, \sum_{k=1}^n v_k \mathbf{a}_k > \\ &= \sum_{k=1}^n v_k < \mathbf{a}_j, \mathbf{a}_k > . \end{aligned} \tag{3.8}$$

Since we know the basis elements, we can easily compute the numbers

$$A_{jk} \equiv < \mathbf{a}_j, \mathbf{a}_k >$$

and

$$b_j \equiv < \mathbf{a}_j, \mathbf{v} > .$$

Therefore, the system (3.8) for the v_k's is a linear algebraic system, which takes the form

$$b_j = \sum_{k=1}^n A_{jk} v_k. \tag{3.9}$$

We can write this set of equations in a more compact form. The set of numbers A_{jk}, $j, k = 1, \ldots n$ are the elements of an $n \times n$ matrix A with A_{jk} being an element in the jth row and kth column. Also, v_j and b_j can be written as column vectors, $\mathbf{v}$ and $\mathbf{b}$, respectively. Thus, system (3.8) can be written in matrix form as

$$A\mathbf{v} = \mathbf{b}.$$

However, if the basis is orthogonal, then the matrix $A_{jk} \equiv < \mathbf{a}_j, \mathbf{a}_k >$ is diagonal and the system is easily solvable. Recall that two vectors are orthogonal if and only if

$$< \mathbf{a}_i, \mathbf{a}_j >= 0, \quad i \neq j. \tag{3.10}$$

Thus, in this case we have that

$$< \mathbf{a}_j, \mathbf{v} >= v_j < \mathbf{a}_j, \mathbf{a}_j >, \quad j = 1, \ldots, n. \tag{3.11}$$

or

$$v_j = \frac{< \mathbf{a}_j, \mathbf{v} >}{< \mathbf{a}_j, \mathbf{a}_j >}. \tag{3.12}$$

In fact, if the basis is orthonormal, i.e., the basis consists of an orthogonal set of unit vectors, then A is the identity matrix and the solution takes on a simpler form:

$$v_j =< \mathbf{a}_j, \mathbf{v} > . \tag{3.13}$$

Example 3.3. *Consider the set of vectors* $\mathbf{a}_1 = \mathbf{i} + \mathbf{j}$ *and* $\mathbf{a}_2 = \mathbf{i} - 2\mathbf{j}$.

1. *Determine the matrix elements* $A_{jk} =< \mathbf{a}_j, \mathbf{a}_k >$.
2. *Is this an orthogonal basis?*
3. *Expand the vector* $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ *in the basis* $\{\mathbf{a}_1, \mathbf{a}_2\}$.

First, we compute the matrix elements of A:

$$\begin{aligned} A_{11} &= < \mathbf{a}_1, \mathbf{a}_1 >= 2 \\ A_{12} &= < \mathbf{a}_1, \mathbf{a}_2 >= -1 \\ A_{21} &= < \mathbf{a}_2, \mathbf{a}_1 >= -1 \\ A_{22} &= < \mathbf{a}_2, \mathbf{a}_2 >= 5 \end{aligned} \tag{3.14}$$

So,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}.$$

Since $A_{12} = A_{21} \neq 0$, *the vectors are not orthogonal. However, they are linearly independent. Obviously, if* $c_1 = c_2 = 0$, *then the linear combination* $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{0}$. *Conversely, we assume that* $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{0}$ *and solve for the coefficients. Inserting the given vectors, we have*

$$\begin{aligned} \mathbf{0} &= c_1(\mathbf{i} + \mathbf{j}) + c_2(\mathbf{i} - 2\mathbf{j}) \\ &= (c_1 + c_2)\mathbf{i} + (c_1 - 2c_2)\mathbf{j}. \end{aligned} \tag{3.15}$$

This implies that

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - 2c_2 &= 0. \end{aligned} \tag{3.16}$$

Solving this system, one has $c_1 = 0$, $c_2 = 0$. *Therefore, the two vectors are linearly independent.*

In order to determine the components of $\mathbf{v}$ *with respect to the new basis, we need to set up the system (3.8) and solve for the* v_k*'s. We have first,*

$$\begin{aligned} \mathbf{b} &= \begin{pmatrix} <\mathbf{a}_1, \mathbf{v}> \\ <\mathbf{a}_2, \mathbf{v}> \end{pmatrix} \\ &= \begin{pmatrix} <\mathbf{i}+\mathbf{j}, 2\mathbf{i}+3\mathbf{j}> \\ <\mathbf{i}-2\mathbf{j}, 2\mathbf{i}+3\mathbf{j}> \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -4 \end{pmatrix}. \end{aligned} \tag{3.17}$$

So, now we have to solve the system $A\mathbf{v} = \mathbf{b}$ *for* $\mathbf{v}$:

$$\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}. \tag{3.18}$$

We can solve this with matrix methods, $\mathbf{v} = A^{-1}\mathbf{b}$, *or rewrite it as a system of two equations and two unknowns. The result is* $v_1 = \frac{7}{3}$, $v_2 = -\frac{1}{3}$. *Thus,* $\mathbf{v} = \frac{7}{3}\mathbf{a}_1 - \frac{1}{3}\mathbf{a}_2$.

3.2 *Linear Transformations*

A MAIN THEME in linear algebra is to study linear transformations between vector spaces. These come in many forms and there are an abundance of applications in physics. For example, the transformation between the spacetime coordinates of observers moving in inertial frames in the theory of special relativity constitute such a transformation.

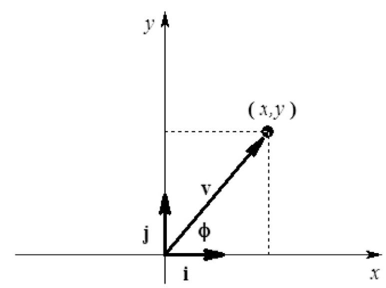


Figure 3.1: Vector $\mathbf{v}$ in a standard coordinate system.

A simple example often encountered in physics courses is the rotation by a fixed angle. This is the description of points in space using two different coordinate bases, one just a rotation of the other by some angle. We begin with a vector $\mathbf{v}$ as described by a set of axes in the standard orientation, as shown in Figure 3.1. Also displayed in this figure are the unit vectors. To find the coordinates (x, y), one needs only draw perpendiculars to the axes and read the coordinates off the axes.

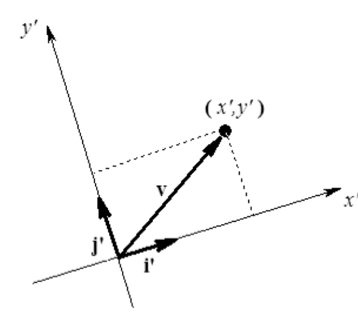


Figure 3.2: Vector $\mathbf{v}$ in a rotated coordinate system.

In order to derive the needed transformation we will make use of polar coordinates. In Figure 3.1 we see that the vector makes an angle of ϕ with respect to the positive x-axis. The components (x, y) of the vector can be determined from this angle and the magnitude of $\mathbf{v}$ as

$$\begin{aligned} x &= v\cos\phi \\ y &= v\sin\phi. \end{aligned} \tag{3.19}$$

We now consider another set of axes at an angle of θ to the old. Such a system is shown in Figure 3.2. We will designate these axes as x' and y'. Note that the basis vectors are different in this system. Projections to the axes are shown. Comparing the coordinates in both systems shown in Figures 3.1-3.2, we see that the primed coordinates are not the same as the unprimed ones.

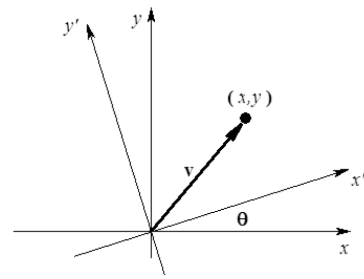


Figure 3.3: Comparison of the coordinate systems.

In Figure 3.3 the two systems are superimposed on each other. The polar form for the primed system is given by

$$
\begin{aligned}
x' &= v\cos(\phi-\theta) \\
y' &= v\sin(\phi-\theta).
\end{aligned} \tag{3.20}
$$

We can use this form to find a relationship between the two systems. Namely, we use the addition formula for trigonometric functions to obtain

$$
\begin{aligned}
x' &= v\cos\phi\cos\theta + v\sin\phi\sin\theta \\
y' &= v\sin\phi\cos\theta - v\cos\phi\sin\theta.
\end{aligned} \tag{3.21}
$$

Passive rotation.

Noting that these expressions involve products of v with $\cos\phi$ and $\sin\phi$, we can use the polar form for x and y to find the desired form:

$$
\begin{aligned}
x' &= x\cos\theta + y\sin\theta \\
y' &= -x\sin\theta + y\cos\theta.
\end{aligned} \tag{3.22}
$$

This is an example of a transformation between two coordinate systems. It is called a rotation by θ. We can designate it generally by

$$(x', y') = \hat{R}_\theta(x, y).$$

It is referred to as a passive transformation, because it does not affect the vector. [Note: We will use the hat for the passive rotation.]

Active rotation.

An active rotation is one in which one rotates the vector, such as shown in Figure 3.4. One can derive a similar transformation for how the coordinate of the vector change under such a transformation. Denoting the new vector as $\mathbf{v}'$ with new coordinates (x'', y''), we have

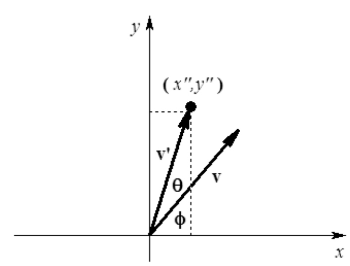


Figure 3.4: Rotation of vector **v**

$$
\begin{aligned}
x'' &= x\cos\theta - y\sin\theta \\
y'' &= x\sin\theta + y\cos\theta.
\end{aligned} \tag{3.23}
$$

We can designate this transformation by

$$(x'', y'') = R_\theta(x, y)$$

and see that the active and passive rotations are related,

$$R_\theta(x, y) = \hat{R}_{-\theta}(x, y).$$

3.3 Matrices

Linear transformations such as the rotation in the last section can be represented by matrices. Such matrix representations often become the core of a linear algebra class to the extent that one loses sight of their meaning. We will review matrix representations and show how they are useful in solving coupled systems of differential equations later in the chapter.

We begin with the rotation transformation as applied to the axes in Equation (3.22). We write vectors like $\mathbf{v}$ as a column matrix

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

We can also write the trigonometric functions in a 2×2 matrix form as

$$\hat{R}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then, the transformation takes the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.24}$$

This can be written in the more compact form

$$\mathbf{v}' = \hat{R}_\theta \mathbf{v}.$$

In using the matrix form of the transformation, we have employed the definition of matrix multiplication. Namely, we have multiplied a 2×2 matrix times a 2×1 matrix. (Note that an $n \times m$ matrix has n rows and m columns.) The multiplication proceeds by selecting the ith row of the first matrix and the jth column of the second matrix. Multiply corresponding elements of each and add them. Then, place the result into the ijth entry of the product matrix. This operation can only be performed if the number of columns of the first matrix is the same as the number of columns of the second matrix.

Example 3.4. *As an example, we multiply a 3×2 matrix times a 2×2 matrix to obtain a 3×2 matrix:*

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 5 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} &= \begin{pmatrix} 1(3) + 2(1) & 1(2) + 2(4) \\ 5(3) + (-1)(1) & 5(2) + (-1)(4) \\ 3(3) + 2(1) & 3(2) + 2(4) \end{pmatrix} \\ &= \begin{pmatrix} 5 & 10 \\ 14 & 6 \\ 11 & 14 \end{pmatrix} \end{aligned} \tag{3.25}$$

In Equation (3.24), we have the row $(\cos\theta, \sin\theta)$ and column $(x, y)^T$. Combining these we obtain $x\cos\theta + y\sin\theta$. This is x'. We perform the same operation for the second row:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}. \tag{3.26}$$

In the last section we also introduced active rotations. These were rotations of vectors keeping the coordinate system fixed. Thus, we start with a vector $\mathbf{v}$ and rotate it by θ to get a new vector $\mathbf{u}$. That transformation can be written as

$$\mathbf{u} = R_\theta \mathbf{v}, \tag{3.27}$$

where

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Now consider a rotation by $-\theta$. Due to the symmetry properties of the sines and cosines, we have

$$R_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

We see that if the 12 and 21 elements of this matrix are interchanged we recover R_θ. This is an example of what is called the *transpose* of R_θ. Given a matrix, A, its transpose A^T is the matrix obtained by interchanging the rows and columns of A. Formally, let A_{ij} be the elements of A. Then

$$A^T_{ij} = A_{ji}.$$

Matrix transpose.

It is also the case that these matrices are inverses of each other. We can understand this in terms of the nature of rotations. We first rotate the vector by θ as $\mathbf{u} = R_\theta \mathbf{v}$ and then rotate $\mathbf{u}$ by $-\theta$ obtaining $\mathbf{w} = R_{-\theta}\mathbf{u}$. Thus, the "composition" of these two transformations leads to

$$\mathbf{w} = R_{-\theta}\mathbf{u} = R_{-\theta}(R_\theta \mathbf{v}). \tag{3.28}$$

We can view this as a net transformation from $\mathbf{v}$ to $\mathbf{w}$ given by

$$\mathbf{w} = (R_{-\theta}R_\theta)\mathbf{v},$$

where the transformation matrix for the composition is given by $R_{-\theta}R_\theta$. Actually, if you think about it, we should end up with the original vector. We can compute the resulting matrix by carrying out the multiplication. We obtain

$$R_{-\theta}R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.29}$$

This is the 2×2 identity matrix. We note that the product of these two matrices yields the identity. This is like the multiplication of numbers. If $ab = 1$, then a and b are multiplicative inverses of each other. So, we see here that R_θ and $R_{-\theta}$ are inverses of each other as well. In fact, we have determined that

$$R_{-\theta} = R_\theta^{-1} = R_\theta^T, \tag{3.30}$$

where the T designates the transpose. We note that matrices satisfying the relation $A^T = A^{-1}$ are called *orthogonal matrices*.

Orthogonal matrices.

We can easily extend this discussion to three dimensions. Such rotations in the xy-plane can be viewed as rotations about the z-axis. Rotating a vector about the z-axis by angle α will leave the z-component fixed. This can be represented by the rotation matrix

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.31}$$

We can also rotate vectors about the other axes, so that we would have two other rotation matrices:

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}. \tag{3.32}$$

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix}. \tag{3.33}$$

As before, passive rotations of the coordinate axes are obtained by replacing the angles above by their negatives; e.g., $\hat{R}_x(\gamma) = R_x(-\gamma)$.[5]

[5] In classical dynamics one describes a general rotation in terms of the so-called Euler angles. These are the angles (ϕ, θ, ψ) such that the combined rotation $\hat{R}_z(\psi)\hat{R}_x(\theta)\hat{R}_z(\phi)$ rotates the initial coordinate system into a new one.

We can generalize what we have seen with the simple example of rotation to other linear transformations. We begin with a vector $\mathbf{v}$ in an n-dimensional vector space. We can consider a transformation L that takes $\mathbf{v}$ into a new vector $\mathbf{u}$ as

$$\mathbf{u} = L(\mathbf{v}).$$

We will restrict ourselves to linear transformations between two n-dimensional vector spaces. A *linear transformation* satisfies the following condition:

$$L(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha L(\mathbf{a}) + \beta L(\mathbf{b}) \tag{3.34}$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$ and scalars α and β.[6]

[6] In section we define a linear operator using two conditions, $L(\mathbf{a} + \mathbf{b}) = L(\mathbf{a}) + L(\mathbf{b})$ and $L(\alpha\mathbf{a}) = \alpha L(\mathbf{a})$. The reader can show that this is equivalent to the condition presented here. Furthermore, all linear transformations take the origin to the origin, $L(\mathbf{0}) = \mathbf{0}$.

Such linear transformations can be represented by matrices. Take any vector $\mathbf{v}$. It can be represented in terms of a basis. Let's use the standard basis $\{\mathbf{e}_i\}$, $i = 1, \ldots n$. Then we have

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i.$$

Now consider the effect of the transformation L on $\mathbf{v}$, using the linearity property:

$$L(\mathbf{v}) = L\left(\sum_{i=1}^{n} v_i \mathbf{e}_i\right) = \sum_{i=1}^{n} v_i L(\mathbf{e}_i). \tag{3.35}$$

Thus, we see that determining how L acts on $\mathbf{v}$ requires that we know how L acts on the basis vectors. Namely, we need $L(\mathbf{e}_i)$. Since $\mathbf{e}_i$ is a vector, this produces another vector in the space. But the resulting vector can be expanded in the basis. Let's assume that the resulting vector takes the form

$$L(\mathbf{e}_i) = \sum_{j=1}^{n} L_{ji} \mathbf{e}_j, \tag{3.36}$$

where L_{ji} is the jth component of $L(\mathbf{e}_i)$ for each $i = 1, \ldots, n$. The matrix of L_{ji}'s is called the matrix representation of the operator L.

Typically, in a linear algebra class you start with matrices and do not see this connection to linear operators. However, there will be times that you will need this connection to understand why matrices are involved. Furthermore, the matrix representation depends on the basis used. We used the standard basis above. However, you could have started with a different basis, such as dictated by another coordinate system. We will not go further into this point at this time and just stick with the standard basis.

Example 3.5. *Consider the linear transformation of* $\mathbf{u} = (u, v)$ *into* $\mathbf{x} = (x, y)$ *by*

$$L(u, v) = (3u - v, v + u) = (x, y).$$

The matrix representation for this transformation is found by considering how L *acts on the basis vectors. We have* $L(1, 0) = (3, 1)$ *and* $L(0, 1) = (-1, 1)$. *Thus, the representation is given as*

$$L = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

Now that we know how L acts on basis vectors, what does this have to say about how L acts on any other vector in the space? We insert expression (3.36) into Equation (3.35). Then we find

$$\begin{aligned} L(\mathbf{v}) &= \sum_{i=1}^{n} v_i L(\mathbf{e}_i) \\ &= \sum_{i=1}^{n} v_i \left(\sum_{j=1}^{n} L_{ji} \mathbf{e}_j\right) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v_i L_{ji}\right) \mathbf{e}_j. \end{aligned} \tag{3.37}$$

Since $L(\mathbf{v}) = \mathbf{u}$, we see that the jth component of $\mathbf{u}$ can be written as

$$u_j = \sum_{i=1}^{n} L_{ji} v_i, \quad j = 1 \ldots n. \tag{3.38}$$

This equation can be written in matrix form as

$$\mathbf{u} = L\mathbf{v},$$

where L now takes the role of a matrix. It is similar to the multiplication of the rotation matrix times a vector as seen in the last section. We will just work with matrix representations from here on.

Example 3.6. *For the transformation $L(u, v) = (3u - v, v + u) = (x, y)$ in the last example, what does $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j}$ get mapped into? We know the matrix representation from the previous example, so we have*

$$\mathbf{u} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}.$$

Next, we can compose transformations like we had done with the two rotation matrices. Let $\mathbf{u} = A(\mathbf{v})$ and $\mathbf{w} = B(\mathbf{u})$ for two transformations A and B. (Thus, $\mathbf{v} \to \mathbf{u} \to \mathbf{w}$.) Then a composition of these transformations is given by

$$\mathbf{w} = B(\mathbf{u}) = B(A\mathbf{v}).$$

This can be viewed as a transformation from $\mathbf{v}$ to $\mathbf{w}$ as

$$\mathbf{w} = BA(\mathbf{v}),$$

where the matrix representation of BA is given by the product of the matrix representations of A and B.

To see this, we look at the ijth element of the matrix representation of BA. We first note that the transformation from $\mathbf{v}$ to $\mathbf{w}$ is given by

$$w_i = \sum_{j=1}^{n} (BA)_{ij} v_j. \tag{3.39}$$

However, if we use the successive transformations, we have

$$\begin{aligned} w_i &= \sum_{k=1}^{n} B_{ik} u_k \\ &= \sum_{k=1}^{n} B_{ik} \left(\sum_{j=1}^{n} A_{kj} v_j \right) \\ &= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} B_{ik} A_{kj} \right) v_j. \end{aligned} \tag{3.40}$$

We have two expressions for w_i as sums over v_j. So, the coefficients must be equal. This leads to our result:

$$(BA)_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj}. \tag{3.41}$$

Thus, we have found the component form of matrix multiplication, which resulted from the composition of two linear transformations. This agrees with our earlier example of matrix multiplication: The ij-th component of the product is obtained by multiplying elements in the ith row of B and the jth column of A and summing.

Example 3.7. *Consider the rotation in two dimensions of the axes by an angle θ. Now apply the scaling transformation*[7]

$$L_s = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

[7] This scaling transformation will rescale x-components by a and y-components by b. If either is negative, it will also provide an additional reflection.

what is the matrix representation of this combination of transformations? The result is a simple product of the matrix representations (in reverse order of application):

$$L_s \hat{R} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a\cos\theta & a\sin\theta \\ -b\sin\theta & b\cos\theta \end{pmatrix}.$$

There are many other properties of matrices and types of matrices that one may encounter. We will list a few.

First of all, there is the $n \times n$ *identity matrix*, I. The identity is defined as that matrix satisfying

Identity matrix.

$$IA = AI = A \tag{3.42}$$

for any $n \times n$ matrix A. The $n \times n$ identity matrix takes the form

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 1 \end{pmatrix} \tag{3.43}$$

A component form is given by the Kronecker delta. Namely, we have that

Kronecker delta, δ_{ij}.

$$I_{ij} = \delta_{ij} \equiv \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{3.44}$$

The *inverse* of matrix A is that matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I. \tag{3.45}$$

There is a systematic method for determining the inverse in terms of cofactors, which we describe a little later. However, the inverse of a 2×2 matrix is easily obtained without learning about cofactors. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now consider the matrix

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Multiplying these matrices, we find that

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}.$$

This is not quite the identity, but it is a multiple of the identity. We just need to divide by $ad - bc$. So, we have found the inverse matrix:

Inverse of a 2×2 matrix.

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We leave it to the reader to show that $A^{-1}A = I$.

The factor $ad - bc$ is the difference in the products of the diagonal and off-diagonal elements of matrix A. This factor is called the *determinant* of A. It is denoted as $\det(A)$, $\det A$ or $|A|$. Thus, we define

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{3.46}$$

For higher dimensional matrices one can write the definition of the determinant. We will for now just indicate the process for 3×3 matrices. We write matrix A as

Detemrinant of a 3×3 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{3.47}$$

The determinant of A can be computed in terms of simpler 2×2 determinants. We define

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned} \tag{3.48}$$

There are many other properties of determinants. For example, if two rows, or columns, of a matrix are multiples of each other, then $\det A = 0$. If one multiplies one row, or column, of a matrix by a constant, k, then the determinant of the matrix is multiplies by k.

If $\det A = 0$, A is called a *singular* matrix. Otherwise, it is called *nonsingular*. If a matrix is nonsingular, then the inverse exists. From our example for a general 2×2 system, the inverse exists if $ad - bc \neq 0$.

Computing the inverse of a larger matrix is a little more complicated. One first constructs the matrix of cofactors. The ij-th cofactor is obtained by computing the determinant of the matrix resulting from eliminating the ith row and jth column of A and multiplying by either $+1$ or -1. Thus,

The matrix of cofactors.

$$C_{ij} = (-1)^{i+j} \det (\alpha_{ij}).$$

Then, the inverse matrix is obtained by dividing the transpose of the matrix of cofactors by the determinant of A. Thus,

$$\left(A^{-1}\right)_{ij} = \frac{C_{ji}}{\det A}.$$

This is best shown by example.

Example 3.8. *Find the inverse of the matrix*

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -2 & 1 \end{pmatrix}.$$

The determinant of this matrix is easily found as

$$\det A = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 14.$$

Next, we construct the matrix of cofactors:

$$C_{ij} = \begin{pmatrix} +\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{pmatrix}.$$

Computing the 2×2 *determinants, we obtain*

$$C_{ij} = \begin{pmatrix} 7 & -2 & -3 \\ 0 & 2 & 4 \\ 7 & -2 & 3 \end{pmatrix}.$$

Finally, we compute the inverse as

$$\begin{aligned}
A^{-1} &= \frac{1}{14}\begin{pmatrix} 7 & -2 & -3 \\ 0 & 2 & 4 \\ 7 & -2 & 3 \end{pmatrix}^T \\
&= \frac{1}{14}\begin{pmatrix} 7 & 0 & 7 \\ -2 & 2 & -2 \\ -3 & 4 & 3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} \\ -\frac{3}{14} & \frac{2}{7} & \frac{3}{14} \end{pmatrix}. \qquad (3.49)
\end{aligned}$$

Another operation that we have seen earlier is the *transpose* of a matrix. The transpose of a matrix is a new matrix in which the rows and columns are interchanged. If write an $n \times m$ matrix A in standard form as

Matrix transpose.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}, \qquad (3.50)$$

then the transpose is defined as

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{mn} \end{pmatrix}. \qquad (3.51)$$

In index form, we have

$$(A^T)_{ij} = A_{ji}, \quad i,j = 1,\dots,n.$$

As we had seen in the last section, a matrix satisfying

$$A^T = A^{-1}, \quad \text{or} \quad AA^T = A^TA = I,$$

is called an orthogonal matrix. One also can show that

$$(AB)^T = B^TA^T.$$

Finally, the *trace* of a square matrix is the sum of its diagonal elements:

Trace of a matrix.

$$\text{Tr}(A) = a_{11} + a_{22} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

We can show that for two square matrices

$$\text{Tr}(AB) = \text{Tr}(BA).$$

A standard application of determinants is the solution of a system of linear algebraic equations using Cramer's Rule. As an example, we consider a simple system of two equations and two unknowns. Let's consider this system of two equations and two unknowns, x and y, in the form

Cramer's Rule for solving algebraic systems of equations.

$$\begin{aligned} ax + by &= e, \\ cx + dy &= f. \end{aligned} \tag{3.52}$$

The standard way to solve this is to eliminate one of the variables. (Just imagine dealing with a bigger system!). So, we can eliminate the x's. Multiply the first equation by c and the second equation by a and subtract. We then get

$$(bc - ad)y = (ec - fa).$$

If $bc - ad \neq 0$, then we can solve to y, getting

$$y = \frac{ec - fa}{bc - ad}$$

. Similarly, we find

$$x = \frac{ed - bf}{ad - bc}.$$

We note the the denominators can be replaced with the determinant of the matrix of coefficients,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In fact, we can also replace each numerator with a determinant. Thus, our solutions may be written as

$$\begin{aligned} x &= \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \\ y &= \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \end{aligned} \tag{3.53}$$

This is Cramer's Rule for writing out solutions of systems of equations. Note that each variable is determined by placing a determinant with e and f placed in the column of the coefficient matrix corresponding to the order of the variable in the equation. The denominator is the determinant of the coefficient matrix. This construction is easily extended to larger systems of equations.

Cramer's Rule can be extended to higher dimensional systems. As an example, we now solve a system of three equations and three unknowns.

Example 3.9. *Solve the system of equations*

$$\begin{aligned} x + 2y - z &= 1, \\ 3y + 2z &= 2 \\ x - 2y + z &= 0. \end{aligned} \tag{3.54}$$

First, one writes the system in the form $L\mathbf{x} = \mathbf{b}$, where L is the coefficient matrix

$$L = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

The solution is generally, $\mathbf{x} = L^{-1}\mathbf{b}$ if L^{-1} exists. So, we check that $\det L = 14 \neq 0$. Thus, L is nonsingular and its inverse exists.

So, the solution of this system of three equations and three unknowns can now be found using Cramer's rule. Thus, we have

$$\begin{aligned} x &= \frac{\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 0 & -2 & 1 \end{vmatrix}}{\det L} = \frac{7}{14} = \frac{1}{2}, \\ y &= \frac{\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{vmatrix}}{\det L} = \frac{6}{14} = \frac{3}{7}, \\ z &= \frac{\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & -2 & 0 \end{vmatrix}}{\det L} = \frac{5}{14}. \end{aligned} \tag{3.55}$$

We end this section by summarizing the rule for the existence of solutions of systems of algebraic equations, $L\mathbf{x} = \mathbf{b}$.

1. If $\det L \neq 0$, then there exists a unique solution, $\mathbf{x} = L^{-1}\mathbf{b}$. In particular, if $\mathbf{b} = \mathbf{0}$, the system is homogeneous and only has the trivial solution, $\mathbf{x} = \mathbf{0}$.

2. If $\det L = 0$, then the system does not have a unique solution. Either there is no solution, or an infinite number of solutions. For example, the system

$$\begin{aligned} 2x + y &= 5, \\ 4x + 2y &= 2, \end{aligned} \tag{3.56}$$

has no solutions, while

$$\begin{aligned} 2x + y &= 0, \\ 4x + 2y &= 0, \end{aligned} \tag{3.57}$$

has an infinite number of solutions ($y = -2x$).

3.4 *Eigenvalue Problems*

3.4.1 *An Introduction to Coupled Systems*

RECALL THAT one of the reasons we have seemingly digressed into topics in linear algebra and matrices is to solve a coupled system of differential equations. The simplest example is a system of linear differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy. \end{aligned} \tag{3.58}$$

We note that this system is coupled. We cannot solve either equation without knowing either $x(t)$ or $y(t)$. A much easier problem would be to solve an uncoupled system like

Uncoupled system.

$$\begin{aligned} \frac{dx}{dt} &= \lambda_1 x \\ \frac{dy}{dt} &= \lambda_2 y. \end{aligned} \tag{3.59}$$

The solutions are quickly found to be

$$\begin{aligned} x(t) &= c_1 e^{\lambda_1 t}, \\ y(t) &= c_2 e^{\lambda_2 t}. \end{aligned} \tag{3.60}$$

Here c_1 and c_2 are two arbitrary constants.

We can determine particular solutions of the system by specifying $x(t_0) = x_0$ and $y(t_0) = y_0$ at some time t_0. Thus,

$$\begin{aligned} x(t) &= x_0 e^{\lambda_1 t}, \\ y(t) &= y_0 e^{\lambda_2 t}. \end{aligned} \tag{3.61}$$

Wouldn't it be nice if we could transform the more general system into one that is not coupled? Let's write these systems in more general form. We write the coupled system as

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x}$$

and the uncoupled system as

$$\frac{d}{dt}\mathbf{y} = \Lambda\mathbf{y},$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We note that Λ is a diagonal matrix.

Now, we seek a transformation between $\mathbf{x}$ and $\mathbf{y}$ that will transform the coupled system into the uncoupled system. Thus, we define the transformation

$$\mathbf{x} = S\mathbf{y}. \tag{3.62}$$

Inserting this transformation into the coupled system we have

$$\begin{aligned} \frac{d}{dt}\mathbf{x} &= A\mathbf{x} \quad \Rightarrow \\ \frac{d}{dt}S\mathbf{y} &= AS\mathbf{y} \quad \Rightarrow \\ S\frac{d}{dt}\mathbf{y} &= AS\mathbf{y}. \end{aligned} \tag{3.63}$$

Multiply both sides by S^{-1}. [We can do this if we are dealing with an invertible transformation; i.e., a transformation in which we can get $\mathbf{y}$ from $\mathbf{x}$ as $\mathbf{y} = S^{-1}\mathbf{x}$.] We obtain

$$\frac{d}{dt}\mathbf{y} = S^{-1}AS\mathbf{y}.$$

Noting that

$$\frac{d}{dt}\mathbf{y} = \Lambda\mathbf{y},$$

we have

$$\Lambda = S^{-1}AS. \tag{3.64}$$

The expression $S^{-1}AS$ is called a *similarity transformation* of matrix A. So, in order to uncouple the system, we seek a similarity transformation that results in a diagonal matrix. This process is called the *diagonalization* of matrix A. We do not know S, nor do we know Λ. We can rewrite this equation as

$$AS = S\Lambda.$$

We can solve this equation if S is *real symmetric*, i.e, $S^T = S$. [In the case of complex matrices, we need the matrix to be Hermitian, $\bar{S}^T = S$ where the bar denotes complex conjugation. Further discussion of diagonalization is left for the end of the chapter.]

We first show that $S\Lambda = \Lambda S$. We look at the ijth component of $S\Lambda$ and rearrange the terms in the matrix product.

$$\begin{aligned}
(S\Lambda)_{ij} &= \sum_{k=1}^{n} S_{ik}\Lambda_{kj} \\
&= \sum_{k=1}^{n} S_{ik}\lambda_j I_{kj} \\
&= \sum_{k=1}^{n} \lambda_j I_{jk} S^T_{ki} \\
&= \sum_{k=1}^{n} \Lambda_{jk} S_{ki} \\
&= (\Lambda S)_{ij}
\end{aligned} \tag{3.65}$$

This result leads us to the fact that S satisfies the equation

$$AS = \Lambda S.$$

Therefore, one has that the columns of S (denoted $\mathbf{v}$) satisfy an equation of the form

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{3.66}$$

This is an equation for vectors $\mathbf{v}$ and numbers λ given matrix A. It is called an *eigenvalue problem*. The vectors are called *eigenvectors* and the numbers, λ, are called *eigenvalues*. In principle, we can solve the eigenvalue problem and this will lead us to solutions of the uncoupled system of differential equations.

3.4.2 *Example of an Eigenvalue Problem*

We will determine the eigenvalues and eigenvectors for

$$A = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}$$

In order to find the eigenvalues and eigenvectors of this equation, we need to solve

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{3.67}$$

Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then the eigenvalue problem can be written out. We have that

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$\begin{aligned}
\begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} v_1 - 2v_2 \\ -3v_1 + 2v_2 \end{pmatrix} &= \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}.
\end{aligned} \tag{3.68}$$

So, we see that the system becomes

$$\begin{aligned}
v_1 - 2v_2 &= \lambda v_1, \\
-3v_1 + 2v_2 &= \lambda v_2.
\end{aligned} \tag{3.69}$$

This can be rewritten as

$$\begin{aligned}
(1-\lambda)v_1 - 2v_2 &= 0, \\
-3v_1 + (2-\lambda)v_2 &= 0.
\end{aligned} \tag{3.70}$$

This is a homogeneous system. We can try to solve it using elimination, as we had done earlier when deriving Cramer's Rule. We find that multiplying the first equation by $2-\lambda$, the second by 2 and adding, we get

$$[(1-\lambda)(2-\lambda) - 6]v_1 = 0.$$

If the factor in the brackets is not zero, we obtain $v_1 = 0$. Inserting this into the system gives $v_2 = 0$ as well. Thus, we find $\mathbf{v}$ is the zero vector. However, this does not get us anywhere. We could have guessed this solution. This simple solution is the solution of all eigenvalue problems and is called the trivial solution. When solving eigenvalue problems, we only look for nontrivial solutions!

So, we have to stipulate that the factor in the brackets is zero. This means that v_1 is still unknown. This situation will always occur for eigenvalue problems. The general eigenvalue problem can be written as

$$A\mathbf{v} - \lambda \mathbf{v} = 0,$$

or by inserting the identity matrix,

$$A\mathbf{v} - \lambda I \mathbf{v} = 0.$$

Finally, we see that we always get a homogeneous system,

$$(A - \lambda I)\mathbf{v} = 0.$$

The factor that has to be zero can be seen now as the determinant of this system. Thus, we require

$$\det(A - \lambda I) = 0. \tag{3.71}$$

We write out this condition for the example at hand. We have that

$$\begin{vmatrix} 1-\lambda & -2 \\ -3 & 2-\lambda \end{vmatrix} = 0.$$

This will always be the starting point in solving eigenvalue problems. Note that the matrix is A with λ's subtracted from the diagonal elements.

Computing the determinant, we have

$$(1-\lambda)(2-\lambda)-6=0,$$

or

$$\lambda^2-3\lambda-4=0.$$

We therefore have obtained a condition on the eigenvalues! It is a quadratic and we can factor it:

$$(\lambda-4)(\lambda+1)=0.$$

So, our eigenvalues are $\lambda = 4, -1$.

The second step is to find the eigenvectors. We have to do this for each eigenvalue. We first insert $\lambda = 4$ into our system:

$$\begin{aligned} -3v_1-2v_2&=0,\\ -3v_1-2v_2&=0. \end{aligned} \tag{3.72}$$

Note that these equations are the same. So, we have one equation in two unknowns. We will not get a unique solution. This is typical of eigenvalue problems. We can pick anything we want for v_2 and then determine v_1. For example, $v_2 - 1$ gives $v_1 = -2/3$. A nicer solution would be $v_2 = 3$ and $v_1 = -2$. These vectors are different, but they point in the same direction in the v_1v_2 plane.

For $\lambda = -1$, the system becomes

$$\begin{aligned} 2v_1-2v_2&=0,\\ -3v_1+3v_2&=0. \end{aligned} \tag{3.73}$$

While these equations do not at first look the same, we can divide out the constants and see that once again we get the same equation,

$$v_1 = v_2.$$

Picking $v_2 = 1$, we get $v_1 = 1$.

In summary, the solution to our eigenvalue problem is

$$\lambda = 4, \quad \mathbf{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\lambda = -1, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3.4.3 Eigenvalue Problems - A Summary

In the last subsection we were introduced to eigenvalue problems as a way to obtain a solution to a coupled system of linear differential equations. Eigenvalue problems appear in many contexts in physical applications. In this section we will summarize the method of solution of eigenvalue problems based upon our discussion in the last section. In the next subsection we will look at another problem that is a bit more geometric and will give us more insight into the process of diagonalization. We will return to our coupled system in a later section and provide more examples of solving eigenvalue problems.

We seek *nontrivial solutions* to the eigenvalue problem

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{3.74}$$

We note that $\mathbf{v} = \mathbf{0}$ is an obvious solution. Furthermore, it does not lead to anything useful. So, it is a trivial solution. Typically, we are given the matrix A and have to determine the eigenvalues, λ, and the associated eigenvectors, $\mathbf{v}$, satisfying the above eigenvalue problem. Later in the course we will explore other types of eigenvalue problems.

For now we begin to solve the eigenvalue problem for $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Inserting this into Equation (3.74), we obtain the homogeneous algebraic system

$$\begin{aligned} (a-\lambda)v_1 + bv_2 &= 0, \\ cv_1 + (d-\lambda)v_2 &= 0. \end{aligned} \tag{3.75}$$

The solution of such a system would be unique if the determinant of the system is not zero. However, this would give the trivial solution $v_1 = 0$, $v_2 = 0$. To get a nontrivial solution, we need to force the determinant to be zero. This yields the eigenvalue equation

$$0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc.$$

This is a quadratic equation for the eigenvalues that would lead to nontrivial solutions. If we expand the right side of the equation, we find that

$$\lambda^2 - (a+d)\lambda + ad - bc = 0.$$

This is the same equation as the characteristic equation for the general constant coefficient differential equation considered in the last chapter as we will later show in Equation (2.113). Thus, the eigenvalues correspond to the solutions of the characteristic polynomial for the system.

Once we find the eigenvalues, then there are possibly an infinite number solutions to the algebraic system. We will see this in the examples.

The method for solving eigenvalue problems, as you have seen, consists of just a few simple steps. We list these steps as follows:

Solving Eigenvalue Problems
a) Write the coefficient matrix;
b) Find the eigenvalues from the equation $\det(A - \lambda I) = 0$; and,
c) Solve the linear system $(A - \lambda I)\mathbf{v} = 0$ for each λ.

3.5 *Matrix Formulation of Planar Systems*

We have investigated several linear systems in the plane in the last chapter. However, we need a deeper insight into the solutions of planar systems. So, in this section we will recast the first order linear systems into matrix form. This will lead to a better understanding of first order systems and allow for extensions to higher dimensions and the solution of nonhomogeneous equations. In particular, we can see that the solutions obtained for planar systems in the last chapters are intimately connected to the underlying eigenvalue problems.

We start with the usual homogeneous system in Equation (2.110). Let the unknowns be represented by the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then we have that

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv A\mathbf{x}.$$

Here we have introduced the *coefficient matrix* A. This is a first order vector differential equation,

$$\mathbf{x}' = A\mathbf{x}.$$

Formerly, we can write the solution as[8]

$$\mathbf{x} = \mathbf{x}_0 e^{At}.$$

We would like to investigate the solution of our system. Our investigations will lead to new techniques for solving linear systems using matrix methods.

[8] The *exponential of a matrix* is defined using the Maclaurin series expansion

$$e^x = \sum_{k=0}^{\infty} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

So, we define

$$e^A = \sum_{k=0}^{\infty} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots. \tag{3.76}$$

In general, it is difficult computing e^A unless A is diagonal.

We begin by recalling the solution to the specific problem (2.117). We obtained the solution to this system as

$$x(t) = c_1 e^t + c_2 e^{-4t},$$
$$y(t) = \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t}. \quad (3.77)$$

This can be rewritten using matrix operations. Namely, we first write the solution in vector form.

$$\begin{aligned}
\mathbf{x} &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\
&= \begin{pmatrix} c_1 e^t + c_2 e^{-4t} \\ \frac{1}{3} c_1 e^t - \frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\
&= \begin{pmatrix} c_1 e^t \\ \frac{1}{3} c_1 e^t \end{pmatrix} + \begin{pmatrix} c_2 e^{-4t} \\ -\frac{1}{2} c_2 e^{-4t} \end{pmatrix} \\
&= c_1 \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} e^{-4t}. \quad (3.78)
\end{aligned}$$

We see that our solution is in the form of a linear combination of vectors of the form

$$\mathbf{x} = \mathbf{v} e^{\lambda t}$$

with $\mathbf{v}$ a constant vector and λ a constant number. This is similar to how we began to find solutions to second order constant coefficient equations. So, for the general problem (3.5) we insert this guess. Thus,

$$\begin{aligned}
\mathbf{x}' &= A\mathbf{x} \Rightarrow \\
\lambda \mathbf{v} e^{\lambda t} &= A \mathbf{v} e^{\lambda t}. \quad (3.79)
\end{aligned}$$

For this to be true for all t, we have that

$$A\mathbf{v} = \lambda \mathbf{v}. \quad (3.80)$$

This is an eigenvalue problem. A is a 2×2 matrix for our problem, but could easily be generalized to a system of n first order differential equations. We will confine our remarks for now to planar systems. However, we need to recall how to solve eigenvalue problems and then see how solutions of eigenvalue problems can be used to obtain solutions to our systems of differential equations.

3.5.1 Solving Constant Coefficient Systems in 2D

Before proceeding to examples, we first indicate the types of solutions that could result from the solution of a homogeneous, constant coefficient system of first order differential equations.

We begin with the linear system of differential equations in matrix form.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x} = A\mathbf{x}. \tag{3.81}$$

The type of behavior depends upon the eigenvalues of matrix A. The procedure is to determine the eigenvalues and eigenvectors and use them to construct the general solution.

If we have an initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$, we can determine the two arbitrary constants in the general solution in order to obtain the particular solution. Thus, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linearly independent solutionspt[-4], then the general solution is given as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

Then, setting $t = 0$, we get two linear equations for c_1 and c_2:

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = \mathbf{x}_0.$$

The major work is in finding the linearly independent solutions. This depends upon the different types of eigenvalues that one obtains from solving the eigenvalue equation, $\det(A - \lambda I) = 0$. The nature of these roots indicate the form of the general solution. On the next page we summarize the classification of solutions in terms of the eigenvalues of the coefficient matrix. We first make some general remarks about the plausibility of these solutions and then provide examples in the following section to clarify the matrix methods for our two dimensional systems.

The construction of the general solution in Case I is straight forward. However, the other two cases need a little explanation.

[-4] Recall that linear independence means $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$ if and only if $c_1, c_2 = 0$. The reader should derive the condition on the $\mathbf{x}_i$ for linear independence.

Classification of the Solutions for Two Linear First Order Differential Equations

1. **Case I: Two real, distinct roots.**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue obtaining two eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then write the general solution as a linear combination $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$

2. **Case II: One Repeated Root**

   Solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$ for one eigenvalue λ, obtaining the first eigenvector $\mathbf{v}_1$. One then needs a second linearly independent solution. This is obtained by solving the nonhomogeneous problem $A\mathbf{v}_2 - \lambda\mathbf{v}_2 = \mathbf{v}_1$ for $\mathbf{v}_2$.

   The general solution is then given by $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$.

3. **Case III: Two complex conjugate roots.**

   Solve the eigenvalue problem $\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$ for one eigenvalue, $\lambda = \alpha + i\beta$, obtaining one eigenvector $\mathbf{v}$. Note that this eigenvector may have complex entries. Thus, one can write the vector

   $$\mathbf{y}(t) = e^{\lambda t}\mathbf{v} = e^{\alpha t}(\cos\beta t + i\sin\beta t)\mathbf{v}.$$

   Now, construct two linearly independent solutions to the problem using the real and imaginary parts of $\mathbf{y}(t)$:

   $$\mathbf{y}_1(t) = Re(\mathbf{y}(t)) \text{ and } \mathbf{y}_2(t) = Im(\mathbf{y}(t)).$$

   Then the general solution can be written as $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

Let's consider Case III. Note that since the original system of equations does not have any i's, then we would expect real solutions. So, we look at the real and imaginary parts of the complex solution. We have that the complex solution satisfies the equation

$$\frac{d}{dt}\left[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))\right] = A[Re(\mathbf{y}(t)) + iIm(\mathbf{y}(t))].$$

Differentiating the sum and splitting the real and imaginary parts of the equation, gives

$$\frac{d}{dt}Re(\mathbf{y}(t)) + i\frac{d}{dt}Im(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))] + iA[Im(\mathbf{y}(t))].$$

Setting the real and imaginary parts equal, we have

$$\frac{d}{dt}Re(\mathbf{y}(t)) = A[Re(\mathbf{y}(t))],$$

and

$$\frac{d}{dt}Im(\mathbf{y}(t)) = A[Im(\mathbf{y}(t))].$$

Therefore, the real and imaginary parts each are linearly independent solutions of the system and the general solution can be written as a linear combination of these expressions.

We now turn to Case II. Writing the system of first order equations as a second order equation for $x(t)$ with the sole solution of the characteristic equation, $\lambda = \frac{1}{2}(a+d)$, we have that the general solution takes the form

$$x(t) = (c_1 + c_2 t)e^{\lambda t}.$$

This suggests that the second linearly independent solution involves a term of the form $\mathbf{v}te^{\lambda t}$. It turns out that the guess that works is

$$\mathbf{x} = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2.$$

Inserting this guess into the system $\mathbf{x}' = A\mathbf{x}$ yields

$$\begin{aligned}
(te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2)' &= A\left[te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2\right]. \\
e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 + \lambda e^{\lambda t}\mathbf{v}_2 &= \lambda te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}A\mathbf{v}_2. \\
e^{\lambda t}\left(\mathbf{v}_1 + \lambda\mathbf{v}_2\right) &= e^{\lambda t}A\mathbf{v}_2.
\end{aligned} \tag{3.82}$$

Noting this is true for all t, we find that

$$\mathbf{v}_1 + \lambda\mathbf{v}_2 = A\mathbf{v}_2. \tag{3.83}$$

Therefore,

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1.$$

We know everything except for $\mathbf{v}_2$. So, we just solve for it and obtain the second linearly independent solution.

3.5.2 Examples of the Matrix Method

HERE WE WILL GIVE SOME EXAMPLES for typical systems for the three cases mentioned in the last section.

Example 3.10. $A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}.$

Eigenvalues: *We first determine the eigenvalues.*

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} \tag{3.84}$$

Therefore,

$$\begin{aligned}
0 &= (4-\lambda)(3-\lambda) - 6 \\
0 &= \lambda^2 - 7\lambda + 6 \\
0 &= (\lambda - 1)(\lambda - 6)
\end{aligned} \tag{3.85}$$

The eigenvalues are then $\lambda = 1, 6$. This is an example of Case I.

Eigenvectors: *Next we determine the eigenvectors associated with each of these eigenvalues. We have to solve the system $A\mathbf{v} = \lambda\mathbf{v}$ in each case.*

Case $\lambda = 1$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{3.86}$$

$$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.87}$$

This gives $3v_1 + 2v_2 = 0$. One possible solution yields an eigenvector of

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Case $\lambda = 6$.

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 6 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{3.88}$$

$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.89}$$

For this case we need to solve $-2v_1 + 2v_2 = 0$. This yields

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

General Solution: *We can now construct the general solution.*

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= c_1 e^{t} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2c_1 e^{t} + c_2 e^{6t} \\ -3c_1 e^{t} + c_2 e^{6t} \end{pmatrix}. \end{aligned} \tag{3.90}$$

Example 3.11. $A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$.

Eigenvalues: *Again, one solves the eigenvalue equation.*

$$0 = \begin{vmatrix} 3-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} \tag{3.91}$$

Therefore,

$$\begin{aligned}
0 &= (3-\lambda)(-1-\lambda)+5 \\
0 &= \lambda^2 - 2\lambda + 2 \\
\lambda &= \frac{-(-2) \pm \sqrt{4-4(1)(2)}}{2} = 1 \pm i.
\end{aligned} \tag{3.92}$$

The eigenvalues are then $\lambda = 1+i, 1-i$. This is an example of Case III.

Eigenvectors: *In order to find the general solution, we need only find the eigenvector associated with $1+i$.*

$$\begin{aligned}
\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= (1+i) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{3.93}$$

We need to solve $(2-i)v_1 - 5v_2 = 0$. Thus,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}. \tag{3.94}$$

Complex Solution: *In order to get the two real linearly independent solutions, we need to compute the real and imaginary parts of $\mathbf{v}e^{\lambda t}$.*

$$\begin{aligned}
e^{\lambda t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} &= e^{(1+i)t} \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t(\cos t + i \sin t) \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t \begin{pmatrix} (2+i)(\cos t + i \sin t) \\ \cos t + i \sin t \end{pmatrix} \\
&= e^t \begin{pmatrix} (2\cos t - \sin t) + i(\cos t + 2\sin t) \\ \cos t + i \sin t \end{pmatrix} \\
&= e^t \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^t \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.
\end{aligned}$$

General Solution: *Now we can construct the general solution.*

$$\begin{aligned}
\mathbf{x}(t) &= c_1 e^t \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} \\
&= e^t \begin{pmatrix} c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ c_1 \cos t + c_2 \sin t \end{pmatrix}.
\end{aligned} \tag{3.95}$$

Note: This can be rewritten as

$$\mathbf{x}(t) = e^t \cos t \begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} + e^t \sin t \begin{pmatrix} 2c_2 - c_1 \\ c_2 \end{pmatrix}.$$

Example 3.12. $A = \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix}$.

Eigenvalues:

$$0 = \begin{vmatrix} 7-\lambda & -1 \\ 9 & 1-\lambda \end{vmatrix} \tag{3.96}$$

Therefore,

$$\begin{aligned} 0 &= (7-\lambda)(1-\lambda)+9 \\ 0 &= \lambda^2 - 8\lambda + 16 \\ 0 &= (\lambda-4)^2. \end{aligned} \tag{3.97}$$

There is only one real eigenvalue, $\lambda = 4$. This is an example of Case II.

Eigenvectors: *In this case we first solve for $\mathbf{v}_1$ and then get the second linearly independent vector.*

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= 4 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.98}$$

Therefore, we have

$$3v_1 - v_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Second Linearly Independent Solution:

Now we need to solve $A\mathbf{v}_2 - \lambda \mathbf{v}_2 = \mathbf{v}_1$.

$$\begin{aligned} \begin{pmatrix} 7 & -1 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - 4 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned} \tag{3.99}$$

Expanding the matrix product, we obtain the system of equations

$$\begin{aligned} 3u_1 - u_2 &= 1 \\ 9u_1 - 3u_2 &= 3. \end{aligned} \tag{3.100}$$

The solution of this system is $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

General Solution: *We construct the general solution as*

$$\mathbf{y}(t) = c_1 e^{\lambda t} \mathbf{v}_1 + c_2 e^{\lambda t} (\mathbf{v}_2 + t\mathbf{v}_1).$$

$$\begin{aligned} &= c_1 e^{4t}\begin{pmatrix}1\\3\end{pmatrix} + c_2 e^{4t}\left[\begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}1\\3\end{pmatrix}\right] \\ &= e^{4t}\begin{pmatrix} c_1 + c_2(1+t) \\ 3c_1 + c_2(2+3t)\end{pmatrix}. \end{aligned} \tag{3.101}$$

3.5.3 Planar Systems - Summary

The reader should have noted by now that there is a connection between the behavior of the solutions of planar systems obtained in Chapter 2 and the eigenvalues found from the coefficient matrices in the previous examples. Here we summarize some of these cases.

Type	Eigenvalues	Stability
Node	Real λ, same signs	$\lambda > 0$, stable
Saddle	Real λ opposite signs	Mostly Unstable
Center	λ pure imaginary	—
Focus/Spiral	Complex λ, $\mathrm{Re}(\lambda) \neq 0$	$\mathrm{Re}(\lambda > 0)$, stable
Degenerate Node	Repeated roots	$\lambda > 0$, stable
Line of Equilibria	One zero eigenvalue	$\lambda > 0$, stable

Table 3.1: List of typical behaviors in planar systems.

The connection, as we have seen, is that the characteristic equation for the associated second order differential equation is the same as the eigenvalue equation of the coefficient matrix for the linear system. However, one should be a little careful in cases in which the coefficient matrix in not diagonalizable. In Table 3.2 are three examples of systems with repeated roots. The reader should look at these systems and look at the commonalities and differences in these systems and their solutions. In these cases one has unstable nodes, though they are degenerate in that there is only one accessible eigenvector.

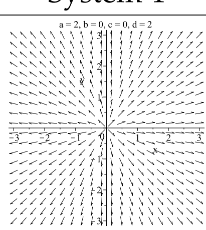
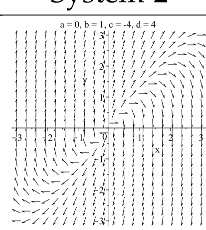
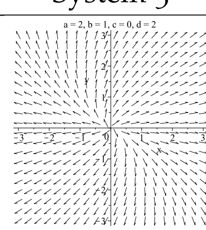
System 1	System 2	System 3
		
$\mathbf{x}' = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\mathbf{x}$	$\mathbf{x}' = \begin{pmatrix}0 & 1\\-4 & 4\end{pmatrix}\mathbf{x}$	$\mathbf{x}' = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}\mathbf{x}$

Table 3.2: Three examples of systems with a repeated root of $\lambda = 2$.

3.6 Applications

In this section we will describe some simple applications leading to systems of differential equations which can be solved using the methods in this chapter. These systems are left for homework problems and the as the start of further explorations for student projects.

3.6.1 Circuits

In the last chapter we investigated simple series LRC circuits. More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. an example of another circuit setup is shown in Figure 3.5. This is not a problem that can be covered in the first year physics course.

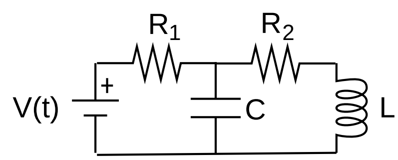


Figure 3.5: A circuit with two loops containing several different circuit elements.

There are two loops, indicated in Figure 3.6 as traversed clockwise. For each loop we need to apply Kirchoff's Loop Rule. There are three oriented currents, labeled I_i, $i = 1, 2, 3$. Corresponding to each current is a changing charge, q_i such that

$$I_i = \frac{dq_i}{dt}, \quad i = 1, 2, 3.$$

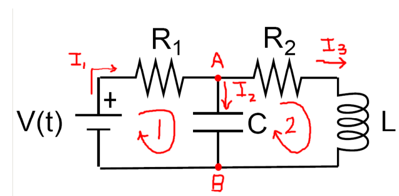


Figure 3.6: The previous parallel circuit with the directions indicated for traversing the loops in Kirchoff's Laws.

For loop one we have

$$I_1 R_1 + \frac{q_2}{C} = V(t). \tag{3.102}$$

For loop two

$$I_3 R_2 + L\frac{dI_3}{dt} = \frac{q_2}{C}. \tag{3.103}$$

We have three unknown functions for the charge. Once we know the charge functions, differentiation will yield the currents. However, we only have two equations. We need a third equation. This is found from Kirchoff's Point (Junction) Rule. Consider the points A and B in Figure 3.6. Any charge (current) entering these junctions must be the same as the total charge (current) leaving the junctions. For point A we have

$$I_1 = I_2 + I_3, \tag{3.104}$$

or

$$\dot{q}_1 = \dot{q}_2 + \dot{q}_3. \tag{3.105}$$

Equations (3.102), (3.103), and (3.105) form a coupled system of differential equations for this problem. There are both first and second order derivatives involved. We can write the whole system in terms of charges as

$$R_1\dot{q}_1 + \frac{q_2}{C} = V(t)$$

$$\begin{aligned} R_2\dot{q}_3 + L\ddot{q}_3 &= \frac{q_2}{C} \\ \dot{q}_1 &= \dot{q}_2 + \dot{q}_3. \end{aligned} \tag{3.106}$$

The question is whether, or not, we can write this as a system of first order differential equations. Since there is only one second order derivative, we can introduce the new variable $q_4 = \dot{q}_3$. The first equation can be solved for $\dot{q}_1$. The third equation can be solved for $\dot{q}_2$ with appropriate substitutions for the other terms. $\dot{q}_3$ is gotten from the definition of q_4 and the second equation can be solved for $\ddot{q}_3$ and substitutions made to obtain the system

$$\begin{aligned} \dot{q}_1 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} \\ \dot{q}_2 &= \frac{V}{R_1} - \frac{q_2}{R_1 C} - q_4 \\ \dot{q}_3 &= q_4 \\ \dot{q}_4 &= \frac{q_2}{LC} - \frac{R_2}{L} q_4. \end{aligned}$$

So, we have a nonhomogeneous first order system of differential equations. In the last section we learned how to solve such systems.

3.6.2 *Love Affairs*

The next application is one that has been studied by several authors as a cute system involving relationships. One considers what happens to the affections that two people have for each other over time. Let R denote the affection that Romeo has for Juliet and J be the affection that Juliet has for Romeo. positive values indicate love and negative values indicate dislike.

One possible model is given by

$$\begin{aligned} \frac{dR}{dt} &= bJ \\ \frac{dJ}{dt} &= cR \end{aligned} \tag{3.107}$$

with $b > 0$ and $c < 0$. In this case Romeo loves Juliet the more she likes him. But Juliet backs away when she finds his love for her increasing.

A typical system relating the combined changes in affection can be modeled as

$$\begin{aligned} \frac{dR}{dt} &= aR + bJ \\ \frac{dJ}{dt} &= cR + dJ. \end{aligned} \tag{3.108}$$

Several scenarios are possible for various choices of the constants. For example, if $a > 0$ and $b > 0$, Romeo gets more and more excited by

Juliet's love for him. If $c > 0$ and $d < 0$, Juliet is being cautious about her relationship with Romeo. For specific values of the parameters and initial conditions, one can explore this match of an overly zealous lover with a cautious lover.

3.6.3 *Predator Prey Models*

Another common model studied is that of competing species. For example, we could consider a population of rabbits and foxes. Left to themselves, rabbits would tend to multiply, thus

$$\frac{dR}{dt} = aR,$$

with $a > 0$. In such a model the rabbit population would grow exponentially. Similarly, a population of foxes would decay without the rabbits to feed on. So, we have that

$$\frac{dF}{dt} = -bF$$

for $b > 0$.

Now, if we put these populations together on a deserted island, they would interact. The more foxes, the rabbit population would decrease. However, the more rabbits, the foxes would have plenty to eat and the population would thrive. Thus, we could model the competing populations as

$$\begin{aligned} \frac{dR}{dt} &= aR - cF, \\ \frac{dF}{dt} &= -bF + dR, \end{aligned} \tag{3.109}$$

where all of the constants are positive numbers. Studying this coupled system would lead to as study of the dynamics of these populations. We will discuss other (nonlinear) systems in the next chapter.

3.6.4 *Mixture Problems*

There are many types of mixture problems. Such problems are standard in a first course on differential equations as examples of first order differential equations. Typically these examples consist of a tank of brine, water containing a specific amount of salt with pure water entering and the mixture leaving, or the flow of a pollutant into, or out of, a lake.

In general one has a rate of flow of some concentration of mixture entering a region and a mixture leaving the region. The goal is to determine how much stuff is in the region at a given time. This is governed by the equation

$$\boxed{\text{Rate of change of substance} = \text{Rate In} - \text{Rate Out.}}$$

This can be generalized to the case of two interconnected tanks. We provide some examples.

Example 3.13. Single Tank Problem

A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. [See Figure 3.7.] At the same time the well-mixed contents drain out at the rate of 5 gallons per minute. Find the amount of salt in the tank at time t. In all such problems one assumes that the solution is well mixed at each instant of time.

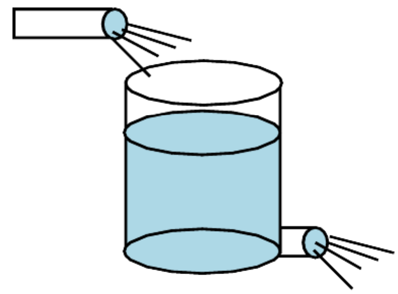


Figure 3.7: A typical mixing problem.

Let $x(t)$ be the amount of salt at time t. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less that leaving the tank. To figure out these rates, one notes that dx/dt has units of pounds per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:

$$\left(2\frac{pounds}{gal}\right)\left(5\frac{gal}{min}\right) = 10\frac{pounds}{min}.$$

Similarly, one can determine the rate out as

$$\left(\frac{x\ pounds}{50\ gal}\right)\left(5\frac{gal}{min}\right) = \frac{x}{10}\frac{pounds}{min}.$$

Thus, we have

$$\frac{dx}{dt} = 10 - \frac{x}{10}.$$

This equation is easily solved using the methods for first order equations.

Example 3.14. Double Tank Problem

One has two tanks connected together, labeled tank X and tank Y, as shown in Figure 3.8.

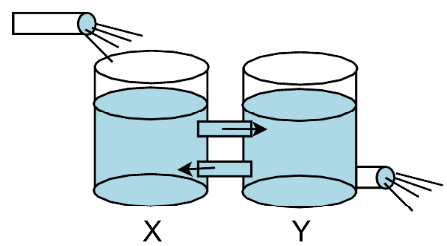


Figure 3.8: The two tank problem.

Let tank X initially have 100 gallons of brine made with 100 pounds of salt. Tank Y initially has 100 gallons of pure water. Now pure water is pumped into tank X at a rate of 2.0 gallons per minute. Some of the mixture of brine and pure water flows into tank Y at 3 gallons per minute. To keep the tank levels the same, one gallon of the Y mixture flows back into tank X at a rate of one gallon per minute and 2.0 gallons per minute drains out. Find the amount of salt at any given time in the tanks. What happens over a long period of time?

In this problem we set up two equations. Let $x(t)$ be the amount of salt in tank X and $y(t)$ the amount of salt in tank Y. Again, we carefully look at the rates into and out of each tank in order to set up the system of differential equations. We obtain the system

$$\begin{aligned}\frac{dx}{dt} &= \frac{y}{100} - \frac{3x}{100}\\ \frac{dy}{dt} &= \frac{3x}{100} - \frac{3y}{100}.\end{aligned} \tag{3.110}$$

This is a linear, homogenous constant coefficient system of two first order equations, which we know how to solve.

3.6.5 Chemical Kinetics

There are many problems that come from studying chemical reactions. The simplest reaction is when a chemical A turns into chemical B. This happens at a certain rate, $k > 0$. This can be represented by the chemical formula

$$\boxed{A \xrightarrow[k]{} B.}$$

In this case we have that the rates of change of the concentrations of A, $[A]$, and B, $[B]$, are given by

$$\begin{aligned} \frac{d[A]}{dt} &= -k[A] \\ \frac{d[B]}{dt} &= k[A] \end{aligned} \tag{3.111}$$

Think about this as it is a key to understanding the next reactions.

A more complicated reaction is given by

$$\boxed{A \xrightarrow[k_1]{} B \xrightarrow[k_2]{} C.}$$

In this case we can add to the above equation the rates of change of concentrations $[B]$ and $[C]$. The resulting system of equations is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{3.112}$$

One can further consider reactions in which a reverse reaction is possible. Thus, a further generalization occurs for the reaction

$$\boxed{A \underset{k_1}{\overset{k_3}{\longleftrightarrow}} B \xrightarrow[k_2]{} C.}$$

The resulting system of equations is

$$\begin{aligned} \frac{d[A]}{dt} &= -k_1[A] + k_3[B], \\ \frac{d[B]}{dt} &= k_1[A] - k_2[B] - k_3[B], \\ \frac{d[C]}{dt} &= k_2[B]. \end{aligned} \tag{3.113}$$

More complicated chemical reactions will be discussed at a later time.

3.6.6 Epidemics

Another interesting area of application of differential equation is in predicting the spread of disease. Typically, one has a population of susceptible people or animals. Several infected individuals are introduced into the population and one is interested in how the infection spreads and if the number of infected people drastically increases or dies off. Such models are typically nonlinear and we will look at what is called the SIR model in the next chapter. In this section we will model a simple linear model.

Let break the population into three classes. First, $S(t)$ are the healthy people, who are susceptible to infection. Let $I(t)$ be the number of infected people. Of these infected people, some will die from the infection and others recover. Let's assume that initially there in one infected person and the rest, say N, are obviously healthy. Can we predict how many deaths have occurred by time t?

Let's try and model this problem using the compartmental analysis we had seen in the mixing problems. The total rate of change of any population would be due to those entering the group less those leaving the group. For example, the number of healthy people decreases due infection and can increase when some of the infected group recovers. Let's assume that the rate of infection is proportional to the number of healthy people,aS. Also, we assume that the number who recover is proportional to the number of infected, rI. Thus, the rate of change of the healthy people is found as

$$\frac{dS}{dt} = -aS + rI.$$

Let the number of deaths be $D(t)$. Then, the death rate could be taken to be proportional to the number of infected people. So,

$$\frac{dD}{dt} = dI$$

Finally, the rate of change of infectives is due to healthy people getting infected and the infectives who either recover or die. Using the corresponding terms in the other equations, we can write

$$\frac{dI}{dt} = aS - rI - dI.$$

This linear system can be written in matrix form.

$$\frac{d}{dt}\begin{pmatrix} S \\ I \\ D \end{pmatrix} = \begin{pmatrix} -a & r & 0 \\ a & -d-r & 0 \\ 0 & d & 0 \end{pmatrix}\begin{pmatrix} S \\ I \\ D \end{pmatrix}. \tag{3.114}$$

The eigenvalue equation for this system is

$$\lambda\left[\lambda^2 + (a+r+d)\lambda + ad\right] = 0.$$

The reader can find the solutions of this system and determine if this is a realistic model.

3.7 Rotations of Conics

EIGENVALUE PROBLEMS show up in applications other than the solution of differential equations. We will see applications of this later in the text. For now, we are content to deal with problems which can be cast into matrix form. One example is the transformation of a simple system through rotation into a more complicated appearing system simply do to the choice of coordinate system. In this section we will explore this through the study of the rotation of conics.

You may have seen the general form for the equation of a conic in Cartesian coordinates in your calculus class. It is given by

$$Ax^2 + 2Bxy + Cy^2 + Ex + Fy = D. \tag{3.115}$$

This equation can describe a variety of conics (ellipses, hyperbolae and parabolae) depending on the constants. The E and F terms result from a translation[9] of the origin and the B term is the result of a rotation of the coordinate system. We leave it to the reader to show that coordinate translations can be made to eliminate the linear terms. So, we will set $E = F = 0$ in our discussion and only consider quadratic equations of the form

$$Ax^2 + 2Bxy + Cy^2 = D.$$

[9] It is easy to see how such terms correspond to translations of conics. Consider the simple example $x^2 + y^2 + 2x - 6y = 0$. By completing the squares in both x and y, this equation can be written as $(x+1)^2 + (y-3)^2 = 10$. Now you recognize that this is a circle whose center has been translated from the origin to $(-1, 3)$.

If $B = 0$, then the resulting equation could be an equation for the standard ellipse or hyperbola with center at the origin. In the case of an ellipse, the semimajor and semiminor axes lie along the coordinate axes. However, you could rotate the ellipse and that would introduce a B term, as we will see.

This conic equation can be written in matrix form. We note that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = Ax^2 + 2Bxy + Cy^2.$$

In short hand matrix form, we thus have for our equation

$$\mathbf{x}^T Q \mathbf{x} = D,$$

where Q is the matrix of coefficients A, B, and C.

We want to determine the transformation that puts this conic into a coordinate system in which there is no B term. Our goal is to obtain an equation of the form

$$A'x'^2 + C'y'^2 = D'$$

in the new coordinates $\mathbf{y}^T = (x', y')$. The matrix form of this equation is given as

$$\mathbf{y}^T \begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix} \mathbf{y} = D'.$$

We will denote the diagonal matrix by Λ.

So, we let

$$\mathbf{x} = R\mathbf{y},$$

where R is a rotation matrix. Inserting this transformation into our equation we find that

$$\begin{aligned} \mathbf{x}^T Q \mathbf{x} &= (R\mathbf{y})^T QR\mathbf{y} \\ &= \mathbf{y}^T (R^T QR)\mathbf{y}. \end{aligned} \tag{3.116}$$

Comparing this result to to desired form, we have

$$\Lambda = R^T QR. \tag{3.117}$$

Recalling that the rotation matrix is an orthogonal matrix, $R^T = R^{-1}$, we have

$$\Lambda = R^{-1} QR. \tag{3.118}$$

Thus, the problem reduces to that of trying to diagonalize the matrix Q. The eigenvalues of Q will lead to the constants in the rotated equation and the eigenvectors, as we will see, will give the directions of the principal axes (the semimajor and semiminor axes). We will first show this in an example.

Example 3.15. *Determine the principle axes of the ellipse given by*

$$13x^2 - 10xy + 13y^2 - 72 = 0.$$

A plot of this conic in Figure 3.9 shows that it is an ellipse. However, we might not know this without plotting it. (Actually, there are some conditions on the coefficients that do allow us to determine the conic. But you may not know this yet.) If the equation were in standard form, we could identify its general shape. So, we will use the method outlined above to find a coordinate system in which the ellipse appears in standard form.

The coefficient matrix for this equation is given by

$$Q = \begin{pmatrix} 13 & -5 \\ -5 & 13 \end{pmatrix}. \tag{3.119}$$

We seek a solution to the eigenvalue problem: $Q\mathbf{v} = \lambda\mathbf{v}$. Recall, the first step is to get the eigenvalue equation from $\det(Q - \lambda I) = 0$. For this problem we have

$$\begin{vmatrix} 13-\lambda & -5 \\ -5 & 13-\lambda \end{vmatrix} = 0. \tag{3.120}$$

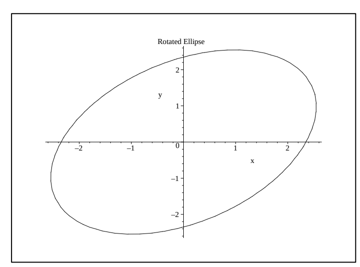


Figure 3.9: Plot of the ellipse given by $13x^2 - 10xy + 13y^2 - 72 = 0$.

So, we have to solve

$$(13-\lambda)^2 - 25 = 0.$$

This is easily solved by taking square roots to get

$$\lambda - 13 = \pm 5,$$

or

$$\lambda = 13 \pm 5 = 18, 8.$$

Thus, the equation in the new system is

$$8x'^2 + 18y'^2 = 72.$$

Dividing out the 72 puts this into the standard form

$$\frac{x'^2}{9} + \frac{y'^2}{4} = 1.$$

Now we can identify the ellipse in the new system. We show the two ellipses in Figure 3.10. We note that the given ellipse is the new one rotated by some angle, which we still need to determine.

Next, we seek the eigenvectors corresponding to each eigenvalue.

Eigenvalue 1: $\lambda = 8$

We insert the eigenvalue into the equation $(Q - \lambda I)\mathbf{v} = 0$. The system for the unknown eigenvector is

$$\begin{pmatrix} 13-8 & -5 \\ -5 & 13-8 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0. \tag{3.121}$$

The first equation is

$$5v_1 - 5v_2 = 0, \tag{3.122}$$

or $v_1 = v_2$. Thus, we can choose our eigenvector to be

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Eigenvalue 2: $\lambda = 18$

In the same way, we insert the eigenvalue into the equation $(Q - \lambda I)\mathbf{v} = 0$ and obtain

$$\begin{pmatrix} 13-18 & -5 \\ -5 & 13-18 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0. \tag{3.123}$$

The first equation is

$$-5v_1 - 5v_2 = 0, \tag{3.124}$$

or $v_1 = -v_2$. Thus, we can choose our eigenvector to be

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

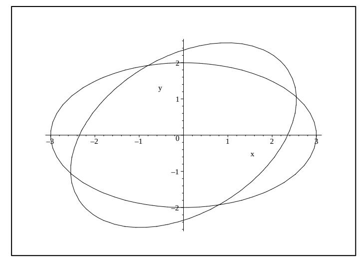


Figure 3.10: Plot of the ellipse given by $13x^2 - 10xy + 13y^2 - 72 = 0$ and the ellipse $\frac{x'^2}{9} + \frac{y'^2}{4} = 1$ showing that the first ellipse is a rotated version of the second ellipse.

In Figure 3.11 we superimpose the eigenvectors on our original ellipse. We see that the eigenvectors point in directions along the semimajor and semiminor axes and indicate the angle of rotation. Eigenvector one is at a 45^o angle. Thus, our ellipse is a rotated version of one in standard position. Or, we could define new axes that are at 45^o to the standard axes and then the ellipse would take the standard form in the new coordinate system.

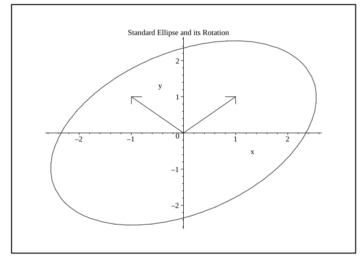


Figure 3.11: Plot of the ellipse given by $13x^2 - 10xy + 13y^2 - 72 = 0$ and the eigenvectors. Note that they are along the semimajor and semiminor axes and indicate the angle of rotation.

A general rotation of any conic can be performed. Consider the general equation:

$$Ax^2 + 2Bxy + Cy^2 + Ex + Fy = D. \tag{3.125}$$

We would like to find a rotation that puts it in the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + E'x' + F'y' = D. \tag{3.126}$$

We use the rotation matrix

$$\hat{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and define $\mathbf{x}' = \hat{R}_\theta^T \mathbf{x}$, or $\mathbf{x} = R_\theta \mathbf{x}'$.

The general equation can be written in matrix form:

$$\mathbf{x}^T Q \mathbf{x} + \mathbf{f}\mathbf{x} = D, \tag{3.127}$$

where Q is the usual matrix of coefficients A, B, and C and $\mathbf{f} = (E, F)$. Transforming this equation gives

$$\mathbf{x}'^T R_\theta^{-1} Q R_\theta \mathbf{x}' + \mathbf{f} R_\theta \mathbf{x}' = D. \tag{3.128}$$

The resulting equation is of the form

$$A'x'^2 + 2B'x'y' + C'y'^2 + E'x' + F'y' = D, \tag{3.129}$$

where

$$B' = 2(C - A)\sin\theta\cos\theta + 2B(2\cos\theta^2 - 1). \tag{3.130}$$

(We only need B' for this discussion). If we want the nonrotated form, then we seek an angle θ such that $B' = 0$. Noting that $2\sin\theta\cos\theta = \sin 2\theta$ and $2\cos\theta^2 - 1 = \cos 2\theta$, this gives

$$\tan(2\theta) = \frac{A - C}{B}. \tag{3.131}$$

Example 3.16. *So, in our previous example, with $A = C = 13$ and $B = -5$, we have $\tan(2\theta) = \infty$. Thus, $2\theta = \pi/2$, or $\theta = \pi/4$.*

Finally, we had noted that knowing the coefficients in the general quadratic is enough to determine the type of conic represented without doing any plotting. This is based on the fact that the determinant of the coefficient matrix is invariant under rotation. We see this from the equation for diagonalization

$$\begin{aligned}\det(\Lambda) &= \det(R_\theta^{-1} Q R_\theta) \\ &= \det(R_\theta^{-1}) \det(Q) \det(R_\theta) \\ &= \det(R_\theta^{-1} R_\theta) \det(Q) \\ &= \det(Q).\end{aligned} \tag{3.132}$$

Therefore, we have

$$\lambda_1 \lambda_2 = AC - B^2.$$

Looking at Equation (3.126), we have three cases:

1. Ellipse $\lambda_1 \lambda_2 > 0$ or $B^2 - AC < 0$.
2. Hyperbola $\lambda_1 \lambda_2 < 0$ or $B^2 - AC > 0$.
3. Parabola $\lambda_1 \lambda_2 = 0$ or $B^2 - AC = 0$. and one eigenvalue is nonzero. Otherwise the equation degenerates to a linear equation.

Example 3.17. *Consider the hyperbola $xy = 6$. We can see that this is a rotated hyperbola by plotting $y = 6/x$. A plot is shown in Figure 3.12. Determine the rotation need to put transform the hyperbola to new coordinates so that its equation will be in standard form.*

The coefficient matrix for this equation is given by

$$A = \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0 \end{pmatrix}. \tag{3.133}$$

The eigenvalue equation is

$$\begin{vmatrix} -\lambda & -0.5 \\ -0.5 & -\lambda \end{vmatrix} = 0. \tag{3.134}$$

Thus,

$$\lambda^2 - 0.25 = 0,$$

or $\lambda = \pm 0.5$.

Once again, $\tan(2\theta) = \infty$, so the new system is at 45^o to the old. The equation in new coordinates is $0.5x^2 + (-0.5)y^2 = 6$, or $x^2 - y^2 = 12$. A plot is shown in Figure 3.13.

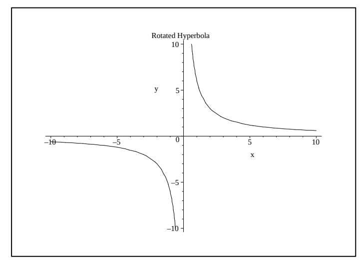


Figure 3.12: Plot of the hyperbola given by $xy = 6$.

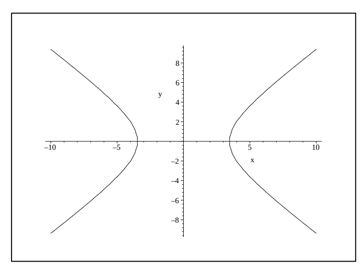


Figure 3.13: Plot of the rotated hyperbola given by $x^2 - y^2 = 12$.

3.8 Appendix: Diagonalization and Linear Systems

As we have seen, the matrix formulation for linear systems can be powerful, especially for n differential equations involving n unknown functions. Our ability to proceed towards solutions depended upon the solution of eigenvalue problems. However, in the case of repeated eigenvalues we saw some additional complications. This all depends deeply on the background linear algebra. Namely, we relied on being able to diagonalize the given coefficient matrix. In this section we will discuss the limitations of diagonalization and introduce the Jordan canonical form.

We begin with the notion of similarity. Matrix A is *similar* to matrix B if and only if there exists a nonsingular matrix P such that

$$B = P^{-1}AP. \tag{3.135}$$

Recall that a nonsingular matrix has a nonzero determinant and is invertible.

We note that the similarity relation is an equivalence relation. Namely, it satisfies the following

1. A is similar to itself.
2. If A is similar to B, then B is similar to A.
3. If A is similar to B and B is similar to C, the A is similar to C.

Also, if A is similar to B, then they have the same eigenvalues. This follows from a simple computation of the eigenvalue equation. Namely,

$$\begin{aligned} 0 &= \det(B - \lambda I) \\ &= \det(P^{-1}AP - \lambda P^{-1}IP) \\ &= \det(P)^{-1}\det(A - \lambda I)\det(P) \\ &= \det(A - \lambda I). \end{aligned} \tag{3.136}$$

Therefore, $\det(A - \lambda I) = 0$ and λ is an eigenvalue of both A and B.

An $n \times n$ matrix A is *diagonalizable* if and only if A is similar to a diagonal matrix D; i.e., there exists a nonsingular matrix P such that

$$D = P^{-1}AP. \tag{3.137}$$

One of the most important theorems in linear algebra is the Spectral Theorem. This theorem tells us when a matrix can be diagonalized. In fact, it goes beyond matrices to the diagonalization of linear operators. We learn in linear algebra that linear operators can be represented by matrices once we pick a particular representation basis. Diagonalization is simplest for finite dimensional vector spaces and requires some generalization for infinite dimensional vectors spaces. Examples of operators to which the spectral theorem applies are self-adjoint operators (more generally normal operators on Hilbert spaces). We will explore some of these ideas later in the course. The spectral theorem provides a canonical decomposition, called the spectral decomposition, or eigendecomposition, of the underlying vector space on which it acts.

The next theorem tells us how to diagonalize a matrix:

Theorem 3.1. *Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors. If so, then*

$$D = P^{-1}AP.$$

If $\{v_1, \ldots, v_n\}$ are the eigenvectors of A and $\{\lambda_1, \ldots, \lambda_n\}$ are the corresponding eigenvalues, then v_j is the jth column of P and $D_{jj} = \lambda_j$.

A simpler determination results by noting

Theorem 3.2. *Let A be an $n \times n$ matrix with n real and distinct eigenvalues. Then A is diagonalizable.*

Therefore, we need only look at the eigenvalues and determine diagonalizability. In fact, one also has from linear algebra the following result.

Theorem 3.3. *Let A be an $n \times n$ real symmetric matrix. Then A is diagonalizable.*

Recall that a symmetric matrix is one whose transpose is the same as the matrix, or $A_{ij} = A_{ji}$.

Example 3.18. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

This is a real symmetric matrix. The characteristic polynomial is found to be

$$\det(A - \lambda I) = -(\lambda - 5)(\lambda - 3)(\lambda + 1) = 0.$$

As before, we can determine the corresponding eigenvectors (for $\lambda = -1, 3, 5$, respectively) as

$$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We can use these to construct the diagonalizing matrix P. Namely, we have

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}. \end{aligned} \tag{3.138}$$

Now diagonalization is an important idea in solving linear systems of first order equations, as we have seen for simple systems. If our system is originally diagonal, that means our equations are completely uncoupled. Let our system take the form

$$\frac{d\mathbf{y}}{dt} = D\mathbf{y}, \tag{3.139}$$

where D is diagonal with entries λ_i, $i = 1, \ldots, n$. The system of equations, $y_i' = \lambda_i y_i$, has solutions

$$y_i(t) = c_c e^{\lambda_i t}.$$

Thus, it is easy to solve a diagonal system.

Let A be similar to this diagonal matrix. Then

$$\frac{d\mathbf{y}}{dt} = P^{-1}AP\mathbf{y}. \tag{3.140}$$

This can be rewritten as

$$\frac{dP\mathbf{y}}{dt} = AP\mathbf{y}.$$

Defining $\mathbf{x} = P\mathbf{y}$, we have

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{3.141}$$

This simple derivation shows that if A is diagonalizable, then a transformation of the original system in $\mathbf{x}$ to new *coordinates*, or a new basis, results in a simpler system in $\mathbf{y}$.

However, it is not always possible to diagonalize a given square matrix. This is because some matrices do not have enough linearly independent vectors, or we have repeated eigenvalues. However, we have the following theorem:

Theorem 3.4. *Every $n \times n$ matrix A is similar to a matrix of the form*

$$J = diag[J_1, J_2, \ldots, J_n],$$

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix} \tag{3.142}$$

We will not go into the details of how one finds this **Jordan Canonical Form** or proving the theorem. In practice you can use a computer algebra system to determine this and the similarity matrix. However, we would still need to know how to use it to solve our system of differential equations.

Example 3.19. *Let's consider a simple system with the 3×3 Jordan block*

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The corresponding system of coupled first order differential equations takes the form

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + x_2, \\ \frac{dx_2}{dt} &= 2x_2 + x_3, \\ \frac{dx_3}{dt} &= 2x_3. \end{aligned} \tag{3.143}$$

The last equation is simple to solve, giving $x_3(t) = c_3 e^{2t}$. Inserting into the second equation, you have a

$$\frac{dx_2}{dt} = 2x_2 + c_3 e^{2t}.$$

Using the integrating factor, e^{-2t}, one can solve this equation to get $x_2(t) = (c_2 + c_3 t)e^{2t}$. Similarly, one can solve the first equation to obtain $x_1(t) = (c_1 + c_2 t + \frac{1}{2} c_3 t^2)e^{2t}$.

This should remind you of a problem we had solved earlier leading to the generalized eigenvalue problem in (3.83). This suggests that there is a more general theory when there are multiple eigenvalues and relating to Jordan canonical forms.

Let's write the solution we just obtained in vector form. We have

$$\mathbf{x}(t) = \left[c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} \frac{1}{2}t^2 \\ t \\ 1 \end{pmatrix} \right] e^{2t}. \tag{3.144}$$

It looks like this solution is a linear combination of three linearly independent solutions,

$$\begin{aligned} \mathbf{x} &= \mathbf{v}_1 e^{2\lambda t} \\ \mathbf{x} &= (t\mathbf{v}_1 + \mathbf{v}_2)e^{\lambda t} \\ \mathbf{x} &= (\frac{1}{2}t^2 \mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3)e^{\lambda t}, \end{aligned} \tag{3.145}$$

where $\lambda = 2$ and the vectors satisfy the equations

$$\begin{aligned} (A - \lambda I)\mathbf{v}_1 &= 0, \\ (A - \lambda I)\mathbf{v}_2 &= \mathbf{v}_1, \\ (A - \lambda I)\mathbf{v}_3 &= \mathbf{v}_2, \end{aligned} \tag{3.146}$$

and

$$\begin{aligned}(A-\lambda I)\mathbf{v}_1 &= 0,\\ (A-\lambda I)^2\mathbf{v}_2 &= 0,\\ (A-\lambda I)^3\mathbf{v}_3 &= 0.\end{aligned} \tag{3.147}$$

It is easy to generalize this result to build linearly independent solutions corresponding to multiple roots (eigenvalues) of the characteristic equation.

Problems

1. Express the vector $\mathbf{v} = (1,2,3)$ as a linear combination of the vectors $\mathbf{a}_1 = (1,1,1)$, $\mathbf{a}_2 = (1,0,-1)$, and $\mathbf{a}_3 = (2,1,0)$.

2. A symmetric matrix is one for which the transpose of the matrix is the same as the original matrix, $A^T = A$. An antisymmetric matrix is one which satisfies $A^T = -A$.

a. Show that the diagonal elements of an $n \times n$ antisymmetric matrix are all zero.

b. Show that a general 3×3 antisymmetric matrix has three independent off-diagonal elements.

c. How many independent elements does a general 3×3 symmetric matrix have?

d. How many independent elements does a general $n \times n$ symmetric matrix have?

e. How many independent elements does a general $n \times n$ antisymmetric matrix have?

3. Consider the matrix representations for two dimensional rotations of vectors by angles α and β, denoted by R_α and R_β, respectively.

a. Find R_α^{-1} and R_α^T. How do they relate?

b. Prove that $R_{\alpha+\beta} = R_\alpha R_\beta = R_\beta R_\alpha$.

4. The Pauli spin matrices in quantum mechanics are given by the matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that

a. $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$.

b. $\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I$, for $i,j = 1,2,3$ and I the 2×2 identity matrix. $\{,\}$ is the anti-commutation operation.

c. $[\sigma_1, \sigma_2] \equiv \sigma_1\sigma_2 - \sigma_2\sigma_1 = 2i\sigma_3$, and similarly for the other pairs. $[,]$ is the commutation operation.

d. Show that an arbitrary 2×2 matrix M can be written as a linear combination of Pauli matrices, $M = a_0 I + \sum_{j=1}^{3} a_j \sigma_j$, where the a_j's are complex numbers.

5. Use Cramer's Rule to solve the system:

$$\begin{aligned} 2x - 5z &= 7 \\ x - 2y &= 1 \\ 3x - 5y - z &= 4. \end{aligned} \tag{3.148}$$

6. Find the eigenvalue(s) and eigenvector(s) for the following:

a. $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$

b. $\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$

c. $\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$

d. $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

7. For the matrices in the last problem, compute the determinants and find the inverses, if they exist.

8. Consider the conic $5x^2 - 4xy + 2y^2 = 30$.

a. Write the left side in matrix form.

b. Diagonalize the coefficient matrix, finding the eigenvalues and eigenvectors.

c. Construct the rotation matrix from the information in part b. What is the angle of rotation needed to bring the conic into standard form?

d. What is the conic?

9. In Equation (3.76) the exponential of a matrix was defined.

a. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Compute e^A.

b. Give a definition of $\cos A$ and compute $\cos \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ in simplest form.

c. Using the definition of e^A, prove $e^{PAP^{-1}} = Pe^A P^{-1}$ for general A.

10. Consider the following systems. For each system determine the coefficient matrix. When possible, solve the eigenvalue problem for each matrix and use the eigenvalues and eigenfunctions to provide solutions to the given systems. Finally, in the common cases which you investigated in Problem 17, make comparisons with your previous answers, such as what type of eigenvalues correspond to stable nodes.

a.

$$\begin{aligned} x' &= 3x - y \\ y' &= 2x - 2y. \end{aligned}$$

b.

$$\begin{aligned} x' &= -y \\ y' &= -5x. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - y \\ y' &= y. \end{aligned}$$

d.

$$\begin{aligned} x' &= 2x + 3y \\ y' &= -3x + 2y. \end{aligned}$$

e.

$$\begin{aligned} x' &= -4x - y \\ y' &= x - 2y. \end{aligned}$$

f.

$$\begin{aligned} x' &= x - y \\ y' &= x + y. \end{aligned}$$

11. Add a third spring connected to mass two in the coupled system shown in Figure 2.17 to a wall on the far right. Assume that the masses are the same and the springs are the same.

a. Model this system with a set of first order differential equations.

b. If the masses are all 2.0 kg and the spring constants are all 10.0 N/m, then find the general solution for the system.

c. Move mass one to the left (of equilibrium) 10.0 cm and mass two to the right 5.0 cm. Let them go. find the solution and plot it as a function of time. Where is each mass at 5.0 seconds?

12. Consider the series circuit in Figure 2.7 with $L = 1.00$ H, $R = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of two first order differential equations for the charge and the current.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when V_0 is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

13. Consider the series circuit in Figure 3.5 with $L = 1.00$ H, $R_1 = R_2 = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Set up the problem as a system of first order differential equations for the charges and the currents in each loop.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when V_0 is applied. Find the current and the charge on the capacitor as functions of time.

c. Plot your solutions and describe how the system behaves over time.

14. Initially a 200 gallon tank is filled with pure water. At time $t = 0$ a salt concentration with 3 pounds of salt per gallon is added to the container at the rate of 4 gallons per minute, and the well-stirred mixture is drained from the container at the same rate.

a. Find the number of pounds of salt in the container as a function of time.

b. How many minutes does it take for the concentration to reach 2 pounds per gallon?

c. What does the concentration in the container approach for large values of time? Does this agree with your intuition?

d. Assuming that the tank holds much more than 200 gallons, and everything is the same except that the mixture is drained at 3 gallons per minute, what would the answers to parts a and b become?

15. You make two gallons of chili for a party. The recipe calls for two teaspoons of hot sauce per gallon, but you had accidentally put in two tablespoons per gallon. You decide to feed your guests the chili anyway. Assume that the guests take 1 cup/min of chili and you replace what was taken with beans and tomatoes without any hot sauce. [1 gal = 16 cups and 1 Tb = 3 tsp.]

a. Write down the differential equation and initial condition for the amount of hot sauce as a function of time in this mixture-type problem.

b. Solve this initial value problem.

c. How long will it take to get the chili back to the recipe's suggested concentration?

16. Consider the chemical reaction leading to the system in (3.113). Let the rate constants be $k_1 = 0.20$ ms^{-1}, $k_2 = 0.05$ ms^{-1}, and $k_3 = 0.10$ ms^{-1}. What do the eigenvalues of the coefficient matrix say about the behavior of the system? Find the solution of the system assuming $[A](0) = A_0 = 1.0$ μmol, $[B](0) = 0$, and $[C](0) = 0$. Plot the solutions for $t = 0.0$ to 50.0 ms and describe what is happening over this time.

17. Consider the epidemic model leading to the system in (3.114). Choose the constants as $a = 2.0$ days^{-1}, $d = 3.0$ days^{-1}, and $r = 1.0$ days^{-1}. What are the eigenvalues of the coefficient matrix? Find the solution of the system assuming an initial population of $1,000$ and one infected individual. Plot the solutions for $t = 0.0$ to 5.0 days and describe what is happening over this time. Is this model realistic?

4
The Harmonics of Vibrating Strings

4.1 Harmonics and Vibrations

"What I am going to tell you about is what we teach our physics students in the third or fourth year of graduate school . . . It is my task to convince you not to turn away because you don't understand it. You see my physics students don't understand it . . . That is because I don't understand it. Nobody does." Richard Feynman (1918-1988)

UNTIL NOW WE HAVE STUDIED oscillations in several physical systems. These lead to ordinary differential equations describing the time evolution of the systems and required the solution of initial value problems. In this chapter we will extend our study include oscillations in space. The typical example is the vibrating string.

When one plucks a violin, or guitar, string, the string vibrates exhibiting a variety of sounds. These are enhanced by the violin case, but we will only focus on the simpler vibrations of the string. We will consider the one dimensional wave motion in the string. Physically, the speed of these waves depends on the tension in the string and its mass density. The frequencies we hear are then related to the string shape, or the allowed wavelengths across the string. We will be interested the harmonics, or pure sinusoidal waves, of the vibrating string and how a general wave on a string can be represented as a sum over such harmonics. This will take us into the field of spectral, or Fourier, analysis.

Such systems are governed by partial differential equations. The vibrations of a string are governed by the one dimensional wave equation. Another simple partial differential equation is that of the heat, or diffusion, equation. This equation governs heat flow. We will consider the flow of heat through a one dimensional rod. The solution of the heat equation also involves the use of Fourier analysis. However, in this case there are no oscillations in time.

There are many applications that are studied using spectral analysis. At the root of these studies is the belief that continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagoreans study of the vibrations of strings, which led to their program of a world of harmony. This idea was carried further by Johannes Kepler (1571-1630) in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis (1616-1703), Brook Taylor (1685-1731) and Jean le Rond d'Alembert (1717-1783).

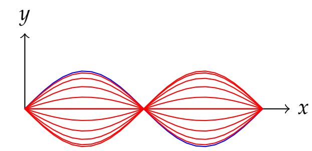


Figure 4.1: Plot of the second harmonic of a vibrating string at different times.

In 1742 d'Alembert solved the wave equation

$$c^2\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, this solution led himself and others, like Leonhard Euler (1707-1783) and Daniel Bernoulli (1700-1782), to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x,0) = h(x)$ has a discontinuous derivative! (We will see how this led to important questions in analysis.)

Solutions of the wave equation, such as the one shown, are solved using the Method of Separation of Variables. Such solutions are studies in courses in partial differential equations and mathematical physics.

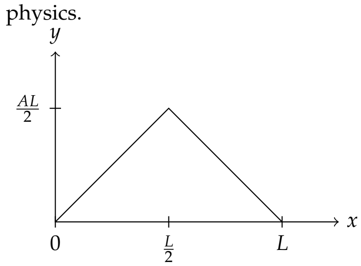


Figure 4.2: Plot of an initial condition for a plucked string.

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin\frac{k\pi x}{L}\cos\frac{k\pi ct}{L},$$

where the string extend over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$.

However, the initial conditions for such superpositions are

$$y(x,0) = \sum_k a_k \sin\frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string given by an initial condition of the form

$$y(x,0) = \begin{cases} Ax, & 0 \le x \le L/2 \\ A(L-x), & L/2 \le x \le L \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's (1768-1830) solution of the heat equation. The use of Fourier expansions has become an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation.

More generally, using a technique called the Method of Separation of Variables, allowed higher dimensional problems to be reduced to one dimensional boundary value problems. However, these studies led to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums of trigonometric functions?
3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

The one dimensional version of the heat equation is a partial differential equation for $u(x,t)$ of the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Solutions satisfying boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, are of the form

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 t/L^2}.$$

In this case, setting $u(x,0) = f(x)$, one has to satisfy the condition

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

This is similar to where we left off with the wave equation example.

There are many other systems in which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. For example, we can consider ocean waves. Ocean waves are affected by the gravitational pull of the moon and the sun and other numerous forces. These lead to the tides, which in turn have their own periods of motion. In an analysis of wave heights, one can separate out the tidal components by making use of Fourier analysis.

4.2 *Boundary Value Problems*

UNTIL THIS POINT we have solved initial value problems. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, \quad x'(0) = 0. \tag{4.1}$$

In the next chapters we will study boundary value problems and various tools for solving such problems. In this chapter we will motivate our interest in boundary value problems by looking into solving the one-dimensional heat equation, which is a partial differential equation. for the rest of the section, we will use this solution to show that in the background of our solution of boundary value problems is a structure based upon linear algebra and analysis leading to the study of inner product spaces. Though technically, we should be lead to Hilbert spaces, which are complete inner product spaces.

For an initial value problem one has to solve a differential equation subject to conditions on the unknown function or its derivatives at more than one value of the independent variable. As an example, we have a slight modification of the above problem: Find the solution $x = x(t)$ for $0 \leq t \leq 1$ that satisfies the problem

$$x'' + x = 2, \quad x(0) = 1, \quad x(1) = 0. \tag{4.2}$$

Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the endpoints. An example would be a horizontal beam supported at the ends, like a bridge. The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of other types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore the effects of different boundary value conditions in our discussions and exercises.

Let's solve the above boundary value problem. As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so we have that the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. The solution of $x'' + x = 0$ is easily found as

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 and c_2 that yield a solution to our problem. The first condition, $x(0) = 0$, gives

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. Using this value for c_1, the second condition, $x(1) = 1$, gives

$$0 = 2 - 2\cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by

$$x(t) = 2\left(1 - \cos t \frac{(\cos 1 - 1)}{\sin 1} \sin t\right).$$

Boundary value problems arise in many physical systems, just as the initial value problems we have seen earlier. We will see in the next sections that boundary value problems for ordinary differential equations often appear in the solutions of partial differential equations. However, there is no guarantee that we will have unique solutions of our boundary value problems as we had found in the example above.

4.3 *Partial Differential Equations*

In this section we will introduce several generic partial differential equations and see how the discussion of such equations leads naturally to the study of boundary value problems for ordinary differential equations. However, we will not derive the particular equations at this time, leaving that for your other courses to cover.

For ordinary differential equations, the unknown functions are functions of a single variable, e.g., $y = y(x)$. Partial differential equations are equations involving an unknown function of several variables, such as $u = u(x, y)$, $u = u(x, y)$, $u = u(x, y, z, t)$, and its (partial) derivatives. Therefore, the derivatives are partial derivatives. We will use the standard notations $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc.

There are a few standard equations that one encounters. These can be studied in one to three dimensions and are all linear differential equations. A list is provided in Table 4.1. Here we have introduced the Laplacian operator, $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems for ordinary differential equations.

Name	2 Vars	3 D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2\nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x, y)$	$\nabla^2 u = F(x, y, z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x, t)u$	$iu_t = \nabla^2 u + F(x, y, z, t)u$

Table 4.1: List of generic partial differential equations.

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L. It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation.

A typical initial-boundary value problem for the heat equation would be that initially one has a temperature distribution $u(x,0) = f(x)$. Placing the bar in an ice bath and assuming the heat flow is only through the ends of the bar, one has the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is plenty of ice to keep that temperature fixed at each end for all time. So, the problem one would need to solve is given as

1D Heat Equation

$$\begin{array}{lcc} \text{PDE} & u_t = ku_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{4.3}$$

Here, k is the heat conduction constant and is determined using properties of the bar.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed. Think of a violin string or a guitar string. Then the string is plucked, giving the string an initial profile. Let $u(x,t)$ be the vertical displacement of the string at position x and time t. The motion of the string is governed by the one dimensional wave equation. The initial-boundary value problem for this problem is given as

1D Wave Equation

$$\begin{array}{lcc} \text{PDE} & u_{tt} = c^2u_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{4.4}$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string and the tension placed on the string.

4.4 *The 1D Heat Equation*

We would like to see how the solution of such problems involving partial differential equations will lead naturally to studying boundary value problems for ordinary differential equations. We will see this as we attempt the solution of the heat equation problem as shown in (4.3). We will employ a method typically used in studying linear partial differential equations, called *the method of separation of variables*.

Solution of the 1D heat equation using the method of separation of variables.

We assume that u can be written as a product of single variable functions of each independent variable,

$$u(x,t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

Dividing both sides by k and $u = XT$, we then get

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}.$$

We have separated the functions of time on one side and space on the other side. The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ :

$$\underbrace{\frac{1}{k}\frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda}_{\text{constant}}.$$

This leads to two equations:

$$T' = k\lambda T, \tag{4.5}$$

$$X'' = \lambda X. \tag{4.6}$$

These are ordinary differential equations. The general solutions to these equations are readily found as

$$T(t) = Ae^{k\lambda t}, \tag{4.7}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{\sqrt{-\lambda}x}. \tag{4.8}$$

We need to be a little careful at this point. The aim is to force our product solutions to satisfy both the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on u lead to conditions on X.

The first condition is $u(0,t) = 0$. This implies that

$$X(0)T(t) = 0$$

for all t. The only way that this is true is if $X(0) = 0$. Similarly, $u(L,t) = 0$ implies that $X(L) = 0$. So, we have to solve the boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0 = X(L). \tag{4.9}$$

We are seeking nonzero solutions, as $X \equiv 0$ is an obvious and uninteresting solution. We call such solutions *trivial solutions*.

There are three cases to consider, depending on the sign of λ.

Case I. $\underline{\lambda > 0}$

In this case we have the exponential solutions

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{\sqrt{-\lambda}x}. \tag{4.10}$$

For $X(0) = 0$, we have

$$0 = c_1 + c_2.$$

We will take $c_2 = -c_1$. Then, $X(x) = c_1(e^{\sqrt{\lambda}x} - e^{\sqrt{-\lambda}x}) = 2c_1 \sinh \sqrt{\lambda}x$. Applying the second condition, $X(L) = 0$ yields

$$c_1 \sinh \sqrt{\lambda}L = 0.$$

This will be true only if $c_1 = 0$, since $\lambda > 0$. Thus, the only solution in this case is $X(x) = 0$. This leads to a trivial solution, $u(x,t) = 0$.

Case II. $\underline{\lambda = 0}$

For this case it is easier to set λ to zero in the differential equation. So, $X'' = 0$. Integrating twice, one finds

$$X(x) = c_1 x + c_2.$$

Setting $x = 0$, we have $c_2 = 0$, leaving $X(x) = c_1 x$. Setting $x = L$, we find $c_1 L = 0$. So, $c_1 = 0$ and we are once again left with a trivial solution.

Case III. $\underline{\lambda < 0}$

In this case is would be simpler to write $\lambda = -\mu^2$. Then the differential equation is

$$X'' + \mu^2 X = 0.$$

The general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

At $x = 0$ we get $0 = c_1$. This leaves $X(x) = c_2 \sin \mu x$. At $x = L$, we find

$$0 = c_2 \sin \mu L.$$

So, either $c_2 = 0$ or $\sin \mu L = 0$. $c_2 = 0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L = n\pi, \quad n = 1, 2, \ldots.$$

Note that $n = 0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we can find solutions to the boundary value problem (4.9) for particular values of λ. The solutions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Product solutions of the heat equation (4.3) satisfying the boundary conditions are therefore

Product solutions.

$$\boxed{u_n(x,t) = b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots, \qquad (4.11)}$$

where b_n is an arbitrary constant. However, these do not necessarily satisfy the initial condition $u(x,0) = f(x)$. What we do get is

$$u_n(x,0) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

So, if our initial condition is in one of these forms, we can pick out the right n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, we can write a linear combination of our product solutions and obtain the *general solution* satisfying the given boundary conditions as

General solution.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}. \qquad (4.12)}$$

The only thing to impose is the initial condition:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, if we are given $f(x)$, can we find the constants b_n? If we can, then we will have the solution to the full initial-boundary value problem. This will be the subject of the next chapter. However, first we will look at the general form of our boundary value problem and relate what we have done to the theory of infinite dimensional vector spaces.

Before moving on to the wave equation, we should note that (4.9) is an eigenvalue problem. We can recast the differential equation as

$$LX = \lambda X,$$

where

$$L = D^2 = \frac{d^2}{dx^2}$$

is a linear differential operator. The solutions, $X_n(x)$, are called eigenfunctions and the λ_n's are the eigenvalues. We will elaborate more on this characterization later in the book.

4.5 The 1D Wave Equation

IN THIS SECTION we will apply the method of separation of variables to the one dimensional wave equation, given by

Solution of the 1D wave equation using the method of separation of variables.

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x} \tag{4.13}$$

and subject to the conditions

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \\ u(0,t) &= 0, \\ u(L,t) &= 0. \end{aligned} \tag{4.14}$$

This problem applies to the propagation of waves on a string of length L with both ends fixed so that they do not move. $u(x,t)$ represents the vertical displacement of the string over time. The derivation of the wave equation assumes that the vertical displacement is small and the string is uniform. The constant c is the wave speed, given by

$$c = \sqrt{\frac{T}{\mu}},$$

where T is the tension in the string and μ is the mass per unit length. We can understand this in terms of string instruments. The tension can be adjusted to produce different tones and the makeup of the string (nylon or steel, thick or thin) also has an effect. In some cases the mass density is changed simply by using thicker strings. Thus, the thicker strings in a piano produce lower frequency notes.

The u_{tt} term gives the acceleration of a piece of the string. The u_{xx} is the concavity of the string. Thus, for a positive concavity the string is curved upward near the point of interest. Thus, neighboring points tend to pull upward towards the equilibrium position. If the concavity is negative, it would cause a negative acceleration.

The solution of this problem is easily found using separation of variables. We let $u(x,t) = X(x)T(t)$. Then we find

$$XT'' = c^2 X'' T,$$

which can be rewritten as

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X}.$$

Again, we have separated the functions of time on one side and space on the other side. Therefore, we set each function equal to a constant.

λ :

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda}_{\text{constant}}.$$

This leads to two equations:

$$T'' = c^2 \lambda T, \tag{4.15}$$

$$X'' = \lambda X. \tag{4.16}$$

As before, we have the boundary conditions on $X(x)$:

$$X(0) = 0, \quad \text{and} \quad X(L) = 0.$$

Again, this gives us that

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

The main difference from the solution of the heat equation is the form of the time function. Namely, from Equation (4.15) we have to solve

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0. \tag{4.17}$$

This equation takes a familiar form. We let

$$\omega_n = \frac{n\pi c}{L},$$

then we have

$$T'' + \omega_n^2 T = 0.$$

The solutions are easily found as

$$T(t) = A_n \cos\omega_n t + B_n \sin\omega_n t. \tag{4.18}$$

Therefore, we have found that the product solutions of the wave equation take the forms $\sin\frac{n\pi x}{L}\cos\omega_n t$ and $\sin\frac{n\pi x}{L}\sin\omega_n t$. The general solution, a superposition of all product solutions, is given by

General solution.

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L}. \tag{4.19}$$

This solution satisfies the wave equation and the boundary conditions. We still need to satisfy the initial conditions. Note that there are two initial conditions, since the wave equation is second order in time.

First, we have $u(x,0) = f(x)$. Thus,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}. \tag{4.20}$$

In order to obtain the condition on the initial velocity, $u_t(x,0) = g(x)$, we need to differentiate the general solution with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left[-A_n \sin \frac{n\pi ct}{L} + B_n \cos \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \tag{4.21}$$

Then, we have

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L}. \tag{4.22}$$

In both cases we have that the given functions, $f(x)$ and $g(x)$, are represented as Fourier sine series. In order to complete the problem we need to determine the constants A_n and B_n for $n = 1, 2, 3, \ldots$. Once we have these, we have the complete solution to the wave equation.

We had seen similar results for the heat equation. In the next section we will find out how to determine the Fourier coefficients for such series of sinusoidal functions.

4.6 *Introduction to Fourier Series*

IN THIS CHAPTER we will look at trigonometric series. In your calculus courses you have probably seen that many functions could have series representations as expansions in powers of x, or $x - a$. This led to MacLaurin or Taylor series. When dealing with Taylor series, you often had to determine the expansion coefficients. For example, given an expansion of $f(x)$ about $x = a$, you learned that the Taylor series was given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n,$$

where the expansion coefficients are determined as

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Then you found that the Taylor series converged for a certain range of x values. (We review Taylor series in the book appendix and later when we study series representations of complex valued functions.)

In a similar way, we will investigate the Fourier trigonometric series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}.$$

We will find expressions useful for determining the Fourier coefficients $\{a_n, b_n\}$ given a function $f(x)$ defined on $[-L, L]$. We will also see if the resulting infinite series reproduces $f(x)$. However, we first begin with some basic ideas involving simple sums of sinusoidal functions.

The natural appearance of such sums over sinusoidal functions is in music, or sound. A pure note can be represented as

$$y(t) = A\sin(2\pi f t),$$

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume of the sound. The larger the amplitude, the louder the sound. In Figure 4.3 we show plots of two such tones with $f = 2$ Hz in the top plot and $f = 5$ Hz in the bottom one.

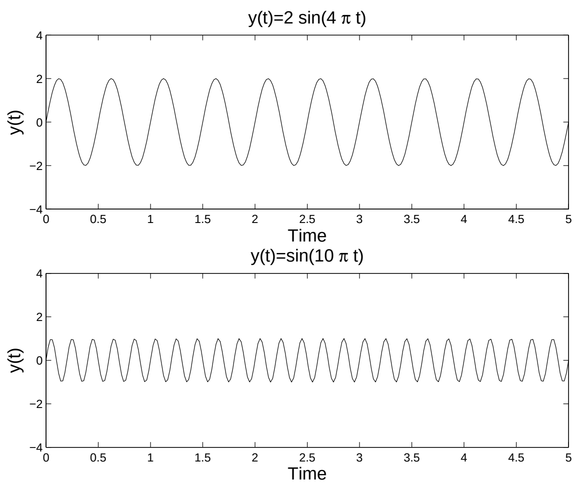


Figure 4.3: Plots of $y(t) = A\sin(2\pi f t)$ on $[0,5]$ for $f = 2$ Hz and $f = 5$ Hz.

In these plots you should notice the difference due to the amplitudes and the frequencies. You can easily reproduce these plots and others in your favorite plotting utility.

As an aside, you should be cautious when plotting functions, or sampling data. The plots you get might not be what you expect, even for a simple sine function. In Figure 4.4 we show four plots of the function $y(t) = 2\sin(4\pi t)$. In the top left you see a proper rendering of this function. However, if you use a different number of points to plot this function, the results may be surprising. In this example we show what happens if you use $N = 200, 100, 101$ points instead of the 201 points used in the first plot. Such disparities are not only possible when plotting functions, but are also present when collecting data. Typically, when you sample a set of data, you only gather a finite amount of information at a fixed rate. This could happen when getting data on ocean wave heights, digitizing music and other audio to put on your computer, or any other process when you attempt to analyze a continuous signal.

Next, we consider what happens when we add several pure tones.

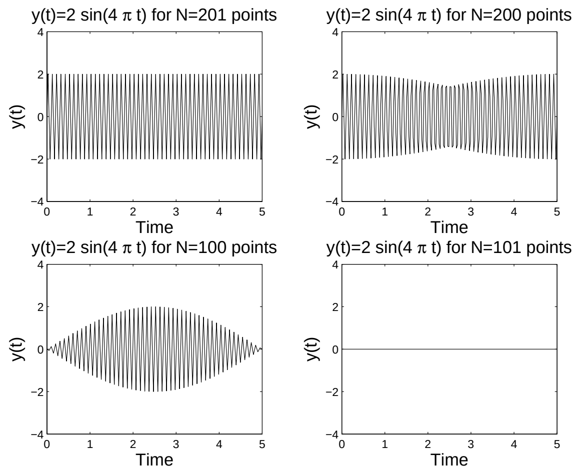


Figure 4.4: Problems can occur while plotting. Here we plot the function $y(t) = 2\sin 4\pi t$ using $N = 201, 200, 100, 101$ points.

After all, most of the sounds that we hear are in fact a combination of pure tones with different amplitudes and frequencies. In Figure 4.5 we see what happens when we add several sinusoids. Note that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. However, we still have a function of time. In this chapter we will ask, "Given a function $f(t)$, can we find a set of sinusoidal functions whose sum converges to $f(t)$?"

Looking at the superpositions in Figure 4.5, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the *period*. We can define this more precisely.

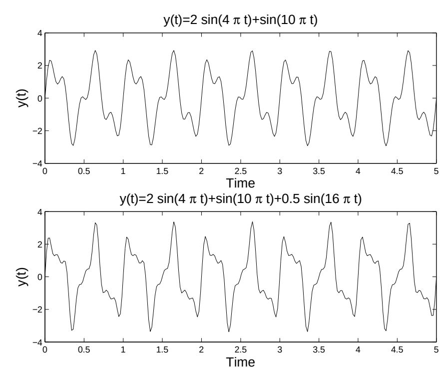


Figure 4.5: Superposition of several sinusoids. Top: Sum of signals with $f = 2$ Hz and $f = 5$ Hz. Bottom: Sum of signals with $f = 2$ Hz, $f = 5$ Hz, and and $f = 8$ Hz.

Definition 4.1. A function is said to be *periodic with period* T if $f(t + T) = f(t)$ for all t and the smallest such positive number T is called the *period*.

For example, we consider the functions used in Figure 4.5. We began with $y(t) = 2\sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 4.3 we can verify this result. (You can count the full number of cycles in the graph and divide this into the total time to get a more accurate value of the period.)

In general, if $y(t) = A\sin(2\pi f t)$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1}, or cycles per second.

Returning to Figure 4.5, the functions $y(t) = 2\sin(4\pi t)$, $y(t) = \sin(10\pi t)$, and $y(t) = 0.5\sin(16\pi t)$ have periods of 0.5s, 0.2s, and 0.125s, respectively. Each superposition in Figure 4.5 retains a period that is the least common multiple of the periods of the signals added. For both plots, this is $1.0\text{s} = 2(0.5)\text{s} = 5(.2)\text{s} = 8(.125)\text{s}$.

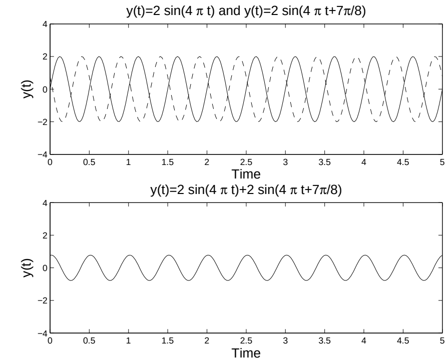


Figure 4.6: Plot of the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum.

Our goal will be to start with a function and then determine the amplitudes of the simple sinusoids needed to sum to that function. We will see that this might involve an infinite number of such terms. T hus, we will be studying an infinite series of sinusoidal functions.

Secondly, we will find that using just sine functions will not be enough either. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We will consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like "Row, Row, Row your Boat" starting at slightly different times.

We can easily add shifted sine functions. In Figure 4.6 we show the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be written as $y(t) = 2\sin(4\pi(t + 7/32))$. Thus, this corresponds to a time shift of $-7/32$.

We should note that the form in the lower plot of Figure 4.6 looks like a simple sinusoidal function for a reason. Let

$$y_1(t) = 2\sin(4\pi t),$$

$$y_2(t) = 2\sin(4\pi t + 7\pi/8).$$

Then,

$$\begin{aligned} y_1 + y_2 &= 2\sin(4\pi t + 7\pi/8) + 2\sin(4\pi t) \\ &= 2[\sin(4\pi t + 7\pi/8) + \sin(4\pi t)] \\ &= 4\cos\frac{7\pi}{16}\sin\left(4\pi t + \frac{7\pi}{16}\right). \end{aligned}$$

This can be confirmed using the identity

$$2\sin x\cos y = \sin(x+y) + \sin(x-y).$$

So, we should account for shifted sine functions in our general sum. Of course, we would then need to determine the unknown time shift as well as the amplitudes of the sinusoidal functions that make up our signal, $f(t)$. While this is one approach that some researchers use to analyze signals, there is a more common approach. This results from another reworking of the shifted function.

Consider the general shifted function

$$y(t) = A\sin(2\pi f t + \phi). \tag{4.23}$$

Note that $2\pi f t + \phi$ is called the *phase* of the sine function and ϕ is called the *phase shift*. We can use the trigonometric identity for the sine of the sum of two angles[1] to obtain

$$y(t) = A\sin(2\pi f t + \phi) = A\sin(\phi)\cos(2\pi f t) + A\cos(\phi)\sin(2\pi f t).$$

Defining $a = A\sin(\phi)$ and $b = A\cos(\phi)$, we can rewrite this as

$$y(t) = a\cos(2\pi f t) + b\sin(2\pi f t).$$

Thus, we see that the signal in Equation (4.23) is a sum of sine and cosine functions with the same frequency and different amplitudes. If we can find a and b, then we can easily determine A and ϕ:

$$A = \sqrt{a^2 + b^2}, \quad \tan\phi = \frac{b}{a}.$$

We are now in a position to state our goal in this chapter.

[1] Recall the identities (4.30)-(4.31)

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \sin y\cos x, \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y. \end{aligned}$$

Goal
Given a signal $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function. This is called *Fourier Analysis.*

4.7 *Fourier Trigonometric Series*

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \tag{4.24}$$

Notice that we have opted to drop the references to the time-frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi f_n t$ when applying these ideas to applications.

The series representation in Equation (4.24) is called a *Fourier trigonometric series.* We will simply refer to this as a *Fourier series* for now. The set of constants a_0, a_n, b_n, $n = 1, 2, \ldots$ are called the *Fourier coefficients.* The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0. Our goal is to find the Fourier series representation given $f(x)$. Having found the Fourier series representation, we will be interested in determining when the Fourier series converges and to what function it converges.

From our discussion in the last section, we see that The Fourier series is periodic. The periods of $\cos nx$ and $\sin nx$ are $\frac{2\pi}{n}$. Thus, the largest period, $T = 2\pi.$, comes from the $n = 1$ terms and the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π.

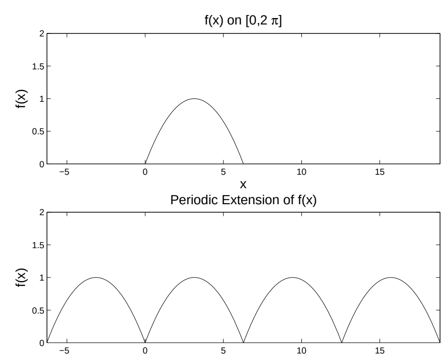


Figure 4.7: Plot of the functions $f(t)$ defined on $[0, 2\pi]$ and its periodic extension.

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 4.7 we show a function defined on $[0, 2\pi]$. In the same figure, we show its periodic extension. These are just copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to $[0, 2\pi]$ will give a representation of the original function. Therefore, we will first consider Fourier series representations of functions defined on this interval. Note that we could just as easily considered functions defined on $[-\pi, \pi]$ or any interval of length 2π.

Fourier Coefficients

Theorem 4.1. *The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$, when it exists, is given by (4.24) with Fourier coefficients*

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{4.25}$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will look at the derivations of the a_n's. First we obtain a_0.

We begin by integrating the Fourier series term by term in Equation (4.24).

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\,dx. \tag{4.26}$$

We assume that we can integrate the infinite sum term by term. Then we need to compute

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2}\,dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos nx\,dx &= \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin nx\,dx &= \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0. \end{aligned} \tag{4.27}$$

From these results we see that only one term in the integrated sum does not vanish leaving

$$\int_0^{2\pi} f(x)\,dx = \pi a_0.$$

This confirms the value for a_0.[2]

Next, we need to find a_n. We will multiply the Fourier series (4.24) by $\cos mx$ for some positive integer m. This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions for different integers m all at the same time. We will see that this will allow us to solve for the a_n's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\begin{aligned} \int_0^{2\pi} f(x)\cos mx\,dx &= \int_0^{2\pi} \frac{a_0}{2}\cos mx\,dx \\ &+ \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\cos mx\,dx. \end{aligned} \tag{4.28}$$

[2] Note that $\frac{a_0}{2}$ is the average of $f(x)$ over the interval $[0, 2\pi]$. Recall from the first semester of calculus, that the average of a function defined on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For $f(x)$ defined on $[0, 2\pi]$, we have

$$f_{\text{ave}} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}.$$

Integrating term by term, the right side becomes

$$\int_0^{2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos mx\,dx + b_n\int_0^{2\pi}\sin nx\cos mx\,dx\right]. \tag{4.29}$$

We have already established that $\int_0^{2\pi}\cos mx\,dx = 0$, which implies that the first term vanishes.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some trigonometric identities. While you have seen such integrals before in your calculus class, we will review how to carry out such integrals. For future reference, we list several useful identities, some of which we will prove along the way.

Useful Trigonometric Identities

$$\sin(x\pm y) = \sin x\cos y \pm \sin y\cos x \tag{4.30}$$
$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y \tag{4.31}$$
$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \tag{4.32}$$
$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \tag{4.33}$$
$$\sin x\sin y = \frac{1}{2}(\cos(x-y)-\cos(x+y)) \tag{4.34}$$
$$\cos x\cos y = \frac{1}{2}(\cos(x+y)+\cos(x-y)) \tag{4.35}$$
$$\sin x\cos y = \frac{1}{2}(\sin(x+y)+\sin(x-y)) \tag{4.36}$$

We first want to evaluate $\int_0^{2\pi}\cos nx\cos mx\,dx$. We do this by using the product identity (4.35). In case you forgot how to derive this identity, we will first review the proof. Recall the addition formulae for cosines:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B,$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Adding these equations gives

$$2\cos A\cos B = \cos(A+B)+\cos(A-B).$$

We can use this identity with $A = mx$ and $B = nx$ to complete the integration. We have

$$\int_0^{2\pi}\cos nx\cos mx\,dx = \frac{1}{2}\int_0^{2\pi}[\cos(m+n)x+\cos(m-n)x]\,dx$$

$$\begin{aligned} &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} \\ &= 0. \end{aligned} \tag{4.37}$$

There is one caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For our problem $m + n \neq 0$, since both m and n are positive integers. However, it is possible for $m = n$. This means that the vanishing of the integral can only happen when $m \neq n$. So, what can we do about the $m = n$ case? One way is to start from scratch with our integration. (Another way is to compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

For $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx\, dx$. This can also be handled using a trigonometric identity. Recall identity (4.33):

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta).$$

Letting $\theta = mx$ and inserting the identity into the integral, we find

$$\begin{aligned} \int_0^{2\pi} \cos^2 mx\, dx &= \frac{1}{2}\int_0^{2\pi}(1+\cos 2mx)\, dx \\ &= \frac{1}{2}\left[x+\frac{1}{2m}\sin 2mx\right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi. \end{aligned} \tag{4.38}$$

To summarize, we have shown that

$$\boxed{\int_0^{2\pi} \cos nx \cos mx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases}} \tag{4.39}$$

This holds true for $m, n = 0, 1, \ldots$. [Why did we include $m, n = 0$?] When we have such a set of functions, they are said to be an orthogonal set over the integration interval.

Definition 4.2. [3] A set of (real) functions $\{\phi_n(x)\}$ is said to be *orthogonal* on $[a, b]$ if $\int_a^b \phi_n(x)\phi_m(x)\, dx = 0$ when $n \neq m$. Furthermore, if we also have that $\int_a^b \phi_n^2(x)\, dx = 1$, these functions are called *orthonormal.*

[3] Definition of an orthogonal set of functions and orthonormal functions.

The set of functions $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by $\sqrt{\pi}$ as indicated by Equation (4.38). This is sometimes referred to normalization of the set of functions.

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x) f(x)\, dx$ is the generalization of the dot product, and is called the scalar product of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. But that is another topic for later.

Returning to the evaluation of the integrals in equation (4.29), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx\, dx$. This can also be evaluated using trigonometric identities. In this case, we need an identity involving products of sines and cosines, (4.36). Such products occur in the addition formulae for sine functions, using (4.30):

$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$

$$\sin(A-B) = \sin A \cos B - \sin B \cos A.$$

Adding these equations, we find that

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$

Setting $A = nx$ and $B = mx$, we find that

$$\begin{aligned} \int_0^{2\pi} \sin nx \cos mx\, dx &= \frac{1}{2} \int_0^{2\pi} [\sin(n+m)x + \sin(n-m)x]\, dx \\ &= \frac{1}{2} \left[\frac{-\cos(n+m)x}{n+m} + \frac{-\cos(n-m)x}{n-m} \right]_0^{2\pi} \\ &= (-1+1) + (-1+1) = 0. \end{aligned} \tag{4.40}$$

So,

$$\boxed{\int_0^{2\pi} \sin nx \cos mx\, dx = 0. \qquad (4.41)}$$

For these integrals we also should be careful about setting $n = m$. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx\, dx = \frac{1}{2} \int_0^{2\pi} \sin 2mx\, dx = \frac{1}{2} \left[\frac{-\cos 2mx}{2m} \right]_0^{2\pi} = 0.$$

Finally, we can finish our evaluation of (4.29). We have determined that all but one integral vanishes. In that case, $n = m$. This leaves us with

$$\int_0^{2\pi} f(x) \cos mx\, dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx\, dx.$$

Since this is true for all $m = 1, 2, \ldots$, we have proven this part of the theorem. The only part left is finding the b_n's This will be left as an exercise for the reader.

We now consider examples of finding Fourier coefficients for given functions. In all of these cases we define $f(x)$ on $[0, 2\pi]$.

Example 4.1. $f(x) = 3\cos 2x$, $x \in [0, 2\pi]$.

We first compute the integrals for the Fourier coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x\,dx = 0. \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \cos nx\,dx = 0, \quad n \neq 2. \\ a_2 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos^2 2x\,dx = 3, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \sin nx\,dx = 0, \forall n. \end{aligned} \tag{4.42}$$

The integrals for a_0, $a_n, n \neq 2$, and b_n are the result of orthogonality. For a_2, the integral evaluation can be done as follows:

$$\begin{aligned} a_2 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos^2 2x\,dx \\ &= \frac{3}{2\pi}\int_0^{2\pi} [1 + \cos 4x]\,dx \\ &= \frac{3}{2\pi}\left[x + \underbrace{\frac{1}{4}\sin 4x}_{\text{This term vanishes!}}\right]_0^{2\pi} = 3. \end{aligned} \tag{4.43}$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3\cos 2x$. Well, we should have known this before doing all of these integrals. So, if we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work.

Example 4.2. $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

We could determine the Fourier coefficients by integrating as in the last example. However, it is easier to use trigonometric identities. We know that

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \ldots$. There is a constant term, implying $a_0/2 = 1/2$. So, $a_0 = 1$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$. So, $a_0 = 1$, $a_2 = -\frac{1}{2}$, and all other Fourier coefficients vanish.

Example 4.3. $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$.

This example will take a little more work. We cannot bypass evaluating any integrals at this time. This function is discontinuous, so we will have to compute each integral by breaking up the integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

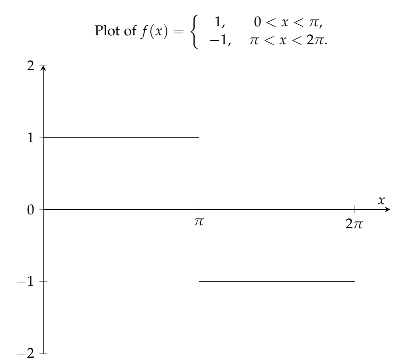


Figure 4.8: Plot of discontinuous function in Example 4.3.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)\,dx \\ &= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi+\pi) = 0. \end{aligned} \tag{4.44}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx - \int_{\pi}^{2\pi}\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\ &= 0. \end{aligned} \tag{4.45}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx - \int_{\pi}^{2\pi}\sin nx\,dx\right] \\ &= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\ &= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\ &= \frac{2}{n\pi}(1-\cos n\pi). \end{aligned} \tag{4.46}$$

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (4.24), we note that $\cos n\pi = (-1)^n$. *Therefore,*

Often we see expressions involving $\cos n\pi = (-1)^n$ and $1 \pm \cos n\pi = 1 \pm (-1)^n$. This is an example showing how to re-index series containing such a factor.

$$1 - \cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd}. \end{cases} \tag{4.47}$$

So, half of the b_n*'s are zero. While we could write the Fourier series representation as*

$$f(x) \sim \frac{4}{\pi}\sum_{n=1,\ odd}^{\infty}\frac{1}{n}\sin nx,$$

we could let $n = 2k-1$ *and write*

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1},$$

But does this series converge? Does it converge to $f(x)$*? We will discuss this question later in the chapter.*

4.8 Fourier Series Over Other Intervals

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$, or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such intervals arise often in applications. For example, one can study vibrations of a one dimensional string of length L and set up the axes with the left end at $x = 0$ and the right end at $x = L$. Another problem would be to study the temperature distribution along a one dimensional rod of length L. Such problems lead to the original studies of Fourier series. As we will see later, symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling our interval. Then we could apply this transformation to the Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

We define $x \in [0, 2\pi]$ and $t \in [0, L]$. A linear transformation relating these intervals is simply $x = \frac{2\pi t}{L}$ as shown in Figure 5.12. So, $t = 0$ maps to $x = 0$ and $t = L$ maps to $x = 2\pi$. Furthermore, this transformation maps $f(x)$ to a new function $g(t) = f(x(t))$, which is defined on $[0, L]$. We will determine the Fourier series representation of this function using the representation for $f(x)$.

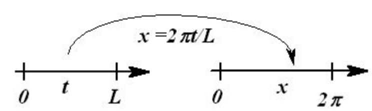


Figure 4.9: A sketch of the transformation between intervals $x \in [0, 2\pi]$ and $t \in [0, L]$.

Recall the form of the Fourier representation for $f(x)$ in Equation (4.24):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \tag{4.48}$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L} \right]. \tag{4.49}$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to determine the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

We need to make a substitution in the integral of $x = \frac{2\pi t}{L}$. We also will need to transform the differential, $dx = \frac{2\pi}{L} dt$. Thus, the resulting form for the Fourier coefficients is

$$a_n = \frac{2}{L} \int_0^L g(t) \cos \frac{2n\pi t}{L} \, dt. \tag{4.50}$$

Similarly, we find that

$$b_n = \frac{2}{L} \int_0^L g(t) \sin \frac{2n\pi t}{L} \, dt. \tag{4.51}$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos \frac{2n\pi t}{L}$ is L/n, which means that the representation for $g(t)$ has a period of L.

At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader. In Table 4.2 we summarize some commonly used Fourier series representations.

We will end our discussion for now with some special cases and an example for a function defined on $[-\pi, \pi]$.

At this point we need to remind the reader about the integration of even and odd functions.

1. **Even Functions:** In this evaluation we made use of the fact that the integrand is an even function. Recall that $f(x)$ is an *even function* if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis as shown in Figure 4.10(A). If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x) \, dx = 2 \int_0^a f(x) \, dx. \tag{4.58}$$

One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned} \int_{-a}^{a} f(x) \, dx &= \int_{-a}^{0} f(x) \, dx + \int_0^a f(x) \, dx \\ &= -\int_a^0 f(-y) \, dy + \int_0^a f(x) \, dx \\ &= \int_0^a f(y) \, dy + \int_0^a f(x) \, dx \\ &= 2 \int_0^a f(x) \, dx. \end{aligned} \tag{4.59}$$

This can be visually verified by looking at Figure 4.10(A).

2. **Odd Functions:** A similar computation could be done for odd functions. $f(x)$ is an *odd function* if $f(-x) = -f(x)$ for all x. The graphs of such functions are symmetric with respect

Table 4.2: Special Fourier Series Representations on Different Intervals

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{4.52}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L} \, dx. \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} \, dx. \quad n = 1, 2, \dots. \end{aligned} \tag{4.53}$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{4.54}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \frac{2n\pi x}{L} \, dx. \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \frac{2n\pi x}{L} \, dx. \quad n = 1, 2, \dots. \end{aligned} \tag{4.55}$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \tag{4.56}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx. \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \quad n = 1, 2, \dots. \end{aligned} \tag{4.57}$$

to the origin as shown in Figure 4.10(B). If one integrates an odd function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\, dx = 0. \tag{4.60}$$

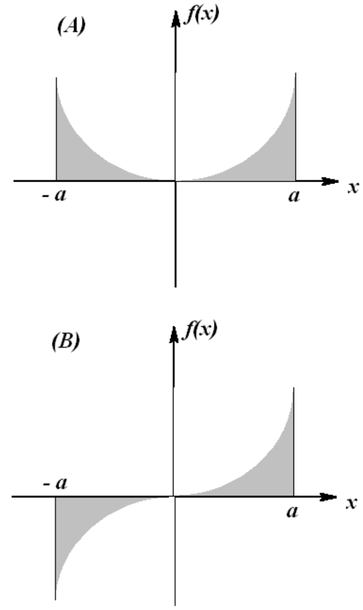


Figure 4.10: Examples of the areas under (A) even and (B) odd functions on symmetric intervals, $[-a, a]$.

Example 4.4. *Let $f(x) = |x|$ on $[-\pi, \pi]$ We compute the coefficients, beginning as usual with a_0. We have, using the fact that $|x|$ is an even function,*

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi \end{aligned} \tag{4.61}$$

We continue with the computation of the general Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx. \tag{4.62}$$

Here we have made use of the fact that $|x| \cos nx$ is an even function. In order to compute the resulting integral, we need to use integration by parts ,

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du,$$

by letting $u = x$ and $dv = \cos nx\, dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n}\sin nx$. Continuing with the computation, we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx. \\ &= \frac{2}{\pi}\left[\frac{1}{n} x \sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx\, dx\right] \\ &= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} \\ &= -\frac{2}{\pi n^2}(1 - (-1)^n). \end{aligned} \tag{4.63}$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n. This leads to a factor $(1 - (-1)^n)$. This factor can be simplified as

$$1 - (-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \tag{4.64}$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n's is simpler. We note that we have to integrate $|x| \sin nx$ from $x = -\pi$ to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n.

Putting this all together, the Fourier series representation of $f(x) = |x|$ on $[-\pi, \pi]$ is given as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,\ odd}^{\infty} \frac{\cos nx}{n^2}. \tag{4.65}$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let $n = 2k - 1$ for $k = 1, 2, 3, \ldots$. Then we only get the odd integers. The series can then be written as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}. \tag{4.66}$$

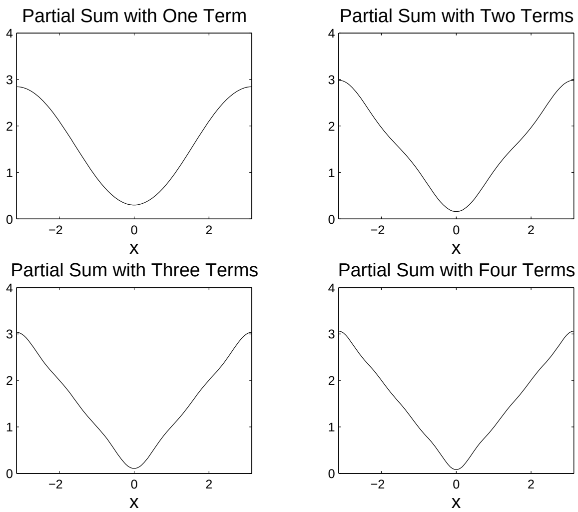


Figure 4.11: Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. Here is an example of an infinite series of functions. What does this series sum to? We show in Figure 4.11 the first few partial sums. They appear to be converging to $f(x) = |x|$ fairly quickly.

Even though $f(x)$ was defined on $[-\pi, \pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 4.12, we see that the representation agrees with $f(x)$ on the interval $[-\pi, \pi]$. Outside this interval we have a periodic extension of $f(x)$ with period 2π.

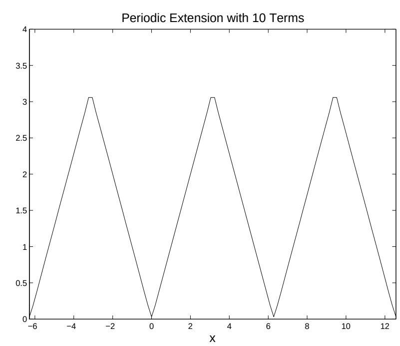


Figure 4.12: Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

Another example is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$ as left for Problem 7. This is determined to be

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{4.67}$$

As seen in Figure 4.13 we again obtain the periodic extension of our function. In this case we needed many more terms. Also, the vertical parts of the first plot are nonexistent. In the second plot we only plot the points and not the typical connected points that most software packages plot as the default style.

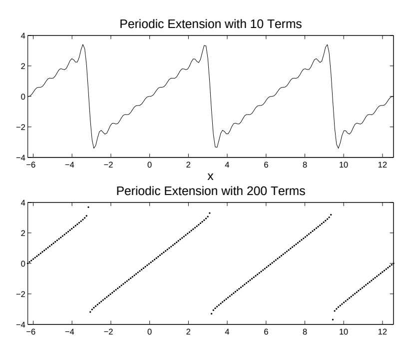


Figure 4.13: Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

Example 4.5. *It is interesting to note that one can use Fourier series to obtain sums of some infinite series. For example, in the last example we found that*

$$x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Now, what if we chose $x = \frac{\pi}{2}$? Then, we have

$$\frac{\pi}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} = 2 \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right].$$

This gives a well known expression for π:

$$\pi = 4 \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right].$$

4.8.1 Fourier Series on $[a, b]$

This section can be skipped on first reading. It is here for completeness and the end result, Theorem 4.2 provides the result of the section.

A FOURIER SERIES REPRESENTATION is also possible for a general interval, $t \in [a, b]$. As before, we just need to transform this interval to $[0, 2\pi]$. Let

$$x = 2\pi \frac{t-a}{b-a}.$$

Inserting this into the Fourier series (4.24) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right]. \tag{4.68}$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, mathematics students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned}
g(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\cos \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} + \sin \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right. \\
&\quad \left. + b_n \left(\sin \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} - \cos \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\cos \frac{2n\pi t}{b-a} \left(a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \right) \right. \\
&\quad \left. + \sin \frac{2n\pi t}{b-a} \left(a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a} \right) \right].
\end{aligned} \tag{4.69}$$

Defining $A_0 = a_0$ and

$$\begin{aligned} A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\ B_n &\equiv a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a}, \end{aligned} \tag{4.70}$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a, b]$.

$$g(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos \frac{2n\pi t}{b-a} + B_n \sin \frac{2n\pi t}{b-a} \right]. \tag{4.71}$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi \frac{t-a}{b-a}$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi(t-a)}{b-a} \, dt, \end{aligned} \tag{4.72}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi(t-a)}{b-a} \, dt. \end{aligned} \tag{4.73}$$

Then, inserting these integrals in A_n, combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$\begin{aligned} A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\ &= \frac{2}{b-a} \int_a^b g(t) \left[\cos \frac{2n\pi(t-a)}{b-a} \cos \frac{2n\pi a}{b-a} - \sin \frac{2n\pi(t-a)}{b-a} \sin \frac{2n\pi a}{b-a} \right] dt \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi t}{b-a} \, dt. \end{aligned} \tag{4.74}$$

A similar computation gives

$$B_n = \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi t}{b-a} \, dt. \tag{4.75}$$

Summarizing, we have shown that:

Theorem 4.2. *The Fourier series representation of $f(x)$ defined on $[a,b]$ when it exists, is given by*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right]. \tag{4.76}$$

with Fourier coefficients

$$\begin{aligned} a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{4.77}$$

4.9 *Sine and Cosine Series*

In the last two examples ($f(x) = |x|$ and $f(x) = x$ on $[-\pi, \pi]$) we have seen Fourier series representations that contain only sine or cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called sine series and those involving just cosines (and the constant term) are called cosine series.

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0, \pi]$. Note from Figures 4.11-4.13 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.
2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0, L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

In Figure 4.14 we show a function defined on $[0, 1]$. To the right is its periodic extension to the whole real axis. This representation has a period of $L = 1$. The bottom left plot is obtained by first reflecting

f about the y-axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

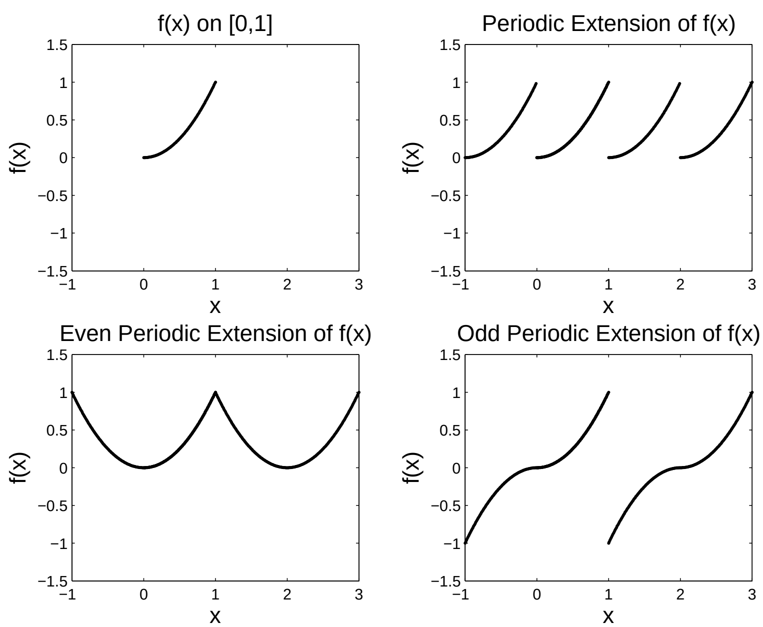


Figure 4.14: This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0,1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the periodic, even and odd extensions. Now, starting with $f(x)$ defined on $[0, L]$, we would like to determine the Fourier series representations leading to these extensions. [For easy reference, the results are summarized in Table 4.3]

We have already seen from Table (4.2) that the periodic extension of $f(x)$, defined on $[0, L]$, is obtained through the Fourier series representation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right], \tag{4.84}$$

where

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{4.85}$$

Given $f(x)$ defined on $[0, L]$, the *even periodic extension* is obtained by simply computing the Fourier series representation for the even

Table 4.3: Fourier Cosine and Sine Series Representations on $[0, L]$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{4.78}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{4.79}$$

Fourier Cosine Series on $[0, L]$

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{4.80}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{4.81}$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{4.82}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{4.83}$$

function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ f(-x) & -L < x < 0. \end{cases} \tag{4.86}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L, L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (4.76) with $a = -L$ and $b = L$.]:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{4.87}$$

with Fourier coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f_e(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{4.88}$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the *Cosine Series Representation* of $f(x)$ for $x \in [0, L]$ is given as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{4.89}$$

where

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{4.90}$$

Similarly, given $f(x)$ defined on $[0, L]$, the *odd periodic extension* is obtained by simply computing the Fourier series representation for the odd function

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \tag{4.91}$$

The resulting series expansion leads to defining the *Sine Series Representation* of $f(x)$ for $x \in [0, L]$ as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{4.92}$$

where

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{4.93}$$

Example 4.6. *In Figure 4.14 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.*

For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is shown in Table 4.4.

Example 4.7. Periodic Extension - Trigonometric Fourier Series *Using the code in Table 4.4, we have that* $a_0 = \frac{2}{3}$ $a_n = \frac{1}{n^2\pi^2}$ *and* $b_n = -\frac{1}{n\pi}$. *Thus, the resulting series is given as*

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi^2} \cos 2n\pi x - \frac{1}{n\pi} \sin 2n\pi x \right].$$

In Figure 4.15 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f. *There appear to be some problems with the convergence around integer values of* x. *We will later see that this is because of the discontinuities in the periodic extension and the resulting overshoot is referred to as the Gibbs phenomenon which is discussed in the appendix to this chapter.*

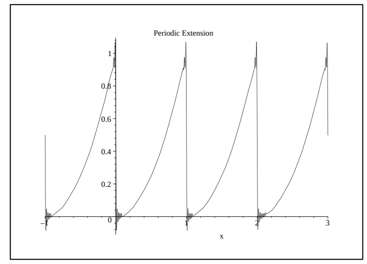


Figure 4.15: The periodic extension of $f(x) = x^2$ on $[0, 1]$.

Example 4.8. Even Periodic Extension - Cosine Series

In this case we compute $a_0 = \frac{2}{3}$ *and* $a_n = \frac{4(-1)^n}{n^2\pi^2}$. *Therefore, we have*

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x.$$

In Figure 4.16 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

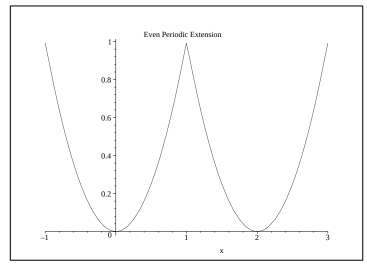


Figure 4.16: The even periodic extension of $f(x) = x^2$ on $[0, 1]$.

Table 4.4: Maple code for computing Fourier coefficients and plotting partial sums of the Fourier series.

```
> restart:
> L:=1:
> f:=x^2:
> assume(n,integer):
> a0:=2/L*int(f,x=0..L);
                              a0 := 2/3
> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);
                                     1
                            an := -------
                                    2   2
                                  n~  Pi
> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);
                                      1
                            bn := - -----
                                    n~ Pi
> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
     -1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title=`Periodic Extension`,
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);
```

Example 4.9. Odd Periodic Extension - Sine Series

Finally, we look at the sine series for this function. We find that $b_n = -\frac{2}{n^3\pi^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)$. *Therefore,*

$$f(x) \sim -\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)\sin n\pi x.$$

Once again we see discontinuities in the extension as seen in Figure 4.17. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 4.14.

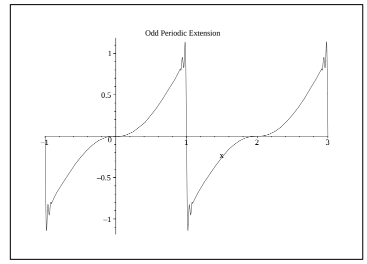


Figure 4.17: The odd periodic extension of $f(x) = x^2$ on $[0, 1]$.

4.10 *Solution of the Heat Equation*

We started out the chapter seeking the solution of an initial-boundary value problem involving the heat equation and the wave equation. In particular, we found the general solution for the problem of heat flow in a one dimensional rod of length L with fixed zero temperature ends. The problem was given by

$$\begin{array}{ccc} \text{PDE} & u_t = ku_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0. \end{array} \tag{4.94}$$

We found the solution using separation of variables. This resulted in a sum over various product solutions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\sin\frac{n\pi x}{L},$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

This equation satisfies the boundary conditions. However, we had only gotten to state initial condition using this solution. Namely,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

We were left with having to determine the constants b_n. Once we know them, we have the solution.

Now we can get the Fourier coefficients when we are given the initial condition, $f(x)$. They are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

We consider a couple of examples with different initial conditions.

Example 1 $f(x) = \sin x$ for $L = \pi$.

In this case the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin nx,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

However, the initial condition takes the form of the first term in the expansion; i.e., the $n = 1$ term. So, we need not carry out the integral because we can immediately write $b_1 = 1$ and $b_n = 0$, $n = 2, 3, \ldots$. Therefore, the solution consists of just one term,

$$u(x,t) = e^{-kt} \sin x.$$

In Figure 4.18 we see that how this solution behaves for $k = 1$ and $t \in [0,1]$.

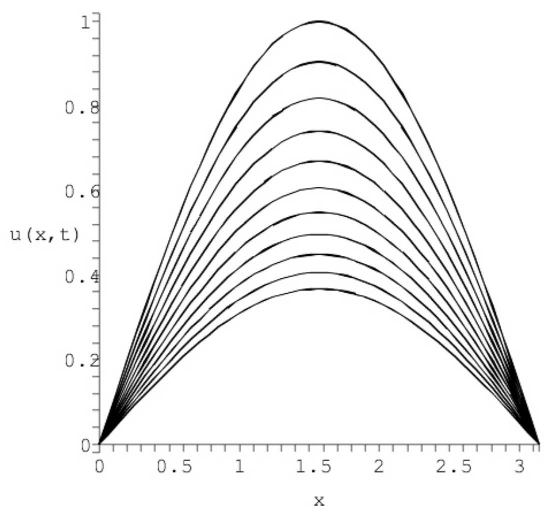


Figure 4.18: The evolution of the initial condition $f(x) = \sin x$ for $L = \pi$ and $k = 1$.

Example 2 $f(x) = x(1-x)$ for $L = 1$.

This example requires a bit more work. The solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt} \sin n\pi x,$$

where

$$b_n = 2\int_0^1 f(x) \sin n\pi x \, dx.$$

This integral is easily computed using integration by parts

$$\begin{aligned} b_n &= 2\int_0^1 x(1-x) \sin n\pi x \, dx \\ &= \left[2x(1-x)\left(-\frac{1}{n\pi}\cos n\pi x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 (1-2x) \cos n\pi x \, dx \end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{n^2\pi^2}\left\{[(1-2x)\sin n\pi x]_0^1 + 2\int_0^1 \sin n\pi x\,dx\right\} \\
&= \frac{4}{n^3\pi^3}\left[\cos n\pi x\right]_0^1 \\
&= \frac{4}{n^3\pi^3}\left(\cos n\pi - 1\right) \\
&= \begin{cases} 0, & n \text{ even} \\ -\frac{8}{n^3\pi^3}, & n \text{ odd} \end{cases}. \qquad (4.95)
\end{aligned}
$$

So, we have that the solution can be written as

$$u(x,t) = \frac{8}{\pi^3}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell-1)^3}e^{-(2\ell-1)^2\pi^2 kt}\sin(2\ell-1)\pi x.$$

In Figure 4.18 we see that how this solution behaves for $k = 1$ and $t \in [0,1]$. Twenty terms were used. We see that this solution diffuses much faster that the last example. Most of the terms damp out quickly as the solution asymptotically approaches the first term.

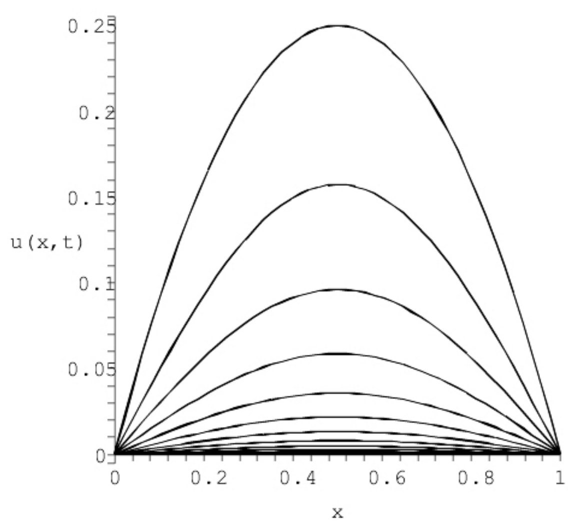


Figure 4.19: The evolution of the initial condition $f(x) = x(1-x)$ for $L = 1$ and $k = 1$.

4.11 Finite Length Strings

WE NOW RETURN to the physical example of wave propagation in a string. We have found that the general solution can be represented as a sum over product solutions. We will restrict our discussion to the special case that the initial velocity is zero and the original profile is given by $u(x,0) = f(x)$. The solution is then

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \qquad (4.96)$$

satisfying

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{4.97}$$

We have learned that the Fourier sine series coefficients are given by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{4.98}$$

Note that we are using A_n's only because of the development of the solution.

We can rewrite this solution in a more compact form. First, we define the wave numbers,

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots,$$

and the angular frequencies,

$$\omega_n = ck_n = \frac{n\pi c}{L}.$$

Then the product solutions take the form

$$\sin k_n x \cos \omega_n t.$$

Using trigonometric identities, these products can be written as

$$\sin k_n x \cos \omega_n t = \frac{1}{2} \left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t) \right].$$

Inserting this expression in our solution, we have

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} A_n \left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t) \right]. \tag{4.99}$$

Since $\omega_n = ck_n$, we can put this into a more suggestive form:

$$u(x,t) = \frac{1}{2} \left[\sum_{n=1}^{\infty} A_n \sin k_n(x + ct) + \sum_{n=1}^{\infty} A_n \sin k_n(x - ct) \right]. \tag{4.100}$$

We see that each sum is simply the sine series for $f(x)$ but evaluated at either $x + ct$ or $x - ct$. Thus, the solution takes the form

$$u(x,t) = \frac{1}{2} \left[f(x + ct) + f(x - ct) \right]. \tag{4.101}$$

If $t = 0$, then we have $u(x,0) = \frac{1}{2}\left[f(x) + f(x)\right] = f(x)$. So, the solution satisfies the initial condition. At $t = 1$, the sum has a term $f(x - c)$. Recall from your mathematics classes that this is simply a shifted version of $f(x)$. Namely, it is shifted to the right. For general times, the function is shifted by ct to the right. For larger values of t, this shift is further to the right. The function (wave) shifts to the right with velocity c. Similarly, $f(x+ct)$ is a wave traveling to the left with velocity $-c$.

Thus, the waves on the string consist of waves traveling to the right and to the left. However, the story does not stop here. We have a problem when needing to shift $f(x)$ across the boundaries. The original problem only defines $f(x)$ on $[0, L]$. If we are not careful, we would think that the function leaves the interval leaving nothing left inside. However, we have to recall that our sine series representation for $f(x)$ has a period of $2L$. So, before we apply this shifting, we need to account for its periodicity. In fact, being a sine series, we really have the odd periodic of $f(x)$ being shifted. The details of such analysis would take us too far from our current goal. However, we can illustrate this with a few figures.

We begin by plucking a string of length L. This can be represented by the function

$$f(x) = \begin{cases} \frac{x}{a} & 0 \le x \le a \\ \frac{L-x}{L-a} & a \le x \le L \end{cases} \tag{4.102}$$

where the string is pulled up one unit at $x = a$. This is shown in Figure 4.20.

Next, we create an odd function by extending the function to a period of $2L$. This is shown in Figure 4.21.

Finally, we construct the periodic extension of this to the entire line. In Figure 4.22 we show in the lower part of the figure copies of the periodic extension, one moving to the right and the other moving to the left. (Actually, the copies are $\frac{1}{2}f(x \pm ct)$.) The top plot is the sum of these solutions. The physical string lies in the interval [0,1].

The time evolution for this plucked string is shown for several times in Figure 4.23. This results in a wave that appears to reflect from the ends as time increases.

The relation between the angular frequency and the wave number, $\omega = ck$, is called a dispersion relation. In this case ω depends on k linearly. If one knows the dispersion relation, then one can find the wave speed as $c = \frac{\omega}{k}$. In this case, all of the harmonics travel at the same speed. In cases where they do not, we have nonlinear dispersion, which we will discuss later.

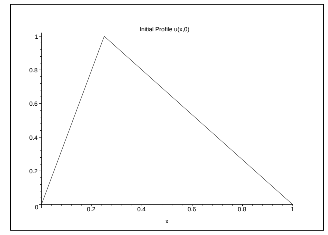


Figure 4.20: The initial profile for a string of length one plucked at $x = 0.25$.

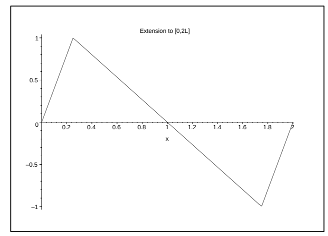


Figure 4.21: Odd extension about the right end of a plucked string.

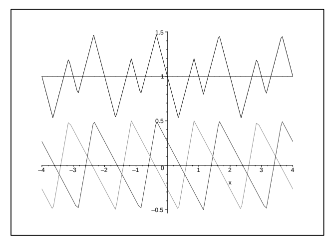
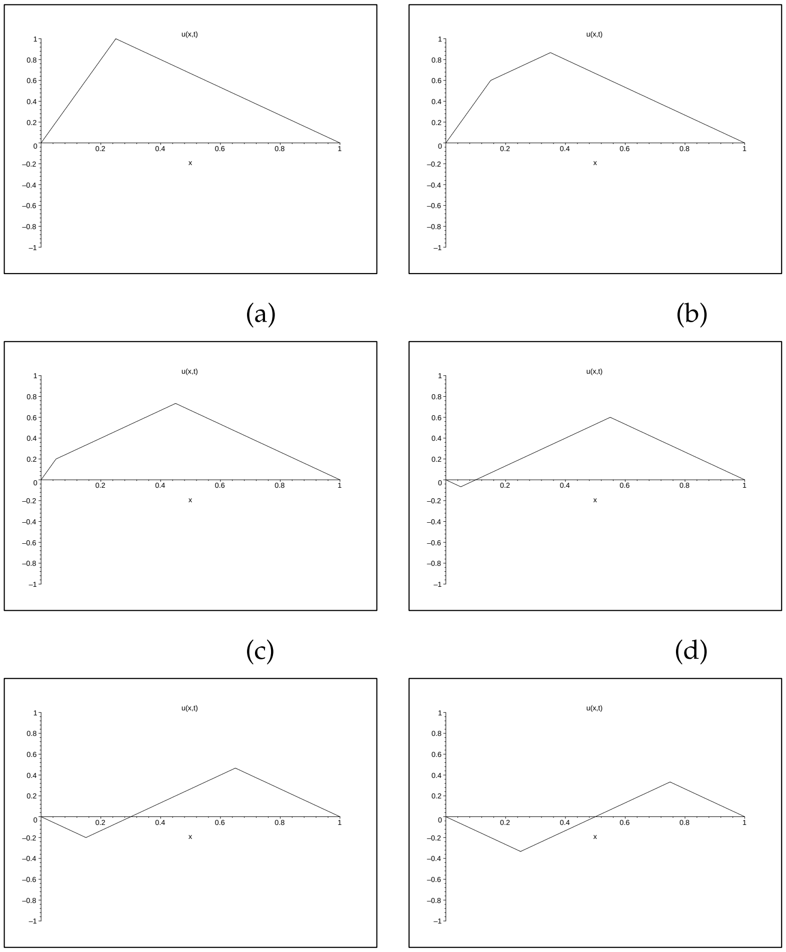


Figure 4.22: Summing the odd periodic extensions. The lower plot shows copies of the periodic extension, one moving to the right and the other moving to the left. The upper plot is the sum.

4.12 *Appendix: The Gibbs Phenomenon*

We have seen the Gibbs phenomenon when there is a jump discontinuity in the periodic extension of a function, whether the function originally had a discontinuity or developed one due to a mismatch in the values of the endpoints. This can be seen in Figures 4.13, 4.15 and 4.17. The Fourier series has a difficult time converging at the



(a) (b) (c) (d) (e) (f)

Figure 4.23: This Figure shows the plucked string at six successive times from (a) to (f).

point of discontinuity and these graphs of the Fourier series show a distinct overshoot which does not go away. This is called the Gibbs phenomenon[4] and the amount of overshoot can be computed.

[4] The Gibbs phenomenon was named after Josiah Willard Gibbs (1839-1903) even though it was discovered earlier by the Englishman Henry Wilbraham (1825-1883). Wilbraham published a soon forgotten paper about the effect in 1848. In 1889 Albert Abraham Michelson (1852-1931), an American physicist,observed an overshoot in his mechanical graphing machine. Shortly afterwards J. Willard Gibbs published papers describing this phenomenon, which was later to be called the Gibbs phenomena. Gibbs was a mathematical physicist and chemist and is considered the father of physical chemistry.

In one of our first examples, Example 4.3, we found the Fourier series representation of the piecewise defined function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

to be

$$f(x) \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

In Figure 4.24 we display the sum of the first ten terms. Note the wiggles, overshoots and under shoots. These are seen more when we plot the representation for $x \in [-3\pi, 3\pi]$, as shown in Figure 4.25.

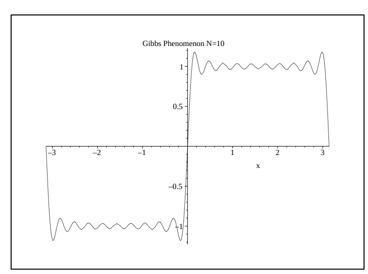


Figure 4.24: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$.

We note that the overshoots and undershoots occur at discontinuities in the periodic extension of $f(x)$. These occur whenever $f(x)$ has a discontinuity or if the values of $f(x)$ at the endpoints of the domain do not agree.

One might expect that we only need to add more terms. In Figure 4.26 we show the sum for twenty terms. Note the sum appears to converge better for points far from the discontinuities. But, the overshoots and undershoots are still present. In Figures 4.27 and 4.28 show magnified plots of the overshoot at $x = 0$ for $N = 100$ and $N = 500$, respectively. We see that the overshoot persists. The peak is at about the same height, but its location seems to be getting closer to the origin. We will show how one can estimate the size of the overshoot.

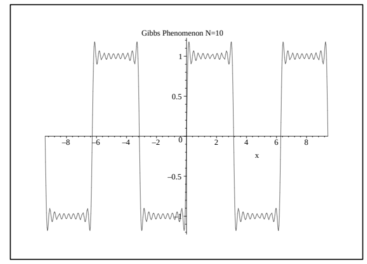


Figure 4.25: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$ plotted on $[-3\pi, 3\pi]$ displaying the periodicity.

We can study the Gibbs phenomenon by looking at the partial sums of general Fourier trigonometric series for functions $f(x)$ defined on the interval $[-L, L]$. Some of this is done discussed in the options section **??** where we talk more about the convergence.

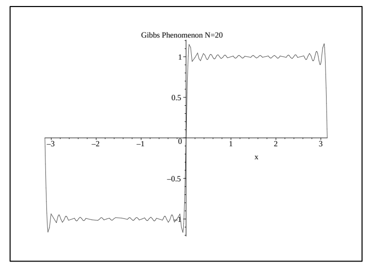


Figure 4.26: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 20$.

Writing out the partial sums, inserting the Fourier coefficients and rearranging, we have

$$\begin{aligned}
S_N(x) &= a_0 + \sum_{n=1}^{N}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \sum_{n=1}^{N}\left[\left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\frac{n\pi y}{L}\,dy\right)\cos\frac{n\pi x}{L}\right. \\
&\quad \left. + \left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\frac{n\pi y}{L}\,dy.\right)\sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{L}\int\limits_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\left(\cos\frac{n\pi y}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L}\right)\right\} f(y)\,dy \\
&= \frac{1}{L}\int\limits_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi(y-x)}{L}\right\} f(y)\,dy \\
&\equiv \frac{1}{L}\int\limits_{-L}^{L} D_N(y-x) f(y)\,dy
\end{aligned}$$

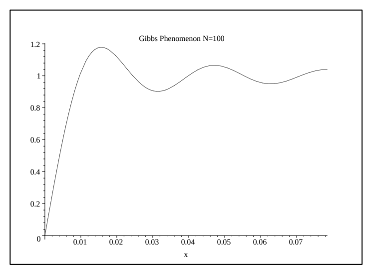


Figure 4.27: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 100$.

We have defined

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi x}{L},$$

which is called the ***N*-th Dirichlet Kernel** . We now prove

Lemma 4.1.

$$D_N(x) = \begin{cases} \frac{\sin((N+\frac{1}{2})\frac{\pi x}{L})}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0, \\ N + \frac{1}{2}, & \sin\frac{\pi x}{2L} = 0. \end{cases}$$

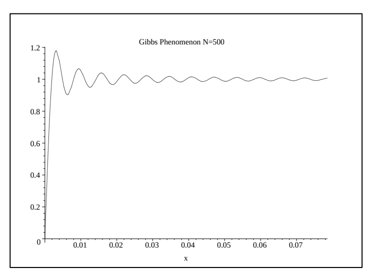


Figure 4.28: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 500$.

Proof. Let $\theta = \frac{\pi x}{L}$ and multiply $D_N(x)$by $2\sin\frac{\theta}{2}$to obtain:

$$\begin{aligned}
2\sin\frac{\theta}{2}D_N(x) &= 2\sin\frac{\theta}{2}\left[\frac{1}{2}+\cos\theta+\cdots+\cos N\theta\right] \\
&= \sin\frac{\theta}{2}+2\cos\theta\sin\frac{\theta}{2}+2\cos 2\theta\sin\frac{\theta}{2}+\cdots+2\cos N\theta\sin\frac{\theta}{2} \\
&= \sin\frac{\theta}{2}+\left(\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}\right)+\left(\sin\frac{5\theta}{2}-\sin\frac{3\theta}{2}\right)+\cdots \\
&\quad +\left[\sin\left(N+\frac{1}{2}\right)\theta-\sin\left(N-\frac{1}{2}\right)\theta\right] \\
&= \sin\left(N+\frac{1}{2}\right)\theta. \qquad (4.103)
\end{aligned}$$

Thus,

$$2\sin\frac{\theta}{2}D_N(x)=\sin\left(N+\frac{1}{2}\right)\theta,$$

or if $\sin\frac{\theta}{2}\neq 0$,

$$D_N(x)=\frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \qquad \theta=\frac{\pi x}{L}.$$

If $\sin\frac{\theta}{2}=0$, then one needs to apply L'Hospital's Rule as $\theta\to 2m\pi$:

$$\begin{aligned}
\lim_{\theta\to 2m\pi}\frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} &= \lim_{\theta\to 2m\pi}\frac{(N+\frac{1}{2})\cos\left(N+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} \\
&= \frac{(N+\frac{1}{2})\cos\left(2m\pi N+m\pi\right)}{\cos m\pi} \\
&= \frac{(N+\frac{1}{2})(\cos 2m\pi N\cos m\pi-\sin 2m\pi N\sin m\pi)}{\cos m\pi} \\
&= N+\frac{1}{2}. \qquad (4.104)
\end{aligned}$$

□

We further note that $D_N(x)$ is periodic with period $2L$ and is an even function.

So far, we have found that

$$S_N(x)=\frac{1}{L}\int_{-L}^{L}D_N(y-x)f(y)\,dy. \qquad (4.105)$$

Now, make the substitution $\xi=y-x$. Then,

$$\begin{aligned}
S_N(x) &= \frac{1}{L}\int_{-L-x}^{L-x}D_N(\xi)f(\xi+x)\,d\xi \\
&= \frac{1}{L}\int_{-L}^{L}D_N(\xi)f(\xi+x)\,d\xi. \qquad (4.106)
\end{aligned}$$

In the second integral we have made use of the fact that $f(x)$ and $D_N(x)$ are periodic with period $2L$ and shifted the interval back to $[-L, L]$.

Now split the integration and use the fact that $D_N(x)$ is an even function. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L}^{0} D_N(\xi)f(\xi+x)\,d\xi + \frac{1}{L}\int_{0}^{L} D_N(\xi)f(\xi+x)\,d\xi \\ &= \frac{1}{L}\int_{0}^{L}\left[f(x-\xi)+f(\xi+x)\right]D_N(\xi)\,d\xi. \end{aligned} \tag{4.107}$$

We can use this result to study the Gibbs phenomenon whenever it occurs. In particular, we will only concentrate on our earlier example. For this case, we have

$$S_N(x) = \frac{1}{\pi}\int_{0}^{\pi}\left[f(x-\xi)+f(\xi+x)\right]D_N(\xi)\,d\xi \tag{4.108}$$

for

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos nx.$$

Also, one can show that

$$f(x-\xi)+f(\xi+x) = \begin{cases} 2, & 0 \le \xi < x, \\ 0, & x \le \xi < \pi - x, \\ -2, & \pi - x \le \xi < \pi. \end{cases}$$

Thus, we have

$$\begin{aligned} S_N(x) &= \frac{2}{\pi}\int_{0}^{x} D_N(\xi)\,d\xi - \frac{2}{\pi}\int_{\pi-x}^{\pi} D_N(\xi)\,d\xi \\ &= \frac{2}{\pi}\int_{0}^{x} D_N(z)\,dz + \frac{2}{\pi}\int_{0}^{x} D_N(\pi-z)\,dz. \end{aligned} \tag{4.109}$$

Here we made the substitution $z = \pi - \xi$ in the second integral. The Dirichlet kernel for $L = \pi$ is given by

$$D_N(x) = \frac{\sin(N+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$

For N large, we have $N + \frac{1}{2} \approx N$, and for small x, we have $\sin\frac{x}{2} \approx \frac{x}{2}$. So, under these assumptions,

$$D_N(x) \approx \frac{\sin Nx}{x}.$$

Therefore,

$$S_N(x) \to \frac{2}{\pi}\int_{0}^{x}\frac{\sin N\xi}{\xi}\,d\xi \qquad \text{for large } N, \text{ and small } x.$$

If we want to determine the locations of the minima and maxima, where the undershoot and overshoot occur, then we apply the first derivative test for extrema to $S_N(x)$. Thus,

$$\frac{d}{dx}S_N(x) = \frac{2}{\pi}\frac{\sin Nx}{x} = 0.$$

The extrema occur for $Nx = m\pi$, $m = \pm 1, \pm 2, \ldots$. One can show that there is a maximum at $x = \pi/N$ and a minimum for $x = 2\pi/N$. The value for the overshoot can be computed as

$$\begin{aligned} S_N(\pi/N) &= \frac{2}{\pi}\int_0^{\pi/N} \frac{\sin N\xi}{\xi}\, d\xi \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\, dt \\ &= \frac{2}{\pi}\mathrm{Si}(\pi) \\ &= 1.178979744\ldots. \end{aligned} \tag{4.110}$$

Note that this value is independent of N and is given in terms of the sine integral,

$$\mathrm{Si}(x) \equiv \int_0^x \frac{\sin t}{t}\, dt.$$

Problems

1. Solve the following boundary value problem:

$$x'' + x = 2, \quad x(0) = 0, \quad x'(1) = 0.$$

2. Find product solutions, $u(x,t) = b(t)\phi(x)$, to the heat equation satisfying the boundary conditions $u_x(0,t) = 0$ and $u(L,t) = 0$. Use these solutions to find a general solution of the heat equation satisfying these boundary conditions.

3. Consider the following boundary value problems. Determine the eigenvalues, λ, and eigenfunctions, $y(x)$ for each problem.[5]

a. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$

b. $y'' - \lambda y = 0, \quad y(-\pi) = 0, \quad y'(\pi) = 0.$

c. $x^2 y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0.$

d. $(x^2 y')' + \lambda y = 0, \quad y(1) = 0, \quad y'(e) = 0.$

[5] In problem d you will not get exact eigenvalues. Show that you obtain a transcendental equation for the eigenvalues in the form $\tan z = 2z$. Find the first three eigenvalues numerically.

4. Consider the boundary value problem for the deflection of a horizontal beam fixed at one end,

$$\frac{d^4 y}{dx^4} = C, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0.$$

Solve this problem assuming that C is a constant.

5. Write $y(t) = 3\cos 2t - 4\sin 2t$ in the form $y(t) = A\cos(2\pi f t + \phi)$.

6. Derive the coefficients b_n in Equation(4.25).

7. For the following sets of functions: i) show that each is orthogonal on the given interval, and ii) determine the corresponding orthonormal set. [See page 183]

a. $\{\sin 2nx\}, \quad n = 1, 2, 3, \ldots, \quad 0 \le x \le \pi.$

b. $\{\cos n\pi x\}, \quad n = 0, 1, 2, \ldots, \quad 0 \le x \le 2.$

c. $\{\sin \frac{n\pi x}{L}\}, \quad n = 1, 2, 3, \ldots, \quad x \in [-L, L].$

8. Consider $f(x) = 4\sin^3 2x$.

a. Derive the trigonometric identity giving $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$ using DeMoivre's Formula.

b. Find the Fourier series of $f(x) = 4\sin^3 2x$ on $[0, 2\pi]$ without computing any integrals.

9. Find the Fourier series of the following:

a. $f(x) = x$, $x \in [0, 2\pi]$.

b. $f(x) = \frac{x^2}{4}$, $|x| < \pi$.

c. $f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi, \\ -\frac{\pi}{2}, & \pi < x < 2\pi. \end{cases}$

10. Find the Fourier Series of each function $f(x)$ of period 2π. For each series, plot the Nth partial sum,

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos nx + b_n \sin nx\right],$$

for $N = 5, 10, 50$ and describe the convergence (is it fast? what is it converging to, etc.) [Some simple Maple code for computing partial sums is shown in the notes.]

a. $f(x) = x$, $|x| < \pi$.

b. $f(x) = |x|$, $|x| < \pi$.

c. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

11. Find the Fourier series of $f(x) = x$ on the given interval. Plot the Nth partial sums and describe what you see.

a. $0 < x < 2$.

b. $-2 < x < 2$.

c. $1 < x < 2$.

12. The result in problem 9b above gives a Fourier series representation of $\frac{x^2}{4}$. By picking the right value for x and a little arrangement of the series, show that [See Example 4.5.]

a.
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

b.
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Hint: Consider how the series in part a. can be used to do this.

13. Sketch (by hand) the graphs of each of the following functions over four periods. Then sketch the extensions each of the functions as both an even and odd periodic function. Determine the corresponding Fourier sine and cosine series and verify the convergence to the desired function using Maple.

a. $f(x) = x^2$, $0 < x < 1$.

b. $f(x) = x(2-x)$, $0 < x < 2$.

c. $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$

d. $f(x) = \begin{cases} \pi, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$

5
Non-sinusoidal Harmonics and Special Functions

"To the pure geometer the radius of curvature is an incidental characteristic - like the grin of the Cheshire cat. To the physicist it is an indispensable characteristic. It would be going too far to say that to the physicist the cat is merely incidental to the grin. Physics is concerned with interrelatedness such as the interrelatedness of cats and grins. In this case the "cat without a grin" and the "grin without a cat" are equally set aside as purely mathematical phantasies." Sir Arthur Stanley Eddington (1882-1944)

In this chapter we provide a glimpse into generalized Fourier series in which the normal modes of oscillation are not sinusoidal. In particular, we will explore Legendre polynomials and Bessel functions which will later arise in problems having cylindrical or spherical symmetry. For vibrating strings, we saw that the harmonics were sinusoidal basis functions for a large, infinite dimensional, function space. Now, we will extend these ideas to non-sinusoidal harmonics and explore the underlying structure behind these ideas.

The background for the study of generalized Fourier series is that of function spaces. We begin by exploring the general context in which one finds oneself when discussing Fourier series and (later) Fourier transforms. We can view the sine and cosine functions in the Fourier trigonometric series representations as basis vectors in an infinite dimensional function space. A given function in that space may then be represented as a linear combination over this infinite basis. With this in mind, we might wonder

- Do we have enough basis vectors for the function space?
- Are the infinite series expansions convergent?
- For other other bases, what functions can be represented by such expansions?

In the context of the boundary value problems which typically appear in physics, one is led to the study of boundary value problems in the form of Sturm-Liouville eigenvalue problems. These lead to an appropriate set of basis vectors for the function space under consideration. We will touch a little on these ideas, leaving some of the deeper results for more advanced courses in mathematics. For now, we will turn to the ideas of functions spaces and explore some typical basis functions who origins lie deep in physical problems. The common basis functions are often referred to as special functions in physics. Examples are the classical orthogonal polynomials (Legendre, Hermite, Laguerre, Tchebychef) and Bessel functions. But first we will introduce function spaces.

5.1 *Function Spaces*

EARLIER WE STUDIED finite dimensional vector spaces. Given a set of basis vectors, $\{\mathbf{a}_k\}_{k=1}^n$, in vector space V, we showed that we can expand any vector $\mathbf{v} \in \mathbf{V}$ in terms of this basis, $\mathbf{v} = \sum_{k=1}^n v_k \mathbf{a}_k$. We then spent some time looking at the simple case of extracting the components v_k of the vector. The keys to doing this simply were to have a scalar product and an orthogonal basis set. These are also the key ingredients that we will need in the infinite dimensional case. In fact, we had already done this when we studied Fourier series.

Recall when we found Fourier trigonometric series representations of functions, we started with a function (vector?) that we wanted to expand in a set of trigonometric functions (basis?) and we sought the Fourier coefficients (components?). In this section we will extend our notions from finite dimensional spaces to infinite dimensional spaces and we will develop the needed background in which to think about more general Fourier series expansions. This conceptual framework is very important in other areas in mathematics (such as ordinary and partial differential equations) and physics (such as quantum mechanics and electrodynamics).

We note that the above determination of vector components for finite dimensional spaces is precisely what we had done to compute the Fourier coefficients using trigonometric bases. Reading further, you will see how this works.

We will consider various infinite dimensional function spaces. Functions in these spaces would differ by what properties they satisfy. For example, we could consider the space of continuous functions on [0,1], the space of differentiably continuous functions, or the set of functions integrable from a to b. As you can see that there are many types of function spaces . In order to view these spaces as vector spaces, we will need to be able to add functions and multiply them by scalars in such as way that they satisfy the definition of a vector space as defined in Chapter 3.

We will also need a scalar product defined on this space of functions. There are several types of scalar products, or inner products, that we can define. For a real vector space, we define

Definition 5.1. An inner product $<,>$ on a real vector space V is a mapping from $V \times V$ into R such that for $u, v, w \in V$ and $\alpha \in R$ one has

1. $< v, v > \geq 0$ and $< v, v >= 0$ iff $v = 0$.
2. $< v, w >=< w, v >$.
3. $< \alpha v, w >= \alpha < v, w >$.
4. $< u + v, w >=< u, w > + < v, w >$.

A real vector space equipped with the above inner product leads to what is called a real inner product space. A more general definition with the third property replaced with $< v, w >= \overline{< w, v >}$ is needed for complex inner product spaces.

For the time being, we will only deal with real valued functions and, thus, we will need an inner product appropriate for such spaces. One such definition is the following. Let $f(x)$ and $g(x)$ be functions defined on $[a, b]$ and introduce the *weight function* $\sigma(x) > 0$. Then, we define the inner product, if the integral exists, as

$$< f, g >= \int_a^b f(x)g(x)\sigma(x)\, dx. \tag{5.1}$$

Spaces in which $< f, f >< \infty$ under this inner product are called the space of square integrable functions on (a, b) under weight σ and denoted as $L^2_\sigma(a, b)$. In what follows, we will assume for simplicity that $\sigma(x) = 1$. This is possible to do by using a change of variables.

The space of square integrable functions.

Now that we have functions spaces equipped with an inner product, we seek a basis for the space? For an n-dimensional space we need n basis vectors. For an infinite dimensional space, how many will we need? How do we know when we have enough? We will provide some answers to these questions later.

Let's assume that we have a basis of functions $\{\phi_n(x)\}_{n=1}^\infty$. Given a function $f(x)$, how can we go about finding the components of f in this basis? In other words, let

$$f(x) = \sum_{n=1}^\infty c_n \phi_n(x).$$

How do we find the c_n's? Does this remind you of the problem we had earlier for finite dimensional spaces? [You may want to review the discussion at the end of Section 3.1 as you read the next derivation.]

Formally, we take the inner product of f with each ϕ_j and use the properties of the inner product to find

$$\begin{aligned} < \phi_j, f > &= < \phi_j, \sum_{n=1}^\infty c_n \phi_n > \\ &= \sum_{n=1}^\infty c_n < \phi_j, \phi_n > . \end{aligned} \tag{5.2}$$

If the basis is an orthogonal basis, then we have

$$< \phi_j, \phi_n >= N_j \delta_{jn}, \tag{5.3}$$

where δ_{jn} is the Kronecker delta. Recall from Chapter 3 that the Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \tag{5.4}$$

Continuing with the derivation, we have

For the generalized Fourier series expansion $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, we have determined the generalized Fourier coefficients to be $c_j = \frac{<\phi_j, f>}{<\phi_j, \phi_j>}$.

$$\begin{aligned} <\phi_j, f> &= \sum_{n=1}^{\infty} c_n <\phi_j, \phi_n> \\ &= \sum_{n=1}^{\infty} c_n N_j \delta_{jn} \\ &= c_1 N_j \delta_{j1} + c_2 N_j \delta_{j2} + \ldots + c_j N_j \delta_{jj} + \ldots \\ &= c_j N_j. \end{aligned} \tag{5.5}$$

So, the expansion coefficients are

$$c_j = \frac{<\phi_j, f>}{N_j} = \frac{<\phi_j, f>}{<\phi_j, \phi_j>} \quad j = 1, 2, \ldots.$$

We summarize this important result:

Generalized Basis Expansion

Let $f(x)$ be represented by an expansion over a basis of orthogonal functions, $\{\phi_n(x)\}_{n=1}^{\infty}$,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Then, the expansion coefficients are formally determined as

$$c_n = \frac{<\phi_n, f>}{<\phi_n, \phi_n>}.$$

This will be referred to as the general Fourier series expansion and the c_j's are called the Fourier coefficients. Technically, equality only holds when the infinite series converges to the given function on the interval of interest.

Example 5.1. *Find the coefficients of the Fourier sine series expansion of $f(x)$, given by*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad x \in [-\pi, \pi].$$

In the last chapter we already established that the set of functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ is orthogonal on the interval $[-\pi, \pi]$. Recall that using trigonometric identities, we have for $n \neq m$

$$<\phi_n, \phi_m> = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\pi}^{\pi}[\cos(n-m)x - \cos(n+m)x]\,dx \\
&= \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} \\
&= 0.
\end{aligned} \tag{5.6}
$$

[1] *So, we have determined that the set* $\phi_n(x) = \sin nx$ *for* $n = 1, 2, \ldots$ *is an orthogonal set of functions on the interval* $[-\pi, \pi]$. *Just as with vectors in three dimensions, we can normalize these basis functions to arrive at an orthonormal basis. This is simply done by dividing by the length of the vector. Recall that the length of a vector is obtained as* $v = \sqrt{\mathbf{v}\cdot\mathbf{v}}$. *In the same way, we define the norm of a function by*

$$\|f\| = \sqrt{< f, f >}.$$

Note, there are many types of norms, but this induced norm will be sufficient for us.

For the above basis of sine functions, we want to first compute the norm of each function. Then we find a new basis from the original basis such that each new basis function has unit length. Of course, this is just an orthonormal basis. We first compute

$$
\begin{aligned}
\|\phi_n\|^2 &= \int_{-\pi}^{\pi}\sin^2 nx\,dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi}[1-\cos 2nx]\,dx \\
&= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi.
\end{aligned} \tag{5.7}
$$

We have found for our example that

$$< \phi_j, \phi_n >= \pi\delta_{jn} \tag{5.8}$$

and that $\|\phi_n\| = \sqrt{\pi}$. *Defining* $\psi_n(x) = \frac{1}{\sqrt{\pi}}\phi_n(x)$, *we have normalized the* ϕ_n*'s and have obtained an orthonomal basis of functions on* $[-\pi, \pi]$.

Now, we can determine the expansion coefficients using

$$b_n = \frac{< \phi_n, f >}{N_n} = \frac{< \phi_n, f >}{< \phi_n, \phi_n >} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Does this result look familiar?

[1] There are many types of norms. The norm defined here is the natural, or induced, norm on the inner product space. Norms are a generalization of the concept of lengths of vectors. Denoting $\|\mathbf{v}\|$ the norm of $\mathbf{v}$, it needs to satisfy the properties

1. $\|\mathbf{v}\| \geq 0$. $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$.
3. $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Examples of common norms are

1. Euclidean norm: $\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$.
2. Taxicab norm: $\|\mathbf{v}\| = |v_1| + \cdots + |v_n|$.
3. L^p norm: $\|f\| = \left(\int [f(x)]^p\,dx\right)^{\frac{1}{p}}$.

5.2 Classical Orthogonal Polynomials

For completeness, we will next discuss series representations of functions using different bases. In this section we introduce the classical orthogonal polynomials. We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstraß Approximation Theorem[2] this set is a basis of $L^2_\sigma(a, b)$, the space of square integrable functions over the interval $[a, b]$ relative to weight $\sigma(x)$. However, we will show that the sequence of functions $\{1, x, x^2, \ldots\}$ does not provide an orthogonal basis for these spaces. We will then proceed to find an appropriate orthogonal basis of functions.

[2] **Stone-Weierstraß Approximation Theorem** Suppose f is a continuous function defined on the interval $[a, b]$. For every $\epsilon > 0$, there exists a polynomial function $P(x)$ such that for all $x \in [a, b]$, we have $|f(x) - P(x)| < \epsilon$. Therefore, every continuous function defined on $[a, b]$ can be uniformly approximated as closely as we wish by a polynomial function.

We are familiar with being able to expand functions over the basis $\{1, x, x^2, \ldots\}$, since these expansions are just power series representation of the functions,[3]

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

[3] The reader may recognize this series expansion as a Maclaurin series expansion, or Taylor series expansion about $x = 0$. For a review of Taylor series, see the Appendix.

However, this basis is not an orthogonal set of basis functions. One can easily see this by integrating the product of two even, or two odd, basis functions with $\sigma(x) = 1$ and (a, b)=$(-1, 1)$. For example,

$$\int_{-1}^{1} x^0 x^2 \, dx = \frac{2}{3}.$$

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask if it is possible to obtain an orthogonal basis involving powers of x. Of course, finite combinations of these basis elements are just polynomials!

OK, we will ask. “Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?" The answer is yes. We recall from introductory linear algebra, which mostly covers finite dimensional vector spaces, that there is a method for carrying this out called the **Gram-Schmidt Orthogonalization Process.** We will review this process for finite dimensional vectors and then generalize to function spaces.

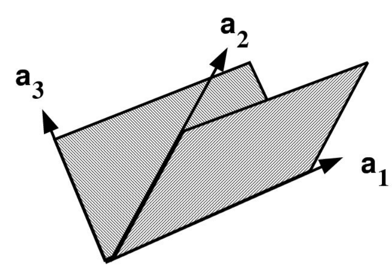


Figure 5.1: The basis $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$.

Let's assume that we have three vectors that span $\mathbf{R}^3$, given by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ and shown in Figure 5.1. We seek an orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

It is sometimes useful to normalize these basis vectors, denoting such a normalized vector with a 'hat':

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1},$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take another element of the original basis, $\mathbf{a}_2$. In Figure 5.2 we show the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. We can now write $\mathbf{a}_2$ as the sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{5.9}$$

Recall the projection of one vector onto another from your vector calculus class.

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{5.10}$$

This is easily proven by writing the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$. Using the definition of the dot product, $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, the projection formula follows.

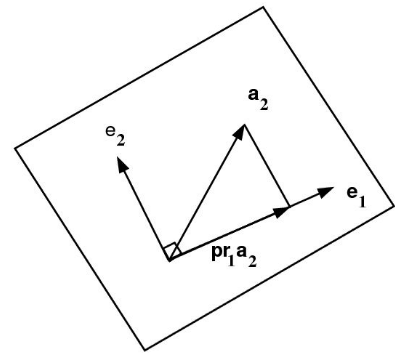


Figure 5.2: A plot of the vectors $\mathbf{e}_1$, $\mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of $\mathbf{a}_2$, on $\mathbf{e}_1$.

Combining Equations (5.9)-(5.10), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{5.11}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

$$\begin{aligned} \mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0. \end{aligned} \tag{5.12}$$

Next, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ with the new vector. This is shown in Figure 5.3. Thus, we can see that

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2}\mathbf{e}_2. \tag{5.13}$$

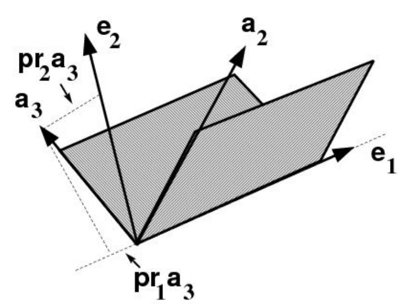


Figure 5.3: A plot of vectors for determining $\mathbf{e}_3$.

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

We can easily generalize this procedure to the N-dimensional case. Let $\mathbf{a}_n$, $n = 1, ..., N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and for $n > 1$,

$$\boxed{\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2}\mathbf{e}_j. \qquad (5.14)}$$

Now, we can generalize this idea to (real) function spaces. Let $f_n(x)$, $n \in N_0 = \{0, 1, 2, \ldots\}$, be a linearly independent sequence of continuous functions defined for $x \in [a, b]$. Then, an orthogonal basis of functions, $\phi_n(x)$, $n \in N_0$ can be found and is given by

$$\phi_0(x) = f_0(x)$$

and

$$\boxed{\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{< f_n, \phi_j >}{\|\phi_j\|^2} \phi_j(x), \quad n = 1, 2, \ldots. \qquad (5.15)}$$

Here we are using inner products relative to weight $\sigma(x)$,

$$< f, g >= \int_a^b f(x)g(x)\sigma(x)\,dx. \tag{5.16}$$

Note the similarity between the orthogonal basis in (5.15) and the expression for the finite dimensional case in Equation (5.14).

Example 5.2. *Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n$, $n \in N_0$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.*

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x)\,dx = 2.$$

We could use this result to fix the normalization of our new basis, but we will hold off on doing that for now.

Now, we compute the second basis element:

$$\begin{aligned}
\phi_1(x) &= f_1(x) - \frac{< f_1, \phi_0 >}{\|\phi_0\|^2} \phi_0(x) \\
&= x - \frac{< x, 1 >}{\|1\|^2} 1 = x,
\end{aligned} \tag{5.17}$$

since $< x, 1 >$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned}
\phi_2(x) &= f_2(x) - \frac{< f_2, \phi_0 >}{\|\phi_0\|^2} \phi_0(x) - \frac{< f_2, \phi_1 >}{\|\phi_1\|^2} \phi_1(x) \\
&= x^2 - \frac{< x^2, 1 >}{\|1\|^2} 1 - \frac{< x^2, x >}{\|x\|^2} x \\
&= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} \\
&= x^2 - \frac{1}{3}.
\end{aligned} \tag{5.18}$$

So far, we have the orthogonal set $\{1, x, x^2 - \frac{1}{3}\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre[4] polynomials, $P_n(x)$. Thus,

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

Note that this normalization is different than the usual one. In fact, we see the $P_2(x)$ does not have a unit norm,

$$\|P_2\|^2 = \int_{-1}^{1} P_2^2(x)\,dx = \frac{2}{5}.$$

[4] Adrien-Marie Legendre (1752-1833) was a French mathematician who made many contributions to analysis and algebra.

The set of Legendre polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many of these special functions had originally appeared as solutions of important boundary value problems in physics. They all have similar properties and we will just elaborate some of these for the Legendre functions in the next section. Others in this group are shown in Table 5.1.

Table 5.1: Common classical orthogonal polynomials with the interval and weight function used to define them.

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty,\infty)$	e^{-x^2}
Laguerre	$L_n^\alpha(x)$	$[0,\infty)$	e^{-x}
Legendre	$P_n(x)$	(-1,1)	1
Gegenbauer	$C_n^\lambda(x)$	(-1,1)	$(1-x^2)^{\lambda-1/2}$
Tchebychef of the 1st kind	$T_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Tchebychef of the 2nd kind	$U_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Jacobi	$P_n^{(\nu,\mu)}(x)$	(-1,1)	$(1-x)^\nu(1-x)^\mu$

5.3 *Fourier-Legendre Series*

In the last chapter we saw how useful Fourier series expansions were for solving the heat and wave equations. In Chapter 9 we will investigate partial differential equations in higher dimensions and find that problems with spherical symmetry may lead to the series representations in terms of a basis of Legendre polynomials. For example, we could consider the steady state temperature distribution inside a hemispherical igloo, which takes the form

$$\phi(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

in spherical coordinates. Evaluating this function at the surface $r = a$ as $\phi(a,\theta) = f(\theta)$, leads to a Fourier-Legendre series expansion of function f :

$$f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta),$$

where $c_n = A_n a^n$

In this section we would like to explore Fourier-Legendre series expansions of functions $f(x)$ defined on $(-1,1)$:

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x). \tag{5.19}$$

As with Fourier trigonometric series, we can determine the expansion coefficients by multiplying both sides of Equation (5.19) by $P_m(x)$ and integrating for $x \in [-1, 1]$. Orthogonality gives the usual form for the generalized Fourier coefficients,

$$c_n = \frac{< f, P_n >}{\|P_n\|^2}, n = 0, 1, \ldots .$$

We will later show that

$$\|P_n\|^2 = \frac{2}{2n+1}.$$

Therefore, the Fourier-Legendre coefficients are

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{5.20}$$

Rodrigues Formula

We can do examples of Fourier-Legendre expansions given just a few facts about Legendre polynomials. The first property that the Legendre polynomials have is the *Rodrigues formula*:

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n \in N_0.} \tag{5.21}$$

From the Rodrigues formula, one can show that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function.

Example 5.3. *Determine $P_2(x)$ from Rodrigues formula:*

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!} \frac{d^2}{dx^2} (x^2 - 1)^2 \\ &= \frac{1}{8} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) \\ &= \frac{1}{8} \frac{d}{dx} (4x^3 - 4x) \\ &= \frac{1}{8} (12x^2 - 4) \\ &= \frac{1}{2} (3x^2 - 1). \end{aligned} \tag{5.22}$$

Note that we get the same result as we found in the last section using orthogonalization.

The first several Legendre polynomials are given in Table 5.2. In Figure 5.4 we show plots of these Legendre polynomials.

Three Term Recursion Formula

All of the classical orthogonal polynomials satisfy a *three term recursion formula* (or, recurrence relation or formula). In the case of the Legendre polynomials, we have

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x), \quad n = 1, 2, \ldots . \tag{5.23}$$

n	$(x^2-1)^n$	$\frac{d^n}{dx^n}(x^2-1)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
0	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}(3x^2-1)$
3	$x^6-3x^4+3x^2-1$	$120x^3-72x$	$\frac{1}{48}$	$\frac{1}{2}(5x^3-3x)$

Table 5.2: Tabular computation of the Legendre polynomials using the Rodrigues formula.

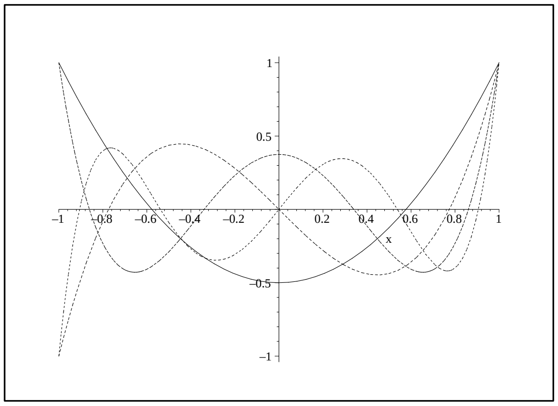


Figure 5.4: Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

This can also be rewritten by replacing n with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{5.24}$$

Example 5.4. *Use the recursion formula to find $P_2(x)$ and $P_3(x)$, given that $P_0(x) = 1$ and $P_1(x) = x$.*

We first begin by inserting $n = 1$ into Equation (5.23):

$$2P_2(x) = 3xP_1(x) - P_0(x) = 3x^2 - 1.$$

So, $P_2(x) = \frac{1}{2}(3x^2-1)$.

For $n = 2$, we have

$$\begin{aligned} 3P_3(x) &= 5xP_2(x) - 2P_1(x) \\ &= \frac{5}{2}x(3x^2-1) - 2x \\ &= \frac{1}{2}(15x^3 - 9x). \end{aligned} \tag{5.25}$$

This gives $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. These expressions agree with the earlier results.

We will prove the three term recursion formula in two ways. First we use the orthogonality properties of Legendre polynomials and the following lemma.

The first proof of the three term recursion formula is based upon the nature of the Legendre polynomials as an orthogonal basis, while the second proof is derived using generating functions.

Lemma 5.1. *The leading coefficient of x^n in $P_n(x)$ is $\frac{1}{2^n n!}\frac{(2n)!}{n!}$.*

Proof. We can prove this using Rodrigues formula. first, we focus on the leading coefficient of $(x^2-1)^n$, which is x^{2n}. The first derivative of x^{2n} is $2nx^{2n-1}$. The second derivative is $2n(2n-1)x^{2n-2}$. The jth derivative is

$$\frac{d^j x^{2n}}{dx^j} = [2n(2n-1)\ldots(2n-j+1)]x^{2n-j}.$$

Thus, the nth derivative is given by

$$\frac{d^n x^{2n}}{dx^n} = [2n(2n-1)\ldots(n+1)]x^n.$$

This proves that $P_n(x)$ has degree n. The leading coefficient of $P_n(x)$ can now be written as

$$\begin{aligned} \frac{1}{2^n n!}[2n(2n-1)\ldots(n+1)] &= \frac{1}{2^n n!}[2n(2n-1)\ldots(n+1)]\frac{n(n-1)\ldots 1}{n(n-1)\ldots 1} \\ &= \frac{1}{2^n n!}\frac{(2n)!}{n!}. \end{aligned} \tag{5.26}$$

□

Theorem 5.1. *Legendre polynomials satisfy the three term recursion formula*

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{5.27}$$

Proof. In order to prove the three term recursion formula we consider the expression $(2n-1)xP_{n-1}(x) - nP_n(x)$. While each term is a polynomial of degree n, the leading order terms cancel. We need only look at the coefficient of the leading order term first expression. It is

$$(2n-1)\frac{1}{2^{n-1}(n-1)!}\frac{(2n-2)!}{(n-1)!} = \frac{1}{2^{n-1}(n-1)!}\frac{(2n-1)!}{(n-1)!} = \frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

The coefficient of the leading term for $nP_n(x)$ can be written as

$$n\frac{1}{2^n n!}\frac{(2n)!}{n!} = n\left(\frac{2n}{2n^2}\right)\left(\frac{1}{2^{n-1}(n-1)!}\right)\frac{(2n-1)!}{(n-1)!}\frac{(2n-1)!}{2^{n-1}\left[(n-1)!\right]^2}.$$

It is easy to see that the leading order terms in $(2n-1)xP_{n-1}(x) - nP_n(x)$ cancel.

The next terms will be of degree $n-2$. This is because the P_n's are either even or odd functions, thus only containing even, or odd, powers of x. We conclude that

$$(2n-1)xP_{n-1}(x) - nP_n(x) = \text{ polynomial of degree } n-2.$$

Therefore, since the Legendre polynomials form a basis, we can write this polynomial as a linear combination of of Legendre polynomials:

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_0P_0(x) + c_1P_1(x) + \ldots + c_{n-2}P_{n-2}(x). \tag{5.28}$$

Multiplying Equation (5.28) by $P_m(x)$ for $m = 0, 1, \ldots, n-3$, integrating from -1 to 1, and using orthogonality, we obtain

$$0 = c_m \|P_m\|^2, \quad m = 0, 1, \ldots, n-3.$$

[Note: $\int_{-1}^{1} x^k P_n(x)\,dx = 0$ for $k \leq n-1$. Thus, $\int_{-1}^{1} x P_{n-1}(x) P_m(x)\,dx = 0$ for $m \leq n-3$.]

Thus, all of these c_m's are zero, leaving Equation (5.28) as

$$(2n-1)xP_{n-1}(x) - nP_n(x) = c_{n-2}P_{n-2}(x).$$

The final coefficient can be found by using the normalization condition, $P_n(1) = 1$. Thus, $c_{n-2} = (2n-1) - n = n-1$. □

Generating Functions

A second proof of the three term recursion formula can be obtained from the *generating function* of the Legendre polynomials. Many special functions have such generating functions. In this case it is given by

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |x| \leq 1, |t| < 1. \tag{5.29}$$

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions. For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system. An even better example, would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position $\mathbf{r}_1$ and the moon at position $\mathbf{r}_2$ as shown in Figure 5.5. The tidal potential Φ is proportional to

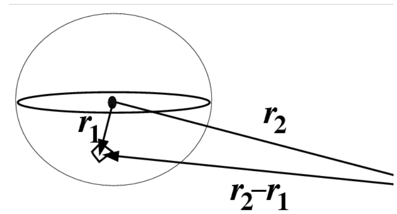


Figure 5.5: The position vectors used to describe the tidal force on the Earth due to the moon.

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}},$$

where θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is much larger than the other. Let's assume that $r_1 \ll r_2$. Then, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}} = \frac{1}{r_2}\frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have that the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n.$$

The first term in the expansion, $\frac{1}{r_2}$, is the gravitational potential that gives the usual force between the Earth and the moon. [Recall that the gravitational potential for mass m at distance r from M is given by $\Phi = -\frac{GMm}{r}$ and that the force is the gradient of the potential, $\mathbf{F} = -\nabla\Phi \propto \nabla\left(\frac{1}{r}\right)$.] The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we can proceed to use it. First of all, the generating function can be used to obtain special values of the Legendre polynomials.

Example 5.5. *Evaluate $P_n(0)$ using the generating function. $P_n(0)$ is found by considering $g(0,t)$. Setting $x = 0$ in Equation (5.29), we have*

$$\begin{aligned} g(0,t) &= \frac{1}{\sqrt{1+t^2}} \\ &= \sum_{n=0}^{\infty} P_n(0)t^n \\ &= P_0(0) + P_1(0)t + P_2(0)t^2 + P_3(0)t^3 + \ldots. \end{aligned} \tag{5.30}$$

We can use the binomial expansion to find the final answer. [See Chapter 1 for a review of the binomial expansion.] Namely, we have

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \ldots.$$

Comparing these expansions, we have the $P_n(0) = 0$ for n odd and for even integers one can show (see Problem 12) that[5]

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{5.31}$$

where $n!!$ is the double factorial,

$$n!! = \begin{cases} n(n-2)\ldots(3)1, & n > 0, odd, \\ n(n-2)\ldots(4)2, & n > 0, even, \\ 1 & n = 0, -1 \end{cases}.$$

[5] This example can be finished by first proving that

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n n!}.$$

Example 5.6. *Evaluate $P_n(-1)$. This is a simpler problem. In this case we have*

$$g(-1,t) = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots.$$

Therefore, $P_n(-1) = (-1)^n$.

Second proof of the three term recursion formula.

Proof of the three term recursion formula using $\frac{\partial g}{\partial t}$.

Proof. We can also use the generating function to find recurrence relations. To prove the three term recursion (5.23) that we introduced above, then we need only differentiate the generating function with respect to t in Equation (5.29) and rearrange the result. First note that

$$\frac{\partial g}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{1-2xt+t^2} g(x,t).$$

Combining this with

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1},$$

we have

$$(x-t)g(x,t) = (1-2xt+t^2) \sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Inserting the series expression for $g(x,t)$ and distributing the sum on the right side, we obtain

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}.$$

Multiplying out the $x-t$ factor and rearranging, leads to three separate sums:

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1} = 0. \tag{5.32}$$

Each term contains powers of t that we would like to combine into a single sum. This is done by reindexing. For the first sum, we could use the new index $k = n-1$. Then, the first sum can be written

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k.$$

Using different indices is just another way of writing out the terms. Note that

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

and

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

actually give the same sum. The indices are sometimes referred to as *dummy indices* because they do not show up in the expanded expression and can be replaced with another letter.

If we want to do so, we could now replace all of the k's with n's. However, we will leave the k's in the first term and now reindex the next sums in Equation (5.32). The second sum just needs the replacement $n = k$ and the last sum we reindex using $k = n + 1$. Therefore, Equation (5.32) becomes

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty}kP_{k-1}(x)t^k = 0. \tag{5.33}$$

We can now combine all of the terms, noting the $k = -1$ term is automatically zero and the $k = 0$ terms give

$$P_1(x) - xP_0(x) = 0. \tag{5.34}$$

Of course, we know this already. So, that leaves the $k > 0$ terms:

$$\sum_{k=1}^{\infty}\left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right]t^k = 0. \tag{5.35}$$

Since this is true for all t, the coefficients of the t^k's are zero, or

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots.$$

While this is the standard form for the three term recurrence relation, the earlier form is obtained by setting $k = n - 1$. □

There are other recursion relations which we list in the box below. Equation (5.36) was derived using the generating function. Differentiating it with respect to x, we find Equation (5.37). Equation (5.38) can be proven using the generating function by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series just as in this last manipulation. This will be left as Problem 4. Combining this result with Equation (5.36), we can derive Equations (5.39)-(5.40). Adding and subtracting these equations yields Equations (5.41)-(5.42).

Recursion Formulae for Legendre Polynomials for $n = 1, 2, \ldots$.

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{5.36}$$

$$(n+1)P'_{n+1}(x) = (2n+1)[P_n(x) + xP'_n(x)] - nP'_{n-1}(x) \tag{5.37}$$

$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) \tag{5.38}$$

$$P'_{n-1}(x) = xP'_n(x) - nP_n(x) \tag{5.39}$$

$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x) \tag{5.40}$$

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x). \tag{5.41}$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{5.42}$$

$$(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x) \tag{5.43}$$

Finally, Equation (5.43) can be obtained using Equations (5.39) and (5.40). Just multiply Equation (5.39) by x,

$$x^2 P_n'(x) - nxP_n(x) = xP_{n-1}'(x).$$

Now use Equation (5.40), but first replace n with $n-1$ to eliminate the $xP_{n-1}'(x)$ term:

$$x^2 P_n'(x) - nxP_n(x) = P_n'(x) - nP_{n-1}(x).$$

Rearranging gives the result.

Legendre Polynomials as Solutions of a Differential Equation

The Legendre polynomials satisfy a second order linear differential equation. This differential equation occurs naturally in the solution of initial-boundary value problems in three dimensions which possess some spherical symmetry. We will see this in the last chapter. There are two approaches we could take in showing that the Legendre polynomials satisfy a particular differential equation. Either we can write down the equations and attempt to solve it, or we could use the above properties to obtain the equation. For now, we will seek the differential equation satisfied by $P_n(x)$ using the above recursion relations.

We begin by differentiating Equation (5.43) and using Equation (5.39) to simplify:

$$\begin{aligned} \frac{d}{dx}\left((x^2-1)P_n'(x)\right) &= nP_n(x) + nxP_n'(x) - nP_{n-1}'(x) \\ &= nP_n(x) + n^2 P_n(x) \\ &= n(n+1)P_n(x). \end{aligned} \tag{5.44}$$

Therefore, Legendre polynomials, or Legendre functions of the first kind, are solutions of the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

As this is a linear second order differential equation, we expect two linearly independent solutions. The second solution, called the Legendre function of the second kind, is given by $Q_n(x)$ and is not well behaved at $x = \pm 1$. For example,

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

We will not need these for physically interesting examples in this book.

A generalization of the Legendre equation is given by $(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$. Solutions to this equation, $P_n^m(x)$ and $Q_n^m(x)$, are called the associated Legendre functions of the first and second kind.

Normalization Constant Another use of the generating function is to obtain the normalization constant. Namely, we want to evaluate

$$\|P_n\|^2 = \int_{-1}^{1} P_n(x)P_n(x)\,dx.$$

This can be done by first squaring the generating function in order to get the products $P_n(x)P_m(x)$, and then integrating over x.

Squaring the generating function has to be done with care, as we need to make proper use of the dummy summation index. So, we first write

$$\begin{aligned}\frac{1}{1-2xt+t^2} &= \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \qquad (5.45)\end{aligned}$$

Integrating from -1 to 1 and using the orthogonality of the Legendre polynomials, we have

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-1}^{1} P_n(x)P_m(x)\,dx \\ &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx. \qquad (5.46)\end{aligned}$$

However, one can show that[6]

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

Expanding this expression about $t=0$, we obtain[7]

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}.$$

Comparing this result with Equation (5.46), we find that

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}. \qquad (5.47)$$

Fourier-Legendre Series

With these properties of Legendre functions we are now prepared to compute the expansion coefficients for the Fourier-Legendre series representation of a given function.

Example 5.7. *Expand $f(x)=x^3$ in a Fourier-Legendre series.*

We simply need to compute

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} x^3 P_n(x)\,dx. \qquad (5.48)$$

We first note that

$$\int_{-1}^{1} x^m P_n(x)\,dx = 0 \quad \text{for } m<n.$$

As a result, we will have for this example that $c_n=0$ for $n>3$. We could just compute $\int_{-1}^{1} x^3 P_m(x)\,dx$ for $m=0,1,2,\ldots$ outright by looking up Legendre

[6] You will need the integral

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)+C.$$

[7] From Appendix A you will need the series expansion

$$\begin{aligned}\ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \\ &= x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots.\end{aligned}$$

polynomials. But, note that x^3 is an odd function, $c_0 = 0$ and $c_2 = 0$. This leaves us with only two coefficients to compute. We refer to Table 5.2 and find that

$$c_1 = \frac{3}{2}\int_{-1}^{1} x^4\,dx = \frac{3}{5}$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} x^3\left[\frac{1}{2}(5x^3-3x)\right]\,dx = \frac{2}{5}.$$

Thus,

$$x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Of course, this is simple to check using Table 5.2:

$$\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x) = \frac{3}{5}x + \frac{2}{5}\left[\frac{1}{2}(5x^3-3x)\right] = x^3.$$

Well, maybe we could have guessed this without doing any integration. Let's see,

$$\begin{aligned} x^3 &= c_1 x + \frac{1}{2}c_2(5x^3-3x) \\ &= (c_1 - \frac{3}{2}c_2)x + \frac{5}{2}c_2 x^3. \end{aligned} \tag{5.49}$$

Equating coefficients of like terms, we have that $c_2 = \frac{2}{5}$ and $c_1 = \frac{3}{2}c_2 = \frac{3}{5}$.

Example 5.8. *Expand the Heaviside[8] function in a Fourier-Legendre series.*

The Heaviside function is defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{5.50}$$

In this case, we cannot find the expansion coefficients without some integration. We have to compute

$$\begin{aligned} c_n &= \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx \\ &= \frac{2n+1}{2}\int_{0}^{1} P_n(x)\,dx. \end{aligned} \tag{5.51}$$

We can make use of the identity

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n > 1. \tag{5.52}$$

We have for $n > 0$

$$c_n = \frac{1}{2}\int_0^1 [P'_{n+1}(x) - P'_{n-1}(x)]\,dx = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

For $n = 0$, we have

$$c_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

[8] Oliver Heaviside (1850-1925) was an English mathematician, physicist and engineer who used complex analysis to study circuits and was a co-founder of vector analysis. The Heaviside function is also called the step function.

This leads to the expansion

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}[P_{n-1}(0) - P_{n+1}(0)]P_n(x).$$

We still need to evaluate the Fourier-Legendre coefficients. Since $P_n(0) = 0$ *for* n *odd, the* c_n*'s vanish for* n *even. Letting* $n = 2k-1$, *we can re-index the sum,*

$$f(x) \sim \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}[P_{2k-2}(0) - P_{2k}(0)]P_{2k-1}(x).$$

We can compute the Fourier coefficients, $c_{2k-1} = \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)]$, *using a result from Problem 12:*

$$P_{2k}(0) = (-1)^k \frac{(2k-1)!!}{(2k)!!}. \tag{5.53}$$

Namely, we have

$$\begin{aligned} c_{2k-1} &= \frac{1}{2}[P_{2k-2}(0) - P_{2k}(0)] \\ &= \frac{1}{2}\left[(-1)^{k-1}\frac{(2k-3)!!}{(2k-2)!!} - (-1)^k\frac{(2k-1)!!}{(2k)!!}\right] \\ &= -\frac{1}{2}(-1)^k\frac{(2k-3)!!}{(2k-2)!!}\left[1 + \frac{2k-1}{2k}\right] \\ &= -\frac{1}{2}(-1)^k\frac{(2k-3)!!}{(2k-2)!!}\frac{4k-1}{2k}. \end{aligned} \tag{5.54}$$

Thus, the Fourier-Legendre series expansion for the Heaviside function is given by

$$f(x) \sim \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-3)!!}{(2n-2)!!}\frac{4n-1}{2n}P_{2n-1}(x). \tag{5.55}$$

The sum of the first 21 terms of this series are shown in Figure 5.6. We note the slow convergence to the Heaviside function. Also, we see that the Gibbs phenomenon is present due to the jump discontinuity at $x = 0$. *[See Section 4.12.]*

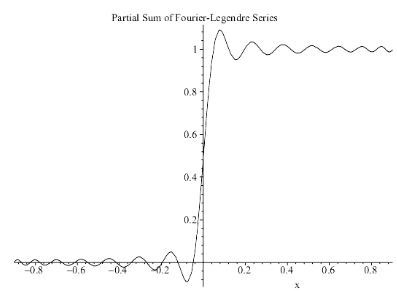


Figure 5.6: Sum of first 21 terms for Fourier-Legendre series expansion of Heaviside function.

5.4 Gamma Function

The name and symbol for the Gamma function were first given by Legendre in 1811. However, the search for a generalization of the factorial extends back to the 1720's when Euler provided the first representation of the factorial as an infinite product, later to be modified by others like Gauß, Weierstraß, and Legendre.

A function that often occurs in the study of special functions is the Gamma function. We will need the Gamma function in the next section on Fourier-Bessel series.

For $x > 0$ we define the Gamma function as

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt, \quad x > 0. \tag{5.56}$$

The Gamma function is a generalization of the factorial function and a plot is shown in Figure 5.7. In fact, we have

$$\Gamma(1) = 1$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

The reader can prove this identity by simply performing an integration by parts. (See Problem 7.) In particular, for integers $n \in Z^+$, we then have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

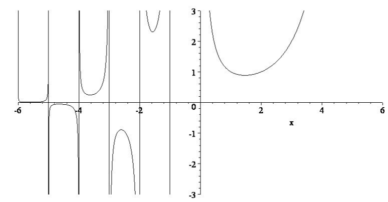


Figure 5.7: Plot of the Gamma function.

We can also define the Gamma function for negative, non-integer values of x. We first note that by iteration on $n \in Z^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}, \quad -n < x < 0$$

Note that the Gamma function is undefined at zero and the negative integers.

Example 5.9. *We now prove that*

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

This is done by direct computation of the integral:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt.$$

Letting $t = z^2$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-z^2}\, dz.$$

Due to the symmetry of the integrand, we obtain the classic integral

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-z^2}\, dz,$$

which can be performed using a standard trick.[9] *Consider the integral*

$$I = \int_{-\infty}^\infty e^{-x^2}\, dx.$$

Then,

$$I^2 = \int_{-\infty}^\infty e^{-x^2}\, dx \int_{-\infty}^\infty e^{-y^2}\, dy.$$

[9] In Example 7.4 we show the more general result:

$$\int_{-\infty}^\infty e^{-\beta y^2}\, dy = \sqrt{\frac{\pi}{\beta}}.$$

Note that we changed the integration variable. This will allow us to write this product of integrals as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy.$$

This is an integral over the entire xy-plane. We can transform this Cartesian integration to an integration over polar coordinates. The integral becomes

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,rdrd\theta.$$

This is simple to integrate and we have $I^2 = \pi$. So, the final result is found by taking the square root of both sides:

$$\Gamma\left(\frac{1}{2}\right) = I = \sqrt{\pi}.$$

In Problem 12 the reader will prove the identity

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$

Another useful relation, which we only state, is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

The are many other important relations, including infinite products, which we will not need at this point. The reader is encouraged to read about these elsewhere. In the meantime, we move on to the discussion of another important special function in physics and mathematics.

5.5 Fourier-Bessel Series

Bessel functions arise in many problems in physics possessing cylindrical symmetry such as the vibrations of circular drumheads and the radial modes in optical fibers. They provide us with another orthogonal set of functions. You might have seen in a course on differential equations that Bessel functions are solutions of the differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{5.57}$$

Solutions to this equation are obtained in the form of series expansions. Namely, one seeks solutions of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$$

by determining the for the coefficients must take. We will leave this for a homework exercise and simply report the results.

The history of Bessel functions, does not originate in the study of partial differential equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to G. N. Watson in his *Treatise on Bessel Functions*, the formulation and solution of Kepler's Problem was discovered by Joseph-Louis Lagrange (1736-1813), in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, E, as functions of time. Lagrange found expressions for the coefficients in the expansions of r and E in trigonometric functions of time. However, he only computed the first few coefficients. In 1816 Friedrich Wilhelm Bessel (1784-1846) had shown that the coefficients in the expansion for r could be given an integral representation. In 1824 he presented a thorough study of these functions, which are now called Bessel functions.

One solution of the differential equation is the *Bessel function of the first kind of order p*, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{5.58}$$

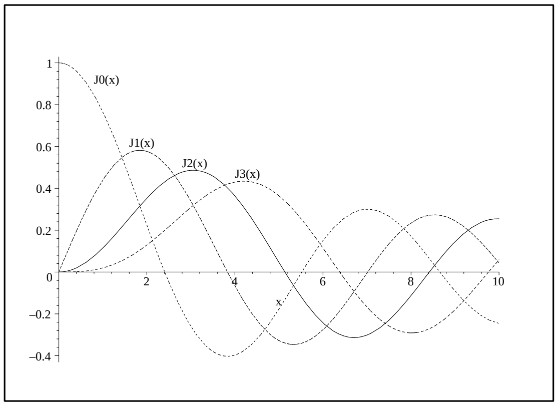


Figure 5.8: Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.

In Figure 5.8 we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

A second linearly independent solution is obtained for p not an integer as $J_{-p}(x)$. However, for p an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when p is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = Y_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{5.59}$$

These functions are called the Neumann functions, or Bessel functions of the second kind of order p.

In Figure 5.9 we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$.

In many applications one desires bounded solutions at $x = 0$. These functions do not satisfy this boundary condition. For example, we will later study one standard problem is to describe the oscillations of a circular drumhead. For this problem one solves the two dimensional wave equation using separation of variables in cylindrical coordinates.

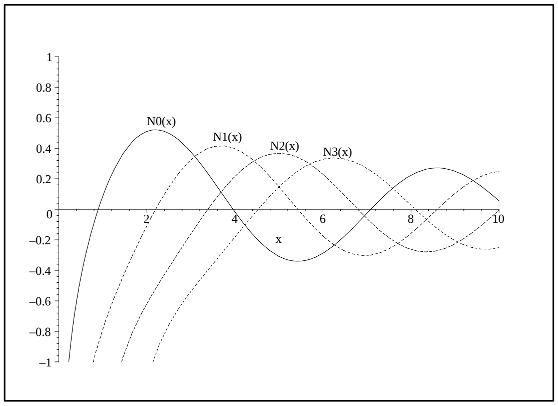


Figure 5.9: Plots of the Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$, and $N_3(x)$.

The r equation leads to a Bessel equation. The Bessel function solutions describe the radial part of the solution and one does not expect a singular solution at the center of the drum. The amplitude of the oscillation must remain finite. Thus, only Bessel functions of the first kind can be used.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind. The reader will have the opportunity to prove these for homework.

Derivative Identities These identities follow directly from the manipulation of the series solution.

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{5.60}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x). \tag{5.61}$$

Recursion Formulae The next identities follow from adding, or subtracting, the derivative identities.

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x). \tag{5.62}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{5.63}$$

Orthogonality As we will see in the next chapter, one can recast the Bessel equation into an eigenvalue problem whose solutions form an orthogonal basis of functions on $L_x^2(0, a)$. Using Sturm-Liouville theory, one can show that

$$\int_0^a x J_p(j_{pn}\frac{x}{a}) J_p(j_{pm}\frac{x}{a})\,dx = \frac{a^2}{2}\left[J_{p+1}(j_{pn})\right]^2 \delta_{n,m}, \tag{5.64}$$

where j_{pn} is the nth root of $J_p(x)$, $J_p(j_{pn}) = 0$, $n = 1, 2, \ldots$. A list of some of these roots are provided in Table 5.3.

n	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
1	2.405	3.832	5.135	6.379	7.586	8.780
2	5.520	7.016	8.147	9.760	11.064	12.339
3	8.654	10.173	11.620	13.017	14.373	15.700
4	11.792	13.323	14.796	16.224	17.616	18.982
5	14.931	16.470	17.960	19.410	20.827	22.220
6	18.071	19.616	21.117	22.583	24.018	25.431
7	21.212	22.760	24.270	25.749	27.200	28.628
8	24.353	25.903	27.421	28.909	30.371	31.813
9	27.494	29.047	30.571	32.050	33.512	34.983

Table 5.3: The zeros of Bessel Functions

Generating Function

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x) t^n, \quad x > 0, t \neq 0. \tag{5.65}$$

Integral Representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta, \quad x > 0, n \in \mathrm{Z}. \tag{5.66}$$

Fourier-Bessel Series

Since the Bessel functions are an orthogonal set of functions of a Sturm-Liouville problem,[10] we can expand square integrable functions in this basis. In fact, the Sturm-Liouville problem is given in the form

$$x^2 y'' + x y' + (\lambda x^2 - p^2) y = 0, \quad x \in [0, a], \tag{5.67}$$

satisfying the boundary conditions: $y(x)$ is bounded at $x = 0$ and $y(a) = 0$. The solutions are then of the form $J_p(\sqrt{\lambda}x)$, as can be shown by making the substitution $t = \sqrt{\lambda}x$ in the differential equation. Namely, we let $y(x) = u(t)$ and note that

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{du}{dt} = \sqrt{\lambda}\frac{du}{dt}.$$

Then,

$$t^2 u'' + t u' + (t^2 - p^2) u = 0,$$

which has a solution $u(t) = J_p(t)$.

[10] In the study of boundary value problems in differential equations, Sturm-Liouville problems are a bountiful source of basis functions for the space of square integrable functions as will be seen in the next section.

Using Sturm-Liouville theory, one can show that $J_p(j_{pn}\frac{x}{a})$ is a basis of eigenfunctions and the resulting *Fourier-Bessel series expansion* of $f(x)$ defined on $x \in [0, a]$ is

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}\frac{x}{a}), \tag{5.68}$$

where the Fourier-Bessel coefficients are found using the orthogonality relation as

$$c_n = \frac{2}{a^2 \left[J_{p+1}(j_{pn})\right]^2} \int_0^a x f(x) J_p(j_{pn}\frac{x}{a})\, dx. \tag{5.69}$$

Example 5.10. *Expand $f(x) = 1$ for $0 < x < 1$ in a Fourier-Bessel series of the form*

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(j_{0n}x)$$

.

We need only compute the Fourier-Bessel coefficients in Equation (5.69):

$$c_n = \frac{2}{[J_1(j_{0n})]^2} \int_0^1 x J_0(j_{0n}x)\, dx. \tag{5.70}$$

From the identity

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{5.71}$$

we have

$$\begin{aligned}
\int_0^1 x J_0(j_{0n}x)\, dx &= \frac{1}{j_{0n}^2} \int_0^{j_{0n}} y J_0(y)\, dy \\
&= \frac{1}{j_{0n}^2} \int_0^{j_{0n}} \frac{d}{dy}\left[y J_1(y)\right]\, dy \\
&= \frac{1}{j_{0n}^2} \left[y J_1(y)\right]_0^{j_{0n}} \\
&= \frac{1}{j_{0n}} J_1(j_{0n}).
\end{aligned} \tag{5.72}$$

As a result, the desired Fourier-Bessel expansion is given as

$$1 = 2\sum_{n=1}^{\infty} \frac{J_0(j_{0n}x)}{j_{0n}J_1(j_{0n})}, \quad 0 < x < 1. \tag{5.73}$$

In Figure 5.10 we show the partial sum for the first fifty terms of this series. Note once again the slow convergence due to the Gibbs phenomenon.

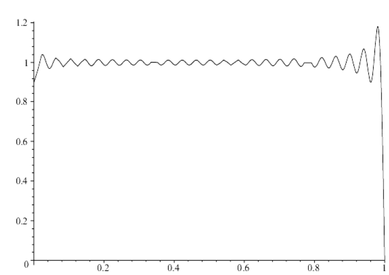


Figure 5.10: Plot of the first 50 terms of the Fourier-Bessel series in Equation (5.73) for $f(x) = 1$ on $0 < x < 1$.

5.6 Sturm-Liouville Eigenvalue Problems

IN THE LAST CHAPTER we explored the solutions of differential equations that led to solutions in the form of trigonometric functions and special functions. Such solutions can be used to represent functions in generalized Fourier series expansions. We would like to generalize some of those techniques we had first used to solve the heat equation in order to solve other boundary value problems. A class of problems to which our previous examples belong and which have eigenfunctions with similar properties are the Sturm-Liouville Eigenvalue Problems. These problems involve self-adjoint (differential) operators which play an important role in the spectral theory of linear operators and the existence of the eigenfunctions. These ideas will be introduced in this section.

5.6.1 Sturm-Liouville Operators

IN PHYSICS MANY PROBLEMS arise in the form of boundary value problems involving second order ordinary differential equations. For example, we will explore the wave equation and the heat equation in three dimensions. Separating out the time dependence leads to a three dimensional boundary value problem in both cases. Further separation of variables leads to a set of boundary value problems involving second order ordinary differential equations.

In general, we might obtain equations of the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{5.74}$$

subject to boundary conditions. We can write such an equation in operator form by defining the differential operator

$$L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x).$$

Then, Equation (5.74) takes the form

$$Ly = f.$$

Recall that we had solved such nonhomogeneous differential equations in Chapter 2. In this chapter we will show that these equations can be solved using eigenfunction expansions. Namely, we seek solutions to the eigenvalue problem

$$L\phi = \lambda\phi$$

with homogeneous boundary conditions on ϕ and then seek a solution of the nonhomogeneous problem, $Ly = f$, as an expansion over these eigenfunctions. Formally, we let

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

However, we are not guaranteed a nice set of eigenfunctions. We need an appropriate set to form a basis in the function space. Also, it would be nice to have orthogonality so that we can easily solve for the expansion coefficients.

It turns out that any linear second order differential operator can be turned into an operator that possesses just the right properties (self-adjointedness) to carry out this procedure. The resulting operator is referred to as a Sturm-Liouville operator. We will highlight some of the properties of such operators and prove a few key theorems, though this will not be an extensive review of Sturm-Liouville theory. The interested reader can review the literature and advanced texts for a more in depth analysis.

We define the *Sturm-Liouville operator* as

The Sturm-Liouville operator.

$$\mathcal{L} = \frac{d}{dx} p(x) \frac{d}{dx} + q(x). \tag{5.75}$$

The *Sturm-Liouville eigenvalue problem* is given by the differential equation

The Sturm-Liouville eigenvalue problem.

$$\mathcal{L}y = -\lambda \sigma(x) y,$$

or

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda\sigma(x)y = 0, \tag{5.76}$$

for $x \in (a, b)$, $y = y(x)$, plus boundary conditions. The functions $p(x)$, $p'(x)$, $q(x)$ and $\sigma(x)$ are assumed to be continuous on (a, b) and $p(x) > 0$, $\sigma(x) > 0$ on $[a, b]$. If the interval is finite and these assumptions on the coefficients are true on $[a, b]$, then the problem is said to be *regular*. Otherwise, it is called *singular*.

We also need to impose the set of homogeneous boundary conditions

Types of boundary conditions.

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= 0, \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0. \end{aligned} \tag{5.77}$$

The α's and β's are constants. For different values, one has special types of boundary conditions. For $\beta_i = 0$, we have what are called *Dirichlet boundary conditions*. Namely, $y(a) = 0$ and $y(b) = 0$. For $\alpha_i = 0$, we have *Neumann boundary conditions*. In this case, $y'(a) = 0$ and $y'(b) = 0$. In terms of the heat equation example, Dirichlet conditions correspond to maintaining a fixed temperature at the ends of the rod. The Neumann boundary conditions would correspond to no heat flow across the ends, or insulating conditions, as there would be no temperature gradient at those points. The more general boundary conditions allow for partially insulated boundaries.

Another type of boundary condition that is often encountered is the *periodic boundary condition*. Consider the heated rod that has been bent to form a circle. Then the two end points are physically the same. So, we would expect that the temperature and the temperature gradient should agree at those points. For this case we write $y(a) = y(b)$ and $y'(a) = y'(b)$. Boundary value problems using these conditions have to be handled differently than the above homogeneous conditions. These conditions leads to different types of eigenfunctions and eigenvalues.

As previously mentioned, equations of the form (5.74) occur often. We now show that Equation (5.74) can be turned into a *differential equation of Sturm-Liouville form:*

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x). \tag{5.78}$$

Another way to phrase this is provided in the theorem:

Theorem 5.2. *Any second order linear operator can be put into the form of the Sturm-Liouville operator (5.76).*

The proof of this is straight forward, as we shall soon show. Consider the equation (5.74). If $a_1(x) = a_2'(x)$, then we can write the equation in the form

$$\begin{aligned} f(x) &= a_2(x)y'' + a_1(x)y' + a_0(x)y \\ &= (a_2(x)y')' + a_0(x)y. \end{aligned} \tag{5.79}$$

This is in the correct form. We just identify $p(x) = a_2(x)$ and $q(x) = a_0(x)$.

However, consider the differential equation

$$x^2y'' + xy' + 2y = 0.$$

In this case $a_2(x) = x^2$ and $a_2'(x) = 2x \neq a_1(x)$. The linear differential operator in this equation is not of Sturm-Liouville type. But, we can change it to a Sturm Liouville operator.

Proof. In the Sturm Liouville operator the derivative terms are gathered together into one perfect derivative. This is similar to what we saw in the Chapter 2 when we solved linear first order equations. In that case we sought an integrating factor. We can do the same thing here. We seek a multiplicative function $\mu(x)$ that we can multiply through (5.74) so that it can be written in Sturm-Liouville form. We first divide out the $a_2(x)$, giving

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{f(x)}{a_2(x)}.$$

Now, we multiply the differential equation by μ :

$$\mu(x)y'' + \mu(x)\frac{a_1(x)}{a_2(x)}y' + \mu(x)\frac{a_0(x)}{a_2(x)}y = \mu(x)\frac{f(x)}{a_2(x)}.$$

The first two terms can now be combined into an exact derivative $(\mu y')'$ if $\mu(x)$ satisfies

$$\frac{d\mu}{dx} = \mu(x)\frac{a_1(x)}{a_2(x)}.$$

This is formally solved to give

$$\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)}\,dx}.$$

Thus, the original equation can be multiplied by factor

$$\frac{\mu(x)}{a_2(x)} = \frac{1}{a_2(x)}e^{\int \frac{a_1(x)}{a_2(x)}\,dx}$$

to turn it into Sturm-Liouville form. □

In summary,

Conversion of a linear second order differential equation to Sturm Liouville form.

Equation (5.74),

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x), \tag{5.80}$$

can be put into the Sturm-Liouville form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = F(x), \tag{5.81}$$

where

$$\begin{aligned} p(x) &= e^{\int \frac{a_1(x)}{a_2(x)}\,dx}, \\ q(x) &= p(x)\frac{a_0(x)}{a_2(x)}, \\ F(x) &= p(x)\frac{f(x)}{a_2(x)}. \end{aligned} \tag{5.82}$$

Example 5.11. *For the example above,*

$$x^2y'' + xy' + 2y = 0.$$

We need only multiply this equation by

$$\frac{1}{x^2}e^{\int \frac{dx}{x}} = \frac{1}{x},$$

to put the equation in Sturm-Liouville form:

$$\begin{aligned} 0 &= xy'' + y' + \frac{2}{x}y \\ &= (xy')' + \frac{2}{x}y. \end{aligned} \tag{5.83}$$

5.6.2 Properties of Sturm-Liouville Eigenvalue Problems

THERE ARE SEVERAL PROPERTIES that can be proven for the (regular) Sturm-Liouville eigenvalue problem in (5.76). However, we will not prove them all here. We will merely list some of the important facts and focus on a few of the properties.

Real, countable eigenvalues.

1. The eigenvalues are real, countable, ordered and there is a smallest eigenvalue. Thus, we can write them as $\lambda_1 < \lambda_2 < \ldots$. However, there is no largest eigenvalue and $n \to \infty$, $\lambda_n \to \infty$.

Oscillatory eigenfunctions.

2. For each eigenvalue λ_n there exists an eigenfunction ϕ_n with $n-1$ zeros on (a, b).

Orthogonality of eigenfunctions.

3. Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function, $\sigma(x)$. Defining the inner product of $f(x)$ and $g(x)$ as
$$< f, g >= \int_a^b f(x)g(x)\sigma(x)\, dx, \tag{5.84}$$
then the orthogonality of the eigenfunctions can be written in the form
$$< \phi_n, \phi_m >=< \phi_n, \phi_n > \delta_{nm}, \quad n, m = 1, 2, \ldots. \tag{5.85}$$

Complete basis of eigenfunctions.

4. The set of eigenfunctions is complete; i.e., any piecewise smooth function can be represented by a generalized Fourier series expansion of the eigenfunctions,
$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$
where
$$c_n = \frac{< f, \phi_n >}{< \phi_n, \phi_n >}.$$
Actually, one needs $f(x) \in L^2_\sigma(a, b)$, the set of *square integrable functions* over $[a, b]$ with weight function $\sigma(x)$. By square integrable, we mean that $< f, f >< \infty$. One can show that such a space is isomorphic to a *Hilbert space*, a complete inner product space. Hilbert spaces play a special role in quantum mechanics.

Rayleigh Quotient.

5. Multiply the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \sigma(x)\phi_n$$

by ϕ_n and integrate. Solve this result for λ_n, to find the *Rayleigh Quotient*

$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b - \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2 \right] dx}{< \phi_n, \phi_n >}$$

The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties.

Example 5.12. *We seek the eigenfunctions of the operator found in Example 5.11. Namely, we want to solve the eigenvalue problem*

$$\mathcal{L}y = (xy')' + \frac{2}{x}y = -\lambda\sigma y \tag{5.86}$$

subject to a set of homogeneous boundary conditions. Let's use the boundary conditions

$$y'(1) = 0, \quad y'(2) = 0.$$

[Note that we do not know $\sigma(x)$ yet, but will choose an appropriate function to obtain solutions.]

Expanding the derivative, we have

$$xy'' + y' + \frac{2}{x}y = -\lambda\sigma y.$$

Multiply through by x to obtain

$$x^2y'' + xy' + (2 + \lambda x\sigma)\, y = 0.$$

Notice that if we choose $\sigma(x) = x^{-1}$, then this equation can be made a Cauchy-Euler type equation. Thus, we have

$$x^2y'' + xy' + (\lambda + 2)\, y = 0.$$

The characteristic equation is

$$r^2 + \lambda + 2 = 0.$$

For oscillatory solutions, we need $\lambda + 2 > 0$. Thus, the general solution is

$$y(x) = c_1 \cos(\sqrt{\lambda + 2}\ln|x|) + c_2 \sin(\sqrt{\lambda + 2}\ln|x|). \tag{5.87}$$

Next we apply the boundary conditions. $y'(1) = 0$ forces $c_2 = 0$. This leaves

$$y(x) = c_1 \cos(\sqrt{\lambda + 2}\ln x).$$

The second condition, $y'(2) = 0$, yields

$$\sin(\sqrt{\lambda+2}\ln 2) = 0.$$

This will give nontrivial solutions when

$$\sqrt{\lambda+2}\ln 2 = n\pi, \quad n = 0, 1, 2, 3 \ldots.$$

In summary, the eigenfunctions for this eigenvalue problem are

$$y_n(x) = \cos\left(\frac{n\pi}{\ln 2}\ln x\right), \quad 1 \le x \le 2$$

and the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 - 2$ for $n = 0, 1, 2, \ldots$.

Note: We include the $n = 0$ case because $y(x) =$ constant is a solution of the $\lambda = -2$ case. More specifically, in this case the characteristic equation reduces to $r^2 = 0$. Thus, the general solution of this Cauchy-Euler equation is

$$y(x) = c_1 + c_2 \ln|x|.$$

Setting $y'(1) = 0$, forces $c_2 = 0$. $y'(2)$ automatically vanishes, leaving the solution in this case as $y(x) = c_1$.

We note that some of the properties listed in the beginning of the section hold for this example. The eigenvalues are seen to be real, countable and ordered. There is a least one, $\lambda_0 = -2$. Next, one can find the zeros of each eigenfunction on [1,2]. Then the argument of the cosine, $\frac{n\pi}{\ln 2}\ln x$, takes values 0 to $n\pi$ for $x \in [1, 2]$. The cosine function has $n - 1$ roots on this interval.

Orthogonality can be checked as well. We set up the integral and use the substitution $y = \pi \ln x / \ln 2$. This gives

$$\begin{aligned} < y_n, y_m > &= \int_1^2 \cos\left(\frac{n\pi}{\ln 2}\ln x\right)\cos\left(\frac{m\pi}{\ln 2}\ln x\right)\frac{dx}{x} \\ &= \frac{\ln 2}{\pi}\int_0^\pi \cos ny \cos my \, dy \\ &= \frac{\ln 2}{2}\delta_{n,m}. \end{aligned} \tag{5.88}$$

5.6.3 *Adjoint Operators*

IN THE STUDY OF THE SPECTRAL THEORY of matrices, one learns about the adjoint of the matrix, $A^\dagger$, and the role that self-adjoint, or Hermitian, matrices play in diagonalization. Also, one needs the concept of adjoint to discuss the existence of solutions to the matrix problem $\mathbf{y} = A\mathbf{x}$. In the same spirit, one is interested in the existence of solutions of the operator equation $Lu = f$ and solutions of the corresponding eigenvalue problem. The study of linear operators on a Hilbert space is a generalization of what the reader had seen in a linear algebra course.

Just as one can find a basis of eigenvectors and diagonalize Hermitian, or self-adjoint, matrices (or, real symmetric in the case of real matrices), we will see that the Sturm-Liouville operator is self-adjoint. In this section we will define the *domain of an operator* and introduce the notion of *adjoint operators*. In the last section we discuss the role the adjoint plays in the existence of solutions to the operator equation $Lu = f$.

We first introduce some definitions.

Definition 5.2. The *domain* of a differential operator L is the set of all $u \in L^2_\sigma(a,b)$ satisfying a given set of homogeneous boundary conditions.

The *adjoint*, $L^\dagger$, of operator L.

Definition 5.3. The *adjoint*, $L^\dagger$, of operator L satisfies

$$< u, Lv > = < L^\dagger u, v >$$

for all v in the domain of L and u in the domain of $L^\dagger$.

Example 5.13. *As an example, we find the adjoint of second order linear differential operator* $L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$.

In order to find the adjoint, we place the operator under an integral. So, we consider the inner product

$$< u, Lv > = \int_a^b u(a_2 v'' + a_1 v' + a_0 v)\, dx.$$

We have to move the operator L from v and determine what operator is acting on u in order to formally preserve the inner product. For a simple operator like $L = \frac{d}{dx}$, *this is easily done using integration by parts. For the given operator, we will need to apply several integrations by parts to the individual terms. We will consider the individual terms.*

First we consider the $a_1 v'$ *term. Integration by parts yields*

$$\int_a^b u(x) a_1(x) v'(x)\, dx = a_1(x) u(x) v(x)\Big|_a^b - \int_a^b (u(x) a_1(x))' v(x)\, dx. \tag{5.89}$$

Now, we consider the $a_2 v''$ *term. In this case it will take two integrations by parts:*

$$\begin{aligned} \int_a^b u(x) a_2(x) v''(x)\, dx &= a_2(x) u(x) v'(x)\Big|_a^b - \int_a^b (u(x) a_2(x))' v(x)'\, dx \\ &= \left[a_2(x) u(x) v'(x) - (a_2(x) u(x))' v(x) \right]\Big|_a^b \\ &\quad + \int_a^b (u(x) a_2(x))'' v(x)\, dx. \end{aligned} \tag{5.90}$$

Combining these results, we obtain

$$< u, Lv > = \int_a^b u(a_2 v'' + a_1 v' + a_0 v)\, dx$$

$$= \left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b + \int_a^b \left[(a_2u)'' - (a_1u)' + a_0u\right] v\,dx. \quad (5.91)$$

Inserting the boundary conditions for v, one has to determine boundary conditions for u such that

$$\left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]\Big|_a^b = 0.$$

This leaves

$$< u, Lv >= \int_a^b \left[(a_2u)'' - (a_1u)' + a_0u\right] v\,dx \equiv < L^\dagger u, v > .$$

Therefore,

$$L^\dagger = \frac{d^2}{dx^2}a_2(x) - \frac{d}{dx}a_1(x) + a_0(x). \quad (5.92)$$

Self-adjoint operators.

When $L^\dagger = L$, the operator is called *formally self-adjoint*. When the domain of L is the same as the domain of $L^\dagger$, the term *self-adjoint* is used. As the domain is important in establishing self-adjointness, we need to do a complete example in which the domain of the adjoint is found.

Example 5.14. *Determine $L^\dagger$ and its domain for operator $Lu = \frac{du}{dx}$ where u satisfies the boundary conditions $u(0) = 2u(1)$ on $[0,1]$.*

We need to find the adjoint operator satisfying $< v, Lu >=< L^\dagger v, u >$. Therefore, we rewrite the integral

$$< v, Lu >= \int_0^1 v\frac{du}{dx}\,dx = uv|_0^1 - \int_0^1 u\frac{dv}{dx}\,dx =< L^\dagger v, u > .$$

From this we have the adjoint problem consisting of an adjoint operator and the associated boundary condition:

1. $L^\dagger = -\frac{d}{dx}$.
2. $uv\Big|_0^1 = 0 \Rightarrow 0 = u(1)[v(1) - 2v(0)] \Rightarrow v(1) = 2v(0)$.

5.6.4 Lagrange's and Green's Identities

Before turning to the proofs that the eigenvalues of a Sturm-Liouville problem are real and the associated eigenfunctions orthogonal, we will first need to introduce two important identities. For the Sturm-Liouville operator,

$$\mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q,$$

we have the two identities:

Lagrange's Identity:	$u\mathcal{L}v - v\mathcal{L}u$	$= [p(uv' - vu')]'.$
Green's Identity:	$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx$	$= [p(uv' - vu')]\vert_a^b.$

Proof. The proof of Lagrange's identity follows by a simple manipulations of the operator:

$$\begin{aligned} u\mathcal{L}v - v\mathcal{L}u &= u\left[\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right] - v\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right] \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right) \\ &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + p\frac{du}{dx}\frac{dv}{dx} - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - p\frac{du}{dx}\frac{dv}{dx} \\ &= \frac{d}{dx}\left[pu\frac{dv}{dx} - pv\frac{du}{dx}\right]. \end{aligned} \tag{5.93}$$

Green's identity is simply proven by integrating Lagrange's identity.

□

5.6.5 Orthogonality and Reality

We are now ready to prove that the eigenvalues of a Sturm-Liouville problem are real and the corresponding eigenfunctions are orthogonal. These are easily established using Green's identity, which in turn is a statement about the Sturm-Liouville operator being self-adjoint.

Theorem 5.3. *The eigenvalues of the Sturm-Liouville problem (5.76) are real.*

Proof. Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n:

$$\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n.$$

The complex conjugate of this equation is

$$\mathcal{L}\overline{\phi}_n = -\overline{\lambda}_n\sigma\overline{\phi}_n.$$

Now, multiply the first equation by $\overline{\phi}_n$ and the second equation by ϕ_n and then subtract the results. We obtain

$$\overline{\phi}_n\mathcal{L}\phi_n - \phi_n\mathcal{L}\overline{\phi}_n = (\overline{\lambda}_n - \lambda_n)\sigma\phi_n\overline{\phi}_n.$$

Integrate both sides of this equation:

$$\int_a^b \left(\overline{\phi}_n\mathcal{L}\phi_n - \phi_n\mathcal{L}\overline{\phi}_n\right)\,dx = (\overline{\lambda}_n - \lambda_n)\int_a^b \sigma\phi_n\overline{\phi}_n\,dx.$$

Apply Green's identity to the left hand side to find

$$[p(\overline{\phi}_n\phi_n' - \phi_n\overline{\phi}_n')]\vert_a^b = (\overline{\lambda}_n - \lambda_n)\int_a^b \sigma\phi_n\overline{\phi}_n\,dx.$$

Using the homogeneous boundary conditions (5.77) for a self-adjoint operator, the left side vanishes to give

$$0 = (\bar{\lambda}_n - \lambda_n) \int_a^b \sigma \|\phi_n\|^2 \, dx.$$

The integral is nonnegative, so we must have $\bar{\lambda}_n = \lambda_n$. Therefore, the eigenvalues are real. □

Theorem 5.4. *The eigenfunctions corresponding to different eigenvalues of the Sturm-Liouville problem (5.76) are orthogonal.*

Proof. This is proven similar to the last theorem. Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n,

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n,$$

and let $\phi_m(x)$ be a solution of the eigenvalue problem associated with $\lambda_m \neq \lambda_n$,

$$\mathcal{L}\phi_m = -\lambda_m \sigma \phi_m,$$

Now, multiply the first equation by ϕ_m and the second equation by ϕ_n. Subtracting the results, we obtain

$$\phi_m \mathcal{L}\phi_n - \phi_n \mathcal{L}\phi_m = (\lambda_m - \lambda_n)\sigma \phi_n \phi_m$$

Similar to the previous proof, we integrate both sides of the equation and use Green's identity and the boundary conditions for a self-adjoint operator. This leaves

$$0 = (\lambda_m - \lambda_n) \int_a^b \sigma \phi_n \phi_m \, dx.$$

Since the eigenvalues are distinct, we can divide by $\lambda_m - \lambda_n$, leaving the desired result,

$$\int_a^b \sigma \phi_n \phi_m \, dx = 0.$$

Therefore, the eigenfunctions are orthogonal with respect to the weight function $\sigma(x)$. □

5.6.6 The Rayleigh Quotient - optional

The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties associated with Sturm-Liouville eigenvalue problems. We begin by multiplying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \sigma(x) \phi_n$$

by ϕ_n and integrating. This gives

$$\int_a^b \left[\phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n^2\right] dx = -\lambda \int_a^b \phi_n^2\, dx.$$

One can solve the last equation for λ to find

$$\lambda = \frac{-\int_a^b \left[\phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) + q\phi_n^2\right] dx}{\int_a^b \phi_n^2 \sigma\, dx}.$$

It appears that we have solved for the eigenvalue and have not needed the machinery we had developed in Chapter 4 for studying boundary value problems. However, we really cannot evaluate this expression because we do not know the eigenfunctions, $\phi_n(x)$ yet. Nevertheless, we will see what we can determine.

One can rewrite this result by performing an integration by parts on the first term in the numerator. Namely, pick $u = \phi_n$ and $dv = \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) dx$ for the standard integration by parts formula. Then, we have

$$\int_a^b \phi_n \frac{d}{dx}\left(p\frac{d\phi_n}{dx}\right) dx = p\phi_n \frac{d\phi_n}{dx}\Big|_a^b - \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2\right] dx.$$

Inserting the new formula into the expression for λ, leads to the *Rayleigh Quotient*

$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi_n}{dx}\right)^2 - q\phi_n^2\right] dx}{\int_a^b \phi_n^2 \sigma\, dx}. \tag{5.94}$$

In many applications the sign of the eigenvalue is important. As we had seen in the solution of the heat equation, $T' + k\lambda T = 0$. Since we expect the heat energy to diffuse, the solutions should decay in time. Thus, we would expect $\lambda > 0$. In studying the wave equation, one expects vibrations and these are only possible with the correct sign of the eigenvalue (positive again). Thus, in order to have nonnegative eigenvalues, we see from (5.94) that

a. $q(x) \leq 0$, and

b. $-p\phi_n \frac{d\phi_n}{dx}\big|_a^b \geq 0$.

Furthermore, if λ is a zero eigenvalue, then $q(x) \equiv 0$ and $\alpha_1 = \alpha_2 = 0$ in the homogeneous boundary conditions. This can be seen by setting the numerator equal to zero. Then, $q(x) = 0$ and $\phi_n'(x) = 0$. The second of these conditions inserted into the boundary conditions forces the restriction on the type of boundary conditions.

One of the (unproven here) properties of Sturm-Liouville eigenvalue problems with homogeneous boundary conditions is that the eigenvalues are ordered, $\lambda_1 < \lambda_2 < \ldots$. Thus, there is a smallest eigenvalue. It turns out that for any continuous function, $y(x)$,

$$\lambda_1 = \min_{y(x)} \frac{-py\frac{dy}{dx}|_a^b + \int_a^b \left[p\left(\frac{dy}{dx}\right)^2 - qy^2 \right] dx}{\int_a^b y^2 \sigma \, dx} \tag{5.95}$$

and this minimum is obtained when $y(x) = \phi_1(x)$. This result can be used to get estimates of the minimum eigenvalue by using trial functions which are continuous and satisfy the boundary conditions, but do not necessarily satisfy the differential equation.

Example 5.15. *We have already solved the eigenvalue problem $\phi'' + \lambda\phi = 0$, $\phi(0) = 0$, $\phi(1) = 0$. In this case, the lowest eigenvalue is $\lambda_1 = \pi^2$. We can pick a nice function satisfying the boundary conditions, say $y(x) = x - x^2$. Inserting this into Equation (5.95), we find*

$$\lambda_1 \leq \frac{\int_0^1 (1-2x)^2 \, dx}{\int_0^1 (x-x^2)^2 \, dx} = 10.$$

Indeed, $10 \geq \pi^2$.

5.6.7 The Eigenfunction Expansion Method - optional

In this section we show how one can solve the nonhomogeneous problem $\mathcal{L}y = f$ using expansions over the basis of Sturm-Liouville eigenfunctions. In this chapter we have seen that Sturm-Liouville eigenvalue problems have the requisite set of orthogonal eigenfunctions. In this section we will apply the eigenfunction expansion method to solve a particular nonhomogenous boundary value problem.

Recall that one starts with a nonhomogeneous differential equation

$$\mathcal{L}y = f,$$

where $y(x)$ is to satisfy given homogeneous boundary conditions. The method makes use of the eigenfunctions satisfying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n$$

subject to the given boundary conditions. Then, one assumes that $y(x)$ can be written as an expansion in the eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

and inserts the expansion into the nonhomogeneous equation. This gives

$$f(x) = \mathcal{L}\left(\sum_{n=1}^{\infty} c_n \phi_n(x)\right) = -\sum_{n=1}^{\infty} c_n \lambda_n \sigma(x) \phi_n(x).$$

The expansion coefficients are then found by making use of the orthogonality of the eigenfunctions. Namely, we multiply the last equation by $\phi_m(x)$ and integrate. We obtain

$$\int_a^b f(x)\phi_m(x)\,dx = -\sum_{n=1}^{\infty} c_n\lambda_n \int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx.$$

Orthogonality yields

$$\int_a^b f(x)\phi_m(x)\,dx = -c_m\lambda_m \int_a^b \phi_m^2(x)\sigma(x)\,dx.$$

Solving for c_m, we have

$$c_m = -\frac{\int_a^b f(x)\phi_m(x)\,dx}{\lambda_m \int_a^b \phi_m^2(x)\sigma(x)\,dx}.$$

Example 5.16. *As an example, we consider the solution of the boundary value problem*

$$(xy')' + \frac{y}{x} = \frac{1}{x}, \quad x \in [1, e], \tag{5.96}$$

$$y(1) = 0 = y(e). \tag{5.97}$$

This equation is already in self-adjoint form. So, we know that the associated Sturm-Liouville eigenvalue problem has an orthogonal set of eigenfunctions. We first determine this set. Namely, we need to solve

$$(x\phi')' + \frac{\phi}{x} = -\lambda\sigma\phi, \quad \phi(1) = 0 = \phi(e). \tag{5.98}$$

Rearranging the terms and multiplying by x, we have that

$$x^2\phi'' + x\phi' + (1 + \lambda\sigma x)\phi = 0.$$

This is almost an equation of Cauchy-Euler type. Picking the weight function $\sigma(x) = \frac{1}{x}$, we have

$$x^2\phi'' + x\phi' + (1 + \lambda)\phi = 0.$$

This is easily solved. The characteristic equation is

$$r^2 + (1 + \lambda) = 0.$$

One obtains nontrivial solutions of the eigenvalue problem satisfying the boundary conditions when $\lambda > -1$. The solutions are

$$\phi_n(x) = A\sin(n\pi \ln x), \quad n = 1, 2, \ldots.$$

where $\lambda_n = n^2\pi^2 - 1$.

It is often useful to normalize the eigenfunctions. This means that one chooses A so that the norm of each eigenfunction is one. Thus, we have

$$\begin{aligned} 1 &= \int_1^e \phi_n(x)^2\sigma(x)\,dx \\ &= A^2\int_1^e \sin(n\pi\ln x)\frac{1}{x}\,dx \\ &= A^2\int_0^1 \sin(n\pi y)\,dy = \frac{1}{2}A^2. \end{aligned} \tag{5.99}$$

Thus, $A=\sqrt{2}$.

We now turn towards solving the nonhomogeneous problem, $\mathcal{L}y=\frac{1}{x}$. *We first expand the unknown solution in terms of the eigenfunctions,*

$$y(x)=\sum_{n=1}^{\infty} c_n\sqrt{2}\sin(n\pi\ln x).$$

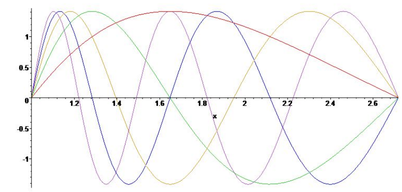


Figure 5.11: Plots of the first five eigenfunctions, $y(x)=\sqrt{2}\sin(n\pi\ln x)$.

Inserting this solution into the differential equation, we have

$$\frac{1}{x}=\mathcal{L}y=-\sum_{n=1}^{\infty} c_n\lambda_n\sqrt{2}\sin(n\pi\ln x)\frac{1}{x}.$$

Next, we make use of orthogonality. Multiplying both sides by $\phi_m(x)=\sqrt{2}\sin(m\pi\ln x)$ *and integrating, gives*

$$\lambda_m c_m=\int_1^e \sqrt{2}\sin(m\pi\ln x)\frac{1}{x}\,dx=\frac{\sqrt{2}}{m\pi}[(-1)^m-1].$$

Solving for c_m, *we have*

$$c_m=\frac{\sqrt{2}}{m\pi}\frac{[(-1)^m-1]}{m^2\pi^2-1}.$$

Finally, we insert our coefficients into the expansion for $y(x)$. *The solution is then*

$$y(x)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\frac{[(-1)^n-1]}{n^2\pi^2-1}\sin(n\pi\ln(x)).$$

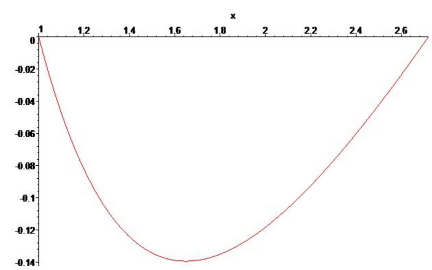


Figure 5.12: Plot of the solution in Example 5.16.

5.7 Appendix: The Least Squares Approximation

IN THE FIRST SECTION of this chapter we showed that we can expand functions over an infinite set of basis functions as

$$f(x)=\sum_{n=1}^{\infty} c_n\phi_n(x)$$

and that the generalized Fourier coefficients are given by

$$c_n=\frac{<\phi_n,f>}{<\phi_n,\phi_n>}.$$

In this section we turn to a discussion of approximating $f(x)$ by the partial sums $\sum_{n=1}^N c_n\phi_n(x)$ and showing that the Fourier coefficients are the best coefficients minimizing the deviation of the partial sum from $f(x)$. This will lead us to a discussion of the convergence of Fourier series.

More specifically, we set the following goal:

Goal
To find the best approximation of $f(x)$ on $[a,b]$ by $S_N(x) = \sum\limits_{n=1}^N c_n\phi_n(x)$ for a set of fixed functions $\phi_n(x)$; i.e., to find the c_n's such that $S_N(x)$ approximates $f(x)$ in the *least squares sense*.

We want to measure the deviation of the finite sum from the given function. Essentially, we want to look at the error made in the approximation. This is done by introducing the *mean square deviation:*

$$E_N = \int_a^b [f(x) - S_N(x)]^2 \rho(x)\,dx,$$

where we have introduced the weight function $\rho(x) > 0$. It gives us a sense as to how close the Nth partial sum is to $f(x)$.

We want to minimize this deviation by choosing the right c_n's. We begin by inserting the partial sums and expand the square in the integrand:

$$\begin{aligned} E_N &= \int_a^b [f(x) - S_N(x)]^2 \rho(x)\,dx \\ &= \int_a^b \left[f(x) - \sum_{n=1}^N c_n\phi_n(x)\right]^2 \rho(x)\,dx \\ &= \int_a^b f^2(x)\rho(x)\;dx - 2\int_a^b f(x)\sum_{n=1}^N c_n\phi_n(x)\rho(x)\;dx \\ &\quad + \int_a^b \sum_{n=1}^N c_n\phi_n(x)\sum_{m=1}^N c_m\phi_m(x)\rho(x)\;dx \end{aligned} \tag{5.100}$$

Looking at the three resulting integrals, we see that the first term is just the inner product of f with itself. The other integrations can be rewritten after interchanging the order of integration and summation. The double sum can be reduced to a single sum using the orthogonality of the ϕ_n's. Thus, we have

$$\begin{aligned} E_N &= <f,f> - 2\sum_{n=1}^N c_n <f,\phi_n> + \sum_{n=1}^N\sum_{m=1}^N c_nc_m <\phi_n,\phi_m> \\ &= <f,f> - 2\sum_{n=1}^N c_n <f,\phi_n> + \sum_{n=1}^N c_n^2 <\phi_n,\phi_n>. \end{aligned} \tag{5.101}$$

We are interested in finding the coefficients, so we will complete the square in c_n. Focusing on the last two terms, we have

$$\begin{aligned}
E_N - <f,f> &= -2\sum_{n=1}^{N} c_n <f,\phi_n> + \sum_{n=1}^{N} c_n^2 <\phi_n,\phi_n> \\
&= \sum_{n=1}^{N} <\phi_n,\phi_n> c_n^2 - 2<f,\phi_n> c_n \\
&= \sum_{n=1}^{N} <\phi_n,\phi_n> \left[c_n^2 - \frac{2<f,\phi_n>}{<\phi_n,\phi_n>} c_n\right] \\
&= \sum_{n=1}^{N} <\phi_n,\phi_n> \left[\left(c_n - \frac{<f,\phi_n>}{<\phi_n,\phi_n>}\right)^2 - \left(\frac{<f,\phi_n>}{<\phi_n,\phi_n>}\right)^2\right].
\end{aligned} \tag{5.102}$$

To this point we have shown that the mean square deviation is given as

$$E_N = <f,f> + \sum_{n=1}^{N} <\phi_n,\phi_n> \left[\left(c_n - \frac{<f,\phi_n>}{<\phi_n,\phi_n>}\right)^2 - \left(\frac{<f,\phi_n>}{<\phi_n,\phi_n>}\right)^2\right].$$

So, E_N is minimized by choosing $c_n = \frac{<f,\phi_n>}{<\phi_n,\phi_n>}$. However, these are the Fourier Coefficients. This minimization is often referred to as **Minimization in Least Squares Sense**.

Inserting the Fourier coefficients into the mean square deviation yields

$$0 \le E_N = <f,f> - \sum_{n=1}^{N} c_n^2 <\phi_n,\phi_n> .$$

Thus, we obtain *Bessel's Inequality*:

$$<f,f> \ge \sum_{n=1}^{N} c_n^2 <\phi_n,\phi_n> .$$

For convergence, we next let N get large and see if the partial sums converge to the function. In particular, we say that the infinite series *converges in the mean* if

$$\int_a^b [f(x) - S_N(x)]^2 \rho(x)\, dx \ \to 0 \text{ as } N \to \infty.$$

Letting N get large in Bessel's inequality shows that $\sum_{n=1}^{N} c_n^2 <\phi_n,\phi_n>$ converges if

$$(<f,f> = \int_a^b f^2(x)\rho(x)\, dx < \infty.$$

The space of all such f is denoted $L^2_\rho(a,b)$, the space of square integrable functions on (a,b) with weight $\rho(x)$.

From the nth term divergence theorem we know that $\sum a_n$ converges implies that $a_n \to 0$ as $n \to \infty$. Therefore, in this problem the terms $c_n^2 < \phi_n, \phi_n >$ approach zero as n gets large. This is only possible if the c_n's go to zero as n gets large. Thus, if $\sum_{n=1}^N c_n\phi_n$ *converges in the mean* to f, then $\int_a^b [f(x) - \sum_{n=1}^N c_n\phi_n]^2 \rho(x)\, dx$ approaches zero as $N \to \infty$. This implies from the above derivation of Bessel's inequality that

$$< f, f > - \sum_{n=1}^{N} c_n^2 (\phi_n, \phi_n) \to 0.$$

This leads to *Parseval's equality*:

$$< f, f >= \sum_{n=1}^{\infty} c_n^2 < \phi_n, \phi_n > .$$

Parseval's equality holds if and only if

$$\lim_{N\to\infty} \int_a^b (f(x) - \sum_{n=1}^{N} c_n\phi_n(x))^2 \rho(x)\; dx = 0.$$

If this is true for every square integrable function in $L^2_\rho(a, b)$, then the set of functions $\{\phi_n(x)\}_{n=1}^\infty$ is said to be **complete**. One can view these functions as an infinite dimensional basis for the space of square integrable functions on (a, b) with weight $\rho(x) > 0$.

One can extend the above limit $c_n \to 0$ as $n \to \infty$, by assuming that $\frac{\phi_n(x)}{\|\phi_n\|}$ is uniformly bounded and that $\int_a^b |f(x)|\rho(x)\; dx < \infty$. This is the **Riemann-Lebesque Lemma**, but will not be proven now.

5.8 *Appendix: The Fredholm Alternative Theorem*

GIVEN THAT $Ly = f$, when can one expect to find a solution? Is it unique? These questions are answered by the Fredholm Alternative Theorem. This theorem occurs in many forms from a statement about solutions to systems of algebraic equations to solutions of boundary value problems and integral equations. The theorem comes in two parts, thus the term "alternative". Either the equation has exactly one solution for all f, or the equation has many solutions for some f's and none for the rest.

The reader is familiar with the statements of the Fredholm Alternative for the solution of systems of algebraic equations. One seeks solutions of the system $Ax = b$ for A an $n \times m$ matrix. Defining the matrix adjoint, A^* through $< Ax, y >=< x, A^*y >$ for all $x, y, \in \mathcal{C}^n$, then either

Theorem 5.5. First Alternative
The equation $Ax = b$ has a solution if and only if $< b, v >= 0$ for all v such that $A^* v = 0$.

or

Theorem 5.6. Second Alternative
A solution of $Ax = b$, if it exists, is unique if and only if $x = 0$ is the only solution of $Ax = 0$.

The second alternative is more familiar when given in the form: The solution of a nonhomogeneous system of n equations and n unknowns is unique if the only solution to the homogeneous problem is the zero solution. Or, equivalently, A is invertible, or has nonzero determinant.

Proof. We prove the second theorem first. Assume that $Ax = 0$ for $x \neq 0$ and $Ax_0 = b$. Then $A(x_0 + \alpha x) = b$ for all α. Therefore, the solution is not unique. Conversely, if there are two different solutions, x_1 and x_2, satisfying $Ax_1 = b$ and $Ax_2 = b$, then one has a nonzero solution $x = x_1 - x_2$ such that $Ax = A(x_1 - x_2) = 0$.

The proof of the first part of the first theorem is simple. Let $A^* v = 0$ and $Ax_0 = b$. Then we have

$$< b, v >=< Ax_0, v >=< x_0, A^* v >= 0.$$

For the second part we assume that $< b, v >= 0$ for all v such that $A^* v = 0$. Write b as the sum of a part that is in the range of A and a part that in the space orthogonal to the range of A, $b = b_R + b_O$. Then, $0 =< b_O, Ax >=< A^* b, x >$ for all x. Thus, $A^* b_O$. Since $< b, v >= 0$ for all v in the nullspace of A^*, then $< b, b_O >= 0$.

Therefore, $< b, v >= 0$ implies that

$$0 =< b, b_O >=< b_R + b_O, b_O >=< b_O, b_O > .$$

This means that $b_O = 0$, giving $b = b_R$ is in the range of A. So, $Ax = b$ has a solution. □

Example 5.17. *Determine the allowed forms of* $\mathbf{b}$ *for a solution of* $A\mathbf{x} = \mathbf{b}$ *to exist, where*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

First note that $A^* = \overline{A}^T$. This is seen by looking at

$$\begin{aligned} < A\mathbf{x}, \mathbf{y} > &= < \mathbf{x}, A^* \mathbf{y} > \\ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \bar{y}_i &= \sum_{j=1}^{n} x_j \sum_{j=1}^{n} a_{ij} \bar{y}_i \\ &= \sum_{j=1}^{n} x_j \overline{\sum_{j=1}^{n} (\bar{a}^T)_{ji} \, y_i}. \end{aligned} \tag{5.103}$$

For this example,

$$A^* = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

We next solve $A^*\mathbf{v} = 0$. *This means,* $v_1 + 3v_2 = 0$. *So, the nullspace of* A^* *is spanned by* $\mathbf{v} = (3, -1)^T$. *For a solution of* $A\mathbf{x} = \mathbf{b}$ *to exist,* $\mathbf{b}$ *would have to be orthogonal to* $\mathbf{v}$. *Therefore, a solution exists when*

$$\mathbf{b} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

So, what does this say about solutions of boundary value problems? We need a more general theory for linear operators. A more general statement would be

Theorem 5.7. *If* L *is a bounded linear operator on a Hilbert space, then* $Ly = f$ *has a solution if and only if* $< f, v >= 0$ *for every* v *such that* $L^\dagger v = 0$.

The statement for boundary value problems is similar. However, we need to be careful to treat the boundary conditions in our statement. As we have seen, after several integrations by parts we have that

$$< \mathcal{L}u, v >= S(u, v) + < u, \mathcal{L}^\dagger v >,$$

where $S(u, v)$ involves the boundary conditions on u and v. Note that for nonhomogeneous boundary conditions, this term may no longer vanish.

Theorem 5.8. *The solution of the boundary value problem* $\mathcal{L}u = f$ *with boundary conditions* $Bu = g$ *exists if and only if*

$$< f, v > -S(u, v) = 0$$

for all v *satisfying* $\mathcal{L}^\dagger v = 0$ *and* $B^\dagger v = 0$.

Example 5.18. *Consider the problem*

$$u'' + u = f(x), \quad u(0) - u(2\pi) = \alpha, u'(0) - u'(2\pi) = \beta.$$

Only certain values of α *and* β *will lead to solutions. We first note that*

$$L = L^\dagger = \frac{d^2}{dx^2} + 1.$$

Solutions of

$$L^\dagger v = 0, \quad v(0) - v(2\pi) = 0, v'(0) - v'(2\pi) = 0$$

are easily found to be linear combinations of $v = \sin x$ *and* $v = \cos x$.

Next one computes

$$\begin{aligned} S(u,v) &= \left[u'v - uv'\right]_0^{2\pi} \\ &= u'(2\pi)v(2\pi) - u(2\pi)v'(2\pi) - u'(0)v(0) + u(0)v'(0). \end{aligned} \tag{5.104}$$

For $v(x) = \sin x$, this yields

$$S(u, \sin x) = -u(2\pi) + u(0) = \alpha.$$

Similarly,

$$S(u, \cos x) = \beta.$$

Using $< f, v > -S(u,v) = 0$, this leads to the conditions that we were seeking,

$$\int_0^{2\pi} f(x) \sin x \, dx = \alpha,$$

$$\int_0^{2\pi} f(x) \cos x \, dx = \beta.$$

Problems

1. Consider the set of vectors $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.

a. Use the Gram-Schmidt process to find an orthonormal basis for R^3 using this set in the given order.

b. What do you get if you do reverse the order of these vectors?

2. Use the Gram-Schmidt process to find the first four orthogonal polynomials satisfying the following:

a. Interval: $(-\infty, \infty)$ Weight Function: e^{-x^2}.

b. Interval: $(0, \infty)$ Weight Function: e^{-x}.

3. Find $P_4(x)$ using

a. The Rodrigues Formula in Equation (5.21).

b. The three term recursion formula in Equation (5.23).

4. In Equations (5.36)-(5.43) we provide several identities for Legendre polynomials. Derive the results in Equations (5.37)-(5.43) as decribed in the text. Namely,

a. Differentiating Equation (5.36) with respect to x, derive Equation (5.37).

b. Derive Equation (5.38) by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series.

c. Combining the last result with Equation (5.36), derive Equations (5.39)-(5.40).

d. Adding and subtracting Equations (5.39)-(5.40), obtain Equations (5.41)-(5.42).

e. Derive Equation (5.43) using some of the other identities.

5. Use the recursion relation (5.23) to evaluate $\int_{-1}^{1} xP_n(x)P_m(x)\,dx$, $n \leq m$.

6. Expand the following in a Fourier-Legendre series for $x \in (-1, 1)$.

a. $f(x) = x^2$.

b. $f(x) = 5x^4 + 2x^3 - x + 3$.

c. $f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$

d. $f(x) = \begin{cases} x, & -1 < x < 0, \\ 0, & 0 < x < 1. \end{cases}$

7. Use integration by parts to show $\Gamma(x+1) = x\Gamma(x)$.

8. Prove the double factorial identities:

$$(2n)!! = 2^n n!$$

and

$$(2n-1)!! = \frac{(2n)!}{2^n n!}.$$

9. Express the following as Gamma functions. Namely, noting the form $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$ and using an appropriate substitution, each expression can be written in terms of a Gamma function.

a. $\int_0^\infty x^{2/3}e^{-x}\,dx$.

b. $\int_0^\infty x^5 e^{-x^2}\,dx$

c. $\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^n\,dx$

10. The coefficients C_k^p in the binomial expansion for $(1+x)^p$ are given by

$$C_k^p = \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

a. Write C_k^p in terms of Gamma functions.

b. For $p = 1/2$ use the properties of Gamma functions to write $C_k^{1/2}$ in terms of factorials.

c. Confirm you answer in part b by deriving the Maclaurin series expansion of $(1+x)^{1/2}$.

11. The Hermite polynomials, $H_n(x)$, satisfy the following:

i. $< H_n, H_m >= \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi} 2^n n! \delta_{n,m}$.

ii. $H_n'(x) = 2nH_{n-1}(x)$.

iii. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

iv. $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$.

Using these, show that

a. $H_n'' - 2xH_n' + 2nH_n = 0$. [Use properties ii. and iii.]

b. $\int_{-\infty}^{\infty} xe^{-x^2} H_n(x) H_m(x)\, dx = \sqrt{\pi} 2^{n-1} n! \left[\delta_{m,n-1} + 2(n+1)\delta_{m,n+1}\right]$. [Use properties i. and iii.]

c. $H_n(0) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^m \frac{(2m)!}{m!}, & n = 2m. \end{cases}$ [Let $x = 0$ in iii. and iterate. Note from iv. that $H_0(x) = 1$ and $H_1(x) = 2x$.]

12. In Maple one can type **simplify(LegendreP(2*n-2,0)-LegendreP(2*n,0));** to find a value for $P_{2n-2}(0) - P_{2n}(0)$. It gives the result in terms of Gamma functions. However, in Example 5.8 for Fourier-Legendre series, the value is given in terms of double factorials! So, we have

$$P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)} = (-1)^n \frac{(2n-3)!!}{(2n-2)!!} \frac{4n-1}{2n}.$$

You will verify that both results are the same by doing the following:

a. Prove that $P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$ using the generating function and a binomial expansion.

b. Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$ using $\Gamma(x) = (x-1)\Gamma(x-1)$ and iteration.

c. Verify the result from Maple that $P_{2n-2}(0) - P_{2n}(0) = \frac{\sqrt{\pi}(4n-1)}{2\Gamma(n+1)\Gamma\left(\frac{3}{2}-n\right)}$.

d. Can either expression for $P_{2n-2}(0) - P_{2n}(0)$ be simplified further?

13. A solution Bessel's equation, $x^2y'' + xy' + (x^2 - n^2)y = 0$, , can be found using the guess $y(x) = \sum_{j=0}^{\infty} a_j x^{j+n}$. One obtains the recurrence relation $a_j = \frac{-1}{j(2n+j)} a_{j-2}$. Show that for $a_0 = (n!2^n)^{-1}$ we get the Bessel function of the first kind of order n from the even values $j = 2k$:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}.$$

14. Use the infinite series in the last problem to derive the derivative identities (5.71) and (5.61):

a. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

b. $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$.

15. Prove the following identities based on those in the last problem.

a. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x)$.

b. $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$.

16. Use the derivative identities of Bessel functions,(5.71)-(5.61), and integration by parts to show that

$$\int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) + C.$$

17. Use the generating function to find $J_n(0)$ and $J_n'(0)$.

18. Bessel functions $J_p(\lambda x)$ are solutions of $x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0$. Assume that $x \in (0,1)$ and that $J_p(\lambda) = 0$ and $J_p(0)$ is finite.

a. Show that this equation can be written in the form

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (\lambda^2 x - \frac{p^2}{x})y = 0.$$

This is the standard Sturm-Liouville form for Bessel's equation.

b. Prove that

$$\int_0^1 xJ_p(\lambda x)J_p(\mu x)\,dx = 0, \quad \lambda \neq \mu$$

by considering

$$\int_0^1 \left[J_p(\mu x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\lambda x)\right) - J_p(\lambda x)\frac{d}{dx}\left(x\frac{d}{dx}J_p(\mu x)\right)\right]dx.$$

Thus, the solutions corresponding to different eigenvalues (λ, μ) are orthogonal.

c. Prove that

$$\int_0^1 x\left[J_p(\lambda x)\right]^2 dx = \frac{1}{2}J_{p+1}^2(\lambda) = \frac{1}{2}J_p'^2(\lambda).$$

19. We can rewrite Bessel functions, $J_\nu(x)$, in a form which will allow the order to be non-integer by using the gamma function. You will need the results from Problem 12b for $\Gamma\left(k+\frac{1}{2}\right)$.

a. Extend the series definition of the Bessel function of the first kind of order ν, $J_\nu(x)$, for $\nu \geq 0$ by writing the series solution for $y(x)$ in Problem 13 using the gamma function.

b. Extend the series to $J_{-\nu}(x)$, for $\nu \geq 0$. Discuss the resulting series and what happens when ν is a positive integer.

c. Use these results to obtain the closed form expressions

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

d. Use the results in part c with the recursion formula for Bessel functions to obtain a closed form for $J_{3/2}(x)$.

20. In this problem you will derive the expansion

$$x^2 = \frac{c^2}{2} + 4\sum_{j=2}^{\infty} \frac{J_0(\alpha_j x)}{\alpha_j^2 J_0(\alpha_j c)}, \quad 0 < x < c,$$

where the $\alpha_j's$ are the positive roots of $J_1(\alpha c) = 0$, by following the below steps.

a. List the first five values of α for $J_1(\alpha c) = 0$ using the Table 5.3 and Figure 5.8. [Note: Be careful determining α_1.]

b. Show that $\|J_0(\alpha_1 x)\|^2 = \frac{c^2}{2}$. Recall,

$$\|J_0(\alpha_j x)\|^2 = \int_0^c x J_0^2(\alpha_j x)\, dx.$$

c. Show that $\|J_0(\alpha_j x)\|^2 = \frac{c^2}{2}\left[J_0(\alpha_j c)\right]^2$, $j = 2, 3, \ldots$. (This is the most involved step.) First note from Problem 18 that $y(x) = J_0(\alpha_j x)$ is a solution of

$$x^2 y'' + xy' + \alpha_j^2 x^2 y = 0.$$

i. Verify the Sturm-Liouville form of this differential equation: $(xy')' = -\alpha_j^2 xy$.

ii. Multiply the equation in part i. by $y(x)$ and integrate from $x = 0$ to $x = c$ to obtain

$$\begin{aligned} \int_0^c (xy')' y\, dx &= -\alpha_j^2 \int_0^c xy^2\, dx \\ &= -\alpha_j^2 \int_0^c x J_0^2(\alpha_j x)\, dx. \end{aligned} \tag{5.105}$$

iii. Noting that $y(x) = J_0(\alpha_j x)$, integrate the left hand side by parts and use the following to simplify the resulting equation.

1. $J_0'(x) = -J_1(x)$ from Equation (5.61).
2. Equation (5.64).
3. $J_2(\alpha_j c) + J_0(\alpha_j c) = 0$ from Equation (5.62).

iv. Now you should have enough information to complete this part.

d. Use the results from parts b and c to derive the expansion coefficients for
$$x^2 = \sum_{j=1}^{\infty} c_j J_0(\alpha_j x)$$
in order to obtain the desired expansion.

21. Prove the if $u(x)$ and $v(x)$ satisfy the general homogeneous boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \beta_1 u'(a) &= 0, \\ \alpha_2 u(b) + \beta_2 u'(b) &= 0 \end{aligned} \tag{5.106}$$

at $x = a$ and $x = b$, then

$$p(x)[u(x)v'(x) - v(x)u'(x)]_{x=a}^{x=b} = 0.$$

22. Prove Green's Identity $\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx = [p(uv' - vu')]|_a^b$ for the general Sturm-Liouville operator $\mathcal{L}$.

23. Find the adjoint operator and its domain for $Lu = u'' + 4u' - 3u$, $u'(0) + 4u(0) = 0$, $u'(1) + 4u(1) = 0$.

24. Show that a Sturm-Liouville operator with periodic boundary conditions on $[a, b]$ is self-adjoint if and only if $p(a) = p(b)$. [Recall, periodic boundary conditions are given as $u(a) = u(b)$ and $u'(a) = u'(b)$.]

25. The Hermite differential equation is given by $y'' - 2xy' + \lambda y = 0$. Rewrite this equation in self-adjoint form. From the Sturm-Liouville form obtained, verify that the differential operator is self adjoint on $(-\infty, \infty)$. Give the integral form for the orthogonality of the eigenfunctions.

26. Find the eigenvalues and eigenfunctions of the given Sturm-Liouville problems.

a. $y'' + \lambda y = 0$, $y'(0) = 0 = y'(\pi)$.
b. $(xy')' + \frac{\lambda}{x} y = 0$, $y(1) = y(e^2) = 0$.

27. The eigenvalue problem $x^2 y'' - \lambda x y' + \lambda y = 0$ with $y(1) = y(2) = 0$ is not a Sturm-Liouville eigenvalue problem. Show that none of the eigenvalues are real by solving this eigenvalue problem.

28. In Example 5.15 we found a bound on the lowest eigenvalue for the given eigenvalue problem.

a. Verify the computation in the example.
b. Apply the method using
$$y(x) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x < 1. \end{cases}$$
Is this an upper bound on λ_1

c. Use the Rayleigh quotient to obtain a good upper bound for the lowest eigenvalue of the eigenvalue problem: $\phi'' + (\lambda - x^2)\phi = 0$, $\phi(0) = 0$, $\phi'(1) = 0$.

29. Use the method of eigenfunction expansions to solve the problem:

$$y'' + 4y = x^2, \quad y(0) = y(1) = 0.$$

30. Determine the solvability conditions for the nonhomogeneous boundary value problem: $u'' + 4u = f(x)$, $u(0) = \alpha$, $u'(1) = \beta$.

6
Complex Representations of Functions

"He is not a true man of science who does not bring some sympathy to his studies, and expect to learn something by behavior as well as by application. It is childish to rest in the discovery of mere coincidences, or of partial and extraneous laws. The study of geometry is a petty and idle exercise of the mind, if it is applied to no larger system than the starry one. Mathematics should be mixed not only with physics but with ethics; that is mixed mathematics. The fact which interests us most is the life of the naturalist. The purest science is still biographical." Henry David Thoreau (1817-1862)

6.1 Complex Representations of Waves

We have seen that we can seek the frequency content of a function $f(t)$ defined on an interval $[0,T]$ by looking for the Fourier coefficients in the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}.$$

The coefficients take forms like

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi nt}{T}\,dt.$$

However, trigonometric functions can be written in a complex exponential form. This is based on *Euler's formula* (or, Euler's identity):[1]

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The complex conjugate is found by replacing i with $-i$ to obtain

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding these expressions, we have

$$2\cos\theta = e^{i\theta} + e^{-i\theta}.$$

Subtracting the exponentials leads to an expression for the sine function. Thus, we have the important result that sines and cosines can be written as complex exponentials:

[1] Euler's formula can be obtained using the Maclaurin expansion of e^x:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{x^n}{n!} + \cdots.$$

We formally set $x = i\theta$ Then,

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} \\
&= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\
&= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \frac{(\theta)^4}{4!} + \cdots \\
&= \left(1 - \frac{(\theta)^2}{2!} + \frac{(\theta)^4}{4!} + \cdots\right) \\
&\quad + i\left(i\theta - \frac{(\theta)^3}{3!} + \frac{(\theta)^5}{5!} + \cdots\right) \\
&= \cos\theta + i\sin\theta. \qquad (6.1)
\end{aligned}$$

$$\boxed{\begin{aligned} \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}, \\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i}. \end{aligned}} \tag{6.2}$$

So, we can write

$$\cos\frac{2\pi nt}{T} = \frac{1}{2}(e^{\frac{2\pi int}{T}} + e^{-\frac{2\pi int}{T}}).$$

Later we will see that we can use this information to rewrite our series as a sum over complex exponentials in the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi int}{T}}$$

where the Fourier coefficients now take the form

$$c_n = \int_0^T f(t) e^{-\frac{2\pi int}{T}}.$$

In fact, in order to connect our analysis to ideal signals over an infinite interval and containing a continuum of frequencies, we will see the above sum become an integral and we will naturally find ourselves needing to work with functions of complex variables and perform complex integrals.

We can extend these ideas to develop a complex representation for waves. Recall from our discussion in Section 4.11 on finite length strings the solution

$$u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} A_n \sin k_n(x+ct) + \sum_{n=1}^{\infty} A_n \sin k_n(x-ct)\right]. \tag{6.3}$$

We can replace the sines with their complex forms as

$$\begin{aligned} u(x,t) &= \frac{1}{4i}\left[\sum_{n=1}^{\infty} A_n \left(e^{ik_n(x+ct)} - e^{-ik_n(x+ct)}\right)\right. \\ &\quad \left. + \sum_{n=1}^{\infty} A_n \left(e^{ik_n(x-ct)} - e^{-ik_n(x-ct)}\right)\right]. \end{aligned} \tag{6.4}$$

Now, defining $k_{-n} = -k_n$, we can rewrite this solution in the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} \left[c_n e^{ik_n(x+ct)} + d_n e^{ik_n(x-ct)}\right]. \tag{6.5}$$

Such representations are also possible for waves propagating over the entire real line. In such cases we are not restricted to discrete frequencies and wave numbers. The sum of the harmonics will then be a sum over a continuous range, which means that our sums become integrals. So, we are then lead to the complex representation

$$u(x,t) = \int_{-\infty}^{\infty} \left[c(k)e^{ik(x+ct)} + d(k)e^{ik(x-ct)} \right] dk. \tag{6.6}$$

The forms $e^{ik(x+ct)}$ and $e^{ik(x-ct)}$ are complex representations of what are called plane waves in one dimension. The integral represents a general wave form consisting of a sum over plane waves, typically representing wave packets. The Fourier coefficients in the representation can be complex valued functions and the evaluation of the integral may be done using methods from complex analysis. We would like to be able to compute such integrals.

With the above ideas in mind, we will now take a tour of complex analysis. We will first review some facts about complex numbers and then introduce complex functions. This will lead us to the calculus of functions of a complex variable, including differentiation and complex integration.

6.2 *Complex Numbers*

COMPLEX NUMBERS WERE FIRST INTRODUCED in order to solve some simple problems. The history of complex numbers only extends about five hundred years. In essence, it was found that we need to find the roots of equations such as $x^2 + 1 = 0$. The solution is $x = \pm\sqrt{-1}$. Due to the usefulness of this concept, which was not realized at first, a special symbol was introduced - the imaginary unit, $i = \sqrt{-1}$. In particular, Girolamo Cardano $(1501 - 1576)$ was one of the first to use square roots of negative numbers when providing solutions of cubic equations. However, complex numbers did not become an important part of mathematics or science until the late seventh and eighteenth centuries after people like de Moivre, the Bernoulli[2] family and Euler took them seriously.

[2] The Bernoulli's were a family of Swiss mathematicians spanning three generations. It all started with Jakob Bernoulli (1654-1705) and his brother Johann Bernoulli (1667-1748). Jakob had a son, Nikolaus Bernoulli (1687-1759) and Johann (1667-1748) had three sons, Nikolaus Bernoulli II (1695-1726), Daniel Bernoulli (1700-1872), and Johann Bernoulli II (1710-1790). The last generation consisted of Johann II's sons, Johann Bernoulli III (1747-1807) and Jakob Bernoulli II (1759-1789). Johann, Jakob and Daniel Bernoulli were the most famous of the Bernoulli's. Jakob studied with Leibniz, Johann studied under his older brother and later taught Leonhard Euler and Daniel Bernoulli, who is known for his work in hydrodynamics.

A complex number is a number of the form $z = x + iy$, where x and y are real numbers. x is called the real part of z and y is the imaginary part of z. Examples of such numbers are $3 + 3i$, $-1i = -i$, $4i$ and 5. Note that $5 = 5 + 0i$ and $4i = 0 + 4i$.

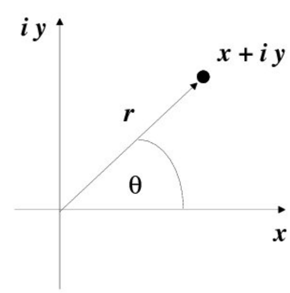


Figure 6.1: The Argand diagram for plotting complex numbers in the complex z-plane.

There is a geometric representation of complex numbers in a two dimensional plane, known as the complex plane C. This is given by the Argand diagram as shown in Figure 6.1. Here we can think of the complex number $z = x + iy$ as a point (x, y) in the z-complex plane or as a vector. The magnitude, or length, of this vector is called the *complex modulus* of z, denoted by $|z| = \sqrt{x^2 + y^2}$.

We can also use the geometric picture to develop a polar representation of complex numbers. From Figure 6.1 we can see that in terms of r and θ we have that

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned} \qquad (6.7)$$

Thus,

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \qquad (6.8)$$

Complex numbers can be represented in rectangular (Cartesian), $z = x + iy$, or polar form, $z = re^{i\theta}$. Her we define the argument, θ, and modulus, $|z| = r$ of complex numbers.

So, given r and θ we have $z = re^{i\theta}$. However, given the Cartesian form, $z = x + iy$, we can also determine the polar form, since

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \tan\theta &= \frac{y}{x}. \end{aligned} \qquad (6.9)$$

Note that $r = |z|$.

Locating $1 + i$ in the complex plane, it is possible to immediately determine the polar form from the angle and length of the "complex vector". This is shown in Figure 6.2. It is obvious that $\theta = \frac{\pi}{4}$ and $r = \sqrt{2}$.

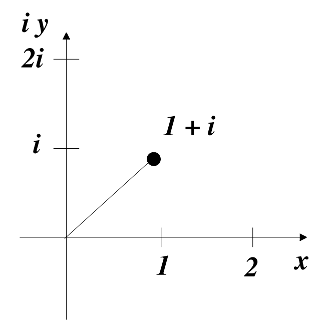


Figure 6.2: Locating $1 + i$ in the complex z-plane.

Example 6.1. *Write $1 + i$ in polar form.If one did not see the polar form from the plot in the z-plane, then one could systematically determine the results. First, write +$1 + i$ in polar form: $1 + i = re^{i\theta}$ for some r and θ. Using the above relations, we have $r = \sqrt{x^2 + y^2} = \sqrt{2}$ and $\tan\theta = \frac{y}{x} = 1$. This gives $\theta = \frac{\pi}{4}$. So, we have found that*

$$1 + i = \sqrt{2}e^{i\pi/4}.$$

We can also use define binary operations of addition, subtraction, multiplication and division of complex numbers to produce a new complex number. The addition of two complex numbers is simply done by adding the real parts and the imaginary parts of each number. So,

We can easily add, subtract, multiply and divide complex numbers.

$$(3 + 2i) + (1 - i) = 4 + i.$$

Subtraction is just as easy,

$$(3 + 2i) - (1 - i) = 2 + 3i.$$

We can multiply two complex numbers just like we multiply any binomials, though we now can use the fact that $i^2 = -1$. For example, we have

$$(3 + 2i)(1 - i) = 3 + 2i - 3i + 2i(-i) = 5 - i.$$

We can even divide one complex number into another one and get a complex number as the quotient. Before we do this, we need to introduce the *complex conjugate*, $\bar{z}$, of a complex number. The complex conjugate of $z = x + iy$, where x and y are real numbers, is given as

$$\overline{z} = x - iy.$$

The complex conjugate of $z = x + iy$, is given as
$$\overline{z} = x - iy.$$

Complex conjugates satisfy the following relations for complex numbers z and w and real number x.

$$\begin{aligned} \overline{z+w} &= \overline{z} + \overline{w} \\ \overline{zw} &= \overline{z}\overline{w} \\ \overline{\overline{z}} &= z \\ \overline{x} &= x. \end{aligned} \tag{6.10}$$

One consequence is that the complex conjugate of $re^{i\theta}$ is

$$\overline{re^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = re^{-i\theta}.$$

Another consequence is that

$$z\overline{z} = re^{i\theta}re^{-i\theta} = r^2.$$

Thus, the product of a complex number with its complex conjugate is a real number. We can also prove this result using the Cartesian form

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

Now we are in a position to write the quotient of two complex numbers in the standard form of a real plus an imaginary number. As an example, we want to divide $3 + 2i$ by $1 - i$. This is accomplished by multiplying the numerator and denominator of this expression by the complex conjugate of the denominator:

$$\frac{3+2i}{1-i} = \frac{3+2i}{1-i}\frac{1+i}{1+i} = \frac{1+5i}{2}.$$

Therefore, the quotient is a complex number, $\frac{1}{2} + \frac{5}{2}i$.

We can also consider powers of complex numbers. For example,

$$(1 + i)^2 = 2i,$$

$$(1 + i)^3 = (1 + i)(2i) = 2i - 2.$$

But, what is $(1 + i)^{1/2} = \sqrt{1+i}$?

In general, we want to find the nth root of a complex number. Let $t = z^{1/n}$. To find t in this case is the same as asking for the solution of

$$z = t^n$$

given z. But, this is the root of an nth degree equation, for which we expect n roots. If we write z in polar form, $z = re^{i\theta}$, then we would naively compute

$$\begin{aligned} z^{1/n} &= \left(re^{i\theta}\right)^{1/n} \\ &= r^{1/n}e^{i\theta/n} \\ &= r^{1/n}\left[\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right]. \end{aligned} \tag{6.11}$$

For example,

$$(1+i)^{1/2} = \left(\sqrt{2}e^{i\pi/4}\right)^{1/2} = 2^{1/4}e^{i\pi/8}.$$

But this is only one solution. We expected two solutions for $n = 2$..

The function $f(z) = z^{1/n}$ is multivalued. $z^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}$, $k = 0, 1, \ldots, n-1$.

The reason we only found one solution is that the polar representation for z is not unique. We note that

$$\boxed{e^{2k\pi i} = 1, \quad k = 0, \pm 1, \pm 2, \ldots.}$$

So, we can rewrite z as $z = re^{i\theta}e^{2k\pi i} = re^{i(\theta+2k\pi)}$. Now, we have that

$$\boxed{z^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}.}$$

Note that there are only different values for $k = 0, 1, \ldots, n-1$. Consider $k = n$. Then one finds that

$$e^{i(\theta+2\pi in)/n} = e^{i\theta/n}e^{2\pi i} = e^{i\theta/n}.$$

So, we have recovered the $n = 0$ value. Similar results can be shown for other k values larger than n.

Now, we can finish the example.

$$\begin{aligned} (1+i)^{1/2} &= \left(\sqrt{2}e^{i\pi/4}e^{2k\pi i}\right)^{1/2}, \quad k = 0, 1, \\ &= 2^{1/4}e^{i(\pi/8+k\pi)}, \quad k = 0, 1, \\ &= 2^{1/4}e^{i\pi/8}, 2^{1/4}e^{9\pi i/8}. \end{aligned} \tag{6.12}$$

Finally, what is $\sqrt[n]{1}$? Our first guess would be $\sqrt[n]{1} = 1$. But, we now know that there should be n roots. These roots are called the *nth roots of unity*. Using the above result with $r = 1$ and $\theta = 0$, we have that

The nth roots of unity, $\sqrt[n]{1}$.

$$\sqrt[n]{1} = \left[\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right], \quad k = 0, \ldots, n-1.$$

For example, we have

$$\sqrt[3]{1} = \left[\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right], \quad k = 0, 1, 2.$$

These three roots can be written out as

$$\sqrt[3]{1} = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

We can locate these cube roots of unity in the complex plane. In Figure 6.3 we see that these points lie on the unit circle and are at the vertices of an equilateral triangle. In fact, all nth roots of unity lie on the unit circle and are the vertices of a regular n-gon with one vertex at $z = 1$.

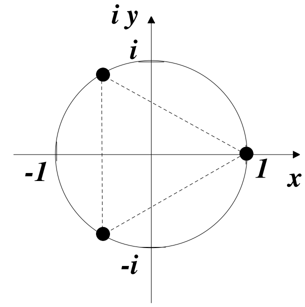


Figure 6.3: Locating the cube roots of unity in the complex z-plane.

6.3 Complex Valued Functions

We would like to next explore complex functions and the calculus of complex functions. We begin by defining a function that takes complex numbers into complex numbers, $f : C \to C$. It is difficult to visualize such functions. For real functions of one variable, $f : R \to R$. We graph these functions by first drawing two intersecting copies of R and then proceeding to map the domain into the range of f.

It would be more difficult to do this for complex functions. Imagine placing together two orthogonal copies of C. One would need a four dimensional space in order to complete the visualization. Instead, typically uses two copies of the complex plane side by side in order to indicate how such functions behave. We will assume that the domain lies in the z-plane and the image lies in the w-plane and write $w = f(z)$. We show these planes in Figure 6.4.

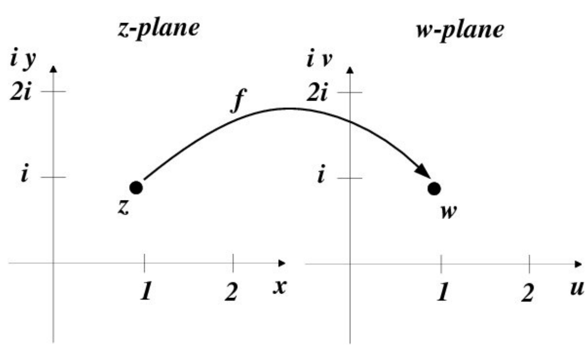


Figure 6.4: Defining a complex valued function, $w = f(z)$, on C for $z = x + iy$ and $w = u + iv$.

Letting $z = x + iy$ and $w = u + iv$, we can write the real and imaginary parts of $f(z)$:

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

We see that one can view this function as a function of z or a function of x and y. Often, we have an interest in writing out the real and imaginary parts of the function, which are functions of two variables.

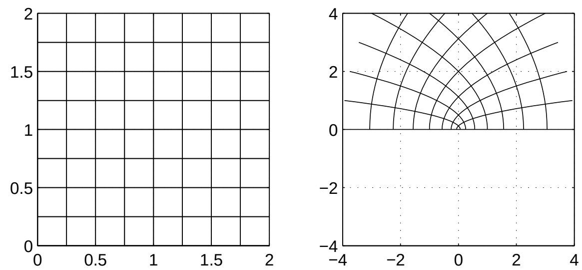


Figure 6.5: 2D plot showing how the function $f(z) = z^2$ maps a grid in the z-plane into the w-plane.

Example 6.2. $f(z) = z^2$.

For example, we can look at the simple function $f(z) = z^2$. It is a simple matter to determine the real and imaginary parts of this function. Namely, we have

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Therefore, we have that

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

In Figure 6.5 we show how a grid in the z-plane is mapped by $f(z) = z^2$ into the w-plane. For example, the horizontal line $x = 1$ is mapped to $u(1, y) = 1 - y^2$ and $v(1, y) = 2y$. Eliminating the "parameter" y between these two equations, we have $u = 1 - v^2/4$. This is a parabolic curve. Similarly, the horizontal line $y = 1$ results in the curve $u = v^2/4 - 1$.

If we look at several curves, $x =$const and $y =$const, then we get a family of intersecting parabolae, as shown in Figure 6.5.

Example 6.3. $f(z) = e^z$.

For this case, we make use of Euler's Formula.

$$\begin{aligned} f(z) &= e^z \\ &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y). \end{aligned} \tag{6.13}$$

Thus, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. In Figure 6.6 we show how a grid in the z-plane is mapped by $f(z) = e^z$ into the w-plane.

Example 6.4. $f(z) = \ln z$.

In this case we make use of the polar form, $z = re^{i\theta}$. Our first thought would be to simply compute

$$\ln z = \ln r + i\theta.$$

However, the natural logarithm is multivalued, just like the nth root. Recalling that $e^{2\pi ik} = 1$ for k an integer, we have $z = re^{i(\theta+2\pi k)}$. Therefore,

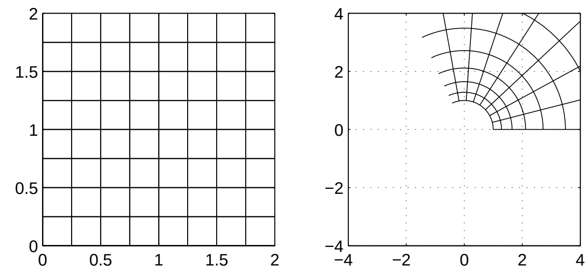


Figure 6.6: 2D plot showing how the function $f(z) = e^z$ maps a grid in the z-plane into the w-plane.

$$\boxed{\ln z = \ln r + i(\theta + 2\pi k), \qquad k = \text{ integer.}}$$

The natural logarithm is a multivalued function. In fact there are an infinite number of values for a given z. Of course, this contradicts the definition of a function that you were first taught.

Thus, one typically will only report the principal value, $\text{Log } z = \ln r + i\theta$*, for* θ *restricted to some interval of length* 2π*, such as* $[0, 2\pi)$*. In order to account for the multivaluedness, one introduces a way to extend the complex plane so as to include all of the branches. This is done by assigning a plane to each branch, using (branch) cuts along lines, and then gluing the planes together at the branch cuts to form what is called a Riemann surface. We will not elaborate upon this any further here and refer the interested reader to more advanced texts. Comparing the multivalued logarithm to the principal value logarithm, we have*

$$\ln z = \text{Log } z + 2n\pi i.$$

We should not that some books use $\log z$ *instead of* $\ln z$*. It should not be confused with the common logarithm.*

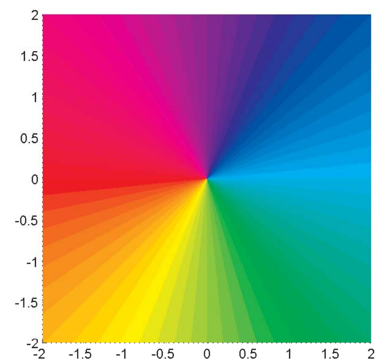


Figure 6.7: Domain coloring of the complex z-plane assigning colors to $\arg(z)$.

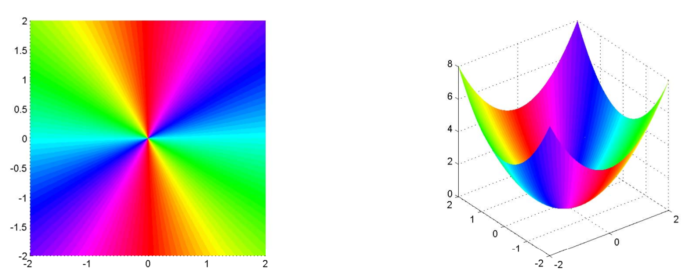


Figure 6.8: Domain coloring for $f(z) = z^2$. The left figure shows the phase coloring. The right figure show the colored surface with height $|f(z)|$.

Another method for visualizing complex functions is *domain coloring*. The idea was described by Frank Ferris. There are a few approaches to this method. The main idea is that one colors each point of the z-plane (the domain) as shown in Figure 6.7. The modulus, $|f(z)|$ is then plotted as a surface. Examples are shown for $f(z) = z^2$ in Figure 6.8 and $f(z) = 1/z(1-z)$ in Figure 6.9.

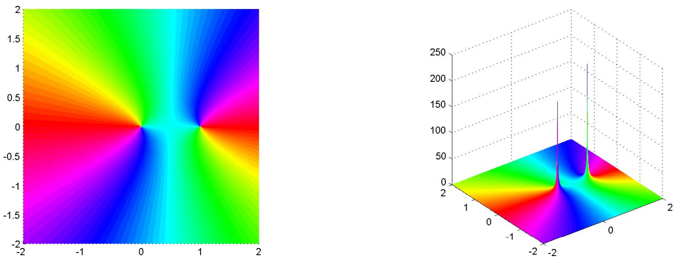


Figure 6.9: Domain coloring for $f(z) = 1/z(1-z)$. The left figure shows the phase coloring. The right figure show the colored surface with height $|f(z)|$.

We would like to put all of this information in one plot. We can do this by adjusting the brightness of the colored domain by using the modulus of the function. In the plots that follow we use the fractional part of $\ln|z|$. In Figure 6.10 we show the effect for the z-plane using $f(z) = z$. In the figures that follow we look at several other functions. In these plots we have chosen to view the functions in a circular window.

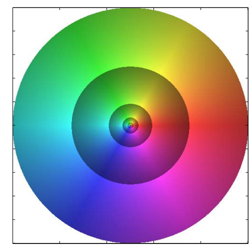


Figure 6.10: Domain coloring for $f(z) = z$ showing a coloring for $\arg(z)$ and brightness based on $|f(z)|$.

One can see the rich behavior hidden in these figures. As you progress in your reading, especially after the next chapter, you should return to these figures and locate the zeros, poles, branch points and branch cuts. A search online will lead you to other colorings and superposition of the uv grid on these figures.

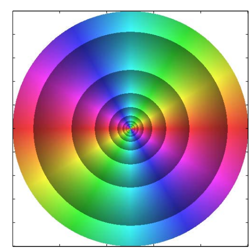


Figure 6.11: Domain coloring for $f(z) = z^2$.

As a final picture, we look at iteration in the complex plane. Consider the function $f(z) = z^2 - 0.75 - 0.2i$. Interesting figures result when studying the iteration in the complex plane. In Figure 6.13 we show $f(z)$ and $f^{20}(z)$, which is the iteration of f twenty times. It leads to an interesting coloring. What happens when one keeps iterating? Such iterations lead to the study of Julia and Mandelbrot sets . In Figure 6.14 we show six iterations of $f(z) = (1 - i/2)\sin x$.

The following code was used in MATLAB to produce these figures.

```
fn = @(x) (1-i/2)*sin(x);
xmin=-2; xmax=2; ymin=-2; ymax=2;
Nx=500;
Ny=500;
x=linspace(xmin,xmax,Nx);
y=linspace(ymin,ymax,Ny);
[X,Y] = meshgrid(x,y); z = complex(X,Y);
tmp=z; for n=1:6
    tmp = fn(tmp);
end Z=tmp;
XX=real(Z);
YY=imag(Z);
R2=max(max(X.^2));
R=max(max(XX.^2+YY.^2));
```

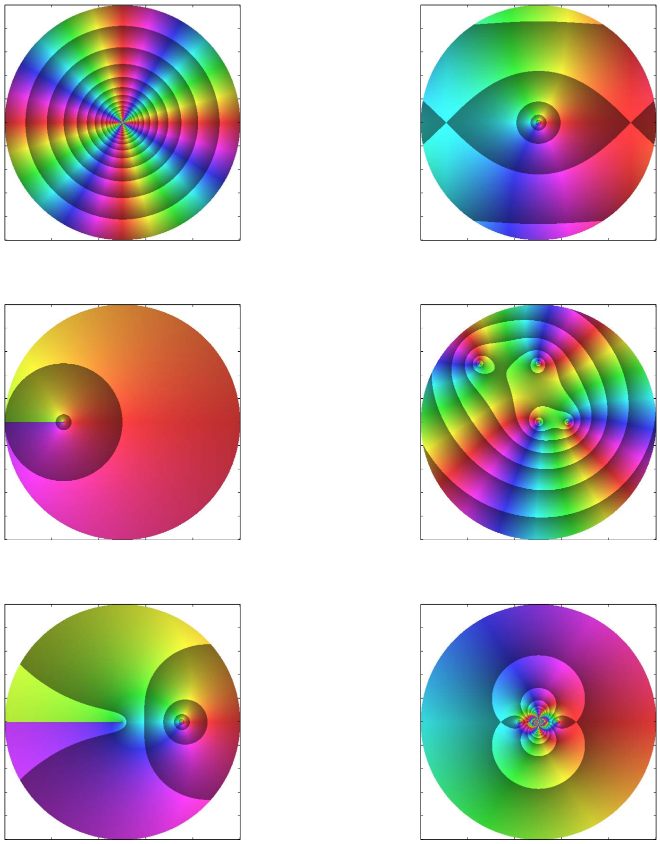


Figure 6.12: Domain coloring for several functions. On the top row the domain coloring is shown for $f(z) = z^4$ and $f(z) = \sin z$. On the second row plots for $f(z) = \sqrt{1+z}$ and $f(z) = \frac{1}{z(1/2-z)(z-i)(z-i+1)}$ are shown. In the last row domain colorings for $f(z) = \ln z$ and $f(z) = \sin(1/z)$ are shown.

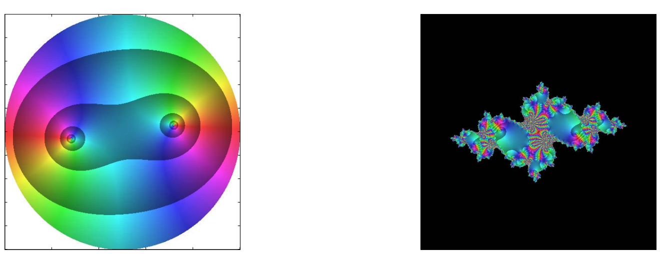


Figure 6.13: Domain coloring for $f(z) = z^2 - 0.75 - 0.2i$. The left figure shows the phase coloring. On the right is the plot for $f^{20}(z)$.

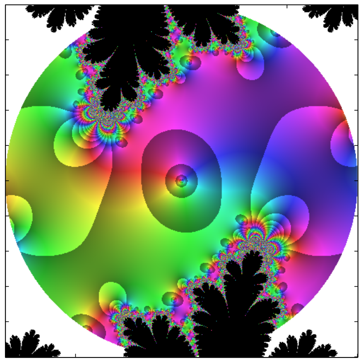


Figure 6.14: Domain coloring for six iterations of $f(z) = (1 - i/2)\sin x$.

```
circle(:,:,1) = X.^2+Y.^2 < R2;
circle(:,:,2)=circle(:,:,1);
circle(:,:,3)=circle(:,:,1);

addcirc(:,:,1)=circle(:,:,1)==0;
addcirc(:,:,2)=circle(:,:,1)==0;
addcirc(:,:,3)=circle(:,:,1)==0;

warning off MATLAB:divideByZero; hsvCircle=ones(Nx,Ny,3);
hsvCircle(:,:,1)=atan2(YY,XX)*180/pi+(atan2(YY,XX)*180/pi<0)*360;
hsvCircle(:,:,1)=hsvCircle(:,:,1)/360; lgz=log(XX.^2+YY.^2)/2;
hsvCircle(:,:,2)=0.75; hsvCircle(:,:,3)=1-(lgz-floor(lgz))/2;
hsvCircle(:,:,1) = flipud((hsvCircle(:,:,1)));

hsvCircle(:,:,2) = flipud((hsvCircle(:,:,2)));

hsvCircle(:,:,3) =flipud((hsvCircle(:,:,3)));

rgbCircle=hsv2rgb(hsvCircle);
rgbCircle=rgbCircle.*circle+addcirc;

image(rgbCircle)
axis square
set(gca,'XTickLabel',{})
set(gca,'YTickLabel',{})
```

6.4 Complex Differentiation

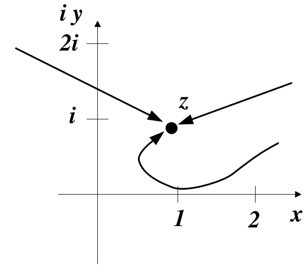


Figure 6.15: There are many paths that approach z as $\Delta z \to 0$.

NEXT WE WANT TO DIFFERENTIATE complex functions . We generalize our definition from single variable calculus,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}, \tag{6.14}$$

provided this limit exists.

The computation of this limit is similar to what we faced in multivariable calculus. Letting $\Delta z \to 0$ means that we get closer to z. There are many paths that one can take that will approach z. [See Figure 6.15.]

It is sufficient to look at two paths in particular. We first consider the path $y =$ constant. Such a path is shown in Figure 6.16 For this path, $\Delta z = \Delta x + i\Delta y = \Delta x$, since y does not change along the path. The derivative, if it exists, is then computed as

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - (u(x,y) + iv(x,y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}. \end{aligned} \tag{6.15}$$

The last two limits are easily identified as partial derivatives of real valued functions of two variables. Thus, we have shown that when $f'(z)$ exists,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{6.16}$$

A similar computation can be made if instead we take a path corresponding to $x =$ constant. In this case $\Delta z = i\Delta y$ and

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - (u(x,y) + iv(x,y))}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}. \end{aligned} \tag{6.17}$$

Therefore,

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{6.18}$$

We have found two different expressions for $f'(z)$ by following two different paths to z. If the derivative exists, then these two expressions

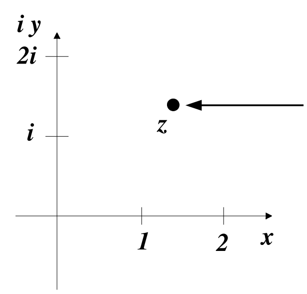


Figure 6.16: A path that approaches z with $y =$ constant.

must be the same. Equating the real and imaginary parts of these expressions, we have

The Cauchy-Riemann Equations.

$$\boxed{\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned} \qquad (6.19)}$$

These are known as the Cauchy-Riemann equations[3].

[3] Augustin-Louis Cauchy (1789-1857) was a French mathematician well known for his work in analysis. Georg Friedrich Bernhard Riemann (1826-1866) was a German mathematician who made major contributions to geometry and analysis.

Theorem 6.1. *$f(z)$ is holomorphic (differentiable) if and only if the Cauchy-Riemann equations are satisfied.*

Example 6.5. $f(z) = z^2$.

In this case we have already seen that $z^2 = x^2 - y^2 + 2ixy$. Therefore, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We first check the Cauchy-Riemann equations.

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= 2y = -\frac{\partial u}{\partial y}.\end{aligned} \tag{6.20}$$

Therefore, $f(z) = z^2$ is differentiable.

We can further compute the derivative using either Equation (6.16) or Equation (6.18). Thus,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i(2y) = 2z.$$

This result is not surprising.

Example 6.6. $f(z) = \bar{z}$.

In this case we have $f(z) = x - iy$. Therefore, $u(x,y) = x$ and $v(x,y) = -y$. But, $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial y} = -1$. Thus, the Cauchy-Riemann equations are not satisfied and we conclude the $f(z) = \bar{z}$ is not differentiable.

Another consequence of the Cauchy-Riemann equations is that both $u(x,y)$ and $v(x,y)$ are *harmonic functions.* A real-valued function $u(x,y)$ is harmonic if it satisfies Laplace's equation in 2D, $\nabla^2 u = 0$, or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Theorem 6.2. *$f(z) = u(x,y) + iv(x,y)$ is differentiable if and only if u and v are harmonic functions.*

This is easily proven using the Cauchy-Riemann equations.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x}$$

$$\begin{aligned} &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \\ &= \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\ &= -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\ &= -\frac{\partial^2 u}{\partial y^2}. \end{aligned} \tag{6.21}$$

Example 6.7. *Is $u(x, y) = x^2 + y^2$ harmonic?*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 \neq 0.$$

No, it is not.

Example 6.8. *Is $u(x, y) = x^2 - y^2$ harmonic?*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

Yes, it is.

Given a harmonic function $u(x, y)$, can one find a function, $v(x, y)$, such $f(z) = u(x, y) + iv(x, y)$ is differentiable? In this case, v are called the *harmonic conjugate* of u.

The harmonic conjugate function.

Example 6.9. *$u(x, y) = x^2 - y^2$ is harmonic, find $v(x, y)$ so that $u + iv$ is differentiable.*

The Cauchy-Riemann equations tell us the following about the unknown function, $v(x, y)$:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y,$$
$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x.$$

We can integrate the first of these equations to obtain

$$v(x, y) = \int 2y\, dx = 2xy + c(y).$$

Here $c(y)$ is an arbitrary function of y. One can check to see that this works by simply differentiating the result with respect to x. However, the second equation must also hold. So, we differentiate our result with respect to y to find that

$$\frac{\partial v}{\partial y} = 2x + c'(y).$$

Since we were supposed to get $2x$, we have that $c'(y) = 0$. Thus, $c(y) = k$ is a constant.

We have just shown that we get an infinite number of functions,

$$v(x, y) = 2xy + k,$$

such that

$$f(z) = x^2 - y^2 + i(2xy + k)$$

is differentiable. In fact, for $k = 0$ *this is nothing other than* $f(z) = z^2$.

6.5 Complex Integration

In the last chapter we introduced functions of a complex variable. We also established when functions are differentiable as complex functions, or holomorphic. In this chapter we will turn to integration in the complex plane. We will learn how to compute complex path integrals, or contour integrals. We will see that contour integral methods are also useful in the computation of some of the real integrals that we will face when exploring Fourier transforms in the next chapter.

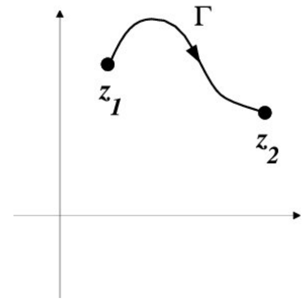


Figure 6.17: We would like to integrate a complex function $f(z)$ over the path Γ in the complex plane.

6.5.1 Complex Path Integrals

In this section we will investigate the computation of complex path integrals. Given two points in the complex plane, connected by a path Γ, we would like to define the integral of $f(z)$ along Γ,

$$\int_\Gamma f(z)\, dz.$$

A natural procedure would be to work in real variables, by writing

$$\int_\Gamma f(z)\, dz = \int_\Gamma [u(x,y) + iv(x,y)]\,(dx + idy).$$

In order to carry out the integration, we then have to find a parametrization of the path and use methods from a multivariate calculus class.

Before carrying this out with some examples, we first provide some definitions.

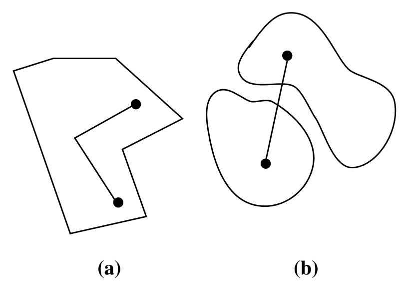


Figure 6.18: Examples of (a) a connected set and (b) a disconnected set.

Definition 6.1. A set D is *connected* if and only if for all z_1, and z_2 in D there exists a piecewise smooth curve connecting z_1 to z_2 and lying in D. Otherwise it is called *disconnected.* Examples are shown in Figure 6.18

Definition 6.2. A set D is *open* if and only if for all z_0 in D there exists an open disk $|z - z_0| < \rho$ in D.

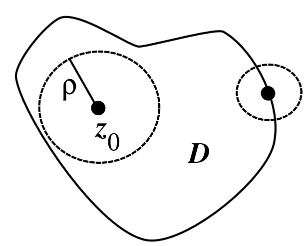


Figure 6.19: Locations of open disks inside and on the boundary of a region.

In Figure 6.19 we show a region with two disks.

For all points on the interior of the region one can find at least one disk contained entirely in the region. The closer one is to the boundary, the smaller the radii of such disks. However, for a point on the boundary, every such disk would contain points inside and outside the disk. Thus, an open set in the complex plane would not contain any of its boundary points.

Definition 6.3. D is called a *domain* if it is both open and connected.

Definition 6.4. Let u and v be continuous in domain D, and Γ a piecewise smooth curve in D. Let $(x(t), y(t))$ be a parametrization of Γ for $t_0 \le t \le t_1$ and $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$. Then

$$\boxed{\int_\Gamma f(z)\, dz = \int_{t_0}^{t_1} [u(x(t), y(t)) + iv(x(t), y(t))] \, (\frac{dx}{dt} + i\frac{dy}{dt})dt. \qquad (6.22)}$$

Note that we have used

$$dz = dx + idy = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt.$$

This definition gives us a prescription for computing path integrals. Let's see how this works with a couple of examples.

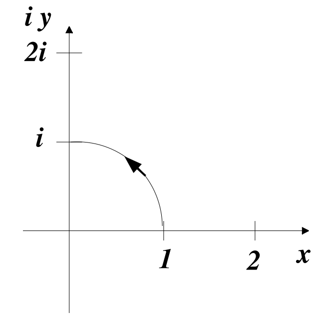


Figure 6.20: Contour for Example 6.10.

Example 6.10. $\int_C z^2\, dz$, *C = the arc of the unit circle in the first quadrant as shown in Figure 6.20.*

We first specify the parametrization . There are two ways we could do this. First, we note that the standard parametrization of the unit circle is

$$(x(\theta), y(\theta)) = (\cos\theta, \sin\theta), \quad 0 \le \theta \le 2\pi.$$

For a quarter circle in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$, *we let* $z = \cos\theta + i\sin\theta$. *Therefore,* $dz = (-\sin\theta + i\cos\theta)\, d\theta$ *and the path integral becomes*

$$\int_C z^2\, dz = \int_0^{\frac{\pi}{2}} (\cos\theta + i\sin\theta)^2(-\sin\theta + i\cos\theta)\, d\theta.$$

We can expand the integrand and integrate, having to perform some trigonometric integrations:[4]

$$\int_0^{\frac{\pi}{2}} [\sin^3\theta - 3\cos^2\theta\sin\theta + i(\cos^3\theta - 3\cos\theta\sin^2\theta)]\, d\theta.$$

[4] The reader should work out these trigonometric integrations and confirm the result. For example, you can use

$$\sin^3\theta = \sin\theta(1 - \cos^2\theta))$$

to write the real part of the integrand as

$$\sin\theta - 4\cos^2\theta\sin\theta.$$

The resulting antiderivative becomes

$$-\cos\theta + \frac{4}{3}\cos^3\theta.$$

The imaginary integrand can be integrated in a similar fashion.

While this is doable, there is a simpler procedure. We first note that $z = e^{i\theta}$ *on C. So,* $dz = ie^{i\theta}d\theta$. *The integration then becomes*

$$\begin{aligned} \int_C z^2\, dz &= \int_0^{\frac{\pi}{2}} (e^{i\theta})^2 ie^{i\theta}\, d\theta \\ &= i\int_0^{\frac{\pi}{2}} e^{3i\theta}\, d\theta \\ &= \left.\frac{ie^{3i\theta}}{3i}\right|_0^{\pi/2} \\ &= -\frac{1+i}{3}. \end{aligned} \qquad (6.23)$$

Example 6.11. $\int_\Gamma z\, dz$, $\Gamma = \gamma_1 \cup \gamma_2$ *is the path shown in Figure 6.21.*

In this problem we have path that is a piecewise smooth curve. We can compute the path integral by computing the values along the two segments of the path and adding up the results. Let the two segments be called γ_1 and γ_2 as shown in Figure 6.21.

Over γ_1 we note that $y = 0$. Thus, $z = x$ for $x \in [0,1]$. It is natural to take x as the parameter. So, $dz = dx$ and we have

$$\int_{\gamma_1} z\,dz = \int_0^1 x\,dx = \frac{1}{2}.$$

For path γ_2 we have that $z = 1 + iy$ for $y \in [0,1]$. Inserting z and $dz = i\,dy$, the integral becomes

$$\int_{\gamma_2} z\,dz = \int_0^1 (1+iy)\,idy = i - \frac{1}{2}.$$

Combining these results, we have $\int_\Gamma z\,dz = \frac{1}{2} + (i - \frac{1}{2}) = i$.

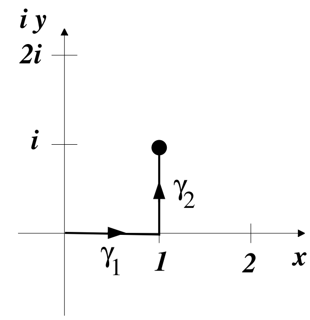


Figure 6.21: Contour for Example 6.11 with $\Gamma = \gamma_1 \cup \gamma_2$.

Example 6.12. *$\int_{\gamma_3} z\,dz$, γ_3 is the path shown in Figure 6.22.*

In this case we take a path from $z = 0$ to $z = 1 + i$ along a different path. Let $\gamma_3 = \{(x,y)|y = x^2, x \in [0,1]\} = \{z|z = x + ix^2, x \in [0,1]\}$. Then, $dz = (1 + 2ix)\,dx$.

The integral becomes

$$\begin{aligned}
\int_{\gamma_3} z\,dz &= \int_0^1 (x+ix^2)(1+2ix)\,dx \\
&= \int_0^1 (x + 3ix^2 - 2x^3)\,dx = \\
&= \left[\frac{1}{2}x^2 + ix^3 - \frac{1}{2}x^4\right]_0^1 = i. \qquad (6.24)
\end{aligned}$$

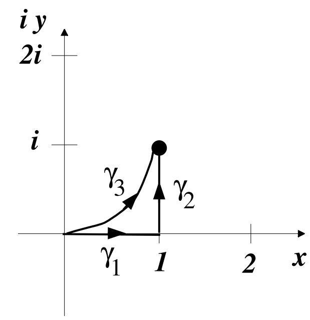


Figure 6.22: Contour for Example 6.12.

In the last case we found the same answer as in Example 6.11. But we should not take this as a general rule for all complex path integrals. In fact, it is not true that integrating over different paths always yields the same results. We will now look into this notion of path independence.

Definition 6.5. The integral $\int f(z)\,dz$ is *path independent* if

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

for all paths from z_1 to z_2.

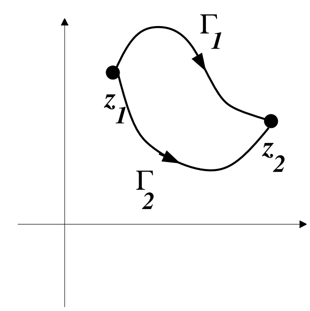


Figure 6.23: $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ for all paths from z_1 to z_2 when the integral of $f(z)$ is path independent.

If $\int f(z)\,dz$ is path independent, then the integral of $f(z)$ over all closed loops is zero,

$$\int_{\text{closed loops}} f(z)\,dz = 0.$$

A common notation for integrating over closed loops is $\oint_C f(z)\,dz$. But first we have to define what we mean by a closed loop.

Definition 6.6. A *simple closed contour* is a path satisfying

a The end point is the same as the beginning point. (This makes the loop closed.)

b The are no self-intersections. (This makes the loop simple.)

A loop in the shape of a figure eight is closed, but it is not simple.

Now, consider an integral over the closed loop C shown in Figure 6.24. We pick two points on the loop breaking it into two contours, C_1 and C_2. Then we make use of the path independence by defining C_2^- to be the path along C_2 but in the opposite direction. Then,

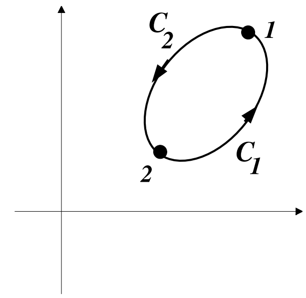


Figure 6.24: The integral $\oint_C f(z)\,dz$ around C is zero if the integral $\int_\Gamma f(z)\,dz$ is path independent.

$$\begin{aligned}\oint_C f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{C_2^-} f(z)\,dz. \qquad (6.25)\end{aligned}$$

Assuming that the integrals from point 1 to point 2 are path independent, then the integrals over C_1 and C_2^- are equal. Therefore, we have $\oint_C f(z)\,dz = 0$.

Example 6.13. *Consider the integral $\oint_C z\,dz$ for C the closed contour shown in Figure 6.22 starting at $z = 0$ following path γ_1, then γ_2 and returning to $z = 0$. Based on the earlier examples and the fact that going backwards on γ_3 introduces a negative sign, we have*

$$\oint_C z\,dz = \int_{\gamma_1} z\,dz + \int_{\gamma_2} z\,dz - \int_{\gamma_3} z\,dz = \frac{1}{2} + \left(i - \frac{1}{2}\right) - i = 0.$$

6.5.2 *Cauchy's Theorem*

NEXT WE WANT TO INVESTIGATE if we can determine that integrals over simple closed contours vanish without doing all the work of parametrizing the contour. First, we need to establish the direction about which we traverse the contour.

Definition 6.7. A curve with parametrization $(x(t), y(t))$ has a *normal* $(n_x, n_y) = (-\frac{dx}{dt}, \frac{dy}{dt})$.

Recall that the normal is a perpendicular to the curve. There are two such perpendiculars. The above normal points outward and the other normal points towards the interior of a closed curve. We will define a positively oriented contour as one that is traversed with the outward normal pointing to the right. As one follows loops, the interior would then be on the left.

We now consider $\oint_C (u+iv)\,dz$ over a simple closed contour. This can be written in terms of two real integrals in the xy-plane.

$$\begin{aligned}\oint_C (u+iv)\,dz &= \int_C (u+iv)(dx+i\,dy)\\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy. \end{aligned} \tag{6.26}$$

These integrals in the plane can be evaluated using Green's Theorem in the Plane. Recall this theorem from your last semester of calculus:

Green's Theorem in the Plane is one of the major integral theorems of vector calculus. It was discovered by George Green (1793-1841) and published in 1828, about four years before he entered Cambridge as an undergraduate.

Green's Theorem in the Plane.

Theorem 6.3. *Let $P(x,y)$ and $Q(x,y)$ be continuously differentiable functions on and inside the simple closed curve C. Denoting the enclosed region S, we have*

$$\int_C P\,dx + Q\,dy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dxdy. \tag{6.27}$$

Using Green's Theorem to rewrite the first integral in (6.26), we have

$$\int_C u\,dx - v\,dy = \int\int_S \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dxdy.$$

If u and v satisfy the Cauchy-Riemann equations (6.19), then the integrand in the double integral vanishes. Therefore,

$$\int_C u\,dx - v\,dy = 0.$$

In a similar fashion, one can show that

$$\int_C v\,dx + u\,dy = 0.$$

We have thus proven the following theorem:

Cauchy's Theorem

Theorem 6.4. *If u and v satisfy the Cauchy-Riemann equations (6.19) inside and on the simple closed contour C, then*

$$\oint_C (u+iv)\,dz = 0. \tag{6.28}$$

Corollary *$\oint_C f(z)\,dz = 0$ when f is differentiable in domain D with $C \subset D$.*

Either one of these is referred to as **Cauchy's Theorem**.

Example 6.14. *Consider $\oint_{|z-1|=3} z^4\,dz$. Since $f(z) = z^4$ is differentiable inside the circle $|z-1| = 3$, this integral vanishes.*

We can use Cauchy's Theorem to show that we can deform one contour into another, perhaps simpler, contour.

One can deform contours into simpler ones.

Theorem 6.5. *If $f(z)$ is holomorphic between two simple closed contours, C and C', then $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$.*

Proof. We consider the two curves as shown in Figure 6.25. Connecting the two contours with contours Γ_1 and Γ_2 (as shown in the figure), C is seen to split into contours C_1 and C_2 and C' into contours C_1' and C_2'. Note that $f(z)$ is differentiable inside the newly formed regions between the curves. Also, the boundaries of these regions are now simple closed curves. Therefore, Cauchy's Theorem tells us that the integrals of $f(z)$ over these regions are zero.

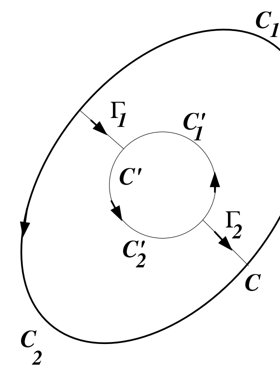


Figure 6.25: The contours needed to prove that $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$ when $f(z)$ is holomorphic between the contours C and C'.

Noting that integrations over contours opposite to the positive orientation are the negative of integrals that are positively oriented, we have from Cauchy's Theorem that

$$\int_{C_1} f(z)\,dz + \int_{\Gamma_1} f(z)\,dz - \int_{C_1'} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0$$

and

$$\int_{C_2} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz - \int_{C_2'} f(z)\,dz - \int_{\Gamma_1} f(z)\,dz = 0.$$

In the first integral we have traversed the contours in the following order: C_1, Γ_1, C_1' backwards, and Γ_2. The second integral denotes the integration over the lower region, but going backwards over all contours except for C_2.

Combining these results by adding the two equations above, we have

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz - \int_{C_1'} f(z)\,dz - \int_{C_2'} f(z)\,dz = 0.$$

Noting that $C = C_1 + C_2$ and $C' = C_1' + C_2'$, we have

$$\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz,$$

as was to be proven. □

Example 6.15. *Compute $\oint_R \frac{dz}{z}$ for R the rectangle $[-2,2] \times [-2i, 2i]$.*

We can compute this integral by looking at four separate integrals over the sides of the rectangle in the complex plane. One simply parametrizes each line segment, perform the integration and sum the four separate results. From the last theorem, we can instead integrate over a simpler contour by deforming the rectangle into a circle as long as $f(z) = \frac{1}{z}$ is differentiable in the region bounded by the rectangle and the circle. So, using the unit circle, as shown in Figure 6.26, the integration might be easier to perform.

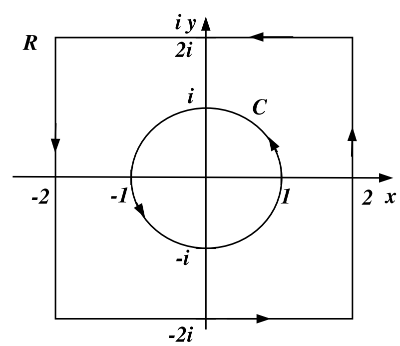


Figure 6.26: The contours used to compute $\oint_R \frac{dz}{z}$. Note that to compute the integral around R we can deform the contour to the circle C since $f(z)$ is differentiable in the region between the contours.

More specifically, the last theorem tells us that

$$\oint_R \frac{dz}{z} = \oint_{|z|=1} \frac{dz}{z}$$

The latter integral can be computed using the parametrization $z = e^{i\theta}$ *for* $\theta \in [0, 2\pi]$. *Thus,*

$$\begin{aligned} \oint_{|z|=1} \frac{dz}{z} &= \int_0^{2\pi} \frac{ie^{i\theta}\, d\theta}{e^{i\theta}} \\ &= i\int_0^{2\pi} d\theta = 2\pi i. \end{aligned} \tag{6.29}$$

Therefore, we have found that $\oint_R \frac{dz}{z} = 2\pi i$ *by deforming the original simple closed contour.*

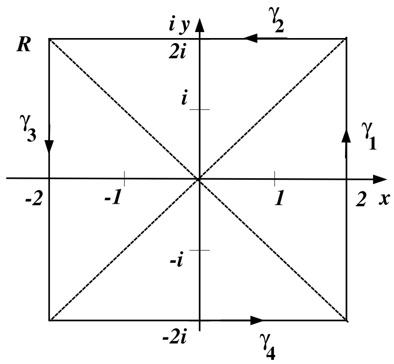


Figure 6.27: The contours used to compute $\oint_R \frac{dz}{z}$. The added diagonals are for the reader to easily see the arguments used in the evaluation of the limits when integrating over the segments of the square R.

For fun, let's do this the long way to see how much effort was saved. We will label the contour as shown in Figure 6.27. The lower segment, γ_4 *of the square can be simple parametrized by noting that along this segment* $z = x - 2i$ *for* $x \in [-2, 2]$. *Then, we have*

$$\begin{aligned} \oint_{\gamma_4} \frac{dz}{z} &= \int_{-2}^{2} \frac{dx}{x-2i} \\ &= \ln|x-2i|_{-2}^{2} \\ &= \left(\ln(2\sqrt{2}) - \frac{\pi i}{4}\right) - \left(\ln(2\sqrt{2}) - \frac{3\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{6.30}$$

We note that the arguments of the logarithms are determined from the angles made by the diagonals provided in Figure 6.27.

Similarly, the integral along the top segment, $z = x + 2i$, $x \in [-2, 2]$, *is computed as*

$$\begin{aligned} \oint_{\gamma_2} \frac{dz}{z} &= \int_{2}^{-2} \frac{dx}{x+2i} \\ &= \ln|x+2i|_{2}^{-2} \end{aligned}$$

$$\begin{aligned} &= \left(\ln(2\sqrt{2}) + \frac{3\pi i}{4}\right) - \left(\ln(2\sqrt{2}) + \frac{\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{6.31}$$

The integral over the right side, $z = 2 + iy$, $y \in [-2, 2]$, is

$$\begin{aligned} \oint_{\gamma_1} \frac{dz}{z} &= \int_{-2}^{2} \frac{idy}{2+iy} \\ &= \ln|2+iy|_{-2}^{2} \\ &= \left(\ln(2\sqrt{2}) + \frac{\pi i}{4}\right) - \left(\ln(2\sqrt{2}) - \frac{\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{6.32}$$

Finally, the integral over the left side, $z = -2 + iy$, $y \in [-2, 2]$, is

$$\begin{aligned} \oint_{\gamma_3} \frac{dz}{z} &= \int_{2}^{-2} \frac{idy}{-2+iy} \\ &= \ln|-2+iy|_{-2}^{2} \\ &= \left(\ln(2\sqrt{2}) + \frac{5\pi i}{4}\right) - \left(\ln(2\sqrt{2}) + \frac{3\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{6.33}$$

Therefore, we have that

$$\begin{aligned} \oint_{R} \frac{dz}{z} &= \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} + \int_{\gamma_3} \frac{dz}{z} + \int_{\gamma_4} \frac{dz}{z} \\ &= \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2} \\ &= 4(\frac{\pi i}{2}) = 2\pi i. \end{aligned} \tag{6.34}$$

This gives the same answer we had found using a simple contour deformation.

The converse of Cauchy's Theorem is not true, namely $\oint_C f(z)\,dz = 0$ does not always imply that $f(z)$ is differentiable. What we do have is **Morera's Theorem**(Giacinto Morera, 1856-1909):

Theorem 6.6. *Let f be continuous in a domain D. Suppose that for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$. Then f is differentiable in D.*

The proof is a bit more detailed than we need to go into here. However, this theorem is useful in the next section.

6.5.3 *Analytic Functions and Cauchy's Integral Formula*

In the previous section we saw that Cauchy's Theorem was useful for computing particular integrals without having to parametrize the contours, or to deform contours to simpler ones. The integrand needs to possess certain differentiability properties. In this section, we will generalize our integrand slightly so that we can integrate a larger family of complex functions. This will take the form of what is called *Cauchy's Integral Formula*, which extends Cauchy's Theorem to functions analytic in an annulus. However, first we need to explore the concept of analytic functions.

Definition of an analytic function: the existence of a convergent power series expansion.

Definition 6.8. $f(z)$ is *analytic* in D if for every open disk $|z - z_0| < \rho$ lying in D, $f(z)$ can be represented as a power series in z_0. Namely,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

This series converges uniformly and absolutely inside the circle of convergence, $|z - z_0| < R$, with radius of convergence R. [See the Appendix for a review of convergence.]

Since $f(z)$ can be written as a uniformly convergent power series, we can integrate it term by term over any simple closed contour in D containing z_0. In particular, we have to compute integrals like $\oint_C (z - z_0)^n \, dz$. As we will see in the homework exercises, these integrals evaluate to zero for most n. Thus, we can show that for $f(z)$ analytic in D and any C lying in D, $\oint_C f(z) \, dz = 0$. Also, f is a uniformly convergent sum of continuous functions, so $f(z)$ is also continuous. Thus, by Morera's Theorem, we have that $f(z)$ is differentiable if it is analytic. Often terms like analytic, differentiable and holomorphic are used interchangeably, though there is a subtle distinction due to their definitions.

There are various types of complex-valued functions. A holomorphic function is (complex-)differentiable in a neighborhood of every point in its domain. An analytic function has a convergent Taylor series expansion in a neighborhood of each point in its domain. We see here that analytic functions are holomorphic and vice versa. If a function is holomorphic throughout the complex plane, then it is called an entire function. Finally, a function which is holomorphic on all of its domain except at a set of isolated poles (to be defined later), then it is called a meromorphic function.

Let's recall some manipulations from our study of series of real functions.

Example 6.16. $f(z) = \frac{1}{1+z}$ *for* $z_0 = 0$.

This case is simple. From Chapter 1 we recall that $f(z)$ *is the sum of a geometric series for* $|z| < 1$. *We have*

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n.$$

Thus, this series expansion converges inside the unit circle ($|z| < 1$*) in the complex plane.*

Example 6.17. $f(z) = \frac{1}{1+z}$ *for* $z_0 = \frac{1}{2}$. *We now look into an expansion about a different point. We could compute the expansion coefficients using Taylor' formula for the coefficients. However, we can also make use of the formula for geometric series after rearranging the function. We seek an expansion in* *powers of* $z - \frac{1}{2}$*. So, we rewrite the function in a form that has this term. Thus,*

$$f(z) = \frac{1}{1+z} = \frac{1}{1+(z-\frac{1}{2}+\frac{1}{2})} = \frac{1}{\frac{3}{2}+(z-\frac{1}{2})}.$$

This is not quite in the form we need. It would be nice if the denominator were of the form of one plus something. [Note: This is just like what we show in the Appendix for functions of real variables. See Example A.28.] We can get the denominator into such a form by factoring out the $\frac{3}{2}$*. Then we would have*

$$f(z) = \frac{2}{3}\frac{1}{1+\frac{2}{3}(z-\frac{1}{2})}.$$

The second factor now has the form $\frac{1}{1-r}$*, which would be the sum of a geometric series with first term* $a = 1$ *and ratio* $r = -\frac{2}{3}(z-\frac{1}{2})$ *provided that* $|r|<1$*. Therefore, we have found that*

$$f(z) = \frac{2}{3}\sum_{n=0}^{\infty}\left[-\frac{2}{3}(z-\frac{1}{2})\right]^n$$

for

$$|-\frac{2}{3}(z-\frac{1}{2})| < 1.$$

This convergence interval can be rewritten as

$$|z-\frac{1}{2}| < \frac{3}{2}.$$

This is a circle centered at $z = \frac{1}{2}$ *with radius* $\frac{3}{2}$*.*

In Figure 6.28 we show the regions of convergence for the power series expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. We note that the first expansion gives that $f(z)$ is at least analytic inside the region $|z| < 1$. The second expansion shows that $f(z)$ is analytic in a region even further outside to the region $|z-\frac{1}{2}| < \frac{3}{2}$. We will see later that there are expansions outside of these regions, though some are expansions involving negative powers of $z - z_0$.

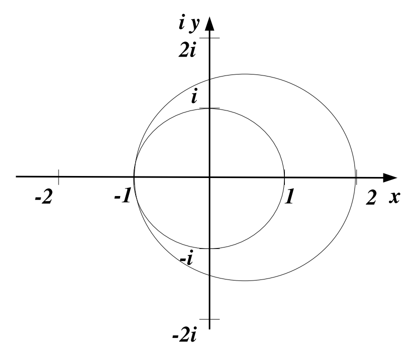


Figure 6.28: Regions of convergence for expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$.

We now present the main theorem of this section:

Cauchy Integral Formula

Theorem 6.7. *Let* $f(z)$ *be analytic in* $|z - z_0| < \rho$ *and let* C *be the boundary (circle) of this disk. Then,*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz. \tag{6.35}$$

Proof. In order to prove this, we first make use of the analyticity of $f(z)$. We insert the power series expansion of $f(z)$ about z_0 into the integrand. Then we have

$$\begin{aligned}\frac{f(z)}{z-z_0} &= \frac{1}{z-z_0}\left[\sum_{n=0}^{\infty} c_n(z-z_0)^n\right]\\ &= \frac{1}{z-z_0}\left[c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \ldots\right]\\ &= \frac{c_0}{z-z_0} + \underbrace{c_1 + c_2(z-z_0) + \ldots}_{\text{analytic function}}. \end{aligned} \tag{6.36}$$

As noted the integrand can be written as

$$\frac{f(z)}{z-z_0} = \frac{c_0}{z-z_0} + h(z),$$

where $h(z)$ is an analytic function, since $h(z)$ is representable as a series expansion about z_0. We have already shown that analytic functions are differentiable, so by Cauchy's Theorem $\oint_C h(z)\,dz = 0$. Noting also that $c_0 = f(z_0)$ is the first term of a Taylor series expansion about $z = z_0$, we have

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_C \left[\frac{c_0}{z-z_0} + h(z)\right]\,dz = f(z_0)\oint_C \frac{1}{z-z_0}\,dz.$$

We need only compute the integral $\oint_C \frac{1}{z-z_0}\,dz$ to finish the proof of Cauchy's Integral Formula. This is done by parametrizing the circle, $|z-z_0| = \rho$, as shown in Figure 6.29. This is simply done by letting

$$z - z_0 = \rho e^{i\theta}.$$

(Note that this has the right complex modulus since $|e^{i\theta}| = 1$. Then $dz = i\rho e^{i\theta}d\theta$. Using this parametrization, we have

$$\oint_C \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\,d\theta}{\rho e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i.$$

Therefore,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{1}{z-z_0}\,dz = 2\pi i f(z_0),$$

as was to be shown. □

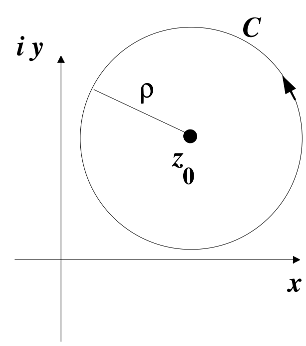


Figure 6.29: Circular contour used in proving the Cauchy Integral Formula.

Example 6.18. *Compute* $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\,dz$.

In order to apply the Cauchy Integral Formula, we need to factor the denominator, $z^2 - 6z + 5 = (z-1)(z-5)$. *We next locate the zeros of the denominator. In Figure 6.30 we show the contour and the points* $z = 1$ *and* $z = 5$. *The only point inside the region bounded by the contour is* $z = 1$. *Therefore, we can apply the Cauchy Integral Formula for* $f(z) = \frac{\cos z}{z-5}$ *to the integral*

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\,dz = \int_{|z|=4} \frac{f(z)}{(z-1)}\,dz = 2\pi i f(1).$$

Therefore, we have

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\, dz = -\frac{\pi i \cos(1)}{2}.$$

We have shown that $f(z_0)$ has an integral representation for $f(z)$ analytic in $|z - z_0| < \rho$. In fact, all derivatives of an analytic function have an integral representation. This is given by

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz. \qquad (6.37)}$$

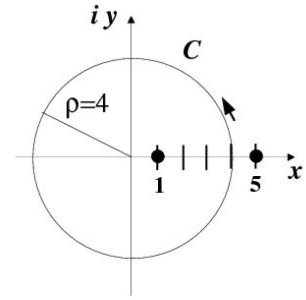


Figure 6.30: Circular contour used in computing $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\, dz$.

This can be proven following a derivation similar to that for the Cauchy Integral Formula. Inserting the Taylor series expansion for $f(z)$ into the integral on the right hand side, we have

$$\begin{aligned} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz &= \sum_{m=0}^{\infty} c_m \oint_C \frac{(z-z_0)^m}{(z-z_0)^{n+1}}\, dz \\ &= \sum_{m=0}^{\infty} c_m \oint_C \frac{dz}{(z-z_0)^{n-m+1}}. \end{aligned} \qquad (6.38)$$

Picking $k = n - m$, the integrals in the sum can be computed by using the following lemma.

Lemma

$$\oint_C \frac{dz}{(z-z_0)^{k+1}} = \begin{cases} 0, & k \neq 0 \\ 2\pi i, & k = 0. \end{cases} \qquad (6.39)$$

This is Problem 3. So, the only nonvanishing integrals are when $k = n - m = 0$, or $m = n$. Therefore,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = 2\pi i c_n.$$

To finish the proof, we recall (from the Appendix) that the coefficients of the Taylor series expansion for $f(z)$ are given by

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

and the result follows.

6.5.4 Laurent Series

UNTIL THIS POINT we have only talked about series whose terms have nonnegative powers of $z - z_0$. It is possible to have series representations in which there are negative powers. In the last section we investigated expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. The regions of convergence for each series was shown in Figure 6.28. Let us reconsider each of these expansions, but for values of z outside the region of convergence previously found..

Example 6.19. $f(z) = \frac{1}{1+z}$ *for* $|z| > 1$.

As before, we make use of the geometric series . Since $|z| > 1$*, we instead rewrite our function as*

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\frac{1}{1+\frac{1}{z}}.$$

We now have the function in a form of the sum of a geometric series with first term $a = 1$ *and ratio* $r = -\frac{1}{z}$. *We note that* $|z| > 1$ *implies that* $|r| < 1$. *Thus, we have the geometric series*

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n.$$

This can be re-indexed[5] *as*

$$f(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \sum_{j=1}^{\infty}(-1)^{j-1}z^{-j}.$$

Note that this series, which converges outside the unit circle, $|z| > 1$*, has negative powers of* z.

[5] Re-indexing a series is often useful in series manipulations. In this case, we have the series

$$\sum_{n=0}^{\infty}(-1)^n z^{-n-1} = z^{-1} - z^{-2} + z^{-3} + \ldots.$$

The index is n. You can see that the index does not appear when the sum is expanded showing the terms. The summation index is sometimes referred to as a dummy index for this reason. Reindexing allows one to rewrite the shorthand summation notation while capturing the same terms. In this example, the exponents are $-n-1$. We can simplify the notation by letting $-n-1 = -j$, or $j = n+1$. Noting that $j = 1$ when $n = 0$, we get the sum $\sum_{j=1}^{\infty}(-1)^{j-1}z^{-j}$.

Example 6.20. $f(z) = \frac{1}{1+z}$ *for* $|z - \frac{1}{2}| > \frac{3}{2}$.

As before, we express this in a form in which we can use a geometric series expansion. We seek powers of $z - \frac{1}{2}$. *So, we add and subtract* $\frac{1}{2}$ *to the* z *to obtain:*

$$f(z) = \frac{1}{1+z} = \frac{1}{1+(z-\frac{1}{2}+\frac{1}{2})} = \frac{1}{\frac{3}{2}+(z-\frac{1}{2})}.$$

Instead of factoring out the $\frac{3}{2}$ *as we had done in Example 6.17, we factor out the* $(z-\frac{1}{2})$ *term. Then, we obtain*

$$f(z) = \frac{1}{1+z} = \frac{1}{(z-\frac{1}{2})}\frac{1}{\left[1+\frac{3}{2}(z-\frac{1}{2})^{-1}\right]}.$$

Now we identify $a = 1$ *and* $r = -\frac{3}{2}(z-\frac{1}{2})^{-1}$. *This leads to the series*

$$\begin{aligned} f(z) &= \frac{1}{z-\frac{1}{2}}\sum_{n=0}^{\infty}\left(-\frac{3}{2}(z-\frac{1}{2})^{-1}\right)^n \\ &= \sum_{n=0}^{\infty}\left(-\frac{3}{2}\right)^n\left(z-\frac{1}{2}\right)^{-n-1}. \end{aligned} \tag{6.40}$$

This converges for $|z-\frac{1}{2}| > \frac{3}{2}$ *and can also be re-indexed to verify that this series involves negative powers of* $z - \frac{1}{2}$.

This leads to the following theorem:

Theorem 6.8. *Let $f(z)$ be analytic in an annulus, $R_1 < |z - z_0| < R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus as shown in Figure 6.31. Then,*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} b_j (z - z_0)^{-j},$$

with

$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{j+1}}\, dz,$$

and

$$b_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-j+1}}\, dz.$$

The above series can be written in the more compact form

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j.$$

Such a series expansion is called a *Laurent series* expansion named after its discoverer Pierre Alphonse Laurent (1813-1854).

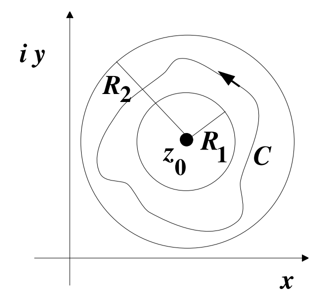


Figure 6.31: This figure shows an annulus, $R_1 < |z - z_0| < R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus.

Example 6.21. *Expand $f(z) = \frac{1}{(1-z)(2+z)}$ in the annulus $1 < |z| < 2$.*

Using partial fractions , we can write this as

$$f(z) = \frac{1}{3}\left[\frac{1}{1-z} + \frac{1}{2+z}\right].$$

We can expand the first fraction, $\frac{1}{1-z}$, as an analytic function in the region $|z| > 1$ and the second fraction, $\frac{1}{2+z}$, as an analytic function in $|z| < 2$. This is done as follows. First, we write

$$\frac{1}{2+z} = \frac{1}{2[1 - (-\frac{z}{2})]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n.$$

Then we write

$$\frac{1}{1-z} = -\frac{1}{z[1 - \frac{1}{z}]} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}.$$

Therefore, in the common region, $1 < |z| < 2$, we have that

$$\begin{aligned}\frac{1}{(1-z)(2+z)} &= \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right] \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{6(2^n)}z^n + \sum_{n=1}^{\infty}\frac{(-1)}{3}z^{-n}. \qquad (6.41)\end{aligned}$$

We note that this indeed is not a Taylor series expansion due to the existence of terms with negative powers in the second sum.

6.5.5 *Singularities and The Residue Theorem*

IN THE LAST SECTION we found that we could integrate functions satisfying some analyticity properties along contours without using detailed parametrizations around the contours. We can deform contours if the function is analytic in the region between the original and new contour. In this section we will extend our tools for performing contour integrals.

The integrand in the Cauchy Integral Formula was of the form $g(z) = \frac{f(z)}{z-z_0}$, where $f(z)$ is well behaved at z_0. The point $z = z_0$ is called a *singularity* of $g(z)$, as $g(z)$ is not defined there. As we saw from the proof of the Cauchy Integral Formula, $g(z)$ has a Laurent series expansion about $z = z_0$,

$$g(z) = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{1}{2}f''(z_0)(z-z_0)^2 + \ldots .$$

We will first classify singularities and then use singularities to aid in computing contour integrals.

Definition 6.9. A *singularity* of $f(z)$ is a point at which $f(z)$ fails to be analytic.

Classification of singularities.

Typically these are isolated singularities. In order to classify the singularities of $f(z)$, we look at the *principal part* of the Laurent series of $f(z)$ about $z = z_0$: $\sum_{j-1}^{\infty} b_j(z-z_0)^{-j}$.

1. If $f(z)$ is bounded near z_0, then z_0 is a **removable singularity**.

2. If there are a finite number of terms in the principal part of the Laurent series of $f(z)$ about $z = z_0$, then z_0 is called a **pole**.

3. If there are an infinite number of terms in the principal part of the Laurent series of $f(z)$ about $z = z_0$, then z_0 is called an **essential singularity**.

Example 6.22. Removable singularity: $f(z) = \frac{\sin z}{z}$.

At first it looks like there is a possible singularity at $z = 0$, *since the denominator is zero at* $z = 0$. *However, we know from the first semester of calculus that* $\lim_{z\to 0} \frac{\sin z}{z} = 1$. *Furthermore, we can expand* $\sin z$ *about* $z = 0$ *and see that*

$$\frac{\sin z}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \ldots) = 1 - \frac{z^2}{3!} + \ldots .$$

Thus, there are only nonnegative powers in the series expansion. So, $z = 0$ *is a removable singularity.*

Example 6.23. Poles $f(z) = \frac{e^z}{(z-1)^n}$.

For $n = 1$ we have $f(z) = \frac{e^z}{z-1}$. This function has a singularity at $z = 1$. The series expansion is found by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{z-1}e^{z-1} = \frac{e}{z-1} + e + \frac{e}{2!}(z-1) + \dots .$$

Note that the principal part of the Laurent series expansion about $z = 1$ only has one term, $\frac{e}{z-1}$. Therefore, $z = 1$ is a pole. Since the leading term has an exponent of -1, $z = 1$ is called a pole of order one, or a simple pole.

Simple pole.

For $n = 2$ we have $f(z) = \frac{e^z}{(z-1)^2}$. The series expansion is found again by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{(z-1)^2}e^{z-1} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2!} + \frac{e}{3!}(z-1) + \dots .$$

Note that the principal part of the Laurent series has two terms involving $(z-1)^{-2}$ and $(z-1)^{-1}$. Since the leading term has an exponent of -2, $z = 1$ is called a pole of order 2, or a double pole.

Double pole.

Example 6.24. Essential Singularity $f(z) = e^{\frac{1}{z}}$.

In this case we have the series expansion about $z = 0$ given by

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

We see that there are an infinite number of terms in the principal part of the Laurent series. So, this function has an essential singularity at $z = 0$.

In the above examples we have seen poles of order one (a simple pole) and two (a double pole). In general, we can define poles of order k.

Definition 6.10. $f(z)$ has a *pole of order k* at z_0 if and only if $(z - z_0)^k f(z)$ has a removable singularity at z_0, but $(z - z_0)^{k-1} f(z)$ for $k > 0$ does not.

Example 6.25. *Determine the order of the pole at $z = 0$ of $f(z) = \cot z \csc z$.*

First we rewrite $f(z)$ in terms of sines and cosines.

$$f(z) = \cot z \csc z = \frac{\cos z}{\sin^2 z}.$$

We note that the denominator vanishes at $z = 0$. However, how do we know that the pole is not a simple pole? Well, we check to see if $(z-0)f(z)$ has a removable singularity at $z = 0$:

$$\begin{aligned} \lim_{z\to 0}(z-0)f(z) &= \lim_{z\to 0} \frac{z\cos z}{\sin^2 z} \\ &= \left(\lim_{z\to 0} \frac{z}{\sin z}\right)\left(\lim_{z\to 0} \frac{\cos z}{\sin z}\right) \\ &= \lim_{z\to 0} \frac{\cos z}{\sin z}. \end{aligned} \tag{6.42}$$

We see that this limit is undefined. So, now we check to see if $(z-0)^2 f(z)$ *has a removable singularity at* $z=0$:

$$\begin{aligned}
\lim_{z\to 0}(z-0)^2 f(z) &= \lim_{z\to 0}\frac{z^2\cos z}{\sin^2 z} \\
&= \left(\lim_{z\to 0}\frac{z}{\sin z}\right)\left(\lim_{z\to 0}\frac{z\cos z}{\sin z}\right) \\
&= \lim_{z\to 0}\frac{z}{\sin z}\cos(0) = 1. \qquad (6.43)
\end{aligned}$$

In this case, we have obtained a finite, nonzero, result. So, $z=0$ *is a pole of order 2.*

We could have also relied on series expansions. So, we expand both the sine and cosine in a Taylor series expansion:

$$f(z) = \frac{\cos z}{\sin^2 z} = \frac{1-\frac{1}{2!}z^2+\ldots}{(z-\frac{1}{3!}z^3+\ldots)^2}.$$

Factoring a z from the expansion in the denominator,

$$f(z) = \frac{1}{z^2}\frac{1-\frac{1}{2!}z^2+\ldots}{(1-\frac{1}{3!}z+\ldots)^2} = \frac{1}{z^2}\left(1+O(z^2)\right),$$

we can see that the leading term will be a $1/z^2$*, indicating a pole of order 2.*

We will see how knowledge of the poles of a function can aid in the computation of contour integrals. We now show that if a function, $f(z)$, has a pole of order k, then

Integral of a function with a simple pole inside C.

$$\boxed{\oint_C f(z)\,dz = 2\pi i \operatorname{Res}[f(z);z_0],}$$

where we have defined $\operatorname{Res}[f(z);z_0]$ as the **residue of** $f(z)$ **at** $z=z_0$**.** In particular, for a pole of order k the residue is given by

Residues of a function with poles of order k.

Residues - Poles of order k

$$\operatorname{Res}[f(z);z_0] = \lim_{z\to z_0}\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]. \qquad (6.44)$$

Proof. Let $\phi(z) = (z-z_0)^k f(z)$ be analytic. Then $\phi(z)$ has a Taylor series expansion about z_0. As we had seen in the last section, we can write the integral representation of derivatives of ϕ as

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i}\oint_C \frac{\phi(z)}{(z-z_0)^k}\,dz.$$

Inserting the definition of $\phi(z)$ we then have

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i}\oint_C f(z)\,dz.$$

Solving for the integral, we have the result

$$\begin{aligned}\oint_C f(z)\,dz &= \frac{2\pi i}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]_{z=z_0} \\ &\equiv 2\pi i \operatorname{Res}[f(z);z_0] \end{aligned} \tag{6.45}$$

□

The residue for a simple pole.

Note: If z_0 is a simple pole, the residue is easily computed as

$$Res[f(z);z_0] = \lim_{z\to z_0}(z-z_0)f(z).$$

In fact, one can show (Problem 18) that for g and h analytic functions at z_0, with $g(z_0)\neq 0$, $h(z_0)=0$, and $h'(z_0)\neq 0$,

$$\operatorname{Res}\left[\frac{g(z)}{h(z)};z_0\right] = \frac{g(z_0)}{h'(z_0)}.$$

Example 6.26. *Find the residues of* $f(z) = \frac{z-1}{(z+1)^2(z^2+4)}$.

$f(z)$ has poles at $z=-1$, $z=2i$, and $z=-2i$. The pole at $z=-1$ is a double pole (pole of order 2). The other poles are simple poles. We compute those residues first:

$$\begin{aligned} Res[f(z);2i] &= \lim_{z\to 2i}(z-2i)\frac{z-1}{(z+1)^2(z+2i)(z-2i)} \\ &= \lim_{z\to 2i}\frac{z-1}{(z+1)^2(z+2i)} \\ &= \frac{2i-1}{(2i+1)^2(4i)} = -\frac{1}{50}-\frac{11}{100}i. \end{aligned} \tag{6.46}$$

$$\begin{aligned} Res[f(z);-2i] &= \lim_{z\to -2i}(z+2i)\frac{z-1}{(z+1)^2(z+2i)(z-2i)} \\ &= \lim_{z\to -2i}\frac{z-1}{(z+1)^2(z-2i)} \\ &= \frac{-2i-1}{(-2i+1)^2(-4i)} = -\frac{1}{50}+\frac{11}{100}i. \end{aligned} \tag{6.47}$$

For the double pole, we have to do a little more work.

$$\begin{aligned} Res[f(z);-1] &= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z-1}{(z+1)^2(z^2+4)}\right] \\ &= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z-1}{z^2+4}\right] \\ &= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z^2+4-2z(z-1)}{(z^2+4)^2}\right] \\ &= \lim_{z\to -1}\frac{d}{dz}\left[\frac{-z^2+2z+4}{(z^2+4)^2}\right] \\ &= \frac{1}{25}. \end{aligned} \tag{6.48}$$

Example 6.27. *Find the residue of $f(z) = \cot z$ at $z = 0$.*

We write $f(z) = \cot z = \frac{\cos z}{\sin z}$ and note that $z = 0$ is a simple pole. Thus,

$$Res[\cot z; z = 0] = \lim_{z \to 0} \frac{z \cos z}{\sin z} = \cos(0) = 1.$$

Example 6.28. $\oint_{|z|=1} \frac{dz}{\sin z}$.

We begin by looking for the singularities of the integrand. These are located at values of z for which $\sin z = 0$. Thus, $z = 0, \pm\pi, \pm 2\pi, \ldots,$ are the singularities. However, only $z = 0$ lies inside the contour, as shown in Figure 6.32. We note further that $z = 0$ is a simple pole, since

$$\lim_{z \to 0} (z - 0) \frac{1}{\sin z} = 1.$$

Therefore, the residue is one and we have

$$\oint_{|z|=1} \frac{dz}{\sin z} = 2\pi i.$$

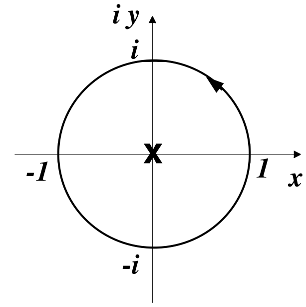


Figure 6.32: Contour for computing $\oint_{|z|=1} \frac{dz}{\sin z}$.

In general, we could have several poles of different orders. For example, we will be computing

$$\oint_{|z|=2} \frac{dz}{z^2 - 1}.$$

The integrand has singularities at $z^2 - 1 = 0$, or $z = \pm 1$. Both poles are inside the contour, as seen in Figure 6.34. One could do a partial fraction decomposition and have two integrals with one pole each. However, in cases in which we have many poles, we can use the following theorem, known as the Residue Theorem.

The Residue Theorem.

The Residue Theorem

Theorem 6.9. *Let $f(z)$ be a function which has poles z_j, $j = 1, \ldots, N$ inside a simple closed contour C and no other singularities in this region. Then,*

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{N} Res[f(z); z_j], \tag{6.49}$$

where the residues are computed using Equation (6.44).

The proof of this theorem is based upon the contours shown in Figure 6.33. One constructs a new contour C' by encircling each pole, as show in the figure. Then one connects a path from C to each circle. In the figure two paths are shown only to indicate the direction followed on the cut. The new contour is then obtained by following C and crossing each cut as it is encountered. Then one goes around a circle in the negative sense and returns along the cut to proceed around C. The sum of the contributions to the contour integration involve two integrals for each cut, which will cancel due to the opposing directions. Thus, we are left with

$$\oint_{C'} f(z)\,dz = \oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \oint_{C_3} f(z)\,dz = 0.$$

Of course, the sum is zero because $f(z)$ is analytic in the enclosed region, since all singularities have be cut out. Solving for $\oint_C f(z)\,dz$, one has that this integral is the sum of the integrals around the separate poles, which can be evaluated with single residue computations. Thus, the result is that $\oint_C f(z)\,dz$ is $2\pi i$ times the sum of the residues.

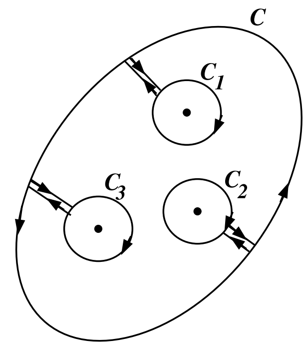


Figure 6.33: A depiction of how one cuts out poles to prove that the integral around C is the sum of the integrals around circles with the poles at the center of each.

Example 6.29. $\oint_{|z|=2} \frac{dz}{z^2-1}$.

We first note that there are two poles in this integral since

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}.$$

In Figure 6.34 we plot the contour and the two poles, denoted by an "x". Since both poles are inside the contour, we need to compute the residues for each one. They are both simple poles, so we have

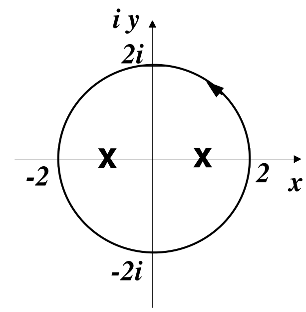


Figure 6.34: Contour for computing $\oint_{|z|=2} \frac{dz}{z^2-1}$.

$$\begin{aligned} Res\left[\frac{1}{z^2-1}; z=1\right] &= \lim_{z\to 1}(z-1)\frac{1}{z^2-1} \\ &= \lim_{z\to 1}\frac{1}{z+1} = \frac{1}{2}, \end{aligned} \tag{6.50}$$

and

$$\begin{aligned} Res\left[\frac{1}{z^2-1}; z=-1\right] &= \lim_{z\to -1}(z+1)\frac{1}{z^2-1} \\ &= \lim_{z\to -1}\frac{1}{z-1} = -\frac{1}{2}. \end{aligned} \tag{6.51}$$

Then,

$$\oint_{|z|=2} \frac{dz}{z^2-1} = 2\pi i\left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

Example 6.30. $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\,dz$.

In this example there are two poles $z = 1, -2$ inside the contour. $z = 1$ is a second order pole and $z = -2$ is a simple pole. [See Figure 6.35]. Therefore, we need the residues at each pole of $f(z) = \frac{z^2+1}{(z-1)^2(z+2)}$:

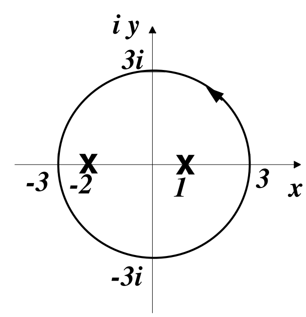


Figure 6.35: Contour for computing $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\,dz$.

$$\begin{aligned} Res[f(z); z=1] &= \lim_{z\to 1}\frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\frac{z^2+1}{(z-1)^2(z+2)}\right] \\ &= \lim_{z\to 1}\left(\frac{z^2+4z-1}{(z+2)^2}\right) \\ &= \frac{4}{9}. \end{aligned} \tag{6.52}$$

$$\begin{aligned} Res[f(z); z = -2] &= \lim_{z \to -2} (z+2) \frac{z^2+1}{(z-1)^2(z+2)} \\ &= \lim_{z \to -2} \frac{z^2+1}{(z-1)^2} \\ &= \frac{5}{9}. \end{aligned} \tag{6.53}$$

The evaluation of the integral is found by computing $2\pi i$ times the sum of the residues:

$$\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\, dz = 2\pi i \left(\frac{4}{9} + \frac{5}{9}\right) = 2\pi i.$$

Example 6.31. $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

Here we have a real integral in which there are no signs of complex functions. In fact, we could apply methods from our calculus class to do this integral, attempting to write $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$. However, we do not get very far.

Computation of integrals of functions of sines and cosines, $f(\cos\theta, \sin\theta)$.

One trick, useful in computing integrals whose integrand is in the form $f(\cos\theta, \sin\theta)$, is to transform the integration to the complex plane through the transformation $z = e^{i\theta}$. Then,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\frac{i}{2}\left(z - \frac{1}{z}\right).$$

Under this transformation, $z = e^{i\theta}$, the integration now takes place around the unit circle in the complex plane. Noting that $dz = ie^{i\theta}\, d\theta = iz\, d\theta$, we have

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} &= \oint_{|z|=1} \frac{\frac{dz}{iz}}{2 + \frac{1}{2}\left(z + \frac{1}{z}\right)} \\ &= -i \oint_{|z|=1} \frac{dz}{2z + \frac{1}{2}\left(z^2+1\right)} \\ &= -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1}. \end{aligned} \tag{6.54}$$

We can apply the Residue Theorem to the resulting integral. The singularities occur for $z^2 + 4z + 1 = 0$. Using the quadratic formula, we have the roots $z = -2 \pm \sqrt{3}$. The location of these poles are shown in Figure 6.36. Only $z = -2 + \sqrt{3}$ lies inside the integration contour. We will therefore need the residue of $f(z) = \frac{-2i}{z^2+4z+1}$ at this simple pole:

$$\begin{aligned} Res[f(z); z = -2+\sqrt{3}] &= \lim_{z \to -2+\sqrt{3}} (z - (-2+\sqrt{3})) \frac{-2i}{z^2+4z+1} \\ &= -2i \lim_{z \to -2+\sqrt{3}} \frac{z - (-2+\sqrt{3})}{(z - (-2+\sqrt{3}))(z - (-2-\sqrt{3}))} \end{aligned}$$

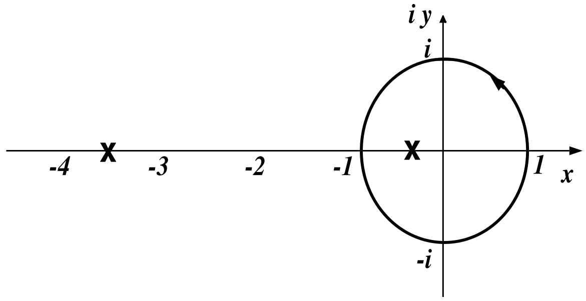


Figure 6.36: Contour for computing $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

$$\begin{aligned} &= -2i \lim_{z \to -2+\sqrt{3}} \frac{1}{z-(-2-\sqrt{3})} \\ &= \frac{-2i}{-2+\sqrt{3}-(-2-\sqrt{3})} \\ &= \frac{-i}{\sqrt{3}} \\ &= \frac{-i\sqrt{3}}{3}. \end{aligned} \tag{6.55}$$

Therefore, we have

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1} = 2\pi i \left(\frac{-i\sqrt{3}}{3} \right) = \frac{2\pi\sqrt{3}}{3}. \tag{6.56}$$

The Weierstraß substitution method.

Before moving on to further applications, we note that there is another way to compute the integral in the last example. Weierstraß introduced a substitution method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t = \tan\frac{\theta}{2}$ and converts the integrand into a rational function of t. You can show that this substitution implies that

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2},$$

and

$$d\theta = \frac{2dt}{1+t^2}.$$

The interested reader can show this in Problem 8 and apply the method. In order to see how it works, we will redo the last problem.

Example 6.32. *Apply the Weierstraß substitution method to compute* $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} &= \int_{-\infty}^{\infty} \frac{1}{2+\frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2} \\ &= 2\int_{-\infty}^{\infty} \frac{dt}{t^2+3} \\ &= \frac{2}{3}\sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}t\right)\Big|_{-\infty}^{\infty} = \frac{2\pi\sqrt{3}}{3}. \end{aligned} \tag{6.57}$$

6.5.6 *Infinite Integrals*

As our final application of complex integration techniques, we will turn to the evaluation of infinite integrals of the form $\int_{-\infty}^{\infty} f(x)\, dx$. These types of integrals will appear later in the text and will help to tie in what seems to be a digression in our study of physics. In this section we will see that such integrals may be computed by extending the integration to a contour in the complex plane.

Recall that such integrals are improper integrals and you had seen them in your calculus classes. The way that one determines if such integrals exist, or converge, is to compute the integral using a limit:

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx.$$

For example,

$$\int_{-\infty}^{\infty} x\, dx = \lim_{R\to\infty} \int_{-R}^{R} x\, dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{(-R)^2}{2} \right) = 0.$$

However, the integrals $\int_0^{\infty} x\, dx$ and $\int_{-\infty}^{0} x\, dx$ do not exist. Note that

$$\int_0^{\infty} x\, dx = \lim_{R\to\infty} \int_0^{R} x\, dx = \lim_{R\to\infty} \left(\frac{R^2}{2} \right) = \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_0^{\infty} f(x)\, dx$$

does not exist while $\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$ does exist. We will be interested in computing the latter type of integral. Such an integral is called the *Cauchy Principal Value Integral* and is denoted with either a P, PV, or a bar through the integral:

The Cauchy principal value integral.

$$P \int_{-\infty}^{\infty} f(x)\, dx = PV \int_{-\infty}^{\infty} f(x)\, dx = ⨍_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx. \tag{6.58}$$

If there is a discontinuity in the integral, one can further modify this definition of principal value integral to bypass the singularity. For example, if $f(x)$ is continuous on $a \le x \le b$ and not defined at $x = x_0$, then

$$\int_a^b f(x)\, dx = \lim_{\epsilon\to 0} \left(\int_a^{x_0-\epsilon} f(x)\, dx + \int_{x_0+\epsilon}^{b} f(x)\, dx \right).$$

In our discussions we will be computing integrals over the real line in the Cauchy principal value sense.

Example 6.33. *Compute $\int_{-1}^{1} \frac{dx}{x^3}$ in the Cauchy Principal Value sense. In this case, $f(x) = \frac{1}{x^3}$ is not defined at $x = 0$. So, we have*

$$\begin{aligned} \int_{-1}^{1} \frac{dx}{x^3} &= \lim_{\epsilon\to 0}\left(\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3}\right) \\ &= \lim_{\epsilon\to 0}\left(-\frac{1}{2x^2}\bigg|_{-1}^{-\epsilon} - \frac{1}{2x^2}\bigg|_{\epsilon}^{1}\right) = 0. \qquad (6.59) \end{aligned}$$

We now proceed to the evaluation of such principal value integrals using complex integration methods. We want to evaluate the integral $\int_{-\infty}^{\infty} f(x)\,dx$. We will extend this into an integration in the complex plane. We extend $f(x)$ to $f(z)$ and assume that $f(z)$ is analytic in the upper half plane ($\text{Im}(z) > 0$) except at isolated poles. We then consider the integral $\int_{-R}^{R} f(x)\,dx$ as an integral over the interval $(-R, R)$. We view this interval as a piece of a contour C_R obtained by completing the contour with a semicircle Γ_R of radius R extending into the upper half plane as shown in Figure 6.37. Note, a similar construction is sometimes needed extending the integration into the lower half plane ($\text{Im}(z) < 0$) as we will later see.

Computation of real integrals by embedding the problem in the complex plane.

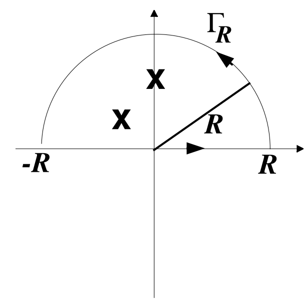


Figure 6.37: Contours for computing $P\int_{-\infty}^{\infty} f(x)\,dx$.

The integral around the entire contour C_R can be computed using the Residue Theorem and is related to integrations over the pieces of the contour by

$$\oint_{C_R} f(z)\,dz = \int_{\Gamma_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz. \qquad (6.60)$$

Taking the limit $R \to \infty$ and noting that the integral over $(-R, R)$ is the desired integral, we have

$$P\int_{-\infty}^{\infty} f(x)\,dx = \oint_C f(z)\,dz - \lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz, \qquad (6.61)$$

where we have identified C as the limiting contour as R gets large.

Now the key to carrying out the integration is that the second integral vanishes in the limit. This is true if $R|f(z)| \to 0$ along Γ_R as $R \to \infty$. This can be seen by the following argument. We can parametrize the contour Γ_R using $z = Re^{i\theta}$. Then, when $|f(z)| < M(R)$,

$$\begin{aligned} \left|\int_{\Gamma_R} f(z)\,dz\right| &= \left|\int_0^{2\pi} f(Re^{i\theta})Re^{i\theta}\,d\theta\right| \\ &\le R\int_0^{2\pi} \left|f(Re^{i\theta})\right|\,d\theta \\ &< RM(R)\int_0^{2\pi} d\theta \\ &= 2\pi RM(R). \qquad (6.62) \end{aligned}$$

So, if $\lim_{R\to\infty} RM(R) = 0$, then $\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = 0$.

We show how this applies some examples.

Example 6.34. $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

We already know how to do this integral from our calculus course. We have that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \left(2\tan^{-1} R\right) = 2\left(\frac{\pi}{2}\right) = \pi.$$

We will apply the methods of this section and confirm this result. The needed contours are shown in Figure 6.38 and the poles of the integrand are at $z = \pm i$.

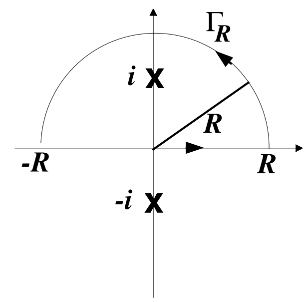


Figure 6.38: Contour for computing $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

We first note that $f(z) = \frac{1}{1+z^2}$ *goes to zero fast enough on* Γ_R *as* R *gets large.*

$$R|f(z)| = \frac{R}{|1+R^2e^{2i\theta}|} = \frac{R}{\sqrt{1+2R^2\cos\theta + R^4}}.$$

Thus, as $R\to\infty$, $R|f(z)|\to 0$. *So,*

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint_C \frac{dz}{1+z^2}.$$

We need only compute the residue at the enclosed pole, $z = i$.

$$Res[f(z); z=i] = \lim_{z\to i}(z-i)\frac{1}{1+z^2} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}.$$

Then, using the Residue Theorem, we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i\left(\frac{1}{2i}\right) = \pi.$$

Example 6.35. $P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$.

There are several new techniques that have to be introduced in order to carry out this integration. We need to handle the pole at $z = 0$ *in a special way and we need something called Jordan's Lemma to guarantee that integral over the contour* Γ_R *vanishes.*

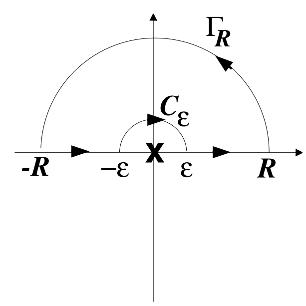


Figure 6.39: Contour for computing $P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$.

For this example the integral is unbounded at $z = 0$. *Constructing the contours as before we are faced for the first time with a pole lying on the contour. We cannot ignore this fact. We can proceed with our computation by carefully going around the pole with a small semicircle of radius* ϵ, *as shown in Figure 6.39. Then our principal value integral computation becomes*

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \lim_{\epsilon\to 0, R\to\infty}\left(\int_{-R}^{-\epsilon} \frac{\sin x}{x}\,dx + \int_{\epsilon}^{R} \frac{\sin x}{x}\,dx\right). \qquad (6.63)$$

We will also need to rewrite the sine function in term of exponentials in this integral.

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2i}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\,dx\right). \qquad (6.64)$$

We now employ **Jordan's Lemma**.

Jordan's Lemma

If $f(z)$ converges uniformly to zero as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{ikz}\, dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$.

A similar result applies for $k < 0$, but one closes the contour in the lower half plane. [See Section 6.5.8 for the proof of Jordan's Lemma.]

We now put these ideas together to compute the given integral. According to Jordan's lemma, we will need to compute the above exponential integrals using two different contours. We first consider $P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx$. We use the contour in Figure 6.39. Then we have

$$\oint_{C_R} \frac{e^{iz}}{z}\, dz = \int_{\Gamma_R} \frac{e^{iz}}{z}\, dz + \int_{-R}^{-\epsilon} \frac{e^{iz}}{z}\, dz + \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz + \int_{\epsilon}^{R} \frac{e^{iz}}{z}\, dz.$$

The integral $\oint_{C_R} \frac{e^{iz}}{z}\, dz$ vanishes since there are no poles enclosed in the contour! The integral over Γ_R will vanish as R gets large according to Jordan's Lemma. The sum of the second and fourth integrals is the integral we seek as $\epsilon \to 0$ and $R \to \infty$.

The remaining integral around the small circle has to be done separately.[6] *We have*

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = \int_{\pi}^{0} \frac{\exp(i\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta = -\int_0^{\pi} i \exp(i\epsilon e^{i\theta})\, d\theta.$$

Taking the limit as ϵ goes to zero, the integrand goes to i and we have

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = -\pi i.$$

So far, we have that

$$P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = -\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\, dz = \pi i.$$

We can compute $P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx$ in a similar manner, being careful with the sign changes due to the orientations of the contours as shown in Figure 6.40. In this case, we find the same value

$$P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx = \pi i.$$

Finally, we can compute the original integral as

$$\begin{aligned} P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx &= \frac{1}{2i}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx \right) \\ &= \frac{1}{2i}(\pi i + \pi i) \\ &= \pi. \end{aligned} \tag{6.65}$$

[6] Note that we have not previously done integrals in which a singularity lies on the contour. One can show, as in this example, that points like this can be accounted for by using using half of a residue (times $2\pi i$). For the semicircle C_ϵ you can verify this. The negative sign comes from going clockwise around the semicircle.

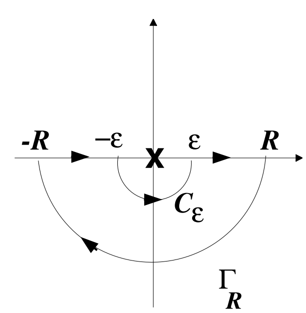


Figure 6.40: Contour in the lower half plane for computing $P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx$.

Example 6.36. *Evaluate* $\oint_{|z|=1} \frac{dz}{z^2+1}$.

In this example there are two simple poles, $z = \pm i$ lying on the contour, as seen in Figure 6.41. This problem is similar to Problem 1c, except we will do it using contour integration instead of a parametrization. We bypass the two poles by drawing small semicircles around them. Since the poles are not included in the closed contour, then the Residue Theorem tells us that the integral nd the path vanishes. We can write the full integration as a sum over three paths, $C_\pm$ for the semicircles and C for the original contour with the poles cut out. Then we take the limit as the semicircle radii go to zero. So,

$$0 = \int_C \frac{dz}{z^2+1} + \int_{C_+} \frac{dz}{z^2+1} + \int_{C_-} \frac{dz}{z^2+1}.$$

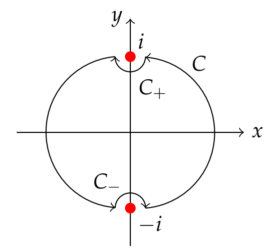


Figure 6.41: Example with poles on contour.

The integral over the semicircle around i can be done using the parametrization $z = i + \epsilon e^{i\theta}$. Then $z^2 + 1 = 2i\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}$. This gives

$$\int_{C_+} \frac{dz}{z^2+1} = \lim_{\epsilon\to 0} \int_0^{-\pi} \frac{i\epsilon e^{i\theta}}{2i\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}}\, d\theta = \frac{1}{2}\int_0^{-\pi} d\theta = -\frac{\pi}{2}.$$

As in the last example, we note that this is just πi times the residue, $Res\left[\frac{1}{z^2+1}; z = i\right] = \frac{1}{2i}$. Since the path is traced clockwise, we find the contribution is $-\pi i Res = -\frac{\pi}{2}$, which is what we obtained above. A Similar computation will give the contribution from $z = -i$ as $\frac{\pi}{2}$. Adding these values gives the total contribution from $C_\pm$ as zero. So, the final result is that

$$\oint_{|z|=1} \frac{dz}{z^2+1} = 0.$$

Example 6.37. *Evaluate* $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx$, *for* $0 < a < 1$.

In dealing with integrals involving exponentials or hyperbolic functions it is sometimes useful to use different types of contours. This example is one such case. We will replace x with z and integrate over the contour in Figure 6.42. Letting $R \to \infty$, the integral along the real axis is the integral that we desire. The integral along the path for $y = 2\pi$ leads to a multiple of this integral since $z = x + 2\pi i$ along this path. Integration along the vertical paths vanish as $R \to \infty$.

Thus, we are left with the following computation:

$$\begin{aligned}\oint_C \frac{e^{az}}{1+e^z}dz &= \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx - e^{2\pi i a}\int_{-R}^{R}\frac{e^{ax}}{1+e^x}dx\right)\\ &= (1-e^{2\pi i a})\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx. \qquad (6.66)\end{aligned}$$

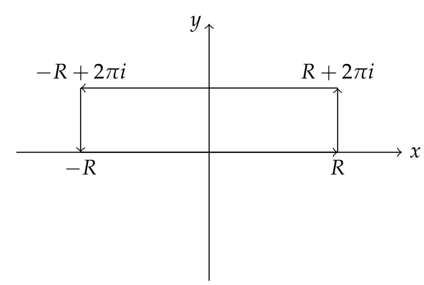


Figure 6.42: Example using a rectangular contour.

We need only evaluate the left contour integral using the Residue Theorem. The poles are found from

$$1 + e^z = 0.$$

Within the contour, this is satisfied by $z = i\pi$. So,

$$Res\left[\frac{e^{az}}{1+e^z}; z = i\pi\right] = \lim_{z\to i\pi}(z - i\pi)\frac{e^{az}}{1+e^z} = -e^{i\pi a}.$$

Applying the Residue Theorem, we have

$$(1-e^{2\pi ia})\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx = -2\pi i e^{i\pi a}.$$

Therefore, we have found that

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx = \frac{-2\pi i e^{i\pi a}}{1-e^{2\pi ia}} = \frac{\pi}{\sin \pi a}, \quad 0<a<1.$$

6.5.7 *Integration Over Multivalued Functions*

In the last chapter we found that some complex functions inherently possess multivaluedness; i.e., such functions do not evaluate to a single value, but have many values. The key examples were $f(z) = z^{1/n}$ and $f(z) = \ln z$. The nth roots have n distinct values and logarithms have an infinite number of values as determined by the range of the resulting arguments. We mentioned that the way to handle multivaluedness is to assign different branches to these functions, introduce a branch cut and glue them together at the branch cuts to form Riemann surfaces. In this way we can draw continuous paths along the Riemann surfaces as we move from on Riemann sheet to another.

Before we do examples of contour integration involving multivalued functions, lets first try to get a handle on multivaluedness in a simple case. We will consider the square root function,

$$w = z^{1/2} = r^{1/2}e^{i(\frac{\theta}{2}+k\pi)}, \quad k=0,1.$$

There are two branches, corresponding to each k value. If we follow a path not containing the origin, then we stay in the same branch, so the final argument (θ) will be equal to the initial argument. However, if we follow a path that encloses the origin, this will not be true. In particular, for an initial point on the unit circle, $z_0 = e^{i\theta_0}$, we have its image as $w_0 = e^{i\theta_0/2}$. However, if we go around a full revolution, $\theta = \theta_0 + 2\pi$, then

$$z_1 = e^{i\theta_0 + 2\pi i} = e^{i\theta_0},$$

but

$$w_1 = e^{(i\theta_0+2\pi i)/2} = e^{i\theta_0/2}e^{\pi i} \neq w_0.$$

Here we obtain a final argument (θ) that is not be equal to the initial argument! Somewhere, we have crossed from one branch to another. Points, such as the origin in this example, are called branch points. Actually, there are two branch points, because we can view the closed path around the origin as a closed path around complex infinity in the compactified complex plane. However, we will not go into that at this time.

We can show this in the following figures. In Figure 6.43 we show how the points A-E are mapped from the z-plane into the w-plane under the square root function for the principal branch, $k = 0$. As we trace out the unit circle in the z-plane, we only trace out a semicircle in the w-plane. If we consider the branch $k = 1$, we then trace out a semicircle in the lower half plane, as shown in Figure 6.44 following the points from F to J.

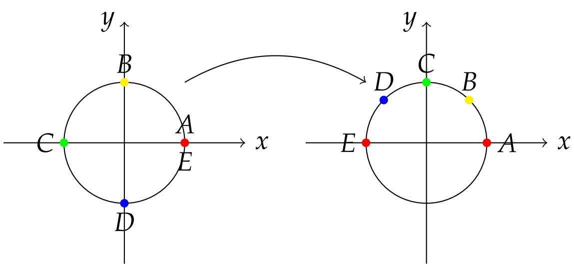


Figure 6.43: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the square root function.

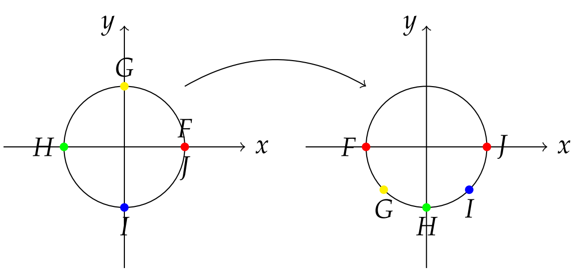


Figure 6.44: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the square root function.

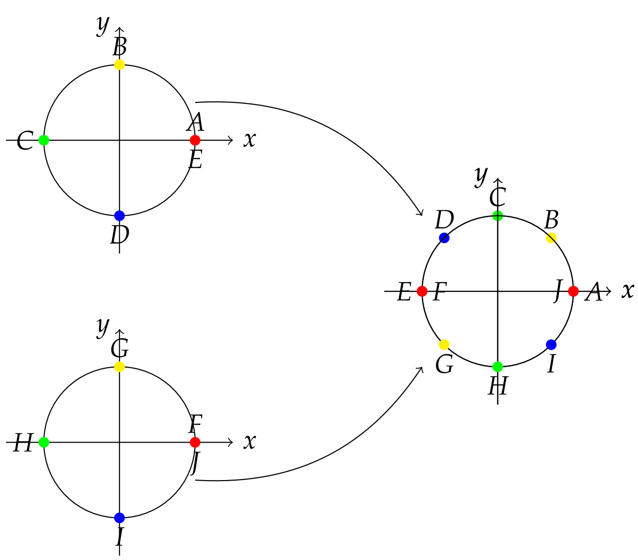


Figure 6.45: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the square root function.

We can combine these into one mapping depicting how the two complex planes corresponding to each branch provide a mapping to the w-plane. This is shown in Figure 6.45. A common way to draw this domain, which looks like two separate complex planes, would be to glue them together. Imagine cutting each plane along the positive x-axis, extending between the two branch points, $z = 0$ and $z = \infty$. As one approaches the cut on the principal branch, then one can move onto the glued second branch. Then one continues around the origin on this branch until one once again reaches the cut. This cut is glued to the principal branch in such a way that the path returns to its starting point. The resulting surface we obtain is the Riemann surface shown in Figure 6.46. Note that there is nothing that forces us to place the branch cut at a particular place. For example, the branch cut could be along the positive real axis, the negative real axis, or any path connecting the origin and complex infinity.

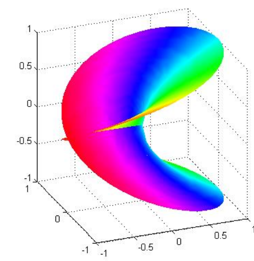


Figure 6.46: Riemann surface for $f(z) = z^{1/2}$.

We now turn to a couple of examples of integrals of multivalued functions.

Example 6.38. *Evaluate* $\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$.

We consider the contour integral $\oint_C \frac{\sqrt{z}}{1+z^2}\,dz$.

The first thing we can see in this problem is the square root function in the integrand. Being there is a multivalued function, we locate the branch point and determine where to draw the branch cut. In Figure 6.47 we show the contour that we will use in this problem. Note that we picked the branch cut along the positive x-axis.

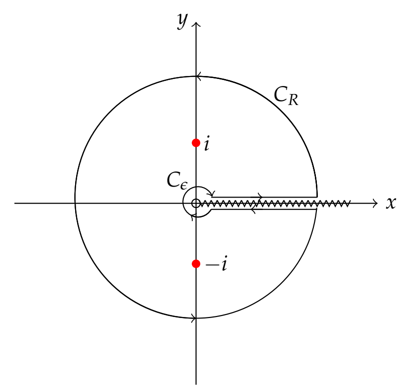


Figure 6.47: An example of a contour which accounts for a branch cut.

We take the contour C *to be positively oriented, being careful to enclose the two poles and to hug the branch cut. It consists of two circles. The outer circle* C_R *is a circle of radius* R *and the inner circle* C_ϵ *will have a radius of* ϵ. *The sought answer will be obtained by letting* $R \to \infty$ *and* $\epsilon \to 0$. *On the large circle we have that the integrand goes to zero fast enough as* $R \to \infty$. *The integral around the small circle vanishes as* $\epsilon \to 0$. *We can see this by parametrizing the circle as* $z = \epsilon e^{i\theta}$ *for* $\theta \in [0, 2\pi]$:

$$\begin{aligned}\oint_{C_\epsilon} \frac{\sqrt{z}}{1+z^2}\,dz &= \int_0^{2\pi} \frac{\sqrt{\epsilon e^{i\theta}}}{1+(\epsilon e^{i\theta})^2} i\epsilon e^{i\theta}\,d\theta \\ &= i\epsilon^{3/2} \int_0^{2\pi} \frac{e^{3i\theta/2}}{1+(\epsilon^2 e^{2i\theta})}\,d\theta. \qquad (6.67)\end{aligned}$$

It should now be easy to see that as $\epsilon \to 0$ *this integral vanishes.*

The integral above the branch cut is the one we are seeking, $\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$. *The integral under the branch cut, where* $z = re^{2\pi i}$, *is*

$$\begin{aligned}\int \frac{\sqrt{z}}{1+z^2}\,dz &= \int_\infty^0 \frac{\sqrt{re^{2\pi i}}}{1+r^2e^{4\pi i}}\,dr \\ &= \int_0^\infty \frac{\sqrt{r}}{1+r^2}\,dr. \qquad (6.68)\end{aligned}$$

We note that this is the same as that above the cut.

Up to this point, we have that the contour integral, as $R \to \infty$ and $\epsilon \to 0$ is

$$\oint_C \frac{\sqrt{z}}{1+z^2} dz = 2\int_0^\infty \frac{\sqrt{x}}{1+x^2} dx.$$

In order to finish this problem, we need the residues at the two simple poles.

$$Res\left[\frac{\sqrt{z}}{1+z^2}; z = i\right] = \frac{\sqrt{i}}{2i} = \frac{\sqrt{2}}{4}(1+i),$$

$$Res\left[\frac{\sqrt{z}}{1+z^2}; z = -i\right] = \frac{\sqrt{-i}}{-2i} = \frac{\sqrt{2}}{4}(1-i).$$

So,

$$2\int_0^\infty \frac{\sqrt{x}}{1+x^2} dx = 2\pi i\left(\frac{\sqrt{2}}{4}(1+i) + \frac{\sqrt{2}}{4}(1-i)\right) = \pi\sqrt{2}.$$

Finally, we have the value of the integral that we were seeking,

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2} dx = \frac{\pi\sqrt{2}}{2}.$$

Example 6.39. *Compute $\int_a^\infty f(x)\, dx$ using contour integration involving logarithms.*[7]

In this example we will apply contour integration to the integral

$$\oint_C f(z) \ln(a-z)\, dz$$

for the contour shown in Figure 6.48.

We will assume that $f(z)$ is single valued and vanishes as $|z| \to \infty$. We will choose the branch cut to span from the origin along the positive real axis. Employing the Residue Theorem and breaking up the integrals over the pieces of the contour in Figure 6.48, we have schematically that

$$2\pi i \sum Res[f(z)\ln(a-z)] = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) f(z)\ln(a-z)\, dz.$$

First of all, we need to assume that $f(z)$ is well behaved at $z = a$ and vanishes fast enough as $|z| = R \to \infty$ Then the integrals over C_2 and C_4 will vanish. For example, for the path C_4, we let $z = a + \epsilon e^{i\theta}$, $0 < \theta < 2\pi$. Then,

$$\int_{C_4} f(z)\ln(a-z)\, dz. = \lim_{\epsilon\to 0}\int_{2\pi}^0 f(a+\epsilon e^{i\theta})\ln(\epsilon e^{i\theta}) i\epsilon e^{i\theta}\, d\theta.$$

If $f(a)$ is well behaved, then we only need to show that $\lim_{\epsilon\to 0} \epsilon \ln \epsilon = 0$. This is left to the reader.

Similarly, we consider the integral over C_2 as R gets large,

$$\int_{C_2} f(z)\ln(a-z)\, dz = \lim_{R\to\infty}\int_0^{2\pi} f(Re^{i\theta})\ln(Re^{i\theta}) iRe^{i\theta}\, d\theta.$$

[7] This was originally published in Neville, E. H., 1945, "Indefinite integration by means of residues". *The Mathematical Student*, **13**, 16-35, and discussed in Duffy, D. G., *Transform Methods for Solving Partial Differential Equations*, 1994.

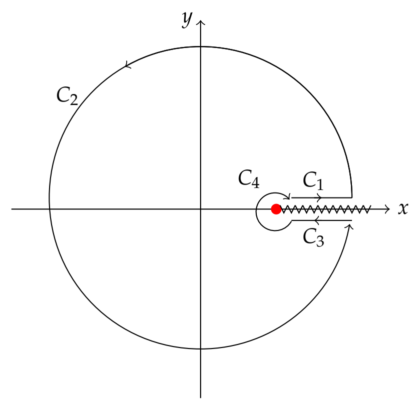


Figure 6.48: Contour needed to compute $\oint_C f(z)\ln(a-z)\, dz$.

Thus, we need only require that

$$\lim_{R\to\infty} R\ln R|f(Re^{i\theta})| = 0.$$

Next, we consider the two straight line pieces. For C_1, the integration of the real axis occurs for $z = x$, so

$$\int_{C_1} f(z)\ln(a-z)\,dz = \int_a^\infty f(x)\ln(a-x)\,dz.$$

However, integration over C_3 requires noting that we need the branch for the logarithm such that $\ln z = \ln(a-x) + 2\pi i$. Then,

$$\int_{C_3} f(z)\ln(a-z)\,dz = \int_\infty^a f(x)[\ln(a-x) + 2\pi i]\,dz.$$

Combining these results, we have

$$\int_a^\infty f(x)\,dx = -\sum Res[f(z)\ln(a-z)].$$

Example 6.40. *Compute $\int_1^\infty \frac{dx}{4x^2-1}$.*

We can apply the last example to this case. Namely,

$$\begin{aligned}\int_1^\infty \frac{dx}{4x^2-1} &= -Res\left[\frac{\ln(1-z)}{4z^2-1}; z=\frac{1}{2}\right] - Res\left[\frac{\ln(1-z)}{4z^2-1}; z=-\frac{1}{2}\right]\\ &= -\frac{\ln\frac{1}{2}}{4} + \frac{\ln\frac{3}{2}}{4} = \frac{\ln 3}{4}. \end{aligned} \tag{6.69}$$

6.5.8 Appendix: Jordan's Lemma

FOR COMPLETENESS, we prove Jordan's Lemma.

Theorem 6.10. *If $f(z)$ converges uniformly to zero as $z \to \infty$, then*

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{ikz}\,dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$.

Proof. We consider the integral

$$I_R = \int_{C_R} f(z)e^{ikz}\,dz,$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$ in the complex plane. Let $z = Re^{i\theta}$ be a parametrization of C_R. Then,

$$I_R = \int_0^\pi f(Re^{i\theta})e^{ikR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\,d\theta.$$

Since

$$\lim_{|z|\to\infty} f(z) = 0, \quad 0 \le \arg z \le \pi,$$

then for large $|R|$, $|f(z)| < \epsilon$ for some $\epsilon > 0$. Then,

$$\begin{aligned}
|I_R| &= \left| \int_0^{\pi} f(Re^{i\theta}) e^{ikR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta \right| \\
&\leq \int_0^{\pi} \left| f(Re^{i\theta}) \right| \left| e^{ikR\cos\theta} \right| \left| e^{-aR\sin\theta} \right| \left| iRe^{i\theta} \right| d\theta \\
&\leq \epsilon R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta \\
&= 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta. \qquad (6.70)
\end{aligned}$$

The last integral still cannot be computed, but we can get a bound on it over the range $\theta \in [0, \pi/2]$. Note that

$$\sin\theta \geq \frac{\pi}{2}\theta, \quad \theta \in [0, \pi/2].$$

Therefore, we have

$$|I_R| \leq 2\epsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\, d\theta = \frac{2\epsilon R}{2aR/\pi}(1 - e^{-aR}).$$

For large R we have

$$\lim_{R\to\infty} |I_R| \leq \frac{\pi\epsilon}{a}.$$

So, as $\epsilon \to 0$, the integral vanishes.

□

Problems

1. Write the following in standard form.

a. $(4 - 7i)(-2 + 3i)$.

b. $(1 - i)^3$.

c. $\frac{5+2i}{1+i}$.

2. Write the following in polar form, $z = re^{i\theta}$.

a. $i - 1$.

b. $-2i$.

c. $\sqrt{3} + 3i$.

3. Write the following in rectangular form, $z = a + ib$.

a. $10e^{i\pi/6}$.

b. $\sqrt{2}e^{5i\pi/4}$.

c. $(1 - i)^{100}$.

4. Find all z such that $z^4 = 16i$. Write the solutions in rectangular form, $z = a + ib$, with no decimal approximation or trig functions.

5. Show that $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ using trigonometric identities and the exponential forms of these functions.

6. Find all z such that $\cos z = 2$, or explain why there are none. You will need to consider $\cos(x + iy)$ and equate real and imaginary parts of the resulting expression similar to problem 5.

7. Find the principal value of i^i. Rewrite the base, i, as an exponential first.

8. Consider the circle $|z - 1| = 1$.

a. Rewrite the equation in rectangular coordinates by setting $z = x + iy$.

b. Sketch the resulting circle using part a.

c. Consider the image of the circle under the mapping $f(z) = z^2$, given by $|z^2 - 1| = 1$.

i. By inserting $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, find the equation of the image curve in polar coordinates.

ii. Sketch the image curve. You may need to refer to your Calculus II text for polar plots. [Maple might help.]

9. Find the real and imaginary parts of the functions:

a. $f(z) = z^3$.

b. $f(z) = \sinh(z)$.

c. $f(z) = \cos \bar{z}$.

10. Find the derivative of each function in Problem 9 when the derivative exists. Otherwise, show that the derivative does not exist.

11. Let $f(z) = u + iv$ be differentiable. Consider the vector field given by $\mathbf{F} = v\mathbf{i} + u\mathbf{j}$. Show that the equations $\nabla \cdot \mathbf{F} = \mathbf{0}$ and $\nabla \times \mathbf{F} = \mathbf{0}$ are equivalent to the Cauchy-Riemann equations. [You will need to recall from multivariable calculus the del operator, $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$.]

12. What parametric curve is described by the function

$$\gamma(t) = (3t + 4) + i(t - 6),$$

$0 \leq t \leq 1$? [Hint: What would you do if you were instead considering the parametric equations $x = 3t + 4$ and $y = t - 6$?]

13. Write the equation that describes the circle of radius 2 which is centered at $z = 3 - 2i$ in a) Cartesian form (in terms of x and y); b) polar form (in terms of θ and r); c) complex form (in terms of z, r, and $e^{i\theta}$).

14. Consider the function $u(x, y) = x^3 - 3xy^2$.

a. Show that $u(x, y)$ is harmonic; i.e., $\nabla^2 u = 0$.

b. Find its harmonic conjugate, $v(x, y)$.

c. Find a differentiable function, $f(z)$, for which $u(x, y)$ is the real part.

d. Determine $f'(z)$ for the function in part c. [Use $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and rewrite your answer as a function of z.]

15. Evaluate the following integrals:

a. $\int_C \bar{z}\, dz$, where C is the parabola $y = x^2$ from $z = 0$ to $z = 1 + i$.

b. $\int_C f(z)\, dz$, where $f(z) = z + 2\bar{z}$ and C is the path from $z = 0$ to $z = 1 + 2i$ consisting of two line segments from $z = 0$ to $z = 1$ and then $z = 1$ to $z = 1 + 2i$.

c. $\int_C \frac{1}{z^2+4}\, dz$ for C the positively oriented circle, $|z| = 2$. [Hint: Parametrize the circle as $z = 2e^{i\theta}$, multiply numerator and denominator by $e^{-i\theta}$, and put in trigonometric form.]

16. Let C be the ellipse $9x^2 + 4y^2 = 36$ traversed once in the counterclockwise direction. Define

$$g(z_0) = \int_C \frac{z^2 + z + 1}{z - z_0}\, dz.$$

Find $g(i)$ and $g(4i)$. [Hint: Sketch the ellipse in the complex plane. Use the Cauchy Integral Theorem with an appropriate $f(z)$, of Cauchy's Theorem if z_0 is outside the contour.]

17. Show that

$$\int_C \frac{dz}{(z - 1 - i)^{n+1}} = \begin{cases} 0, & n \neq 0, \\ 2\pi i, & n = 0, \end{cases}$$

for C the boundary of the square $0 \leq x \leq 2$, $0 \leq y \leq 2$ taken counterclockwise. [Hint: Use the fact that contours can be deformed into simpler shapes (like a circle) as long as the integrand is analytic in the region between them. After picking a simpler contour, integrate using parametrization.]

18. Show that for g and h analytic functions at z_0, with $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$,

$$\text{Res}\left[\frac{g(z)}{h(z)}; z_0\right] = \frac{g(z_0)}{h'(z_0)}.$$

19. For the following determine if the given point is a removable singularity, an essential singularity, or a pole (indicate its order).

a. $\frac{1-\cos z}{z^2}$, $\quad z=0$.
b. $\frac{\sin z}{z^2}$, $\quad z=0$.
c. $\frac{z^2-1}{(z-1)^2}$, $\quad z=1$.
d. $ze^{1/z}$, $\quad z=0$.
e. $\cos\frac{\pi}{z-\pi}$, $\quad z=\pi$.

20. Find the Laurent series expansion for $f(z)=\frac{\sinh z}{z^3}$ about $z=0$. [Hint: You need to first do a MacLaurin series expansion for the hyperbolic sine.]

21. Find series representations for all indicated regions.

a. $f(z)=\frac{z}{z-1}$, $|z|<1$, $|z|>1$.
b. $f(z)=\frac{1}{(z-i)(z+2)}$, $|z|<1$, $1<|z|<2$, $|z|>2$. [Hint: Use partial fractions to write this as a sum of two functions first.]

22. Find the residues at the given points:

a. $\frac{2z^2+3z}{z-1}$ at $z=1$.
b. $\frac{\ln(1+2z)}{z}$ at $z=0$.
c. $\frac{\cos z}{(2z-\pi)^3}$ at $z=\frac{\pi}{2}$.

23. Consider the integral $\int_0^{2\pi}\frac{d\theta}{5-4\cos\theta}$.

a. Evaluate this integral by making the substitution $2\cos\theta=z+\frac{1}{z}$, $z=e^{i\theta}$ and using complex integration methods.
b. In the 1800's Weierstrass introduced a method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t=\tan\frac{\theta}{2}$ and converts the integrand into a rational function of t. Note that the integration around the unit circle corresponds to $t\in(-\infty,\infty)$.

i. Show that
$$\sin\theta=\frac{2t}{1+t^2},\quad \cos\theta=\frac{1-t^2}{1+t^2}.$$

ii. Show that
$$d\theta=\frac{2dt}{1+t^2}$$

iii. Use the Weierstrass substitution to compute the above integral.

24. Do the following integrals.

a.
$$\oint_{|z-i|=3}\frac{e^z}{z^2+\pi^2}\,dz.$$

b.
$$\oint_{|z-i|=3} \frac{z^2-3z+4}{z^2-4z+3}\,dz.$$

c.
$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4}\,dx.$$

[Hint: This is $\operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\,dx$.]

25. Evaluate the integral $\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx$.

[Hint: Replace x with $z = e^t$ and use the rectangular contour in Figure 6.49 with $R \to \infty$.

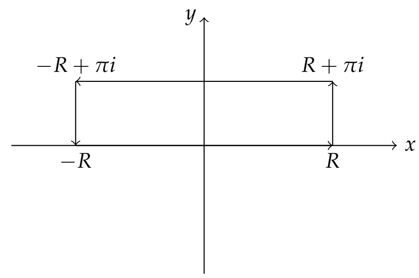


Figure 6.49: Example using a rectangular contour.

26. Do the following integrals for fun!

a. For C the boundary of the square $|x| \leq 2$, $|y| \leq 2$,
$$\oint_C \frac{dz}{z(z-1)(z-3)^2}.$$

b.
$$\int_0^\pi \frac{\sin^2\theta}{13-12\cos\theta}\,d\theta.$$

c.
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+5x+6}.$$

d.
$$\int_0^\infty \frac{\cos \pi x}{1-9x^2}\,dx.$$

e.
$$\int_0^\infty \frac{dx}{(x^2+9)(1-x)^2}.$$

f.
$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx.$$

g.
$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx.$$

7
Transform Techniques in Physics

"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.", Nikolai Lobatchevsky (1792-1856)

7.1 Introduction

Some of the most powerful tools for solving problems in physics are transform methods. The idea is that one can transform the problem at hand to a new problem in a different space, hoping that the problem in the new space is easier to solve. Such transforms appear in many forms.

In this chapter we will explore the use of integral transforms. Given a function $f(x)$, we define an integral transform to a new function $F(k)$ as

$$F(k) = \int_a^b f(x)K(x,k)\,dx.$$

Here $K(x,k)$ is called the kernel of the transform. We will concentrate specifically on Fourier transforms,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx,$$

and Laplace transforms

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt.$$

As we had seen in Chapter 3 and will see later in the book, the solutions of a linear partial differential equations can be found by using the method of separation of variables to reduce solving PDEs to solving ODEs. We can also use transform methods to transform the given PDE into ODEs or algebraic equations. Solving these equations, we then construct solutions of the PDE (or ODE) using an inverse transform. A schematic of these processes is shown below and we will describe in this chapter how one can use Fourier and Lapace transforms to this effect.

$$\begin{tikzcd}
PDE \arrow[r] & ODE \arrow[l, bend left] \arrow[r] & AlgEQ \arrow[l, bend left] \arrow[ll, bend left]
\end{tikzcd}$$

7.1.1 Example 1 - The Linearized KdV Equation

As a relatively simple example, we consider the linearized Kortweg-deVries (KdV) equation:

$$u_t + cu_x + \beta u_{xxx} = 0, \quad -\infty < x < \infty. \tag{7.1}$$

This equation governs the propagation of some small amplitude water waves. Its nonlinear counterpart has been at the center of attention in the last 40 years as a generic nonlinear wave equation.

The nonlinear counterpart to this equation is the Kortweg-deVries (KdV) equation: $u_t + 6uu_x + u_{xxx} = 0$. This equation was derived by Diederik Johannes Korteweg (1848-1941) and his student Gustav de Vries (1866-1934). This equation governs the propagation of traveling waves called solitons. These were first observed by John Scott Russell (1808-1882) and were the source of a long debate on the existence of such waves. The history of this debate is interesting and the KdV turned up as a generic equation in many other fields in the latter part of the last century leading to many papers on nonlinear evolution equations.

We seek solutions that oscillate in space. So, we assume a solution of the form

$$u(x,t) = A(t)e^{ikx}. \tag{7.2}$$

Such behavior was seen in Chapters 3 and 6 for the wave equation for vibrating strings. In that case, we found plane wave solutions of the form $e^{ik(x\pm ct)}$, which we could write as $e^{i(kx\pm\omega t)}$ by defining $\omega = kc$. We further note that one often seeks complex solutions as a linear combination of such forms and then takes the real part in order to obtain physical solutions. In this case, we will find plane wave solutions for which the angular frequency $\omega = \omega(k)$ is a function of the wavenumber.

Inserting the guess into the linearized KdV equation, we find that

$$\frac{dA}{dt} + i(ck - \beta k^3)A = 0. \tag{7.3}$$

Thus, we have converted the problem of seeking a solution of the partial differential equation into seeking a solution to an ordinary differential equation. This new problem is easier to solve. In fact, given an initial value, $A(0)$, we have

$$A(t) = A(0)e^{-i(ck-\beta k^3)t}. \tag{7.4}$$

Therefore, the solution of the partial differential equation is

$$u(x,t) = A(0)e^{ik(x-(c-\beta k^2)t)}. \tag{7.5}$$

We note that this solution takes the form $e^{i(kx-\omega t)}$, where

$$\omega = ck - \beta k^3.$$

In general, the equation $\omega = \omega(k)$ gives the angular frequency as a function of the wave number, k, and is called a *dispersion relation*. For $\beta = 0$, we see that c is nothing but the wave speed. For $\beta \neq 0$, the wave speed is given as

A dispersion relation is an expression giving the angular frequency as a function of the wave number, $\omega = \omega(k)$.

$$v = \frac{\omega}{k} = c - \beta k^2.$$

This suggests that waves with different wave numbers will travel at different speeds. Recalling that wave numbers are related to wavelengths, $k = \frac{2\pi}{\lambda}$, this means that waves with different wavelengths will travel at different speeds. For example, an initial localized wave packet will not maintain its shape. It is said to disperse, as the component waves of differing wavelengths will tend to part company.

For a general initial condition, we write the solutions to the linearized KdV as a superposition of plane waves. We can do this since the partial differential equation is linear. This should remind you of what we had done when using separation of variables. We first sought product solutions and then took a linear combination of the product solutions to obtain the general solution.

For this problem, we will sum over all wave numbers. The wave numbers are not restricted to discrete values. We instead have a continuous range of values. Thus, "summing" over k means that we have to integrate over the wave numbers. Thus, we have the general solution[1]

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k,0) e^{ik(x-(c-\beta k^2)t)}\, dk. \tag{7.6}$$

[1] The extra 2π has been introduced to be consistent with the definition of the Fourier transform which is given later in the chapter.

Note that we have indicated that A a function of k. This is similar to introducing the A_n's and B_n's in the series solution for waves on a string.

How do we determine the $A(k,0)$'s? We introduce an initial condition. Let $u(x,0) = f(x)$. Then, we have

$$f(x) = u(x,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k,0) e^{ikx}\, dk. \tag{7.7}$$

Thus, given $f(x)$, we seek $A(k,0)$. In this chapter we will see that

$$A(k,0) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx.$$

This is what is called the Fourier transform of $f(x)$. It is just one of the so-called integral transforms that we will consider in this chapter.

In Figure 7.1 we summarize the transform scheme. One can use methods like separation of variables to solve the partial differential equation directly, evolving the initial condition $u(x,0)$ into the solution $u(x,t)$ at a later time.

$$\begin{array}{ccccc}
u(x,0) & \longrightarrow & \boxed{\text{FT}} & \longrightarrow & A(k,0) \\
\Big| & & & & \Big| \\
\text{PDE} & & & & \text{ODE} \\
\downarrow & & & & \downarrow \\
u(x,t) & \longleftarrow & \boxed{\text{IFT}} & \longleftarrow & A(k,t)
\end{array}$$

Figure 7.1: Schematic of using Fourier transforms to solve a linear evolution equation.

The transform method works as follows. Starting with the initial condition, one computes its Fourier Transform (FT) as[2]

$$A(k,0) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx.$$

[2] Note: The Fourier transform as used in this section and the next section are defined slightly differently than how we will define them later. The sign of the exponentials has been reversed.

Applying the transform on the partial differential equation, one obtains an ordinary differential equation satisfied by $A(k,t)$ which is simpler to solve than the original partial differential equation. Once $A(k,t)$ has been found, then one applies the Inverse Fourier Transform (IFT) to $A(k,t)$ in order to get the desired solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k,t)e^{ikx}\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k,0)e^{ik(x-(c-\beta k^2)t)}\,dk. \end{aligned} \tag{7.8}$$

7.1.2 Example 2 - The Free Particle Wave Function

A MORE FAMILIAR EXAMPLE IN PHYSICS comes from quantum mechanics. The Schrödinger equation gives the wave function $\Psi(x,t)$ for a particle under the influence of forces, represented through the corresponding potential function $V(x)$. The one dimensional time dependent Schrödinger equation is given by

The one dimensional time dependent Schrödinger equation.

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi. \tag{7.9}$$

We consider the case of a free particle in which there are no forces, $V = 0$. Then we have

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}. \tag{7.10}$$

Taking a hint from the study of the linearized KdV equation, we will assume that solutions of Equation (7.10) take the form

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k,t)e^{ikx}\,dk.$$

[Here we have opted to use the more traditional notation, $\phi(k,t)$ instead of $A(k,t)$ as above.]

Inserting the expression for $\Psi(x,t)$ into (7.10), we have

$$i\hbar\int_{-\infty}^{\infty} \frac{d\phi(k,t)}{dt}e^{ikx}\,dk = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \phi(k,t)(ik)^2 e^{ikx}\,dk.$$

Since this is true for all t, we can equate the integrands, giving

$$i\hbar\frac{d\phi(k,t)}{dt} = \frac{\hbar^2 k^2}{2m}\phi(k,t).$$

As with the last example, we have obtained a simple ordinary differential equation. The solution of this equation is given by

$$\phi(k,t) = \phi(k,0)e^{-i\frac{\hbar k^2}{2m}t}.$$

Applying the inverse Fourier transform, the general solution to the time dependent problem for a free particle is found as

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k,0) e^{ik(x-\frac{\hbar k}{2m}t)}\, dk.$$

We note that this takes the familiar form

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k,0) e^{i(kx-\omega t)}\, dk,$$

where the dispersion relation is found as

$$\omega = \frac{\hbar k^2}{2m}.$$

The wave speed is given as

$$v = \frac{\omega}{k} = \frac{\hbar k}{2m}.$$

As a special note, we see that this is not the particle velocity! Recall that the momentum is given as $p = \hbar k$.[3] So, this wave speed is $v = \frac{p}{2m}$, which is only half the classical particle velocity! A simple manipulation of this result will clarify the "problem".

[3] Since $p = \hbar k$, we also see that the dispersion relation is given by $\omega = \frac{\hbar k^2}{2m} = \frac{p^2}{2m\hbar} = \frac{E}{\hbar}$.

We assume that particles can be represented by a localized wave function. This is the case if the major contributions to the integral are centered about a central wave number, k_0. Thus, we can expand $\omega(k)$ about k_0:

$$\omega(k) = \omega_0 + \omega_0'(k-k_0)t + \ldots. \tag{7.11}$$

Here $\omega_0 = \omega(k_0)$ and $\omega_0' = \omega'(k_0)$. Inserting this expression into the integral representation for $\Psi(x,t)$, we have

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k,0) e^{i(kx-\omega_0 t-\omega_0'(k-k_0)t+\ldots)}\, dk,$$

We now make the change of variables, $s = k - k_0$ and rearrange the resulting factors to find

$$\begin{aligned}
\Psi(x,t) &\approx \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k_0+s,0) e^{i((k_0+s)x-(\omega_0+\omega_0's)t)}\, ds \\
&= \frac{1}{2\pi} e^{i(-\omega_0 t+k_0\omega_0' t)} \int_{-\infty}^{\infty} \phi(k_0+s,0) e^{i(k_0+s)(x-\omega_0' t)}\, ds \\
&= e^{i(-\omega_0 t+k_0\omega_0' t)}\Psi(x-\omega_0' t,0).
\end{aligned} \tag{7.12}$$

What we have found is that for an initially localized wave packet, $\Psi(x,0)$ with wave numbers grouped around k_0 the wave function,$\Psi(x,t)$, is a translated version of the initial wave function, up to a phase factor. In quantum mechanics we are more interested in the probability density for locating a particle, so from

$$|\Psi(x,t)|^2 = |\Psi(x-\omega_0' t,0)|^2$$

we see that the "velocity of the wave packet" is found to be

$$\omega_0' = \frac{d\omega}{dk}\Big|_{k=k_0} = \frac{\hbar k}{m}.$$

This corresponds to the classical velocity of the particle ($v_{\text{part}} = p/m$). Thus, one usually defines ω_0' to be the *group velocity*,

Group and phase velocities, $v_g = \frac{d\omega}{dk}$, $v_p = \frac{\omega}{k}$.

$$v_g = \frac{d\omega}{dk}$$

and the former velocity as the *phase velocity*,

$$v_p = \frac{\omega}{k}.$$

7.1.3 *Transform Schemes*

THESE EXAMPLES HAVE ILLUSTRATED one of the features of transform theory. Given a partial differential equation, we can transform the equation from spatial variables to wave number space, or time variables to frequency space. In the new space the time evolution is simpler. In these cases, the evolution is governed by an ordinary differential equation. One solves the problem in the new space and then transforms back to the original space. This is depicted in Figure 7.2 for the Schrödinger equation and was shown in Figure 7.1 for the linearized KdV equation.

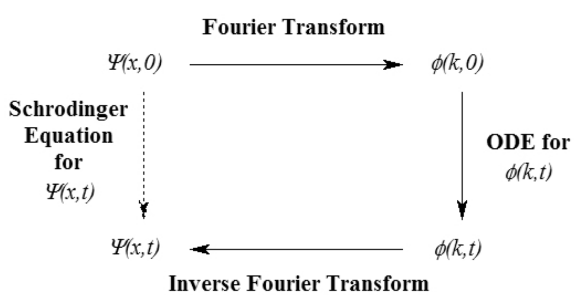


Figure 7.2: The scheme for solving the Schrödinger equation using Fourier transforms. The goal is to solve for $\Psi(x,t)$ given $\Psi(x,0)$. Instead of a direct solution in coordinate space (on the left side), one can first transform the initial condition obtaining $\phi(k,0)$ in wave number space. The governing equation in the new space is found by transforming the PDE to get an ODE. This simpler equation is solved to obtain $\phi(k,t)$. Then an inverse transform yields the solution of the original equation.

This is similar to the solution of the system of ordinary differential equations in Chapter 3, $\dot{\mathbf{x}} = A\mathbf{x}$. In that case we diagonalized the system using the transformation $\mathbf{x} = S\mathbf{y}$. This lead to a simpler system $\dot{\mathbf{y}} = \Lambda\mathbf{y}$. Solving for $\mathbf{y}$, we inverted the solution to obtain $\mathbf{x}$. Similarly, one can apply this diagonalization to the solution of linear algebraic systems of equations. The general scheme is shown in Figure 7.3.

Similar transform constructions occur for many other type of problems. We will end this chapter with a study of Laplace transforms, which are useful in the study of initial value problems, particularly for linear ordinary differential equations with constant coefficients. A similar scheme for using Laplace transforms is depicted in Figure 7.25.

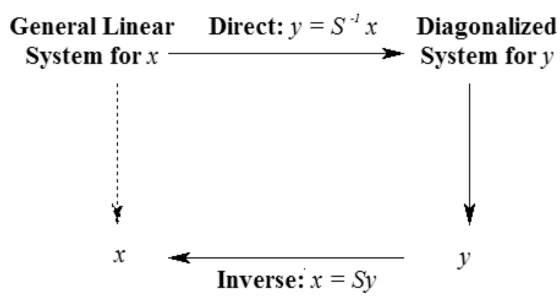


Figure 7.3: The scheme for solving the linear system $A\mathbf{x} = \mathbf{b}$. One finds a transformation between $\mathbf{x}$ and $\mathbf{y}$ of the form $\mathbf{x} = S\mathbf{y}$ which diagonalizes the system. The resulting system is easier to solve for $\mathbf{y}$. Then one uses the inverse transformation to obtain the solution to the original problem. Also, this scheme applies to solving the ODE system $\dot{\mathbf{x}} = A\mathbf{x}$ as we had seen in Chapter 3.

In this chapter we will turn to the study of Fourier transforms. These will provide an integral representation of functions defined on the real line. Such functions can also represent analog signals. Analog signals are continuous signals which can be represented as a sum over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series were used to represent in an earlier chapter. We will then investigate a related transform, the Laplace transform, which is useful in solving initial value problems such as those encountered in ordinary differential equations.

7.2 *Complex Exponential Fourier Series*

WE FIRST RECALL from Chapter 4 the trigonometric Fourier series representation of a function defined on $[-\pi, \pi]$ with period 2π. The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \tag{7.13}$$

where the Fourier coefficients were found as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 0, 1, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{7.14}$$

In order to derive the exponential Fourier series, we replace the trigonometric functions with exponential functions and collect like exponential terms. This gives

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{inx} + e^{-inx}}{2}\right) + b_n\left(\frac{e^{inx} - e^{-inx}}{2i}\right)\right]$$

$$= \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{inx} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2} \right) e^{-inx}. \quad (7.15)$$

The coefficients of the complex exponentials can be rewritten by defining

$$c_n = \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots. \quad (7.16)$$

This implies that

$$\bar{c}_n = \frac{1}{2}(a_n - ib_n), \quad n = 1, 2, \ldots. \quad (7.17)$$

So far the representation is rewritten as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \bar{c}_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Re-indexing the first sum, by introducing $k = -n$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=-1}^{-\infty} \bar{c}_{-k} e^{-ikx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Since k is a dummy index, we replace it with a new n as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=-1}^{-\infty} \bar{c}_{-n} e^{-inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

We can now combine all of the terms into a simple sum. We first define c_n for negative n's by

$$c_n = \bar{c}_{-n}, \quad n = -1, -2, \ldots.$$

Letting $c_0 = \frac{a_0}{2}$, we can write the *complex exponential Fourier series* representation as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \quad (7.18)$$

where

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots \\ c_n &= \frac{1}{2}(a_{-n} - ib_{-n}), \quad n = -1, -2, \ldots \\ c_0 &= \frac{a_0}{2}. \end{aligned} \quad (7.19)$$

Given such a representation, we would like to write out the integral forms of the coefficients, c_n. So, we replace the a_n's and b_n's with their integral representations and replace the trigonometric functions with complex exponential functions. Doing this, we have for $n = 1, 2, \ldots.$

$$c_n \quad = \quad \frac{1}{2}(a_n + ib_n)$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx}+e^{-inx}}{2}\right)dx + \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx}-e^{-inx}}{2i}\right)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \qquad (7.20)
\end{aligned}
$$

It is a simple matter to determine the c_n's for other values of n. For $n = 0$, we have that

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

For $n = -1, -2, \ldots$, we find that

$$c_n = \bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{-inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

Therefore, for all n we have obtained the complex exponential series for $f(x)$ defined on $[-\pi, \pi]$.

Complex Exponential Series for $f(x)$ defined on $[-\pi, \pi]$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \qquad (7.21)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \qquad (7.22)$$

We have converted the trigonometric series for functions defined on $[-\pi, \pi]$ to a complex exponential series in Equation (7.21) with Fourier coefficients given by (7.22). We can easily extend the above analysis to other intervals. For example, for $x \in [-L, L]$ the Fourier trigonometric series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

with Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, \ldots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots.$$

This can be rewritten as an exponential Fourier series of the form

Complex Exponential Series for $f(x)$ defined on $[-L, L]$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \qquad (7.23)$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx. \qquad (7.24)$$

7.3 Exponential Fourier Transform

Both the trigonometric and complex exponential Fourier series provide us with representations of a class of functions of finite period in terms of sums over a discrete set of frequencies. In particular, for functions defined on $x \in [-L, L]$, the period of the Fourier series representation is $2L$. We can write the arguments in the exponentials, $e^{-in\pi x/L}$, in terms of the angular frequency, $\omega_n = n\pi/L$, as $e^{-i\omega_n x}$. We note that the frequencies, ν_n, are then defined through $\omega_n = 2\pi\nu_n = \frac{n\pi}{L}$. Therefore, the complex exponential series is seen to be a sum over a discrete, or countable, set of frequencies.

We would now like to extend the finite interval to an infinite interval, $x \in (-\infty, \infty)$ and to extend the discrete set of (angular) frequencies to a continuous range of frequencies, $\omega \in (-\infty, \infty)$. One can do this rigorously. It amounts to letting L and n get large and keeping $\frac{n}{L}$ fixed.

We first define $\Delta\omega = \frac{\pi}{L}$, so that $\omega_n = n\Delta\omega$. Inserting the Fourier coefficients (7.24) into Equation (7.21), we have

$$\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(\xi) e^{in\pi\xi/L}\, d\xi \right) e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(\xi) e^{i\omega_n \xi}\, d\xi \right) e^{-i\omega_n x}. \end{aligned} \tag{7.25}$$

Now, we let L get large, so that $\Delta\omega$ becomes small and ω_n approaches the angular frequency ω. Then

$$\begin{aligned} f(x) &\sim \lim_{\Delta\omega \to 0, L \to \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-L}^{L} f(\xi) e^{i\omega_n \xi}\, d\xi \right) e^{-i\omega_n x} \Delta\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi) e^{i\omega\xi}\, d\xi \right) e^{-i\omega x}\, d\omega. \end{aligned} \tag{7.26}$$

Looking at this last result, we formally arrive at the definition of the *Fourier transform*

Definitions of the Fourier transform and the inverse Fourier transform.

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx. \tag{7.27}$$

This is a generalization of the Fourier coefficients (7.22). Once we know the Fourier transform, $\hat{f}(\omega)$, then we can *reconstruct* the original function, $f(x)$, using the *inverse Fourier transform*, which is given by

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x}\, d\omega. \tag{7.28}$$

We note that it can be proven that the Fourier transform exists when $f(x)$ is *absolutely integrable*, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Such functions are said to be L_1.

The **Fourier transform** and **inverse Fourier transform** are inverse operations. Defining the Fourier transform as

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx. \tag{7.29}$$

and the inverse Fourier transform as

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x}\, d\omega. \tag{7.30}$$

then

$$F^{-1}[F[f]] = f(x) \tag{7.31}$$

and

$$F[F^{-1}[\hat{f}]] = \hat{f}(\omega). \tag{7.32}$$

We will now prove the first of these equations, (7.31). [The second equation, (7.32), follows in a similar way.]

Proof. The proof is carried out by inserting the definition of the Fourier transform, (7.29), into the inverse transform definition, (7.30), and then interchanging the orders of integration. Thus, we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F[f] e^{-i\omega x}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi) e^{i\omega\xi}\, d\xi\right] e^{-i\omega x}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi) e^{i\omega(\xi - x)}\, d\xi d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\, d\omega\right] f(\xi)\, d\xi. \end{aligned} \tag{7.33}$$

In order to complete the proof, we need to evaluate the inside integral, which does not depend upon $f(x)$. This is an improper integral, so we first define

$$D_\Omega(x) = \int_{-\Omega}^{\Omega} e^{i\omega x}\, d\omega$$

and compute the inner integral as

$$\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\, d\omega = \lim_{\Omega\to\infty} D_\Omega(\xi - x).$$

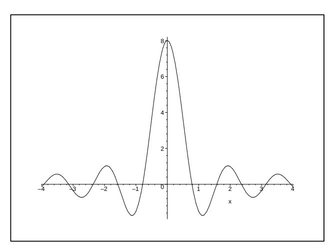


Figure 7.4: A plot of the function $D_\Omega(x)$ for $\Omega = 4$.

We can compute $D_\Omega(x)$. A simple evaluation yields

$$\begin{aligned} D_\Omega(x) &= \int_{-\Omega}^{\Omega} e^{i\omega x}\,d\omega \\ &= \left.\frac{e^{i\omega x}}{ix}\right|_{-\Omega}^{\Omega} \\ &= \frac{e^{ix\Omega}-e^{-ix\Omega}}{2ix} \\ &= \frac{2\sin x\Omega}{x}. \end{aligned} \tag{7.34}$$

A plot of this function is in Figure 7.4 for $\Omega = 4$. For large Ω the peak grows and the values of $D_\Omega(x)$ for $x \neq 0$ tend to zero as show in Figure 7.5. In fact, as x approaches 0, $D_\Omega(x)$ approaches 2Ω. For $x \neq 0$, the $D_\Omega(x)$ function tends to zero.

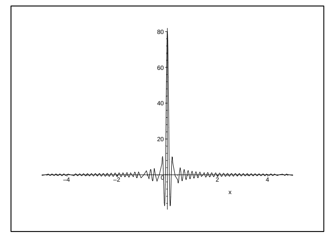


Figure 7.5: A plot of the function $D_\Omega(x)$ for $\Omega = 40$.

We further note that

$$\lim_{\Omega\to\infty} D_\Omega(x) = 0 \quad x \neq 0$$

and $\lim_{\Omega\to\infty} D_\Omega(x)$ is infinite at $x = 0$. However, the area is constant for each Ω. In fact,

$$\int_{-\infty}^{\infty} D_\Omega(x)\,dx = 2\pi.$$

We can show this by recalling the computation in Example 6.35,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.$$

Then,

$$\begin{aligned} \int_{-\infty}^{\infty} D_\Omega(x)\,dx &= \int_{-\infty}^{\infty} \frac{2\sin x\Omega}{x}\,dx \\ &= \int_{-\infty}^{\infty} 2\frac{\sin y}{y}\,dy \\ &= 2\pi. \end{aligned} \tag{7.35}$$

Another way to look at $D_\Omega(x)$ is to consider the sequence of functions $f_n(x) = \frac{\sin nx}{\pi x}$, $n = 1, 2, \ldots$. Then we have shown that this sequence of functions satisfies the two properties,

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \neq 0,$$

$$\int_{-\infty}^{\infty} f_n(x)\,dx = 1.$$

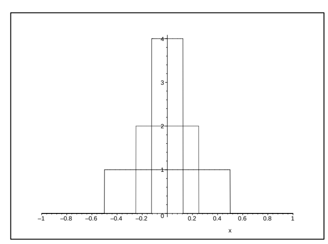


Figure 7.6: A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

This is a key representation of such generalized functions. The limiting value vanishes at all but one point, but the area is finite.

Such behavior can be seen for the limit of other sequences of functions. For example, consider the sequence of functions

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n}, \\ \frac{n}{2}, & |x| < \frac{1}{n}. \end{cases}$$

This is a sequence of functions as shown in Figure 7.6. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one. Thus, the limiting function is zero at most points but has area one.

The limit is not really a function. It is a *generalized function*. It is called the *Dirac delta function*, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

P. A. M. Dirac (1902-1984) introduced the δ function in his book, "The Principles of Quantum Mechanics", 4th Ed., Oxford University Press, 1958, originally published in 1930, as part of his orthogonality statement for a basis of functions in a Hilbert space, $< \xi' | \xi'' >= c\delta(\xi' - \xi'')$ in the same way we introduced discrete orthogonality using the Kronecker delta.

Before returning to the proof that the inverse Fourier transform of the Fourier transform is the identity, we state one more property of the Dirac delta function, which we will prove in the next section. Namely, we will show that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

Returning to the proof, we now have that

$$\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\, d\omega = \lim_{\Omega\to\infty} D_\Omega(\xi - x) = 2\pi\delta(\xi - x).$$

Inserting this into (7.33), we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega\right] f(\xi)\, d\xi. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\xi - x) f(\xi)\, d\xi. \\ &= f(x). \end{aligned} \tag{7.36}$$

Thus, we have proven that the inverse transform of the Fourier transform of f is f. □

Properties of the Dirac δ-function:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(y)\, dy.$$

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \frac{1}{|f'(x_j)|}\delta(x - x_j)\, dx.$$

(For n simple roots.)

These and other properties are often written outside the integral:

$$\delta(ax) = \frac{1}{|a|}\delta(x).$$

$$\delta(-x) = \delta(x).$$

$$\delta((x-a)(x-b)) = \frac{1}{|a-b|}[\delta(x-a) + \delta(x-a)].$$

$$\delta(f(x)) = \sum_j \frac{\delta(x - x_j)}{|f'(x_j)|}, f(x_j) = 0, f'(x_j) \neq 0.$$

7.4 *The Dirac Delta Function*

IN THE LAST SECTION we introduced the Dirac delta function, $\delta(x)$. As noted above, this is one example of what is known as a *generalized function*, or a distribution. Dirac had introduced this function in the 1930's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral. [Note: The Dirac delta function was also discussed in the optional Section **??**.]

Two properties were used in the last section. First one has that the area under the delta function is one,

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Integration over more general intervals gives

$$\int_a^b \delta(x)\,dx = \begin{cases} 1, & 0 \in [a,b], \\ 0, & 0 \notin [a,b]. \end{cases} \tag{7.37}$$

The other property that was used was the *sifting property*:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = \int_{-\infty}^{\infty} \delta(x-a) f(a)\,dx = f(a) \int_{-\infty}^{\infty} \delta(x-a)\,dx = f(a).$$

Another property results from using a scaled argument, ax. In this case we show that

$$\delta(ax) = |a|^{-1}\delta(x). \tag{7.38}$$

As usual, this only has meaning under an integral sign. So, we place $\delta(ax)$ inside an integral and make a substitution $y = ax$:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(ax)\,dx &= \lim_{L\to\infty} \int_{-L}^{L} \delta(ax)\,dx \\ &= \lim_{L\to\infty} \frac{1}{a} \int_{-aL}^{aL} \delta(y)\,dy. \end{aligned} \tag{7.39}$$

If $a > 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\,dy.$$

However, if $a < 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{a} \int_{\infty}^{-\infty} \delta(y)\,dy = -\frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\,dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor $1/a$ to $1/|a|$. Thus,

$$\boxed{\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(y)\,dy. \qquad (7.40)}$$

Example 7.1. *Evaluate* $\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\,dx$*. This is a straight forward integration:*

$$\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\,dx = \frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\,dx = \frac{11}{4}.$$

A more general scaling of the argument takes the form $\delta(f(x))$. The integral of $\delta(f(x))$ can be evaluated depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1)=0$, then one has that

$$\int_{-\infty}^{\infty} \delta(f(x))\,dx = \int_{-\infty}^{\infty} \frac{1}{|f'(x_1)|}\delta(x-x_1)\,dx.$$

This can be proven using the substitution $y=f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x)) = \frac{1}{|f'(x_1)|}\delta(x-x_1),$$

again keeping in mind that this only has meaning when placed under an integral.

Example 7.2. *Evaluate* $\int_{-\infty}^{\infty}\delta(3x-2)x^2\,dx$.

This is not a simple $\delta(x-a)$. *So, we need to find the zeros of* $f(x)=3x-2$. *There is only one,* $x=\frac{2}{3}$. *Also,* $|f'(x)|=3$. *Therefore, we have*

$$\int_{-\infty}^{\infty}\delta(3x-2)x^2\,dx = \int_{-\infty}^{\infty}\frac{1}{3}\delta(x-\frac{2}{3})x^2\,dx = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}.$$

Note that this integral can be evaluated the long way by using the substitution $y=3x-2$. *Then,* $dy=3\,dx$ *and* $x=(y+2)/3$. *This gives*

$$\int_{-\infty}^{\infty}\delta(3x-2)x^2\,dx = \frac{1}{3}\int_{-\infty}^{\infty}\delta(y)\left(\frac{y+2}{3}\right)^2\,dy = \frac{1}{3}\left(\frac{4}{9}\right) = \frac{4}{27}.$$

More generally, one can show that when $f(x_j)=0$ and $f'(x_j)\neq 0$ for x_j, $j=1,2,\ldots,n$, (i.e.; when one has n simple zeros), then

$$\boxed{\delta(f(x)) = \sum_{j=1}^{n}\frac{1}{|f'(x_j)|}\delta(x-x_j).}$$

Example 7.3. *Evaluate* $\int_0^{2\pi}\cos x\,\delta(x^2-\pi^2)\,dx$.

In this case the argument of the delta function has two simple roots. Namely, $f(x)=x^2-\pi^2=0$ *when* $x=\pm\pi$. *Furthermore,* $f'(x)=2x$. *Therefore,* $|f'(\pm\pi)|=2\pi$. *This gives*

$$\delta(x^2-\pi^2) = \frac{1}{2\pi}[\delta(x-\pi)+\delta(x+\pi)].$$

Inserting this expression into the integral and noting that $x=-\pi$ *is not in the integration interval, we have*

$$\begin{aligned}\int_0^{2\pi}\cos x\,\delta(x^2-\pi^2)\,dx &= \frac{1}{2\pi}\int_0^{2\pi}\cos x\,[\delta(x-\pi)+\delta(x+\pi)]\,dx\\ &= \frac{1}{2\pi}\cos\pi = -\frac{1}{2\pi}.\end{aligned} \tag{7.41}$$

Finally, one can show that there is a relationship between the Heaviside function (or, step function) and the Dirac delta function. We define the Heaviside function as

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Then, it is easy to see that $H'(x) = \delta(x)$. In some texts the notation $\theta(x)$ is used for the step function.

7.5 *Properties of the Fourier Transform*

WE NOW RETURN to the Fourier transform. Before actually computing the Fourier transform of some functions, we prove a few of the properties of the Fourier transform.

First we note that there are several forms that one may encounter for the Fourier transform. In applications functions can either be functions of time, $f(t)$, or space, $f(x)$. The corresponding Fourier transforms are then written as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt, \tag{7.42}$$

or

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx. \tag{7.43}$$

ω is called the *angular frequency* and is related to the *frequency* ν by $\omega = 2\pi\nu$. The units of frequency are typically given in Hertz (Hz). Sometimes the frequency is denoted by f when there is no confusion. k is called the *wavenumber*. It has units of inverse length and is related to the wavelength, λ, by $k = \frac{2\pi}{\lambda}$.

1. **Linearity.** For any functions $f(x)$ and $g(x)$ for which the Fourier transform exists and constant a, we have

$$F[f+g] = F[f] + F[g]$$

and

$$F[af] = aF[f].$$

These simply follow from the properties of integration and establish the linearity of the Fourier transform.

2. **Transform of a Derivative.** $F\left[\frac{df}{dx}\right] = -ik\hat{f}(k)$

Here we compute the Fourier transform (7.29) of the derivative by inserting the derivative in the Fourier integral and using integration by parts.

$$F\left[\frac{df}{dx}\right] = \int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx}\, dx$$

$$= \lim_{L\to\infty}\left[f(x)e^{ikx}\right]_{-L}^{L} - ik\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx. \tag{7.44}$$

The limit will vanish if we assume that $\lim_{x\to\pm\infty} f(x) = 0$. The last integral is recognized as the Fourier transform of f, proving the given property.

3. **Higher Order Derivatives.** $F\left[\dfrac{d^n f}{dx^n}\right] = (-ik)^n \hat{f}(k)$

   The proof of this property follows from the last result, or doing several integration by parts. We will consider the case when $n = 2$. Noting that the second derivative is the derivative of $f'(x)$ and applying the last result, we have

$$\begin{aligned} F\left[\frac{d^2 f}{dx^2}\right] &= F\left[\frac{d}{dx}f'\right] \\ &= -ikF\left[\frac{df}{dx}\right] = (-ik)^2\hat{f}(k). \end{aligned} \tag{7.45}$$

   This result will be true if

$$\lim_{x\to\pm\infty} f(x) = 0 \text{ and } \lim_{x\to\pm\infty} f'(x) = 0.$$

   The generalization to the transform of the nth derivative easily follows.

4. $F\left[xf(x)\right] = -i\frac{d}{dk}\hat{f}(k)$

   This property can be shown by using the fact that $\frac{d}{dk}e^{ikx} = ixe^{ikx}$ and the ability to differentiate an integral with respect to a parameter.

$$\begin{aligned} F[xf(x)] &= \int_{-\infty}^{\infty} xf(x)e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{d}{dk}\left(\frac{1}{i}e^{ikx}\right)dx \\ &= -i\frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= -i\frac{d}{dk}\hat{f}(k). \end{aligned} \tag{7.46}$$

   This result can be generalized to $F\left[x^n f(x)\right]$ as an exercise.

5. **Shifting Properties.** For constant a, we have the following shifting properties:

$$f(x-a) \leftrightarrow e^{ika}\hat{f}(k), \tag{7.47}$$

$$f(x)e^{-iax} \leftrightarrow \hat{f}(k-a). \tag{7.48}$$

   Here we have denoted the Fourier transform pairs using a double arrow as $f(x) \leftrightarrow \hat{f}(k)$. These are easily proven by inserting the desired forms into the definition of the Fourier transform (7.29), or inverse Fourier transform (7.30). The first shift property (7.47) is shown by the following argument. We evaluate the Fourier transform.

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{ikx}\,dx.$$

Now perform the substitution $y = x - a$. Then,

$$\begin{aligned} F[f(x-a)] &= \int_{-\infty}^{\infty} f(y)e^{ik(y+a)}\,dy \\ &= e^{ika}\int_{-\infty}^{\infty} f(y)e^{iky}\,dy \\ &= e^{ika}\hat{f}(k). \end{aligned} \tag{7.49}$$

The second shift property (7.48) follows in a similar way.

6. **Convolution** We define the convolution of two functions $f(x)$ and $g(x)$ as

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dx. \tag{7.50}$$

Then the Fourier transform of the convolution is the product of the Fourier transforms of the individual functions:

$$F[f*g] = \hat{f}(k)\hat{g}(k). \tag{7.51}$$

We will return to the proof of this property in Section 7.6.

7.5.1 Fourier Transform Examples

In this section we will compute the Fourier transforms of several functions.

Example 7.4. Gaussian Functions. $f(x) = e^{-ax^2/2}$.

The Fourier transform of a Gaussian is a Gaussian.

This function is called the Gaussian function. It has many applications in areas such as quantum mechanics, molecular theory, probability and heat diffusion. We will compute the Fourier transform of this function and show that the Fourier transform of a Gaussian is a Gaussian. In the derivation we will introduce classic techniques for computing such integrals.

We begin by applying the definition of the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx. \tag{7.52}$$

The first step in computing this integral is to complete the square in the argument of the exponential. Our goal is to rewrite this integral so that a *simple substitution will lead to a classic integral of the form* $\int_{-\infty}^{\infty} e^{\beta y^2}\,dy$*, which we can integrate. The completion of the square follows as usual:*

$$\begin{aligned}
-\frac{a}{2}x^2 + ikx &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x\right] \\
&= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x + \left(-\frac{ik}{a}\right)^2 - \left(-\frac{ik}{a}\right)^2\right] \\
&= -\frac{a}{2}\left(x - \frac{ik}{a}\right)^2 - \frac{k^2}{2a}. \qquad (7.53)
\end{aligned}$$

We now put this expression into the integral and make the substitutions $y = x - \frac{ik}{a}$ *and* $\beta = \frac{a}{2}$.

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx \\
&= e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x-\frac{ik}{a}\right)^2}\,dx \\
&= e^{-\frac{k^2}{2a}} \int_{-\infty-\frac{ik}{a}}^{\infty-\frac{ik}{a}} e^{-\beta y^2}\,dy. \qquad (7.54)
\end{aligned}$$

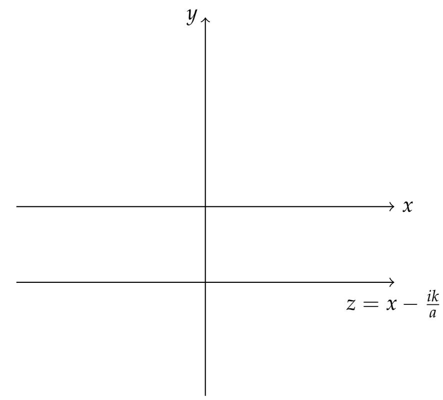


Figure 7.7: Simple horizontal contour.

One would be tempted to absorb the $-\frac{ik}{a}$ *terms in the limits of integration. However, we know from our previous study that the integration takes place over a contour in the complex plane as shown in Figure 7.7.*

In this case we can deform this horizontal contour to a contour along the real axis since we will not cross any singularities of the integrand. So, we now safely write

$$\hat{f}(k) = e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\beta y^2}\,dy.$$

The resulting integral is a classic integral and can be performed using a standard trick. Define I by [4]

$$I = \int_{-\infty}^{\infty} e^{-\beta y^2}\,dy.$$

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-\beta y^2}\,dy \int_{-\infty}^{\infty} e^{-\beta x^2}\,dx.$$

Note that we needed to change the integration variable so that we can write this product as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta(x^2+y^2)}\,dxdy.$$

This is an integral over the entire xy-plane. We now transform to polar coordinates to obtain

$$\begin{aligned}
I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-\beta r^2}\,rdrd\theta \\
&= 2\pi \int_0^{\infty} e^{-\beta r^2}\,rdr \\
&= -\frac{\pi}{\beta}\left[e^{-\beta r^2}\right]_0^{\infty} = \frac{\pi}{\beta}. \qquad (7.55)
\end{aligned}$$

[4] Here we show

$$\int_{-\infty}^{\infty} e^{-\beta y^2}\,dy = \sqrt{\frac{\pi}{\beta}}.$$

Note that we solved the $\beta = 1$ case in Example 5.9, so a simple variable transformation $z = \sqrt{\beta}y$ is all that is needed to get the answer. However, it cannot hurt to see this classic derivation again.

The final result is gotten by taking the square root, yielding

$$I = \sqrt{\frac{\pi}{\beta}}.$$

We can now insert this result to give the Fourier transform of the Gaussian function:

$$\hat{f}(k) = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}. \tag{7.56}$$

Example 7.5. Box or Gate Function. $f(x) = \begin{cases} b, & |x| \le a \\ 0, & |x| > a \end{cases}.$

This function is called the box function, or gate function. It is shown in Figure 7.8. The Fourier transform of the box function is relatively easy to compute. It is given by

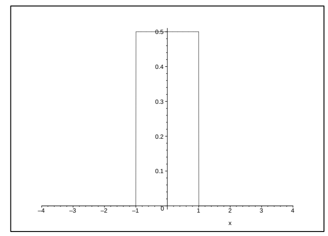


Figure 7.8: A plot of the box function in Example 7.5.

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \\ &= \int_{-a}^{a} b e^{ikx}\, dx \\ &= \left. \frac{b}{ik} e^{ikx} \right|_{-a}^{a} \\ &= \frac{2b}{k} \sin ka. \end{aligned} \tag{7.57}$$

We can rewrite this as

$$\hat{f}(k) = 2ab \frac{\sin ka}{ka} \equiv 2ab \operatorname{sinc} ka.$$

Here we introduced the sinc function,

$$\operatorname{sinc} x = \frac{\sin x}{x}.$$

A plot of this function is shown in Figure 7.9.

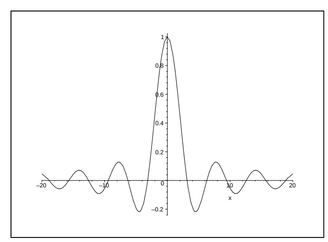


Figure 7.9: A plot of the Fourier transform of the box function in Example 7.5. This is the general shape of the sinc function.

We will now consider special limiting values for the box function and its transform. This will lead us to the Uncertainty Principle for signals, connecting the relationship between the localization properties of a signal and its transform.

1. *$a \to \infty$ and b fixed.*

   In this case, as a gets large the box function approaches the constant function $f(x) = b$. At the same time, we see that the Fourier transform approaches a Dirac delta function. We had seen this function earlier when we first defined the Dirac delta function. Compare Figure 7.9 with Figure 7.4. In fact, $\hat{f}(k) = bD_a(k)$. [Recall the definition of $D_\Omega(x)$ in Equation (7.34).] So, in the limit we obtain $\hat{f}(k) = 2\pi b\delta(k)$. This limit implies fact that the Fourier transform of $f(x) = 1$ is $\hat{f}(k) = 2\pi\delta(k)$. As the width of the box becomes *wider, the Fourier transform becomes more localized. In fact, we have arrived at the result that*

$$\int_{-\infty}^{\infty} e^{ikx} = 2\pi\delta(k). \tag{7.58}$$

2. $b \to \infty$, $a \to 0$, *and* $2ab = 1$.

   In this case the box narrows and becomes steeper while maintaining a constant area of one. This is the way we had found a representation of the Dirac delta function previously. The Fourier transform approaches a constant in this limit. As a approaches zero, the sinc function approaches one, leaving $\hat{f}(k) \to 2ab = 1$. *Thus, the Fourier transform of the Dirac delta function is one. Namely, we have*

$$\int_{-\infty}^{\infty} \delta(x) e^{ikx} = 1. \tag{7.59}$$

   In this case we have that the more localized the function $f(x)$ *is, the more spread out the Fourier transform,* $\hat{f}(k)$, *is. We will summarize these notions in the next item by relating the widths of the function and its Fourier transform.*

3. **The Uncertainty Principle**

   The widths of the box function and its Fourier transform are related as we have seen in the last two limiting cases. It is natural to define the width, Δx *of the box function as*

$$\Delta x = 2a.$$

   The width of the Fourier transform is a little trickier. This function actually extends the entire k-axis. However, as $\hat{f}(k)$ *became more localized, the central peak in Figure 7.9 became narrower. So, we define the width of this function,* Δk *as the distance between the first zeros on either side of the main lobe. This gives*

$$\Delta k = \frac{2\pi}{a}.$$

   Combining these two relations, we find that

$$\Delta x \Delta k = 4\pi.$$

   Thus, the more localized a signal, the less localized its transform. This notion is referred to as the Uncertainty Principle. For general signals, one needs to define the effective widths more carefully, but the main idea holds:

$$\Delta x \Delta k \geq c > 0.$$

More formally, the uncertainty principle for signals is about the relation between duration and bandwidth, which are defined by $\Delta t = \frac{\|tf\|_2}{\|f\|_2}$ and $\Delta\omega = \frac{\|\omega\hat{f}\|_2}{\|\hat{f}\|_2}$, respectively, where $\|f\|_2 = \int_{-\infty}^{\infty} |f(t)|^2\,dt$ and $\|\hat{f}\|_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$. Under appropriate conditions, one can prove that $\Delta t\Delta\omega \geq \frac{1}{2}$. Equality holds for Gaussian signals. Werner Heisenberg (1901-1976) introduced the uncertainty principle into quantum physics in 1926, relating uncertainties in the position (Δx) and momentum (Δp_x) of particles. In this case, $\Delta x\Delta p_x \geq \frac{1}{2}\hbar$. Here, the uncertainties are defined as the positive square roots of the quantum mechanical variances of the position and momentum.

We now turn to other examples of Fourier transforms.

Example 7.6. $f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}, a > 0.$

The Fourier transform of this function is

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= \int_{o}^{\infty} e^{ikx-ax}\,dx \\ &= \frac{1}{a-ik}. \end{aligned} \tag{7.60}$$

Next, we will compute the inverse Fourier transform of this result and recover the original function.

Example 7.7. $\hat{f}(k) = \frac{1}{a-ik}$.

The inverse Fourier transform of this function is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{-ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\,dk.$$

This integral can be evaluated using contour integral methods. We recall Jordan's Lemma from the last chapter:

If $f(z)$ converges uniformly to zero as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{ikz}\,dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$. A similar result applies for $k < 0$, but one closes the contour in the lower half plane.

In this example, we have to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-ixz}}{a-iz}\,dz.$$

According to Jordan's Lemma, we need to enclose the contour with a semicircle in the upper half plane for $x < 0$ and in the lower half plane for $x > 0$ as shown in Figure 7.10.

The integrations along the semicircles will vanish and we will have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\,dk \\ &= \pm\frac{1}{2\pi}\oint_C \frac{e^{-ixz}}{a-iz}\,dz \\ &= \begin{cases} 0, & x < 0 \\ -\frac{1}{2\pi}2\pi i \text{ Res }[z=-ia], & x > 0 \end{cases} \\ &= \begin{cases} 0, & x < 0 \\ e^{-ax}, & x > 0 \end{cases}. \end{aligned} \tag{7.61}$$

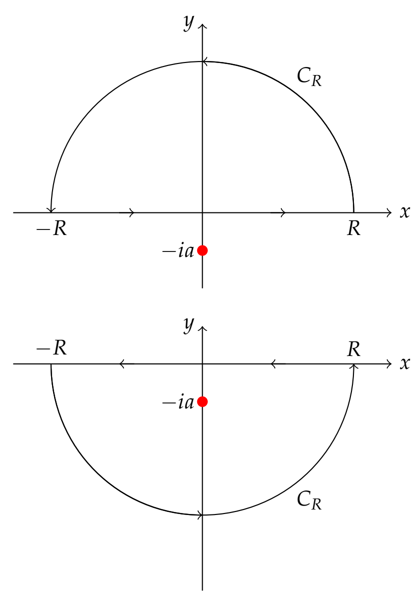


Figure 7.10: Contours for inverting $\hat{f}(k) = \frac{1}{a-ik}$.

Example 7.8. $\hat{f}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

We would like to find the inverse Fourier transform of this function. Instead of carrying out any integration, we will make use of the properties of Fourier transforms. Since the transforms of sums are the sums of transforms, we can look at each term individually. Consider $\delta(\omega - \omega_0)$. This is a shifted function. From the shift theorems in Equations (7.47)-(7.48) we have

$$e^{i\omega_0 t} f(t) \leftrightarrow \hat{f}(\omega - \omega_0).$$

Recalling from a previous example that

$$\int_{-\infty}^{\infty} e^{i\omega t}\, dt = 2\pi\delta(\omega),$$

we have

$$F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} e^{-i\omega_0 t}.$$

The second term can be transformed similarly. Therefore, we have

$$F^{-1}[\pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0] = \frac{1}{2} e^{i\omega_0 t} + \frac{1}{2} e^{-i\omega_0 t} = \cos\omega_0 t.$$

Example 7.9. The Finite Wave Train. $f(t) = \begin{cases} \cos\omega_0 t, & |t| \le a \\ 0, & |t| > a \end{cases}$.

For the last example, we consider the finite wave train, which will reappear in the last chapter on signal analysis. In Figure 7.11 we show a plot of this function.

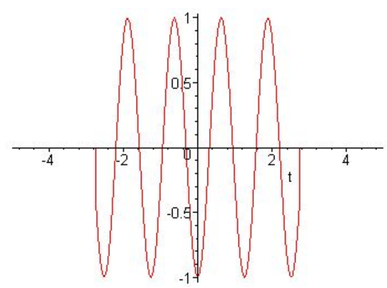


Figure 7.11: A plot of the finite wave train.

A straight forward computation gives

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt \\
&= \int_{-a}^{a} [\cos\omega_0 t + i \sin\omega_0 t] e^{i\omega t}\, dt \\
&= \int_{-a}^{a} \cos\omega_0 t \cos\omega t\, dt + i \int_{-a}^{a} \sin\omega_0 t \sin\omega t\, dt \\
&= \frac{1}{2} \int_{-a}^{a} [\cos((\omega + \omega_0)t) + \cos((\omega - \omega_0)t)]\, dt \\
&= \frac{\sin((\omega + \omega_0)a)}{\omega + \omega_0} + \frac{\sin((\omega - \omega_0)a)}{\omega - \omega_0}.
\end{aligned} \tag{7.62}$$

7.6 *The Convolution Theorem*

IN THE LIST OF PROPERTIES OF THE FOURIER TRANSFORM, we defined the convolution of two functions, $f(x)$ and $g(x)$ to be the integral

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x - t)\, dt. \tag{7.63}$$

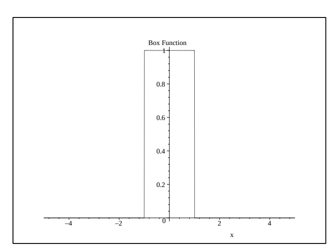


Figure 7.12: A plot of the box function $f(x)$.

In some sense one is looking at a sum of the overlaps of one of the functions and all of the shifted versions of the other function. The German word for convolution is *faltung*, which means "folding".

First, we note that the convolution is commutative: $f * g = g * f$. This is easily shown by replacing $x - t$ with a new variable, y.

$$\begin{aligned}
(g * f)(x) &= \int_{-\infty}^{\infty} g(t) f(x-t)\, dt \\
&= -\int_{\infty}^{-\infty} g(x-y) f(y)\, dy \\
&= \int_{-\infty}^{\infty} f(y) g(x-y)\, dy \\
&= (f * g)(x). \qquad (7.64)
\end{aligned}$$

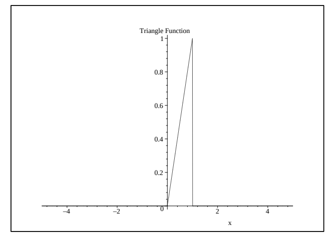


Figure 7.13: A plot of the triangle function.

Example 7.10. *Graphical Convolution.*

In order to understand the convolution operation, we need to apply it to several functions. We will do this graphically for the box function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and the triangular function

$$g(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

as shown in Figures 7.12 and 7.13.

In order to determine the contributions to the integrand, we look at the shifted and reflected function $g(t-x)$ in Equation 7.63 for various values of t. For $t = 0$, we have $g(-x)$. This is a reflection of the triangle function as shown in Figure 7.14.

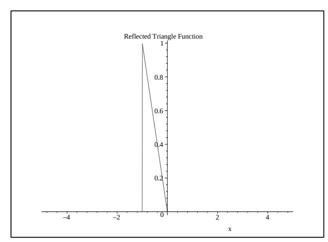


Figure 7.14: A plot of the reflected triangle function.

We then translate this function performing horizontal shifts by t. In Figure 7.15 we show such a shifted and reflected $g(x)$ for $t = 2$. The following figures show other shifts superimposed on $f(x)$. The integrand is the product of $f(x)$ and $g(t-x)$ and the convolution evaluated at t is given by the shaded areas. In Figures 7.16 the area is zero, as there is no overlap of the functions. Intermediate shift values are displayed in Figures 7.17-7.19 and the convolution is shown by the area under the product of the two functions.

$$F[f * g] = \hat{f}(k)\hat{g}(k). \qquad (7.65)$$

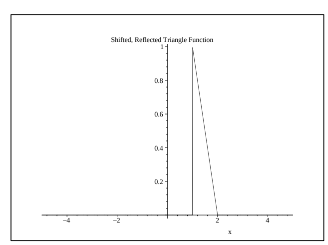


Figure 7.15: A plot of the reflected triangle function shifted by 2 units.

We see that the value of the convolution integral builds up and then quickly drops to zero. The plot of the convolution of the box and triangle functions is given in Figure 7.20.

Next we would like to compute the Fourier transform of the convolution integral. First, we use the definitions of Fourier transform and

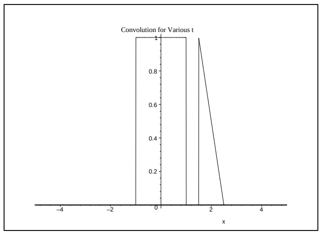


Figure 7.16: A plot of the box and triangle functions with the overlap indicated by the shaded area.

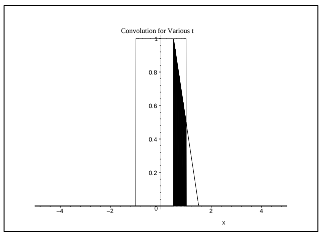


Figure 7.17: A plot of the box and triangle functions with the overlap indicated by the shaded area.

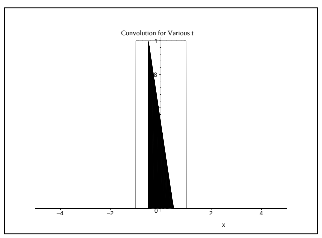


Figure 7.18: A plot of the box and triangle functions with the overlap indicated by the shaded area.

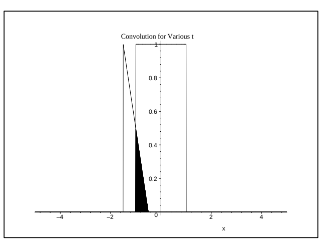


Figure 7.19: A plot of the box and triangle functions with the overlap indicated by the shaded area.

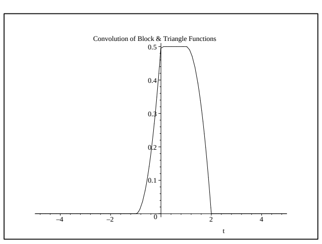


Figure 7.20: A plot of the convolution of the box and triangle functions.

convolution to write the transform as

$$\begin{aligned} F[f*g] &= \int_{-\infty}^{\infty} (f*g)(x)e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)g(x-t)\,dt\right) e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t)e^{ikx}\,dx\right) f(t)\,dt. \end{aligned} \tag{7.66}$$

We now substitute $y = x - t$ on the inside integral and separate the integrals:

$$\begin{aligned} F[f*g] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t)e^{ikx}\,dx\right) f(t)\,dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y)e^{ik(y+t)}\,dy\right) f(t)\,dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y)e^{iky}\,dy\right) f(t)e^{ikt}\,dt. \end{aligned} \tag{7.67}$$

We see that the two integrals are just the Fourier transforms of f and g. Therefore, the Fourier transform of a convolution is the product of the Fourier transforms of the functions involved:

Example 7.11. *Convolution of two Gaussian functions $f(x) = e^{-ax^2}$.*

In this example we will compute the convolution of two Gaussian functions with different widths. Let $f(x) = e^{-ax^2}$ and $g(x) = e^{-bx^2}$. A direct evaluation of the integral would be to compute

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} e^{-at^2-b(x-t)^2}\,dt.$$

This integral can be rewritten as

$$(f*g)(x) = e^{-bx^2}\int_{-\infty}^{\infty} e^{-(a+b)t^2+2bxt}\,dt.$$

One could proceed to complete the square and finish carrying out the integration. However, we will use the Convolution Theorem to evaluate the convolution and leave the evaluation of this integral to Problem 8.

Recalling the Fourier transform of a Gaussian from Example 7.4, we have

$$\hat{f}(k) = F[e^{-ax^2}] = \sqrt{\frac{\pi}{a}}e^{-k^2/4a} \tag{7.68}$$

and

$$\hat{g}(k) = F[e^{-bx^2}] = \sqrt{\frac{\pi}{b}}e^{-k^2/4b}.$$

Denoting the convolution function by $h(x) = (f*g)(x)$, the Convolution Theorem gives

$$\hat{h}(k) = \hat{f}(k)\hat{g}(k) = \frac{\pi}{\sqrt{ab}}e^{-k^2/4a}e^{-k^2/4b}.$$

This is another Gaussian function, as seen by rewriting the Fourier transform of $h(x)$ as

$$\hat{h}(k) = \frac{\pi}{\sqrt{ab}} e^{-\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)k^2} = \frac{\pi}{\sqrt{ab}} e^{-\frac{a+b}{4ab}k^2}. \tag{7.69}$$

In order to complete the evaluation of the convolution of these two Gaussian functions, we need to find the inverse transform of the Gaussian in Equation (7.69). We can do this by looking at Equation (7.68). We have first that

$$F^{-1}\left[\sqrt{\frac{\pi}{a}} e^{-k^2/4a}\right] = e^{-ax^2}.$$

Moving the constants, we then obtain

$$F^{-1}[e^{-k^2/4a}] = \sqrt{\frac{a}{\pi}} e^{-ax^2}.$$

We now make the substitution $\alpha = \frac{1}{4a}$,

$$F^{-1}[e^{-\alpha k^2}] = \sqrt{\frac{1}{4\pi\alpha}} e^{-x^2/4\alpha}.$$

This is in the form needed to invert (7.69). Thus, for $\alpha = \frac{a+b}{4ab}$ we find

$$(f * g)(x) = h(x) = \sqrt{\frac{\pi}{a+b}} e^{-\frac{ab}{a+b}x^2}.$$

7.6.1 Application to Signal Analysis

There are many applications of the convolution operation. One of these areas is the study of analog signals. An analog signal is a continuous signal and may contain either a finite, or continuous set of frequencies. Fourier transforms can be used to represent such signals as a sum over the frequency content. In this section we will describe how convolutions can be used in studying signal analysis.

The first application is filtering. For a given signal there might be some noise in the signal, or some undesirable high frequencies. Or, the device used for recording an analog signal might naturally not be able to record high frequencies. Let $f(t)$ denote the amplitude of a given analog signal and $\hat{f}(\omega)$ be the Fourier transform of this signal. An example is provided in Figure 7.21. Recall that the Fourier transform gives the frequency content of the signal and that $\omega = 2\pi\nu$, where ν is the frequency in Hertz, or cycles per second (cps).

There are many ways to filter out unwanted frequencies. The simplest would be to just drop all of the high frequencies, $|\omega| > \omega_0$ for some cutoff frequency ω_0. The Fourier transform of the filtered signal would then be zero for $|\omega| > \omega_0$. This could be accomplished by multiplying the Fourier transform of the signal by a function that vanishes

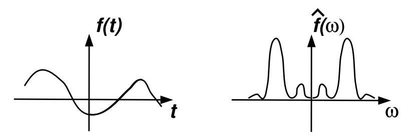


Figure 7.21: Schematic plot of a signal $f(t)$ and its Fourier transform $\hat{f}(\omega)$.

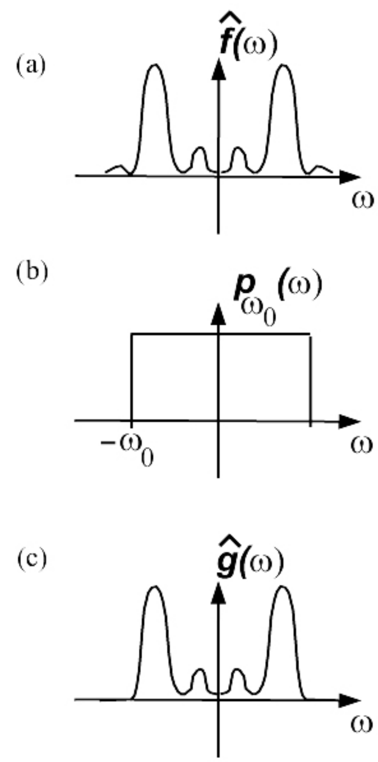


Figure 7.22: (a) Plot of the Fourier transform $\hat{f}(\omega)$ of a signal. (b) The gate function $p_{\omega_0}(\omega)$ used to filter out high frequencies. (c) The product of the functions, $\hat{g}(\omega) = \hat{f}(\omega)p_{\omega_0}(\omega)$, in (a) and (b).

for $|\omega| > \omega_0$. For example, we could consider the gate function

$$p_{\omega_0}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}. \tag{7.70}$$

Figure 7.22 shows how the gate function is used to filter the signal.

In general, we multiply the Fourier transform of the signal by some filtering function $\hat{h}(t)$ to get the Fourier transform of the filtered signal,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{h}(\omega).$$

The new signal, $g(t)$ is then the inverse Fourier transform of this product, giving the new signal as a convolution:

$$g(t) = F^{-1}[\hat{f}(\omega)\hat{h}(\omega)] = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)\,d\tau. \tag{7.71}$$

Such processes occur often in systems theory as well. One thinks of $f(t)$ as the input signal into some filtering device which in turn produces the output, $g(t)$. The function $h(t)$ is called the *impulse response.* This is because it is a response to the impulse function, $\delta(t)$. In this case, one has

$$\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)\,d\tau = h(t).$$

Another application of the convolution is in windowing. This represents what happens when one measures a real signal. Real signals cannot be recorded for all values of time. Instead data is collected over a finite time interval. If the length of time the data is collected is T, then the resulting signal is zero outside this time interval. This can be modeled in the same way as with filtering, except the new signal will be the product of the old signal with the windowing function. The resulting Fourier transform of the new signal will be a convolution of the Fourier transforms of the original signal and the windowing function.

Example 7.12. Finite Wave Train, Revisited.

We return to the finite wave train in Example 7.9 given by

$$h(t) = \begin{cases} \cos\omega_0 t, & |t| \leq a \\ 0, & |t| > a \end{cases}.$$

We can view this as a windowed version of $f(t) = \cos\omega_0 t$ obtained by multiplying $f(t)$ by the gate function

$$g_a(t) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}. \tag{7.72}$$

This is shown in Figure 7.23. Then, the Fourier transform is given as a convolution,

$$\begin{aligned} \hat{h}(\omega) &= (\hat{f} * \hat{g}_a)(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega-\nu)\hat{g}_a(\nu)\,d\nu. \end{aligned} \tag{7.73}$$

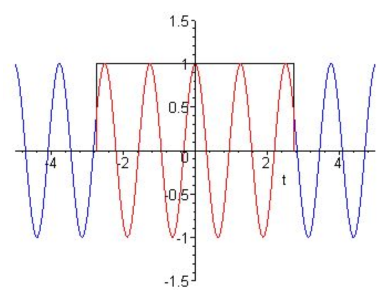


Figure 7.23: A plot of the finite wave train.

Note that the convolution in frequency space requires the extra factor of $1/(2\pi)$.

We need the Fourier transforms of f and g_a in order to finish the computation. The Fourier transform of the box function was already found in Example 7.5 as

$$\hat{g}_a(\omega) = \frac{2}{\omega}\sin\omega a.$$

The Fourier transform of the cosine function, $f(t) = \cos\omega_0 t$, *is*

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) e^{i\omega t}\, dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{i(\omega+\omega_0)t} + e^{i(\omega-\omega_0)t}\right)\, dt \\
&= \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]. \qquad (7.74)
\end{aligned}$$

Note that we had earlier computed the inverse Fourier transform of this function in Example 7.8.

Inserting these results in the convolution integral, we have

$$\begin{aligned}
\hat{h}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega-\nu)\hat{g}_a(\nu)\, d\nu \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi\left[\delta(\omega-\nu+\omega_0) + \delta(\omega-\nu-\omega_0)\right]\frac{2}{\nu}\sin\nu a\, d\nu \\
&= \frac{\sin(\omega+\omega_0)a}{\omega+\omega_0} + \frac{\sin(\omega-\omega_0)a}{\omega-\omega_0}. \qquad (7.75)
\end{aligned}$$

This is the same result we had obtained in Example 7.9.

7.6.2 *Parseval's Equality*

As another example of the convolution theorem, we derive Parseval's Equality (named after Marc-Antoine Parseval (1755-1836)):

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega. \qquad (7.76)$$

This equality has a physical meaning for signals. The integral on the left side is a measure of the energy content of the signal in the time domain. The right side provides a measure of the energy content of the transform of the signal. Parseval's equality, is simply a statement that the energy is invariant under the transform. Parseval's equality is a special case of Plancherel's formula (named after Michel Plancherel).

Let's rewrite the Convolution Theorem in its inverse form

$$F^{-1}[\hat{f}(k)\hat{g}(k)] = (f * g)(t). \qquad (7.77)$$

Then, by the definition of the inverse Fourier transform, we have

$$\int_{-\infty}^{\infty} f(t-u)g(u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{-i\omega t}\,d\omega.$$

Setting $t = 0$,

$$\int_{-\infty}^{\infty} f(-u)g(u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)\,d\omega. \tag{7.78}$$

Now, let $g(t) = \overline{f(-t)}$, or $f(-t) = \overline{g(t)}$. We note that the Fourier transform of $g(t)$ is related to the Fourier transform of $f(t)$:

$$\begin{aligned} \hat{g}(\omega) &= \int_{-\infty}^{\infty} \overline{f(-t)}e^{i\omega t}\,dt \\ &= -\int_{\infty}^{-\infty} \overline{f(\tau)}e^{-i\omega\tau}\,d\tau \\ &= \overline{\int_{-\infty}^{\infty} f(\tau)e^{i\omega\tau}\,d\tau} = \overline{\hat{f}(\omega)}. \end{aligned} \tag{7.79}$$

So, inserting this result into Equation (7.78), we find that

$$\int_{-\infty}^{\infty} f(-u)\overline{f(-u)}\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$$

which yields Parseval's Equality in the form (7.76) after subsituting $t = -u$ on the left.

As noted above, the forms in Equations (7.76) and (7.78) are often referred to as the Plancherel formula or Parseval formula. A more commonly defined Paresval equation is that given for Fourier series. For example, for a function $f(x)$ defined on $[-\pi, \pi]$, which has a Fourier series representation, we have

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx.$$

In general, there is a Parseval identity for functions that can be expanded in a complete sets of orthonormal functions, $\{\phi_n(x)\}$, $n = 1, 2, \ldots,$ which is given by

$$\sum_{n=1}^{\infty} < f, \phi_n >^2 = \|f\|^2.$$

Here $\|f\|^2 =< f, f >$. The Fourier series example is just a special case of this formula.

7.7 *The Laplace Transform*

The Laplace transform is named after Pierre-Simon de Laplace (1749-1827). Laplace made major contributions, especially to celestial mechanics, tidal analysis, and probability.

UP UNTIL THIS POINT we have only explored Fourier exponential transforms as one type of integral transform. The Fourier transform is useful on infinite domains. However, students are often introduced to another integral transform, called the Laplace transform, in their introductory differential equations class. These transforms are defined over semi-infinite domains and are useful for solving ordinary differential equations.

The Fourier and Laplace transforms are examples of a broader class of transforms known as *integral transforms*. For a function $f(x)$ defined on an interval (a, b), we define the integral transform

$$F(k) = \int_a^b K(x,k) f(x)\, dx,$$

where $K(x,k)$ is a specified kernel of the transform. Looking at the Fourier transform, we see that the interval is stretched over the entire real axis and the kernel is of the form, $K(x,k) = e^{ikx}$. In Table 7.1 we show several types of integral transforms.

Laplace Transform	$F(s) = \int_0^\infty e^{-sx} f(x)\, dx$
Fourier Transform	$F(k) = \int_{-\infty}^\infty e^{ikx} f(x)\, dx$
Fourier Cosine Transform	$F(k) = \int_0^\infty \cos(kx) f(x)\, dx$
Fourier Sine Transform	$F(k) = \int_0^\infty \sin(kx) f(x)\, dx$
Mellin Transform	$F(k) = \int_0^\infty x^{k-1} f(x)\, dx$
Hankel Transform	$F(k) = \int_0^\infty x J_n(kx) f(x)\, dx$

Table 7.1: A table of common integral transforms.

Laplace transforms also have proven useful in engineering for solving circuit problems and doing systems analysis. In Figure 7.24 it is shown that a signal $x(t)$ is provided as input to a linear system, indicated by $h(t)$. One is interested in the system output, $y(t)$, which is given by a convolution of the input and system functions. By considering the transforms of $x(t)$ and $h(t)$, the transform of the output is given as a product of the Laplace transforms in the s-domain. In order to obtain the output, one needs to compute a convolution product for Laplace transforms similar to the convolution operation we had seen for Fourier transforms earlier in the chapter. Of course, for us to do this in practice, we have to know how to compute Laplace transforms.

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t) e^{-st}\, dt, \quad s > 0. \tag{7.80}$$

This is an improper integral and one needs

$$\lim_{t\to\infty} f(t) e^{-st} = 0$$

to guarantee convergence.

It is typical that one makes use of Laplace transforms by referring to a Table of transform pairs. A sample of such pairs is given in Table

Time domain

$$\begin{tikzcd}
x(t) \arrow[r] \arrow[d, "\begin{array}{c}\text{Laplace}\\ \text{Transform}\end{array}"] & \boxed{h(t)} \arrow[r] \arrow[d, "\begin{array}{c}\text{Laplace}\\ \text{Transform}\end{array}"] & y(t) = h(t) * x(t) \\
X(s) \arrow[r] & \boxed{H(s)} \arrow[r] & Y(s) = H(s)X(s) \arrow[u, "\begin{array}{c}\text{Inverse Laplace}\\ \text{Transform}\end{array}"']
\end{tikzcd}$$

Frequency domain

Figure 7.24: A schematic depicting the use of Laplace transforms in systems theory.

7.2. Combining some of these simple Laplace transforms with the properties of the Laplace transform, as shown in Table 7.3, we can deal with many applications of the Laplace transform. We will first prove a few of the given Laplace transforms and show how they can be used to obtain new transform pairs. In the next section we will show how these can be used to solve ordinary differential equations.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
c	$\frac{c}{s}$	e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > 0$	$\delta(t-a)$	$e^{-as}, \quad a \geq 0, s > 0$

Table 7.2: Table of selected Laplace transform pairs.

We begin with some simple transforms. These are found by simply using the definition of the Laplace transform.

Example 7.13. $\mathcal{L}[1]$

For this example, we insert $f(t) = 1$ into the integral transform:

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\, dt.$$

This is an improper integral and the computation is understood by introducing an upper limit of a and then letting $a \to \infty$. We will not always write this limit, but it will be understood that this is how one computes such improper *integrals. Proceeding with the computation, we have*

$$\begin{aligned}
\mathcal{L}[1] &= \int_0^\infty e^{-st}\,dt \\
&= \lim_{a\to\infty}\int_0^a e^{-st}\,dt \\
&= \lim_{a\to\infty}\left(-\frac{1}{s}e^{-st}\right)_0^a \\
&= \lim_{a\to\infty}\left(-\frac{1}{s}e^{-sa}+\frac{1}{s}\right) = \frac{1}{s}.
\end{aligned} \tag{7.81}$$

Thus, we have found that the Laplace transform of 1 is $\frac{1}{s}$. This result can be extended to any constant c, using the linearity of the transform. Since the Laplace transform is simply an integral, $\mathcal{L}[c] = c\mathcal{L}[1]$. Therefore,

$$\mathcal{L}[c] = \frac{c}{s}.$$

Example 7.14. $\mathcal{L}[e^{at}]$,

For this example, we can easily compute the transform. Again, we only need to compute the integral of an exponential function.

$$\begin{aligned}
\mathcal{L}[e^{at}] &= \int_0^\infty e^{at}e^{-st}\,dt \\
&= \int_0^\infty e^{(a-s)t}\,dt \\
&= \left(\frac{1}{a-s}e^{(a-s)t}\right)_0^\infty \\
&= \lim_{t\to\infty}\frac{1}{a-s}e^{(a-s)t} - \frac{1}{a-s} = \frac{1}{s-a}.
\end{aligned} \tag{7.82}$$

Note that the last limit was computed as $\lim_{t\to\infty} e^{(a-s)t} = 0$. *This is only true if* $a - s < 0$, *or* $s > a$. *[Actually, a could be complex. In this case we would only need s to be greater than the real part of a,* $s > Re(a) = 0$.*]*

Example 7.15. $\mathcal{L}[\cos at]$ *and* $\mathcal{L}[\sin at]$

For these examples, we could again insert the trigonometric functions directly into the transform and integrate. For example,

$$\mathcal{L}[\cos at] = \int_0^\infty e^{-st}\cos at\,dt.$$

Recall how one evaluates integrals involving the product of a trigonometric function and the exponential function. One integrates by parts two times and then obtains an integral of the original unknown integral. Rearranging the resulting integral expressions, one arrives at the desired result. However, there is a much simpler way to compute these transforms.

Recall that $e^{iat} = \cos at + i\sin at$. *Making use of the linearity of the Laplace transform, we have*

$$\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at] + i\mathcal{L}[\sin at].$$

Thus, transforming this complex exponential will simultaneously provide the Laplace transforms for the sine and cosine functions! The transform is simply computed as

$$\mathcal{L}[e^{iat}] = \int_0^\infty e^{iat}e^{-st}\,dt = \int_0^\infty e^{-(s-ia)t}\,dt = \frac{1}{s-ia}.$$

Note that we could easily have used the result for the transform of an exponential, which was already proven. In this case $s > Re(ia) = 0$.

We now extract the real and imaginary parts of the result using the complex conjugate of the denominator:

$$\frac{1}{s-ia} = \frac{1}{s-ia}\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}.$$

Reading off the real and imaginary parts gives

$$\begin{aligned} \mathcal{L}[\cos at] &= \frac{s}{s^2+a^2} \\ \mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}. \end{aligned} \tag{7.83}$$

Example 7.16. $\mathcal{L}[t]$

For this example we evaluate

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt.$$

This integral can be done using the method of integration by parts. (Pick $u = t$ and $dv = e^{-st}\,dt$. Then, $du = dt$ and $v = -\frac{1}{s}e^{-st}$.) So, we have

$$\begin{aligned} \int_0^\infty te^{-st}\,dt &= -t\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2}. \end{aligned} \tag{7.84}$$

Example 7.17. $\mathcal{L}[t^n]$

We can generalize the last example to integer powers of t greater than $n = 1$. In this case we have to do the integral

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt.$$

Following the previous example, we again integrate by parts:[5]

$$\begin{aligned} \int_0^\infty t^n e^{-st}\,dt &= -t^n\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\,dt \\ &= \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\,dt. \end{aligned} \tag{7.85}$$

We could continue to integrate by parts until the final integral is computed. However, look at the integral that resulted after one integration by parts. It is just the Laplace transform of t^{n-1}. So, we can write the result as

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}].$$

[5] This integral can just as easily be done using differentiation. We note that

$$\left(-\frac{d}{ds}\right)^n \int_0^\infty e^{-st}\,dt = \int_0^\infty t^n e^{-st}\,dt.$$

Since

$$\int_0^\infty e^{-st}\,dt = \frac{1}{s},$$

$$\int_0^\infty t^n e^{-st}\,dt = \left(-\frac{d}{ds}\right)^n \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

We compute $\int_0^\infty t^n e^{-st}\,dt$ using an iterative method.

This is an example of a recursive definition of a sequence. In this case we have a sequence of integrals. Denoting

$$I_n = \mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$$

and noting that $I_0 = \mathcal{L}[1] = \frac{1}{s}$*, we have the following:*

Here we see an example of a first order difference equation and the solution of the corresponding initial value problem.

$$I_n = \frac{n}{s} I_{n-1}, \quad I_0 = \frac{1}{s}. \tag{7.86}$$

This is also what is called a difference equation. It is a first order difference equation with an "initial condition", I_0*. There is a whole theory of difference equations, which we will not get into here.*

Our goal is to solve the above difference equation. It is easy to do by simple iteration. Note that replacing n *with* $n-1$*, we have*

$$I_{n-1} = \frac{n-1}{s} I_{n-2}.$$

So, repeating the process we find

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n}{s}\left(\frac{n-1}{s} I_{n-2}\right) \\ &= \frac{n(n-1)}{s^2} I_{n-2}. \end{aligned} \tag{7.87}$$

We can repeat this process until we get to I_0*, which we know. We have to carefully count the number of iterations. We do this by iterating* k *times and then figure out how many steps will get us to the known initial value. A list of iterates is easily written out:*

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \\ &= \dots \\ &= \frac{n(n-1)(n-2)\dots(n-k+1)}{s^k} I_{n-k}. \end{aligned} \tag{7.88}$$

Since we know $I_0 = \frac{1}{s}$*, we choose to stop at* $k = n$ *obtaining*

$$I_n = \frac{n(n-1)(n-2)\dots(2)(1)}{s^n} I_0 = \frac{n!}{s^{n+1}}.$$

Therefore, we have shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$*. [Such iterative techniques are useful in obtaining a variety of of integrals, such as* $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\,dx$*. See Problem 10]*

As a final note, one can extend this result to cases when n is not an integer. To do this, one introduces what is called the Gamma function, which was discussed in Chapter 5. This function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{7.89}$$

Note the similarity to the Laplace transform of t^{x-1} :

$$\mathcal{L}[t^{x-1}] = \int_0^\infty t^{x-1} e^{-st}\, dt.$$

For $x-1$ an integer and $s=1$, we have that

$$\Gamma(x) = (x-1)!.$$

Thus, the Gamma function can be viewed as a generalization of the factorial and we have shown that

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

for $p > -1$.

Now we are ready to introduce additional properties of the Laplace transform. We will prove a few of the properties in Table 7.3.

Laplace Transform Properties
$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$
$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$
$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$
$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$
$\mathcal{L}[e^{at}f(t)] = F(s-a)$
$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$
$\mathcal{L}[(f*g)(t)] = \mathcal{L}[\int_0^t f(t-u)g(u)\, du] = F(s)G(s)$

Table 7.3: Table of selected Laplace transform properties.

Example 7.18. $\mathcal{L}\left[\frac{df}{dt}\right]$

We have to compute

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st}\, dt.$$

We can move the derivative off of f by integrating by parts. This is similar to what we had done when finding the Fourier transform of the derivative of a function. Letting $u = e^{-st}$ and $v = f(t)$, we have

$$\begin{aligned}
\mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt} e^{-st}\, dt \\
&= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\, dt \\
&= -f(0) + sF(s).
\end{aligned} \tag{7.90}$$

Here we have assumed that $f(t)e^{-st}$ vanishes for large t.

The final result is that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0).$$

Example 6: $\mathcal{L}\left[\frac{d^2f}{dt^2}\right]$

We can compute this Laplace transform using two integrations by parts, or we could make use of the last result. Letting $g(t) = \frac{df(t)}{dt}$, we have

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \mathcal{L}\left[\frac{dg}{dt}\right] = sG(s) - g(0) = sG(s) - f'(0).$$

But,

$$G(s) = \mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0).$$

So,

$$\begin{aligned} \mathcal{L}\left[\frac{d^2f}{dt^2}\right] &= sG(s) - f'(0) \\ &= s\left[sF(s) - f(0)\right] - f'(0) \\ &= s^2F(s) - sf(0) - f'(0). \end{aligned} \tag{7.91}$$

7.8 Further Uses of Laplace Transforms

ALTHOUGH THE LAPLACE TRANSFORM is a very useful transform, it is often encountered only as a method for solving initial value problems in introductory differential equations. In this section we will show how to solve simple differential equations. Along the way we will introduce step and impulse functions and show how the Convolution Theorem plays a role in finding solutions. Also, we will show that there is an inverse Laplace transform, called the Bromwich integral, named after Thomas John l'Anson Bromwich (1875-1929) . This inverse transform is not usually covered in differential equations courses because the integration takes place in the complex plane.

However, we will first explore an unrelated application of Laplace transforms. We will see that the Laplace transform is useful in finding sums of infinite series. Generally, many of the topics in this section are optional and not needed in the rest of the text.

7.8.1 Series Summation Using Laplace Transforms

WE SAW IN CHAPTER 4 that Fourier series can be used to sum series. For example, in Problem 4.13, one gets to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this section we will show how Laplace transforms can be used to sum series. [See Wheelon's book[6].] There is an interesting history of using integral transforms to sum series. For example, Richard Feynman[7] (1918-1988) described how one can use the convolution theorem for Laplace transforms to sum series with denominators that involved products. We will describe this and simpler sums in this section.

[6] Albert D. Wheelon, *Tables of Summable Series and Integrals Involving Bessel Functions*, Holden-Day, 1968.

[7] R. P. Feynman, 1949, *Phys. Rev.* **76**, p. 769

We begin by considering the Laplace transform of a known function,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Inserting this expression into the sum $\sum_n F(n)$ and interchanging the sum and integral, we find

$$\begin{aligned}\sum_{n=0}^\infty F(n) &= \sum_{n=0}^\infty \int_0^\infty f(t)e^{-nt}\,dt \\ &= \int_0^\infty f(t)\sum_{n=0}^\infty \left(e^{-t}\right)^n\,dt \\ &= \int_0^\infty f(t)\frac{1}{1-e^{-t}}\,dt. \qquad (7.92)\end{aligned}$$

The last step was obtained using the sum of a geometric series. The key is being able to carry out the final integral as we show in the next example.

Example 7.19. *Evaluate the sum* $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$.

Since, $\mathcal{L}[1] = 1/s$, *we have*

$$\begin{aligned}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} &= \sum_{n=1}^\infty \int_0^\infty (-1)^{n+1}e^{-nt}\,dt \\ &= \int_0^\infty \frac{e^{-t}}{1+e^{-t}}\,dt \\ &= \int_1^2 \frac{du}{u} = \ln 2. \qquad (7.93)\end{aligned}$$

Example 7.20. *Evaluate the sum* $\sum_{n=1}^\infty \frac{1}{n^2}$.

This is a special case of the Riemann zeta function [8]

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}. \qquad (7.94)$$

[8] A translation of Riemann, Bernhard (1859), "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" is in H. M. Edwards (1974). Riemann's Zeta Function. Academic Press. Riemann had shown that the Riemann zeta function can be obtained through contour integral representation, $2\sin(\pi s)\Gamma\zeta(s) = i\oint_C \frac{(-x)^{s-1}}{e^x-1}\,dx$, for a specific contour C.

This function is important in the study of prime numbers and more recently has seen applications in the study of dynamical systems. The series in this example is $\zeta(2)$. *We have already seen in 4.13that*

$$\zeta(2) = \frac{\pi^2}{6}.$$

Using Laplace transforms, we can provide an integral representation of $\zeta(2)$.

The first step is to find the correct Laplace transform pair. The sum involves the function $F(n) = 1/n^2$. So, we look for a function $f(t)$ whose Laplace transform is $F(s) = 1/s^2$. We know by now that the inverse Laplace transform of $F(s) = 1/s^2$ is $f(t) = t$. As before, we replace each term in the series by a Laplace transform, exchange the summation and integration, and sum the resulting geometric series:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \int_0^{\infty} te^{-nt}\, dt \\ &= \int_0^{\infty} \frac{t}{e^t - 1}\, dt. \end{aligned} \tag{7.95}$$

So, we have that

$$\int_0^{\infty} \frac{t}{e^t - 1}\, dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

Integrals of this type occur often in statistical mechanics in the form of Bose-Einstein integrals. These are of the form

$$G_n(z) = \int_0^{\infty} \frac{x^{n-1}}{z^{-1}e^x - 1}\, dx.$$

Note that $G_n(1) = \Gamma(n)\zeta(n)$.

In general the Riemann zeta function has to be tabulated through other means. In some special cases, one can closed form expressions. For example,

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_n,$$

where the B_n's are the Bernoulli numbers. Bernoulli numbers are defined through the Maclaurin series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

The first few Riemann zeta functions are

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

We can extend this method of using Laplace transforms to summing series whose terms take special general forms. For example, from Feynman's paper we note that

$$\frac{1}{(a+bn)^2} = -\frac{\partial}{\partial a} \int_0^{\infty} e^{-s(a+bn)}\, ds.$$

This identity can be shown easily by first noting

$$\int_0^{\infty} e^{-s(a+bn)}\, ds = \left[\frac{-e^{-s(a+bn)}}{a+bn} \right]_0^{\infty} = \frac{1}{a+bn}.$$

Now, differentiate the result with respect to a and the result follows.

The latter identity can be generalized further as

$$\frac{1}{(a+bn)^{k+1}} = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial a^k}\int_0^\infty e^{-s(a+bn)}\,ds.$$

In Feynman's 1949 paper, he develops methods for handling several other general sums using the convolution theorem. Wheelon gives more examples of these. We will just provide one such result and an example. First, we note that

$$\frac{1}{ab} = \int_0^1 \frac{du}{[a(1-u)+bu]^2}.$$

However,

$$\frac{1}{[a(1-u)+bu]^2} = \int_0^\infty te^{-t[a(1-u)+bu]}\,dt.$$

So, we have

$$\frac{1}{ab} = \int_0^1 du\int_0^\infty te^{-t[a(1-u)+bu]}\,dt.$$

We see in the next example how this representation can be useful.

Example 7.21. *Evaluate* $\sum_{n=0}^\infty \frac{1}{(2n+1)(2n+2)}$. *We compute this as follows:*

$$\begin{aligned}
\sum_{n=0}^\infty \frac{1}{(2n+1)(2n+2)} &= \sum_{n=0}^\infty \int_0^1 \frac{du}{[(2n+1)(1-u)+(2n+2)u]^2}\\
&= \sum_{n=0}^\infty \int_0^1 du\int_0^\infty te^{-t(2n+1+u)}\,dt\\
&= \int_0^\infty \frac{e^{-t}}{1-e^{-2t}}\int_0^1 e^{-tu}\,du\,dt\\
&= \int_0^\infty \frac{te^{-t}}{1-e^{-2t}}\frac{1-e^{-t}}{t}\,dt\\
&= \int_0^\infty \frac{e^{-t}}{1+e^{-t}}\,dt\\
&= -\ln(1+e^{-t})\Big|_0^\infty = \ln 2.
\end{aligned} \tag{7.96}$$

7.8.2 Solution of ODEs Using Laplace Transforms

One of the typical applications of Laplace transforms is the solution of nonhomogeneous linear constant coefficient differential equations. In the following examples we will show how this works.

The general idea is that one transforms the equation for an unknown function $y(t)$ into an algebraic equation for its transform, $Y(t)$. Typically, the algebraic equation is easy to solve for $Y(s)$ as a function of s. Then one transforms back into t-space using Laplace transform tables and the properties of Laplace transforms. The scheme is shown in Figure 7.25.

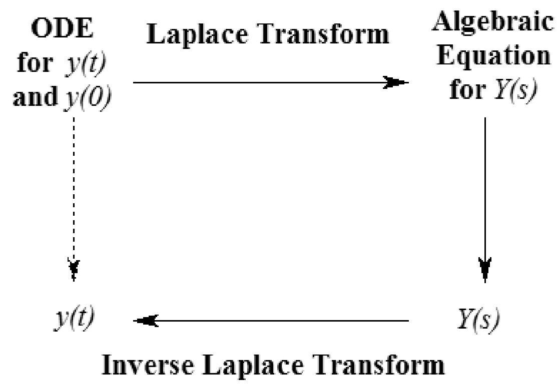


Figure 7.25: The scheme for solving an ordinary differential equation using Laplace transforms. One transforms the initial value problem for $y(t)$ and obtains an algebraic equation for $Y(s)$. Solve for $Y(s)$ and the inverse transform give the solution to the initial value problem.

Example 7.22. *Solve the initial value problem $y' + 3y = e^{2t}$, $y(0) = 1$.*

The first step is to perform a Laplace transform of the initial value problem. The transform of the left side of the equation is

$$\mathcal{L}[y' + 3y] = sY - y(0) + 3Y = (s+3)Y - 1.$$

Transforming the right hand side, we have

$$\mathcal{L}[e^{2t}] = \frac{1}{s-2}.$$

Combining these two results, we obtain

$$(s+3)Y - 1 = \frac{1}{s-2}.$$

The next step is to solve for $Y(s)$:

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)}.$$

Now, we need to find the inverse Laplace transform. Namely, we need to figure out what function has a Laplace transform of the above form. It is easy to do if we only had the first term. The inverse transform of the first term is e^{-3t}.

So far we have not seen anything that looks like the second form in the table of transforms that we have compiled. However, we are not stuck. We know that we can rewrite the second term by using a partial fraction decomposition. Let's recall how to do this.

This is an example of carrying out a partial fraction decomposition.

The goal is to find constants, A and B, such that

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}.$$

We picked this form because we know that recombining the two terms into one term will have the same denominator. We just need to make sure the numerators agree afterwards. So, adding the two terms, we have

$$\frac{1}{(s-2)(s+3)} = \frac{A(s+3) + B(s-2)}{(s-2)(s+3)}.$$

Equating numerators,

$$1 = A(s+3) + B(s-2).$$

This has to be true for all s. Rewriting the equation by gathering terms with common powers of s, we have

$$(A+B)s + 3A - 2B = 1.$$

The only way that this can be true for all s is that the coefficients of the different powers of s agree on both sides. This leads to two equations for A and B:

$$\begin{aligned} A + B &= 0 \\ 3A - 2B &= 1. \end{aligned} \tag{7.97}$$

The first equation gives $A = -B$, *so the second equation becomes* $-5B = 1$. *The solution is then* $A = -B = \frac{1}{5}$.

Returning to the problem, we have found that

$$Y(s) = \frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right).$$

[Of course, we could have tried to guess the form of the partial fraction decomposition as we had done earlier when talking about Laurent series.]

In order to finish the problem at hand, we find a function whose Laplace transform is of this form. We easily see that

$$y(t) = e^{-3t} + \frac{1}{5}\left(e^{2t} - e^{-3t}\right)$$

works. Simplifying, we have the solution of the initial value problem

$$y(t) = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}.$$

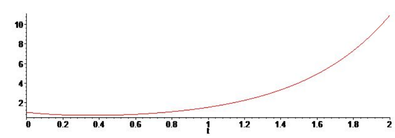


Figure 7.26: A plot of the solution to Example 7.22.

Example 7.23. *Solve the initial value problem* $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

We can probably solve this without Laplace transforms, but it is a simple exercise. Transforming the equation, we have

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) + 4Y \\ &= (s^2+4)Y - s - 3. \end{aligned} \tag{7.98}$$

Solving for Y, *we have*

$$Y(s) = \frac{s+3}{s^2+4}.$$

We now ask if we recognize the transform pair needed. The denominator looks like the type needed for the transform of a sine or cosine. We just need to play with the numerator. Splitting the expression into two terms, we have

$$Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4}.$$

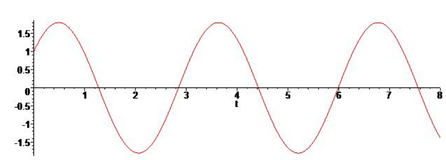


Figure 7.27: A plot of the solution to Example 7.23.

The first term is now recognizable as the transform of $\cos 2t$. *The second term is not the transform of* $\sin 2t$. *It would be if the numerator were a 2. This can be corrected by multiplying and dividing by 2:*

$$\frac{3}{s^2+4} = \frac{3}{2}\left(\frac{2}{s^2+4}\right).$$

The solution is then found as

$$y(t) = \mathcal{L}\left[\frac{s}{s^2+4} + \frac{3}{2}\left(\frac{2}{s^2+4}\right)\right] = \cos 2t + \frac{3}{2}\sin 2t.$$

7.8.3 Step and Impulse Functions

OFTEN THE INITIAL VALUE PROBLEMS that one faces in differential equations courses can be solved using either the Method of Undetermined Coefficients or the Method of Variation of Parameters. However, using the latter can be messy and involves some skill with integration. Many circuit designs can be modeled with systems of differential equations using Kirchoff's Rules. Such systems can get fairly complicated. However, Laplace transforms can be used to solve such systems and electrical engineers have long used such methods in circuit analysis.

In this section we add a couple of more transform pairs and transform properties that are useful in accounting for things like turning on a driving force, using periodic functions like a square wave, or introducing impulse forces.

We first recall the Heaviside step function, given by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{7.99}$$

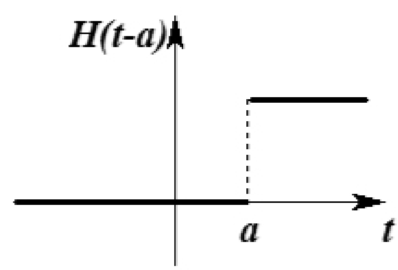


Figure 7.28: A shifted Heaviside function, $H(t-a)$.

A more general version of the step function is the horizontally shifted step function, $H(t-a)$. This function is shown in Figure 7.28. The Laplace transform of this function is found for $a > 0$ as

$$\begin{aligned} \mathcal{L}[H(t-a)] &= \int_0^\infty H(t-a)e^{-st}\,dt \\ &= \int_a^\infty e^{-st}\,dt \\ &= \left.\frac{e^{-st}}{s}\right|_a^\infty = \frac{e^{-as}}{s}. \end{aligned} \tag{7.100}$$

Just like the Fourier transform, the Laplace transform has two shift theorems involving the multiplication of the function, $f(t)$, or its transform, $F(s)$, by exponentials. The first and second shifting properties/theorems are given by

$$\mathcal{L}[e^{at}f(t)] = F(s-a) \tag{7.101}$$

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s). \tag{7.102}$$

We prove the First Shift Theorem and leave the other proof as an exercise for the reader. Namely,

$$\begin{aligned}\mathcal{L}[e^{at}f(t)] &= \int_0^\infty e^{at}f(t)e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a). \end{aligned} \tag{7.103}$$

Example 7.24. *Compute the Laplace transform of $e^{-at}\sin\omega t$.*

This function arises as the solution of the underdamped harmonic oscillator. We first note that the exponential multiplies a sine function. The shift theorem tells us that we need the transform of the sine function. So,

$$F(s) = \frac{\omega}{s^2+\omega^2}.$$

Using this transform, we can obtain the solution to this problem as

$$\mathcal{L}[e^{-at}\sin\omega t] = F(s+a) = \frac{\omega}{(s+a)^2+\omega^2}.$$

More interesting examples can be found in piecewise functions. First we consider the function $H(t)-H(t-a)$. For $t<0$ both terms are zero. In the interval $[0,a]$ the function $H(t)=1$ and $H(t-a)=0$. Therefore, $H(t)-H(t-a)=1$ for $t\in[0,a]$. Finally, for $t>a$, both functions are one and therefore the difference is zero. This function is shown in Figure 7.29.

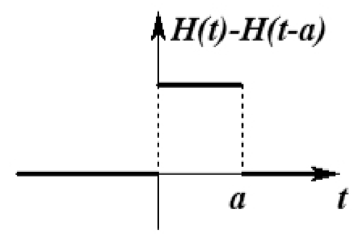


Figure 7.29: The box function, $H(t)-H(t-a)$.

We now consider the piecewise defined function

$$g(t) = \begin{cases} f(t), & 0\le t\le a, \\ 0, & t<0, t>a. \end{cases}$$

This function can be rewritten in terms of step functions. We only need to multiply $f(t)$ by the above box function,

$$g(t) = f(t)[H(t)-H(t-a)].$$

We depict this in Figure 7.30.

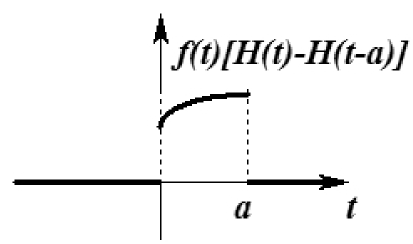


Figure 7.30: Formation of a piecewise function, $f(t)[H(t)-H(t-a)]$.

Even more complicated functions can be written out in terms of step functions. We only need to look at sums of functions of the form $f(t)[H(t-a)-H(t-b)]$ for $b>a$. This is just a box between a and b of height $f(t)$. An example of a square wave function is shown in Figure 7.31. It can be represented as a sum of an infinite number of boxes,

$$f(t) = \sum_{n=-\infty}^{\infty}[H(t-2na)-H(t-(2n+1)a)].$$

Example 7.25. *Laplace Transform of a square wave turned on at $t=0$,*

$$f(t) = \sum_{n=0}^{\infty}[H(t-2na)-H(t-(2n+1)a)].$$

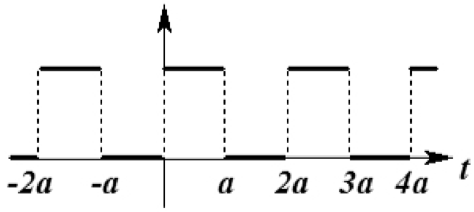


Figure 7.31: A square wave, $f(t) = \sum_{n=-\infty}^{\infty}[H(t-2na) - H(t-(2n+1)a)]$.

Using the properties of the Heaviside function, we have

$$\begin{aligned}
\mathcal{L}[f(t)] &= \sum_{n=0}^{\infty}\left[\mathcal{L}[H(t-2na)] - \mathcal{L}[H(t-(2n+1)a)]\right] \\
&= \sum_{n=0}^{\infty}\left[\frac{e^{-2nas}}{s} - \frac{e^{-(2n+1)as}}{s}\right] \\
&= \frac{1-e^{-as}}{s}\sum_{n=0}^{\infty}\left(e^{-2as}\right)^n \\
&= \frac{1-e^{-as}}{s}\left(\frac{1}{1-e^{-2as}}\right) \\
&= \frac{1-e^{-as}}{s(1-e^{-2as})}. \qquad (7.104)
\end{aligned}$$

Note that the third line in the derivation is a geometric series. We summed this series to get the answer in a compact form.

Another interesting example is the delta function. The delta function represents a point impulse, or point driving force. For example, while a mass on a spring is undergoing simple harmonic motion, one could hit it for an instant at time $t = a$. In such a case, we could represent the force as a multiple of $\delta(t-a)$. One would then need the Laplace transform of the delta function to solve the associated initial value problem.

We find that for $a > 0$

$$\begin{aligned}
\mathcal{L}[\delta(t-a)] &= \int_0^{\infty}\delta(t-a)e^{-st}\,dt \\
&= \int_{-\infty}^{\infty}\delta(t-a)e^{-st}\,dt \\
&= e^{-as}. \qquad (7.105)
\end{aligned}$$

The Dirac delta function can be used to represent a unit impulse. Summing over a number of impulses, or point sources, we can describe a general function. Such a sum of impulses located at points a_i, $i = 1, \ldots, n$ with strengths $f(a_i)$ would be given by

$$f(x) = \sum_{i=1}^{n} f(a_i)\delta(x-a_i).$$

A continuous sum could be written as

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\,d\xi.$$

This is simply an application of the sifting property of the delta function. In the next example we explore the application of a unit impulse to a still harmonic oscillator.

Example 7.26. *Solve the initial value problem* $y'' + 4\pi^2 y = \delta(t-2)$, $y(0) = y'(0) = 0$.

This initial value problem models a spring oscillation with an impulse force. Without the forcing term, given by the delta function, this spring is initially at rest and not stretched. The delta function models a unit impulse at $t = 2$. *Of course, we anticipate that at this time the spring will begin to oscillate. We will solve this problem using Laplace transforms.*

First, transform the differential equation:

$$s^2 Y - sy(0) - y'(0) + 4\pi^2 Y = e^{-2s}.$$

Inserting the initial conditions, we have

$$(s^2 + 4\pi^2)Y = e^{-2s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{e^{-2s}}{s^2 + 4\pi^2}.$$

We now seek the function for which this is the Laplace transform. The form of this function is an exponential times some Laplace transform, $F(s)$. *Thus, we need the Second Shift Theorem.*

First we need to find the $f(t)$ *corresponding to*

$$F(s) = \frac{1}{s^2 + 4\pi^2}.$$

The denominator suggests a sine or cosine. Since the numerator is constant, we pick sine. From the tables of transforms, we have

$$\mathcal{L}[\sin 2\pi t] = \frac{2\pi}{s^2 + 4\pi^2}.$$

So, we write

$$F(s) = \frac{1}{2\pi}\frac{2\pi}{s^2 + 4\pi^2}.$$

This gives $f(t) = (2\pi)^{-1}\sin 2\pi t$.

We now apply the Second Shift Theorem, $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$.

$$\begin{aligned} y(t) &= H(t-2)f(t-2) \\ &= \frac{1}{2\pi}H(t-2)\sin 2\pi(t-2). \end{aligned} \tag{7.106}$$

This solution tells us that the mass is at rest until $t = 2$ *and then begins to oscillate at its natural frequency. A plot of this solution is shown in Figure 7.32*

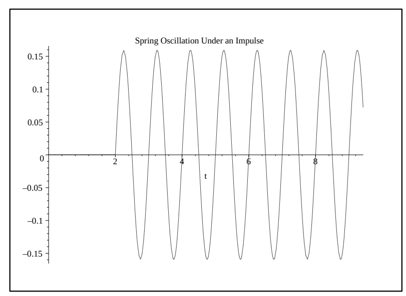


Figure 7.32: A plot of the solution to Example 7.26 in which a spring at rest experiences an impulse force at $t = 2$.

Example 7.27. *Solve the initial value problem* $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$, *where*

$$f(t) = \begin{cases} cos\pi t, & 0 \leq t \leq 2 \\ 0, & otherwise. \end{cases}$$

We need the Laplace transform of $f(t)$. *This function can be written in terms of a Heaviside function,* $f(t) = \cos \pi t H(t-2)$. *In order to apply the Second Shift Theorem, we need a shifted version of the cosine function. However,* $\cos \pi(t-2) = \cos \pi t$. *So,* $f(t) = \cos \pi(t-2)H(t-2)$ *and*

$$F(s) = (1 - e^{-2s})\mathcal{L}[\cos \pi t] = (1 - e^{-2s})\frac{s}{s^2 + \pi^2}.$$

Now we can proceed to solve the initial value problem. Its Laplace transform is

$$(s^2 + 1)Y(s) = (1 - e^{-2s})\frac{s}{s^2 + \pi^2},$$

or

$$Y(s) = (1 - e^{-2s})\frac{s}{(s^2 + \pi^2)(s^2 + 1)}.$$

We can retrieve the solution to the initial value problem using the Second Shift Theorem again. A partial fraction decomposition gives

$$\frac{s}{(s^2 + \pi^2)(s^2 + 1)} = \frac{1}{\pi^2 - 1}\left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + \pi^2}\right].$$

Thus,

$$\mathcal{L}\left[\frac{s}{(s^2 + \pi^2)(s^2 + 1)}\right] = \frac{1}{\pi^2 - 1}\left(\cos t - \cos \pi t\right).$$

The final solution is then

$$y(t) = \frac{1}{\pi^2 - 1}\left[\cos t - \cos \pi t - H(t-2)(\cos(t-2) - \cos \pi t)\right].$$

A plot of this solution is shown in Figure 7.33

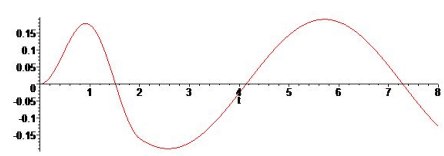


Figure 7.33: A plot of the solution to Example 7.27 in which a spring at rest experiences an piecewise defined force.

7.8.4 *The Convolution Theorem*

FINALLY, WE CONSIDER the convolution of two functions. Often we are faced with having the product of two Laplace transforms that we know and we seek the inverse transform of the product. For example, let's say you end up with $Y(s) = \frac{1}{(s-1)(s-2)}$ while trying to solve an initial value problem. We know how to do this if we only have one of the factors present in the denominator. Of course, we could do a partial fraction decomposition. But, there is another way to find the inverse transform, especially if we cannot perform a partial fraction decomposition.

We define the convolution of two functions defined on $[0, \infty)$ much the same way as we had done for the Fourier transform. The *convolution* $f * g$ is defined as

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du. \tag{7.107}$$

Note that the convolution integral has finite limits as opposed to the Fourier transform case.

The convolution operation has two important properties:

1. The convolution is commutative: $f * g = g * f$

   Proof. The key is to make a substitution $y = t - u$ in the integral. This makes f a simple function of the integration variable.

$$\begin{aligned}(g * f)(t) &= \int_0^t g(u)f(t-u)\,du \\ &= -\int_t^0 g(t-y)f(y)\,dy \\ &= \int_0^t f(y)g(t-y)\,dy \\ &= (f * g)(t). \end{aligned} \tag{7.108}$$

   □

2. **The Convolution Theorem:** The Laplace transform of a convolution is the product of the Laplace transforms of the individual functions:

$$\mathcal{L}[f * g] = F(s)G(s)$$

   Proof. Proving this theorem takes a bit more work. We will make some assumptions that will work in many cases. First, we assume that the functions are causal, $f(t) = 0$ and $g(t) = 0$ for $t < 0$. Secondly, we will assume that we can interchange integrals, which needs more rigorous attention than will be provided here. The first assumption will allow us to write the finite integral as an infinite integral. Then a change of variables will allow us to split the integral into the product of two integrals that are recognized as a product of two Laplace transforms.

$$\begin{aligned}\mathcal{L}[f * g] &= \int_0^\infty \left(\int_0^t f(u)g(t-u)\,du \right) e^{-st}\,dt \\ &= \int_0^\infty \left(\int_0^\infty f(u)g(t-u)\,du \right) e^{-st}\,dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty f(u) \left(\int_0^\infty g(t-u) e^{-st}\, dt \right) du \\
&= \int_0^\infty f(u) \left(\int_0^\infty g(\tau) e^{-s(\tau+u)}\, d\tau \right) du \\
&= \int_0^\infty f(u) e^{-su} \left(\int_0^\infty g(\tau) e^{-s\tau}\, d\tau \right) du \\
&= \left(\int_0^\infty f(u) e^{-su}\, du \right) \left(\int_0^\infty g(\tau) e^{-s\tau}\, d\tau \right) \\
&= F(s)G(s). \qquad (7.109)
\end{aligned}
$$

□

We make use of the Convolution Theorem to do the following example.

Example 7.28. $y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)(s-2)}\right]$.

We note that this is a product of two functions

$$Y(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-1}\frac{1}{s-2} = F(s)G(s).$$

We know the inverse transforms of the factors: $f(t) = e^t$ *and* $g(t) = e^{2t}$.

Using the Convolution Theorem, we find $y(t) = (f * g)(t)$. *We compute the convolution:*

$$
\begin{aligned}
y(t) &= \int_0^t f(u) g(t-u)\, du \\
&= \int_0^t e^u e^{2(t-u)}\, du \\
&= e^{2t} \int_0^t e^{-u}\, du \\
&= e^{2t}[-e^t + 1] = e^{2t} - e^t. \qquad (7.110)
\end{aligned}
$$

One can also confirm this by carrying out a partial fraction decomposition.

Example 7.29. *Consider the initial value problem,* $y'' + 9y = 2\sin 3t$, $y(0) = 1$, $y'(0) = 0$.

The Laplace transform of this problem is given by

$$(s^2+9)Y - s = \frac{6}{s^2+9}.$$

Solving for $Y(s)$, *we obtain*

$$Y(s) = \frac{6}{(s^2+9)^2} + \frac{s}{s^2+9}.$$

The inverse Laplace transform of the second term is easily found as $\cos(3t)$; *however, the first term is more complicated.*

If we look at Table 7.2, we see that the Laplace transform pairs with the denominator $(s^2+\omega^2)^2$ *are*

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2},$$

and

$$\mathcal{L}[t\cos\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

So, we might consider rewriting a partial fraction decomposition as

$$\frac{6}{(s^2+9)^2} = \frac{A6s}{(s^2+9)^2} + \frac{B(s^2-9)}{(s^2+9)^2} + \frac{Cs+D}{s^2+9}.$$

Combining the terms on the right over a common denominator, we find

$$6 = 6As + B(s^2-9) + (Cs+D)(s^2+9).$$

Collecting like powers of s, *we have*

$$Cs^3 + (D+B)s^2 + 6As + (D-B) = 6.$$

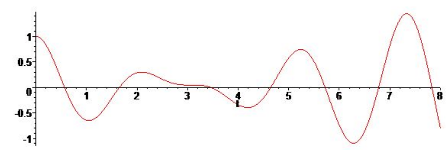


Figure 7.34: Plot of the solution to Example 7.29 showing a resonance.

Therefore, $C=0$, $A=0$, $D+B=0$, *and* $D-B=\frac{2}{3}$. *Solving the last two equations, we find* $D=-B=\frac{1}{3}$.

Using these results, we find

$$Y(s) = -\frac{1}{3}\frac{(s^2-9)}{(s^2+9)^2} + \frac{1}{3}\frac{1}{s^2+9} + \frac{s}{s^2+9}.$$

Therefore, the solution to the initial value problem,

$$y(t) = -\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t + \cos 3t.$$

Note that the amplitude of the solution will grow in time from the first term. You can see this in Figure 7.34. This is known as a resonance.

Example 7.30. *Find* $\mathcal{L}^{-1}[\frac{6}{(s^2+9)^2}]$ *using the Convolution Theorem.*

We can use the Convolution Theorem to find the Laplace transform in the last example. We note that

$$\frac{6}{(s^2+9)^2} = \frac{2}{3}\frac{3}{(s^2+9)}\frac{3}{(s^2+9)}$$

is a product of two Laplace transforms (up to the constant factor). Thus,

$$\mathcal{L}^{-1}[\frac{6}{(s^2+9)^2}] = \frac{2}{3}(f*g)(t),$$

where $f(t)=g(t)=sin3t$. *Evaluating this convolution product, we have*

$$\mathcal{L}^{-1}[\frac{6}{(s^2+9)^2}] \quad = \quad \frac{2}{3}(f*g)(t)$$

$$\begin{aligned}
&= \frac{2}{3}\int_0^t \sin 3u \sin 3(t-u)\,du \\
&= \frac{1}{3}\int_0^t [\cos 3(2u-t) - \cos 3t]\,du \\
&= \frac{1}{3}\left[\frac{1}{6}\sin(6u-3t) - u\cos 3t\right]_0^t \\
&= \frac{1}{9}\sin 3t - \frac{1}{3}t\cos 3t. \qquad (7.111)
\end{aligned}$$

This is the result we had obtained in the last example.

7.8.5 The Inverse Laplace Transform

UP UNTIL THIS POINT we have seen that the inverse Laplace transform can be found by making use of Laplace transform tables and properties of Laplace transforms. This is typically the way Laplace transforms are taught and used. One can do the same for Fourier transforms. However, in that case we introduced an inverse transform in the form of an integral. Does such an inverse exist for the Laplace transform? Yes, it does! In this section we will introduce the inverse Laplace transform integral and show how it is used.

We begin by considering a function $f(t)$ which vanishes for $t < 0$ and define the function $g(t) = f(t)e^{-ct}$. For $g(t)$ absolutely integrable,

$$\int_{-\infty}^{\infty} |g(t)|\,dt = \int_0^{\infty} |f(t)|e^{-ct}\,dt < \infty,$$

we can write the Fourier transform,

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \int_0^{\infty} f(t)e^{i\omega t - ct}dt$$

and the inverse Fourier transform,

$$g(t) = f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)e^{-i\omega t}\,d\omega.$$

Multiplying by e^{ct} and inserting $\hat{g}(\omega)$ into the integral for $g(t)$, we find

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f(\tau)e^{(i\omega - c)\tau}d\tau e^{-(i\omega - c)t}\,d\omega.$$

Letting $s = c - i\omega$ (so $d\omega = ids$), we have

$$f(t) = \frac{i}{2\pi}\int_{c+i\infty}^{c-i\infty}\int_0^{\infty} f(\tau)e^{-s\tau}d\tau e^{st}\,ds.$$

Note that the inside integral is simply $F(s)$. So, we have

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds. \qquad (7.112)}$$

The integral in the last equation is the inverse Laplace transform, called the *Bromwich integral.* This integral is evaluated along a path in the complex plane. The typical way to compute this integral is to chose c so that all poles are to the left of the contour and to close the contour with a semicircle enclosing the poles. One then relies on a generalization of Jordan's lemma to the second and third quadrants.

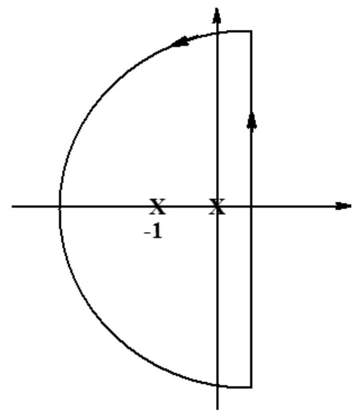


Figure 7.35: The contour used for applying the Bromwich integral to $F(s) = \frac{1}{s(s+1)}$.

Example 7.31. *Find the inverse Laplace transform of* $F(s) = \frac{1}{s(s+1)}$.

The integral we have to compute is

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(s+1)}\, ds.$$

This integral has poles at $s = 0$ *and* $s = -1$. *The contour we will use is shown in Figure 7.35. We enclose the contour with a semicircle to the left of the path in the complex s-plane. One has to verify that the integral over the semicircle vanishes as the radius goes to infinity. Assuming that we have done this, then the result is simply obtained as* $2\pi i$ *times the sum of the residues. The residues in this case are:*

$$Res\left[\frac{e^{zt}}{z(z+1)}; z = 0\right] = \lim_{z\to 0} \frac{e^{zt}}{(z+1)} = 1$$

and

$$Res\left[\frac{e^{zt}}{z(z+1)}; z = -1\right] = \lim_{z\to -1} \frac{e^{zt}}{z} = -e^{-t}.$$

Therefore, we have

$$f(t) = 2\pi i\left[\frac{1}{2\pi i}(1) + \frac{1}{2\pi i}(-e^{-t})\right] = 1 - e^{-t}.$$

We can verify this result using the Convolution Theorem or using a partial fraction decomposition. The decomposition is simplest:

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

The first term leads to an inverse transform of 1 and the second term gives an e^{-t}. *Thus, we have verified the result from doing contour integration.*

Problems

1. In this problem you will show that the sequence of functions

$$f_n(x) = \frac{n}{\pi}\left(\frac{1}{1+n^2x^2}\right)$$

approaches $\delta(x)$ as $n \to \infty$. Use the following to support your argument:

a. Show that $\lim_{n\to\infty} f_n(x) = 0$ for $x \neq 0$.

b. Show that the area under each function is one.

2. Evaluate the following integrals:

a. $\int_0^\pi \sin x \delta\left(x - \frac{\pi}{2}\right)\, dx$.
b. $\int_{-\infty}^{\infty} \delta\left(\frac{x-5}{3}e^{2x}\right)\left(3x^2 - 7x + 2\right)\, dx$.
c. $\int_0^\pi x^2 \delta\left(x + \frac{\pi}{2}\right)\, dx$.
d. $\int_0^\infty e^{-2x}\delta(x^2 - 5x + 6)\, dx$. [See Problem 3.]
e. $\int_{-\infty}^{\infty} (x^2 - 2x + 3)\delta(x^2 - 9)\, dx$. [See Problem 3.]

3. For the case that a function has multiple roots, $f(x_i) = 0$, $i = 1, 2, \ldots$, it can be shown that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

Use this result to evaluate $\int_{-\infty}^{\infty} \delta(x^2 - 5x + 6)(3x^2 - 7x + 2)\, dx$.

4. For $a > 0$, find the Fourier transform, $\hat{f}(k)$, of $f(x) = e^{-a|x|}$.

5. Prove the second shift property in the form

$$F\left[e^{i\beta x} f(x)\right] = \hat{f}(k + \beta).$$

6. A damped harmonic oscillator is given by

$$f(t) = \begin{cases} Ae^{-\alpha t}e^{i\omega_0 t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

.

a. Find $\hat{f}(\omega)$ and
b. the frequency distribution $|\hat{f}(\omega)|^2$.
c. Sketch the frequency distribution.

7. Show that the convolution operation is associative: $(f * (g * h))(t) = ((f * g) * h)(t)$.

8. In this problem you will directly compute the convolution of two Gaussian functions in two steps.

a. Use completing the square to evaluate

$$\int_{-\infty}^{\infty} e^{-\alpha t^2 + \beta t}\, dt.$$

b. Use the result from part a to directly compute the convolution in example 7.11:

$$(f * g)(x) = e^{-bx^2} \int_{-\infty}^{\infty} e^{-(a+b)t^2 + 2bxt}\, dt.$$

9. You will compute the (Fourier) convolution of two box functions of the same width. Recall the box function is given by

$$f_a(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a. \end{cases}$$

Consider $(f_a * f_a)(x)$ for different intervals of x. A few preliminary sketches would help. In Figure 7.36 the factors in the convolution integrand are show for one value of x. The integrand is the product of the first two functions. The convolution at x is the area of the overlap in the third figure. Think about how these pictures change as you vary x. Plot the resulting areas as a function of x. This is the graph of the desired convolution.

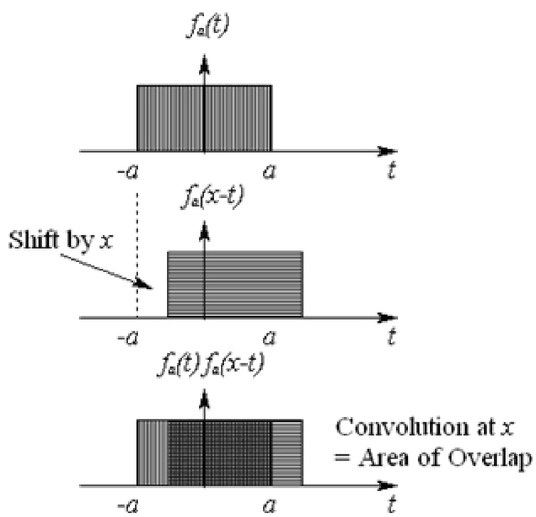


Figure 7.36: Sketch used to compute the convolution of the box function with itself. In the top figure is the box function. The second figure shows the box shifted by x. The last figure indicates the overlap of the functions.

10. Define the integrals $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\,dx$. Noting that $I_0 = \sqrt{\pi}$,

a. Find a recursive relation between I_n and I_{n-1}.

b. Use this relation to determine I_1, I_2 and I_3.

c. Find an expression in terms of n for I_n.

11. Find the Laplace transform of the following functions.

a. $f(t) = 9t^2 - 7$.

b. $f(t) = e^{5t-3}$.

c. $f(t) = \cos 7t$.

d. $f(t) = e^{4t} \sin 2t$.

e. $f(t) = e^{2t}(t + \cosh t)$.

f. $f(t) = t^2 H(t-1)$.

g. $f(t) = \begin{cases} \sin t, & t < 4\pi, \\ \sin t + \cos t, & t > 4\pi \end{cases}$.

h. $f(t) = \int_0^t (t-u)^2 \sin u \, du$.

12. Find the inverse Laplace transform of the following functions using the properties of Laplace transforms and the table of Laplace transform pairs.

a. $F(s) = \frac{18}{s^3} + \frac{7}{s}$.

b. $F(s) = \frac{1}{s-5} - \frac{2}{s^2+4}$.

c. $F(s) = \frac{s+1}{s^2+1}$.

d. $F(s) = \frac{3}{s^2+2s+2}$.

e. $F(s) = \frac{1}{(s-1)^2}$.

f. $F(s) = \frac{e^{-3s}}{s^2-1}$.

13. Compute the convolution $(f * g)(t)$ (in the Laplace transform sense) and its corresponding Laplace transform $\mathcal{L}[f * g]$ for the following functions:

a. $f(t) = t^2$, $g(t) = t^3$.

b. $f(t) = t^2$, $g(t) = \cos 2t$.

c. $f(t) = 3t^2 - 2t + 1$, $g(t) = e^{-3t}$.

b. $f(t) = \delta\left(t - \frac{\pi}{4}\right)$, $g(t) = \sin 5t$.

14. Use the convolution theorem to compute the inverse transform of the following:

a. $F(s) = \frac{2}{s^2(s^2+1)}$.

b. $F(s) = \frac{e^{-3s}}{s^2}$.

15. Find the inverse Laplace transform two different ways: i) Use Tables. ii) Use the Bromwich Integral.

a. $F(s) = \frac{1}{s^3(s+4)^2}$.

b. $F(s) = \frac{1}{s^2-4s-5}$.

c. $F(s) = \frac{s+3}{s^2+8s+17}$.

d. $F(s) = \frac{s+1}{(s-2)^2(s+4)}$.

16. Use Laplace transforms to solve the following initial value problems. Where possible, describe the solution behavior in terms of oscillation and decay.

a. $y'' - 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 0$.

b. $y'' - y = te^{2t}$, $y(0) = 0$, $y'(0) = 1$.

c. $y'' + 4y = \delta(t-1)$, $y(0) = 3$, $y'(0) = 0$.

d. $y'' + 6y' + 18y = 2H(\pi - t)$, $y(0) = 0$, $y'(0) = 0$.

17. Use Laplace transforms to sum the following series.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{1+2n}$.

b. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

c. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+3)}$.

d. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2-a^2}$.

e. $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2-a^2}$.

f. $\sum_{n=1}^{\infty} \frac{1}{n} e^{-an}$.

18. Do the following.

a. Find the first four nonvanishing terms of the Maclaurin series expansion of $f(x) = \frac{x}{e^x - 1}$.

b. Use the result in part a. to determine the first four nonvanishing Bernoulli numbers, B_n.

c. Use these results to compute $\zeta(2n)$ for $n = 1, 2, 3, 4$.

19. Given the following Laplace transforms, $F(s)$, find the function $f(t)$. Note that in each case there are an infinite number of poles, resulting in an infinite series representation.

a. $F(s) = \frac{1}{\cosh s}$.

b. $F(s) = \frac{1}{s \sinh s}$.

c. $F(s) = \frac{\sinh s}{s^2 \cosh s}$.

d. $F(s) = \frac{\sinh(\beta\sqrt{s}x)}{s \sinh(\beta\sqrt{s}L)}$.

8
Vector Analysis and EM Waves

"From a long view of the history of mankind seen from, say, ten thousand years from now, there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics."
The Feynman Lectures on Physics (1964), Richard Feynman (1918-1988)

UP TO THIS POINT we have mainly been confined to problems involving only one or two independent variables. In particular, the heat equation and the wave equation involved one time and one space dimension. However, we live in a world of three spatial dimensions. (Though, some theoretical physicists live in worlds of many more dimensions, or at least they think so.) We will need to extend the study of the heat equation and the wave equation to three spatial dimensions.

Recall that the one-dimensional wave equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{8.1}$$

For higher dimensional problems we will need to generalize the $\frac{\partial^2 u}{\partial x^2}$ term. For the case of electromagnetic waves in a source-free environment, we will derive a three dimensional wave equation for the electric and magnetic fields: It is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{8.2}$$

This is the generic form of the linear wave equation in Cartesian coordinates. It can be written a more compact form using the Laplacian, ∇^2,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{8.3}$$

The introduction of the Laplacian is common when generalizing to higher dimensions. In fact, we have already presented some generic one and three dimensional equations in Table 4.1, which we reproduce in Table 8.1. We have studied the one dimensional wave equation, heat equation, and Schrödinger equation. For steady-state, or equilibrium, heat flow problems, the heat equation no longer involves the time derivative. What is left is called Laplace's equation, which we have also seen in relation to complex functions. Adding an external heat source, Laplace's equation becomes what is known as Poisson's equation.

Name	2 Vars	3 D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2\nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x,y)$	$\nabla^2 u = F(x,y,z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x,t)u$	$iu_t = \nabla^2 u + F(x,y,z,t)u$

Table 8.1: List of generic partial differential equations.

Using the Laplacian allows us not only to write these equations in a more compact form, but also in a coordinate-free representation. Many problems are more easily cast in other coordinate systems. For example, the propagation of electromagnetic waves in an optical fiber are naturally described in terms of cylindrical coordinates. The heat flow inside a hemispherical igloo can be described using spherical coordinates. The vibrations of a circular drumhead can be described using polar coordinates. In each of these cases the Laplacian has to be written in terms of the needed coordinate systems.

The solution of these partial differential equations can be handled using separation of variables or transform methods. In the next chapter we will look at several examples of applying the separation of variables in higher dimensions. This will lead to the study of ordinary differential equations, which in turn leads to new sets of functions, other than the typical sine and cosine solutions.

In this chapter we will review some of the needed vector analysis for the derivation of the three dimensional wave equation from Maxwell's equations. We will review the basic vector operations (the dot and cross products), define the gradient, curl, and divergence and introduce standard vector identities that are often seen in physics courses. Equipped with these vector operations, we will derive the three dimensional waves equation for electromagnetic waves from Maxwell's equations. We will conclude this chapter with a section on curvilinear coordinates and provide the vector differential operators for different coordinate systems.

8.1 Vector Analysis

8.1.1 A Review of Vector Products

AT THIS POINT you might want to reread the first section of Chapter 3. In that chapter we introduced the formal definition of a vector space and some simple properties of vectors. We also discussed one of the common vector products, the dot product, which is defined as

$$\mathbf{u} \cdot \mathbf{v} = uv \cos\theta. \tag{8.4}$$

There is also a component form, which we write as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{k=1}^{3} u_k v_k. \tag{8.5}$$

One of the first physical examples using a cross product is the definition of work. The work done on a body by a constant force $\mathbf{F}$ during a displacement $\mathbf{d}$ is

$$W = \mathbf{F} \cdot \mathbf{d}.$$

In the case of a nonconstant force, we have to add up the incremental contributions to the work, $dW = \mathbf{F} \cdot d\mathbf{r}$ to obtain

$$W = \int_C dW = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{8.6}$$

over the path C. Note how much this looks like a complex path integral. It is a path integral, but the path lies in a real three dimensional space.

Another application of the dot product is the proof of the Law of Cosines. Recall that this law gives the side opposite a given angle in terms of the angle and the other two sides of the triangle:

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{8.7}$$

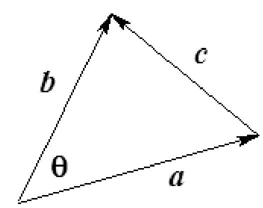


Figure 8.1: $v = r\omega$. The Law of Cosines can be derived using vectors.

Consider the triangle in Figure 8.1. We draw the sides of the triangle as vectors. Note that $\mathbf{b} = \mathbf{c} + \mathbf{a}$. Also, recall that the square of the length any vector can be written as the dot product of the vector with itself. Therefore, we have

$$\begin{aligned} c^2 &= \mathbf{c} \cdot \mathbf{c} \\ &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned} \tag{8.8}$$

We note that this also comes up in writing out inverse square laws in many applications. Namely, the vector $\mathbf{a}$ can locate a mass, or charge, and vector $\mathbf{b}$ points to an observation point. Then the inverse square law would involve vector $\mathbf{c}$, whose length is obtained as $\sqrt{a^2 + b^2 - 2ab\cos\theta}$. Typically, one does not have $\mathbf{a}$'s and $\mathbf{b}$'s, but something like $\mathbf{r}_1$ and $\mathbf{r}_2$, or $\mathbf{r}$ and $\mathbf{R}$. For these problems one is typically interested in approximating the expression of interest in terms of ratios like $\frac{r}{R}$ for $R \gg r$.

Another important vector product is the cross product. The cross product produces a vector, unlike the dot product that results in a scalar. The magnitude of the cross product is given as

$$|\mathbf{a} \times \mathbf{b}| = ab \sin\theta. \tag{8.9}$$

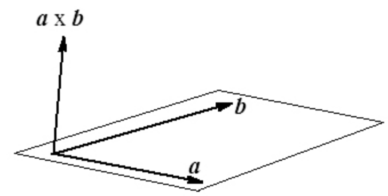


Figure 8.2: The cross product is shown. The direction is obtained using the right hand rule: Curl fingers from **a** through to **b**. The thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

Being a vector, we also have to specify the direction. The cross product produces a vector that is perpendicular to both vectors **a** and **b**. Thus, the vector is normal to the plane in which these vectors live. There are two possible directions. The direction taken is given by the right hand rule. This is shown in Figure 8.2. The direction can also be determined using your right hand. Curl your fingers from **a** through to **b**. The thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

One of the first occurrences of the cross product in physics is in the definition of the torque, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Recall that the torque is the analogue to the force. A net torque will cause an angular acceleration. Consider a rigid body in which a force is applied to to the body at a position **r** from the axis of rotation. (See Figure 8.3.) Then this force produces a torque with respect to the axis. The direction of the torque is given by the right hand rule. Point your fingers in the direction of **r** and rotate them towards **F**. In the figure this would be out of the page. This indicates that the bar would rotate in a counter clockwise direction if this were the only force acting on the bar.

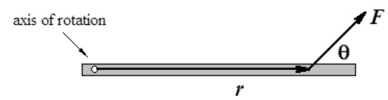


Figure 8.3: A force applied at a point located at **r** from the axis of rotation produces a torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ with respect to the axis.

Another example is that of a body rotating about an axis as shown in Figure 8.4. We locate the body with a position vector pointing from the origin of the coordinate system to the body. The tangential velocity of the body is related to the angular velocity by a cross product $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. The direction of the angular velocity is given be a right hand rule. Curl the fingers of your right hand in the direction of the motion of the rotating mass. Your thumb will point in the direction of $\boldsymbol{\omega}$. Counter clockwise motion produces a positive angular velocity and clockwise will give a negative angular velocity. Note that for the origin at the center of rotation of the mass, we obtain the familiar expression $v = r\omega$.

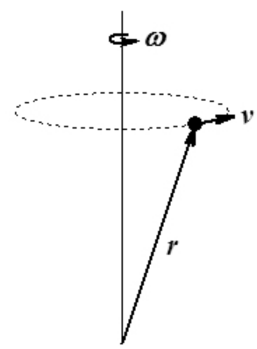


Figure 8.4: A mass rotates at an angular velocity $\boldsymbol{\omega}$about a fixed axis of rotation. The tangential velocity with respect to a given origin is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

There is also a geometric interpretation of the cross product. Consider the vectors **a** and **b** in Figure 8.5. Now draw a perpendicular from the tip of **b** to vector **a**. This forms a triangle of height h. Slide the triangle over to form a rectangle of base a and height h. The area of this triangle is

$$\begin{aligned} A &= ah \\ &= a(b\sin\theta) \\ &= |\mathbf{a} \times \mathbf{b}|. \end{aligned} \tag{8.10}$$

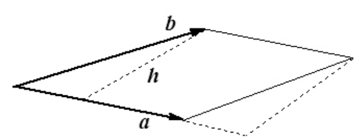


Figure 8.5: The magnitudes of the cross product gives the area of the parallelogram defined by **a** and **b**.

Therefore, the magnitude of the cross product is the area of the triangle formed by the vectors **a** and **b**.

The dot product was shown to have a simple form in terms of the components of the vectors. Similarly, we can write the cross product in component form. Recall that we can expand any vector $\mathbf{v}$ as

$$\mathbf{v} = \sum_{k=1}^{n} v_k \mathbf{e}_k, \tag{8.11}$$

where the $\mathbf{e}_k$'s are the standard basis vectors.

We would like to expand the cross product of two vectors,

$$\mathbf{u} \times \mathbf{v} = \left(\sum_{k=1}^{n} u_k \mathbf{e}_k \right) \times \left(\sum_{k=1}^{n} v_k \mathbf{e}_k \right).$$

In order to do this we need a few properties of the cross product.

Properties of the cross product.

First of all, the cross product is not commutative. In fact, it is anticommutative:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

A simple consequence of this is that $\mathbf{v} \times \mathbf{v} = 0$. Just replace $\mathbf{u}$ with $\mathbf{v}$ in the anticommutativity rule and you have $\mathbf{v} \times \mathbf{v} = -\mathbf{v} \times \mathbf{v}$. Something that is its negative must be zero.

The cross product also satisfies distributive properties:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}),$$

and

$$\mathbf{u} \times (a\mathbf{v}) = (a\mathbf{u}) \times \mathbf{v} = a\mathbf{u} \times \mathbf{v}.$$

Thus, we can expand the cross product in terms of the components of the given vectors. A simple computation shows that $\mathbf{u} \times \mathbf{v}$ can be expressed in terms of sums over $\mathbf{e}_i \times \mathbf{e}_j$:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \left(\sum_{i=1}^{n} u_i \mathbf{e}_i \right) \times \left(\sum_{j=1}^{n} v_j \mathbf{e}_j \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} u_i v_j \mathbf{e}_i \times \mathbf{e}_j. \end{aligned} \tag{8.12}$$

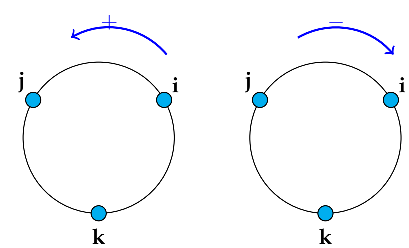


Figure 8.6: The sign for the cross product for basis vectors can be determined from a simple diagram. Arrange the vectors on a circle as above. If the needed computation goes counterclockwise, then the sign is positive. Thus, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$.

The cross products of basis vectors are simple to compute. First of all, the cross products $\mathbf{e}_i \times \mathbf{e}_j$ vanish when $i = j$ by anticommutativity of the cross product. For $i \neq j$, it is not much more difficult. For the typical basis, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, this is simple. Imagine computing $\mathbf{i} \times \mathbf{j}$. This is a vector of length $|\mathbf{i} \times \mathbf{j}| = |\mathbf{i}||\mathbf{j}| \sin 90^\circ = 1$. The vector $\mathbf{i} \times \mathbf{j}$ is perpendicular to both vectors, $\mathbf{i}$ and $\mathbf{j}$. Thus, the cross product is either $\mathbf{k}$ or $-\mathbf{k}$. Using the right hand rule, we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Similarly, we find the following

$$\begin{aligned} \mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} &= \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \end{aligned} \tag{8.13}$$

Inserting these results into the cross product for vectors in R^3, we have

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \tag{8.14}$$

While this form for the cross product is correct and useful, there are other forms that help in verifying identities or making computation simpler with less memorization. However, some of these new expressions can lead to problems for the novice as dealing with indices can be daunting at first sight.

One expression that is useful for computing cross products is the familiar computation using determinants. Namely, we have that

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned} \tag{8.15}$$

A more compact form for the cross product is obtained by introducing the completely antisymmetric symbol, ϵ_{ijk}. This symbol is defined by the relations

The completely antisymmetric symbol, or permutation symbol, ϵ_{ijk}.

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1,$$

and

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1,$$

and all other combinations, like ϵ_{113}, vanish. Note that all indices must differ. Also, if the order is a cyclic permutation of $\{1,2,3\}$, then the value is $+1$. For this reason ϵ_{ijk} is also called the permutation symbol or the Levi-Civita symbol. We can also indicate the index permutation more generally using the following identities:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}.$$

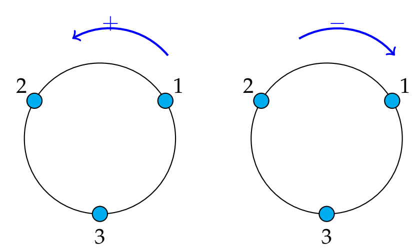


Figure 8.7: The sign for the permutation symbol can be determined from a simple cyclic diagram similar to that for the cross product. Arrange the numbers from 1 to 3 on a circle. If the needed computation goes counterclockwise, then the sign is positive, otherwise it is negative.

Returning to the cross product, we can introduce the standard basis $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, and $\mathbf{e}_3 = \mathbf{k}$. With this notation, we have that

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk}\mathbf{e}_k. \tag{8.16}$$

Example 8.1. *Compute the cross product of the basis vectors $\mathbf{e}_2 \times \mathbf{e}_1$ using the permutation symbol. A straight forward application of the definition of the cross product,*

$$\mathbf{e}_2 \times \mathbf{e}_1 = \sum_{k=1}^{3} \epsilon_{21k}\mathbf{e}_k$$

$$\begin{aligned} &= \epsilon_{211}\mathbf{e}_1 + \epsilon_{212}\mathbf{e}_2 + \epsilon_{213}\mathbf{e}_3 \\ &= -\mathbf{e}_3. \end{aligned} \tag{8.17}$$

It is helpful to write out enough terms in these sums until you get familiar with manipulating the indices. Note that the first two terms vanished because of repeated indices. In the last term we used $\epsilon_{213} = -1$.

We now write out the general cross product as

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j \mathbf{e}_i \times \mathbf{e}_j \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j \left(\sum_{k=1}^{3} \epsilon_{ijk}\mathbf{e}_k\right) \\ &= \sum_{i,j,k=1}^{3} \epsilon_{ijk} u_i v_j \mathbf{e}_k. \end{aligned} \tag{8.18}$$

Note that the last sum is a triple sum over the indices i, j, and k.

Example 8.2. *Let* $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ *and* $\mathbf{v} = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$. *Compute* $\mathbf{u} \times \mathbf{v}$. *We can compute this easily using determinants.*

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 1 & 5 & 4 \end{vmatrix} \\ &= \begin{vmatrix} -3 & 0 \\ 5 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}\mathbf{k} \\ &= -12\mathbf{i} - 8\mathbf{j} + 13\mathbf{k}. \end{aligned} \tag{8.19}$$

Using the permutation symbol to compute this cross product, we have

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \epsilon_{123}u_1v_2\mathbf{k} + \epsilon_{231}u_2v_3\mathbf{i} + \epsilon_{312}u_3v_1\mathbf{j} \\ &\quad + \epsilon_{213}u_2v_1\mathbf{k} + \epsilon_{132}u_1v_3\mathbf{j} + \epsilon_{321}u_3v_2\mathbf{i} \\ &= 2(5)\mathbf{k} + (-3)4\mathbf{i} + (0)1\mathbf{j} - (-3)1\mathbf{k} - (2)4\mathbf{j} - (0)5\mathbf{i} \\ &= -12\mathbf{i} - 8\mathbf{j} + 13\mathbf{k}. \end{aligned} \tag{8.20}$$

Sometimes it is useful to note that the kth component of the cross product is given by

$$\boxed{(\mathbf{u} \times \mathbf{v})_k = \sum_{i,j=1}^{3} \epsilon_{ijk} u_i v_j.}$$

In more advanced texts, or in the case of relativistic computations with tensors, the summation symbol is suppressed. For this case, one writes

$$(\mathbf{u} \times \mathbf{v})_k = \epsilon_{ijk} u_i v_j,$$

where it is understood that summation is performed over repeated indices. This is called the *Einstein summation convention*.

Since the cross product can be written as both a determinant,

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \epsilon_{ij1}u_iv_j\mathbf{i} + \epsilon_{ij2}u_iv_j\mathbf{j} + \epsilon_{ij3}u_iv_j\mathbf{k}. \end{aligned} \tag{8.21}$$

Einstein summation convention is used to suppress summation notation. In general relativity, one also needs to employ raised indices, so that vector components are written in the form u^i. The convention then requires that one only sums over a combination of one lower and one upper index. Thus, we would write $\epsilon_{ijk}u^iv^j$. We will forgo the need for raised indices.

and using the permutation symbol,

$$\mathbf{u}\times\mathbf{v} = \epsilon_{ijk}u_iv_j\mathbf{e}_k,$$

we can define the determinant as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}a_{1i}a_{2j}a_{3k}. \tag{8.22}$$

Here we added the triple sum in order to emphasize the hidden summations.

Example 8.3. *Compute the determinant* $\begin{vmatrix} 1 & 0 & 2 \\ 0 & -3 & 4 \\ 2 & 4 & -1 \end{vmatrix}$.

We insert the components of each row into the expression for the determinant:

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 2 \\ 0 & -3 & 4 \\ 2 & 4 & -1 \end{vmatrix} &= \epsilon_{123}(1)(-3)(-1) + \epsilon_{231}(0)(4)(2) + \epsilon_{312}(2)(0)(4) \\ &\quad +\epsilon_{213}(0)(0)(-1) + \epsilon_{132}(1)(4)(4) + \epsilon_{321}(2)(-3)(2) \\ &= 3+0+0-0-14-(-12) \\ &= 15. \end{aligned} \tag{8.23}$$

Note that if one adds copies of the first two columns, as shown in Figure 8.8, then the products of the first three diagonals, downward to the right (blue), give the positive terms in the determinant computation and the products of the last three diagonals, downward to the left (red), give the negative terms.

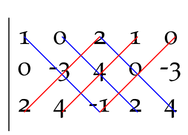


Figure 8.8: Diagram for computing determinants.

One useful identity is

$$\boxed{\epsilon_{jki}\epsilon_{j\ell m} = \delta_{k\ell}\delta_{im} - \delta_{km}\delta_{i\ell},}$$

Product identity satisfied by the permutation symbol, ϵ_{ijk}.

where δ_{ij} is the Kronecker delta. Note that the Einstein summation convention is used in this identity; i.e., summing over j is understood. So, the left side is really a sum of three terms:

$$\epsilon_{jki}\epsilon_{j\ell m} = \epsilon_{1ki}\epsilon_{1\ell m} + \epsilon_{2ki}\epsilon_{2\ell m} + \epsilon_{3ki}\epsilon_{3\ell m}.$$

This identity is simple to understand. For nonzero values of the Levi-Civita symbol, we have to require that all indices differ for each factor on the left side of the equation: $j \neq k \neq i$ and $j \neq \ell \neq m$. Since the first two slots are the same j, and the indices only take values $1, 2$, or 3, then either $k = \ell$ or $k = m$. This will give terms with factors of $\delta_{k\ell}$ or δ_{km}. If the former is true, then there is only one possibility for the third slot, $i = m$. Thus, we have a term $\delta_{k\ell}\delta_{im}$. Similarly, the other case yields the second term on the right side of the identity. We just need to get the signs right. Obviously, changing the order of ℓ and m will introduce a minus sign. A little care will show that the identity gives the correct ordering.

Other identities involving the permutation symbol are

$$\epsilon_{mjk}\epsilon_{njk} = 2\delta_{mn},$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6.$$

We will end this section by recalling triple products. There are only two ways to construct triple products. Starting with the cross product $\mathbf{b} \times \mathbf{c}$, which is a vector, we can multiply the cross product by a $\mathbf{a}$ to either obtain a scalar or a vector.

In the first case we have the triple scalar product, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Actually, we do not need the parentheses. Writing $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ could only mean one thing. If we computed $\mathbf{a} \cdot \mathbf{b}$ first, we would get a scalar. Then the result would be a multiple of $\mathbf{c}$, which is not a scalar. So, leaving off the parentheses would mean that we want the triple scalar product by convention.

Let's consider the component form of this product. We will use the Einstein summation convention and the fact that the permutation symbol is cyclic in ijk. Using $\epsilon_{jki} = \epsilon_{ijk}$,

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{a}_i(\mathbf{b} \times \mathbf{c})_i \\ &= \epsilon_{jki}a_i b_j c_k \\ &= \epsilon_{ijk}a_i b_j c_k \\ &= (\mathbf{a} \times \mathbf{b})_k c_k \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \end{aligned} \tag{8.24}$$

We have proven that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Now, imagine how much writing would be involved if we had expanded everything out in terms of all of the components.

Note that this result suggests that the triple scalar product can be computed by just computing a determinant:

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \epsilon_{ijk}a_ib_jc_k \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \end{aligned} \tag{8.25}$$

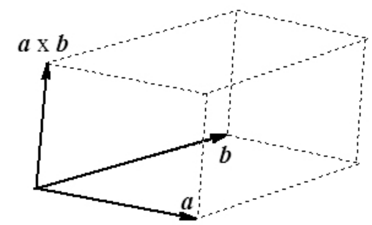


Figure 8.9: Three non-coplanar vectors define a parallelepiped. The volume is given by the triple scalar product, $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$.

There is a geometric interpretation of the scalar triple product. Consider the three vectors drawn as in Figure 8.9. If they do not all lie in a plane, then they form the sides of a parallelepiped. The cross product $\mathbf{a}\times\mathbf{b}$ gives the area of the base as we had seen earlier. The cross product is perpendicular to this base. The dot product of $\mathbf{c}$ with this cross product gives the height of the parallelepiped. So, the volume of the parallelepiped is the height times the base, or the triple scalar product. In general, one gets a signed volume, as the cross product could be pointing below the base.

The second type of triple product is the triple cross product,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \epsilon_{mnj}\epsilon_{ijk}a_ib_mc_n\mathbf{e}_k.$$

In this case we cannot drop the parentheses as this would lead to a real ambiguity. Lets think a little about this product. The vector $\mathbf{b}\times\mathbf{c}$ is a vector that is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$. Computing the triple cross product would then produce a vector perpendicular to $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$. But the later vector is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ already. Therefore, the triple cross product must lie in the plane spanned by these vectors. In fact, there is an identity that tells us exactly the right combination of vectors $\mathbf{b}$ and $\mathbf{c}$. It is given by

The BAC-CAB rule.

$$\boxed{\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \qquad (8.26)}$$

This rule is called the BAC-CAB rule because of the order of the right side of this equation.

Example 8.4. *Prove that* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$.

We can prove the BAC-CAB rule the permutation symbol and some identities. We first use the cross products $\mathbf{e}_i\times\mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k$ *and* $\mathbf{b}\times\mathbf{c} = \epsilon_{mnj}b_mc_n\mathbf{e}_j$:

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (a_i\mathbf{e}_i)\times((\mathbf{b}\times\mathbf{c})_j\mathbf{e}_j) \\ &= a_i(\mathbf{b}\times\mathbf{c})_j(\mathbf{e}_i\times\mathbf{e}_j) \\ &= a_i(\mathbf{b}\times\mathbf{c})_j\epsilon_{ijk}\mathbf{e}_k \\ &= \epsilon_{mnj}\epsilon_{ijk}a_ib_mc_n\mathbf{e}_k \end{aligned} \tag{8.27}$$

Now, we use the identity

$$\epsilon_{mnj}\epsilon_{ijk} = \delta_{mk}\delta_{ni} - \delta_{mi}\delta_{nk},$$

the properties of the Kronecker delta functions, and then rearrange the results to finish the proof:

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \epsilon_{mnj}\epsilon_{ijk}a_i b_m c_n \mathbf{e}_k \\ &= a_i b_m c_n \left(\delta_{mk}\delta_{ni} - \delta_{mi}\delta_{nk}\right) \mathbf{e}_k \\ &= a_n b_m c_n \mathbf{e}_m - a_m b_m c_n \mathbf{e}_n \\ &= (b_m \mathbf{e}_m)(c_n a_n) - (c_n \mathbf{e}_n)(a_m b_m) \\ &= \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \end{aligned} \tag{8.28}$$

8.1.2 Differentiation and Integration of Vectors

You have already been introduced to the idea that vectors can be differentiated and integrated in your introductory physics course. These ideas are also the major theme encountered in a multivariate calculus class, or Calculus III. We review some of these topics in the next sections. We first recall the differentiation and integration of vector functions.

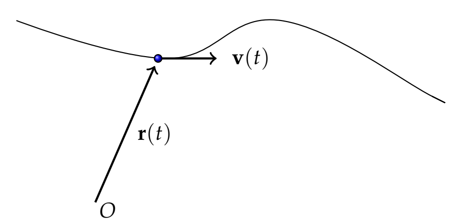


Figure 8.10: Position and velocity vectors of moving particle.

The position vector can change in time, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + x(t)\mathbf{k}$. The rate of change of this vector is the velocity,

$$\begin{aligned} \mathbf{v}(t) &= \frac{d\mathbf{r}}{dt} \\ &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \\ &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\ &= v_x\mathbf{i} + v_y\mathbf{k} + v_z\mathbf{k}. \end{aligned} \tag{8.29}$$

The velocity vector is tangent to the path, $\mathbf{r}(t)$, as seen in Figure 8.1.2. The magnitude of this vector gives the speed,

$$|\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

Moreover, differentiating this vector gives the acceleration, $\mathbf{a}(t) = \mathbf{v}'(t)$.

In general, one can differentiate an arbitrary time-dependent vector $\mathbf{v}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ as

$$\frac{d\mathbf{v}}{dt} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}. \tag{8.30}$$

Example 8.5. *A simple example is given by the motion on a circle. A circle in the xy-plane can be parametrized as* $\mathbf{r}(t) = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}$. *Then the velocity is found as*

$$\mathbf{v}(t) = -r\omega\sin(\omega t)\mathbf{i} + r\omega\cos(\omega t)\mathbf{j}.$$

Its speed is $v = r\omega$, *which is easily recognized as the tangential speed. The acceleration is*

$$\mathbf{a}(t) = -\omega^2 r\cos(\omega t)\mathbf{i} - \omega^2 r\sin(\omega t)\mathbf{j}.$$

The magnitude gives the centripetal acceleration, $a = \omega^2 r$ *and the acceleration vector is pointing towards the center of the circle.*

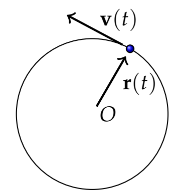


Figure 8.11: Particle on circular path.

Once one can differentiate time-dependent vectors, one can prove some standard properties.

a. $\dfrac{d}{dt}[\mathbf{u}+\mathbf{v}] = \dfrac{d\mathbf{u}}{dt} + \dfrac{d\mathbf{v}}{dt}.$

b. $\dfrac{d}{dt}[c\mathbf{u}] = c\dfrac{d\mathbf{u}}{dt}.$

c. $\dfrac{d}{dt}[f(t)\mathbf{u}] = f'(t)\mathbf{u} + f(t)\dfrac{d\mathbf{u}}{dt}.$

d. $\dfrac{d}{dt}[\mathbf{u}\cdot\mathbf{v}] = \dfrac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\dfrac{d\mathbf{v}}{dt}.$

e. $\dfrac{d}{dt}[\mathbf{u}\times\mathbf{v}] = \dfrac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\dfrac{d\mathbf{v}}{dt}.$

f. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = \dfrac{d\mathbf{u}}{df}\dfrac{df}{dt}.$

Example 8.6. *Let* $|\mathbf{r}(t)|$ *=const. Then,* $\mathbf{r}'(t)$ *is perpendicular* $\mathbf{r}(t)$.

Since $|\mathbf{r}|$ *=const,* $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{r}$ *=const. Differentiating this expression, one has* $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0$. *Therefore,* $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0$, *as was to be shown.*

In this discussion, we have referred to t as the time. However, when parametrizing spacecurves, t could represent any parameter. For example, the circle could be parametrized for t the angle swept out along any arc of the circle, $\mathbf{r}(t) = r\cos t\mathbf{i} + r\sin t\mathbf{j}$, for $t_1 \le t \le t_2$. We can still differentiate with respect to this parameter. It not longer has the meaning of velocity. another standard parameter is that of arclength. The arclength of a path is the distance along the path from some starting point. In deriving an expression for arclength, one first considers incremental distances along paths. Moving from point (x, y, z) to point $(x+\Delta x, y+\Delta y, z+\Delta z)$, one has gone a distance of

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.$$

Given a curve parametrized by t, such as the time, one can rewrite this as

$$\Delta s = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\,\Delta t.$$

Letting Δt get small, as well as the other increments, we are led to

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt. \tag{8.31}$$

We note that the square root is $|\mathbf{r}'(t)|$. So,

$$ds = |\mathbf{r}'(t)|dt,$$

or

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

In order to find the total arclength, we need only integrate over the parameter range,

$$s = \int_{t_1}^{t_2} |\mathbf{r}'(t)|\,dt.$$

If t is time and $\mathbf{r}(t)$ is the position vector of a particle, then $|\mathbf{r}'(t)|$ is the particle speed and we have that the distance traveled is simply an integral of the speed,

$$s = \int_{t_1}^{t_2} v\,dt.$$

If one is interested in knowing the distance traveled from point $\mathbf{r}(t_1)$ to an arbitrary point $\mathbf{r}(t)$, one can define the arclength function

$$s(t) = \int_{t_1}^{t} |\mathbf{r}'(\tau)|\,d\tau.$$

Example 8.7. *Determine the length of the parabolic path described by* $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j}$, $t \in [0,1]$.

We want to determine the length, $L = \int_0^1 |\mathbf{r}'(t)|\,dt$, *of a path. First, we have* $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$. *Then,* $|\mathbf{r}'(t)| = \sqrt{1+4t^2}$. *Using*

$$\int \sqrt{t^2+a^2}\,dt = \frac{1}{2}\left(t\sqrt{t^2+a^2} + a^2\ln(t+\sqrt{t^2+a^2})\right),$$

$$\begin{aligned} s &= \int_0^1 |\mathbf{r}'(t)|\,dt \\ &= \int_0^1 \sqrt{1+4t^2}\,dt \\ &= \left[x\sqrt{x^2+\frac{1}{4}} + \frac{1}{4}\ln\left(x+\sqrt{x^2+\frac{1}{4}}\right)\right]_0^1 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5}). \end{aligned} \tag{8.32}$$

Line integrals are defined as integrals of functions along a path, or curve, in space. Let $f(x,y,z)$ be the function, and C a parametrized path. Then we are interested in computing $\int_C f(x,y,z)\,ds$, where s is the arclength parameter. This integral looks similar to the contour integrals that we had studied in Chapter 5. We can compute such integrals in a similar manner by introducing the parametrization:

$$\int_C f(x,y,z)\,ds = \int_C f(x(t),y(t),z(t))|\mathbf{r}'(t)|\,dt.$$

Example 8.8. *Compute* $\int_C (x^2+y^2+z^2)\,ds$ *for the helical path* $\mathbf{r} = (\cos t, \sin t, t)$, $t \in [0, 2\pi]$.

In order to do this integral, we have to integrate over the given range of t *values. So, we replace* ds *with* $|\mathbf{r}'(t)|dt$. *In this problem* $|\mathbf{r}'(t)| = \sqrt{2}$. *Also, we insert the parametric forms for* $x(t) = \cos t$, $y(t) = \sin t$, *and* $z = t$ *into* $f(x,y,z)$. *Thus,*

$$\int_C (x^2+y^2+z^2)\,ds = \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt = 2\sqrt{2}\pi\left(1+\frac{4\pi^2}{3}\right). \quad (8.33)$$

One can also integrate vector functions. Given the vector function $\mathbf{v}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, we can do a straight forward term by term integration,

$$\int_a^b \mathbf{v}(t)\,dt = \int_a^b f(t)\,dt\mathbf{i} + \int_a^b g(t)\,dt\mathbf{j} + \int_a^b h(t)\,dt\mathbf{k}.$$

If $\mathbf{v}(t)$ is the velocity and t is the time, then

$$\int_a^b \mathbf{v}(t)\,dt = \int_a^b \frac{d\mathbf{r}}{dt}\,dt = \mathbf{r}(b) - \mathbf{r}(a).$$

We can thus interpret this integral as giving the displacement of a particle between times $t = a$ and $t = b$.

At the beginning of this chapter we had recalled the work done on a body by a nonconstant force $\mathbf{F}$ over a path C,

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} \quad (8.34)$$

If the path is parametrized by t, then we can write $d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt$. Thus the prescription for computing line integrals such as this is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt.$$

There are other forms that such line integrals can take. Let $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$. Noting that $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{y} + dz\mathbf{k}$, then we can write

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

Example 8.9. *Compute the work done by the force* $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ *on a particle as it moves around the circle* $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$*, for* $0 \leq t \leq \pi$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\,dx - x\,dy.$$

One way to complete this is to note that $dx = -\sin t\,dt$ *and* $dy = \cos t\,dt$*. Then*

$$\int_C y\,dx - x\,dy = \int_0^{\pi} (-\sin^2 t - \cos^2 t)\,dt = -\pi.$$

8.1.3 Div, Grad, Curl

Throughout physics we see functions which vary in both space and time. A function $f(x, y, z, t)$ is called a scalar function when the output is a scalar, or number. An example of such a function is the temperature. A function $\mathbf{F}(x, y, z, t)$ is called a vector (or vector valued) function if the output of the function is a vector. Let $\mathbf{v}(x, y, z, t)$ represent the velocity of a fluid at position (x, y, z) at time t. This is an example of a vector function. Typically when we assign a number, or a vector, to every point in a domain, we refer to this as a scalar, or vector, field. In this section we discuss how fields change from one point in space to another. Namely, we look at derivatives of multivariate functions with respect to their independent variables and the meanings of these derivatives in a physical context.

In studying functions of one variable in calculus, one is introduced to the derivative, $\frac{df}{dx}$: The derivative has several meanings. The standard mathematical meaning is that the derivative gives the slope of the graph of $f(x)$ at x. The derivative also tells us how rapidly $f(x)$ varies when x is changed by dx. Recall that dx is called a differential. We can think of the differential dx as an infinitesimal increment in x. Then changing x by an amount dx results in a change in $f(x)$ by

$$df = \frac{df}{dx}\,dx.$$

We can extend this idea to functions of several variables. Consider the temperature $T(x, y, z)$ at a point in space. The change in temperature depends on the direction in which one moves in space. Extending the above relation between differentials of the dependent and independent variables, we have

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz. \tag{8.35}$$

Note that if we only changed x, keeping y and z fixed, then we recover the form $dT = \frac{dT}{dx}\,dx$.

Introducing the vectors,

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}, \tag{8.36}$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}, \tag{8.37}$$

The gradient of a function,

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}, .$$

we can write Equation (8.35) as

$$dT = \nabla T \cdot d\mathbf{r} \tag{8.38}$$

Equation (8.37) defines the gradient of a scalar function, T. Equation (8.38) gives the change in T as one moves in the direction $d\mathbf{r}$.

Using the definition of the dot product, we also have

$$dT = |\nabla T||d\mathbf{r}|\cos\theta.$$

Note that by fixing $|d\mathbf{r}|$ and varying θ, the maximum value of dT is obtained when $\cos\theta = 1$. Thus, the maximum value of dT is in the direction of the gradient. Similarly, since $\cos\pi = -1$, the minimum value of dT is in a direction 180° from the gradient.

The greatest change is a function is in the direction of its gradient.

Example 8.10. *Let $f(x,y,z) = x^2y + ze^{xy}$. Compute ∇f.*

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \\ &= (2xy + yze^{xy})\mathbf{i} + (x^2 + xze^{xy})\mathbf{j} + e^{xy}\mathbf{k}. \end{aligned} \tag{8.39}$$

From this analysis, we see that the rate of change of a function, such as $T(x,y,z,)$, depends on the direction one heads away from a given point. So, if one moves an infinitesimal distance ds in some direction $d\mathbf{r}$, then how does T change with respect to s? Another way to ask this is to ask what is the directional derivative of T in direction $\mathbf{n}$? We define this directional derivative as

The directional derivative of a function, $D_{\mathbf{n}}T = \frac{dT}{ds} = \nabla T \cdot \mathbf{n}$.

$$D_{\mathbf{n}}T = \frac{dT}{ds}. \tag{8.40}$$

We can develop an operational definition of the directional derivative. From Equation (8.38) we have

$$\frac{dT}{ds} = \nabla T \cdot \frac{d\mathbf{r}}{ds}. \tag{8.41}$$

We note that

$$\frac{d\mathbf{r}}{ds} = \left(\frac{dx}{ds}\right)\mathbf{i} + \left(\frac{dy}{ds}\right)\mathbf{j} + \left(\frac{dz}{ds}\right)\mathbf{k}$$

and

$$\left|\frac{d\mathbf{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = 1.$$

Thus, $\mathbf{n} = \frac{d\mathbf{r}}{ds}$ is a unit vector pointing in the direction of interest and the directional derivative of $T(x,y,z)$ in direction $\mathbf{n}$ can be written as

$$D_{\mathbf{n}}T = \nabla T \cdot \mathbf{n}. \tag{8.42}$$

Example 8.11. *Let the temperature in a rectangular plate be given by* $T(x,y) = 5.0\sin\frac{3\pi x}{2}\sin\frac{\pi y}{2}$. *Determine the directional derivative at* $(x,y) = (1,1)$ *in the following directions: (a)* $\mathbf{i}$, *(b)* $3\mathbf{i}+4\mathbf{j}$.

In part (a) we have

$$D_{\mathbf{i}}T = \nabla T \cdot \mathbf{i} = \frac{\partial T}{\partial x}.$$

So,

$$D_{\mathbf{i}}T\bigg|_{(1,1)} = \frac{15}{2}\cos\frac{3\pi}{2}\sin\frac{\pi}{2} = 0.$$

In part (b) the direction given is not a unit vector, $|3\mathbf{i}+4\mathbf{j}| = 5$. *Dividing by the length of the vector, we obtain a unit normal vector,* $\mathbf{n} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$. *The directional derivative can now be computed:*

$$\begin{aligned} D_{\mathbf{n}}T &= \nabla T \cdot \mathbf{n} \\ &= \frac{3}{5}\frac{\partial T}{\partial x} + \frac{4}{5}\frac{\partial T}{\partial y} \\ &= \frac{9\pi}{2}\cos\frac{3\pi x}{2}\sin\frac{\pi y}{2} + 2\pi\sin\frac{3\pi x}{2}\cos\frac{\pi y}{2}. \end{aligned} \tag{8.43}$$

Evaluating this result at $(x,y) = (1,1)$, *we have*

$$D_{\mathbf{n}}T\bigg|_{(1,1)} = \frac{9\pi}{2}\cos\frac{3\pi}{2}\sin\frac{\pi}{2} + 2\pi\sin\frac{3\pi}{2}\cos\frac{\pi}{2} = 0.$$

We can write the gradient in the form

$$\nabla T = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)T. \tag{8.44}$$

Thus, we see that the gradient can be viewed as an operator acting on T. The operator,

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k},$$

is called the *del*, or *gradient*, operator. It is a differential vector operator. It can act on scalar functions to produce a vector field. Recall, if the gravitational potential is given by $\Phi(\mathbf{r})$, then the gravitational force is found as $\mathbf{F} = -\nabla\Phi$.

We can also allow the del operator to act on vector fields. Recall that a *vector field* is simply a vector valued function. For example, a force field is a function defined at points in space indicating the force that would act on a mass placed at that location. We could denote it as $\mathbf{F}(x,y,z)$. Again, think about the gravitational force above. The force acting on a mass in the Earth's gravitational field is a given by a vector field. At each point in space one would see that the force vector takes on different magnitudes and directions depending upon the location of the mass in space.

How can we combine the (vector) del operator and a vector field? Well, we could "multiply" them. We could either compute the dot product, $\nabla \cdot \mathbf{F}$, or we could compute the cross product $\nabla \times \mathbf{F}$. The first expression is called the *divergence* of the vector field and the second is called the *curl* of the vector field. These are typically encountered in a third semester calculus course. In some texts they are denoted by div $\mathbf{F}$ and curl $\mathbf{F}$.

The divergence is computed the same as any other dot product. Writing the vector field in component form,

The divergence, div $\mathbf{F} = \nabla \cdot \mathbf{F}$.

$$\mathbf{F} = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k},$$

we find the divergence is simply given as

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \end{aligned} \tag{8.45}$$

Similarly, we can compute the curl of $\mathbf{F}$. Using the determinant form, we have

The curl $\mathbf{F} = \nabla \times \mathbf{F}$.

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \end{aligned} \tag{8.46}$$

Example 8.12. *Compute the divergence and curl of the vector field:* $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

$$\nabla \cdot \mathbf{F} = \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} = 0.$$

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ y & -x & 0 \end{vmatrix} \\ &= \left(-\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right)\mathbf{k} = -2. \end{aligned} \tag{8.47}$$

These operations also have interpretations. The divergence measures how much the vector field $\mathbf{F}$ spreads from a point. When the divergence of a vector field is nonzero around a point, that is an indication that there is a source (div $\mathbf{F} > 0$) or a sink (div $\mathbf{F} < 0$). For example, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ indicates that there are sources contributing to the electric field. For a single charge, the field lines are radially pointing towards (sink) or away from (source) the charge. A field in which the divergence is zero is called divergenceless or solenoidal.

The curl is an indication of a rotational field. It is a measure of how much a field curls around a point. Consider the flow of a stream. The velocity of each element of fluid can be represented by a velocity field. If the curl of the field is nonzero, then when we drop a leaf into the stream we will see it begin to rotate about some point. A field that has zero curl is called irrotational.

The last common differential operator is the Laplace operator. This is the common second derivative operator, the divergence of the gradient,

$$\nabla^2 f = \nabla \cdot \nabla f.$$

The Laplace operator, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

It is easily computed as

$$\begin{aligned} \nabla^2 f &= \nabla \cdot \nabla f \\ &= \nabla \cdot \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned} \tag{8.48}$$

8.1.4 The Integral Theorems

MAXWELL'S EQUATIONS ARE GIVEN LATER IN THIS CHAPTER in differential form and only describe electric and magnetic fields locally. At times we would like to also provide global information, or information over an finite region. In this case one can derive various integral theorems. These are the finale in a three semester calculus sequence. We will not delve into these theorems here, as this will take us away from our goal of deriving a three dimensional wave equation. However, these integral theorems are important and useful in deriving local conservation laws.

These theorems are all different versions of a generalized Fundamental Theorem of Calculus:

(a) $\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$, The Fundamental Theorem of Calculus in 1D.

(b) $\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{r} = T(\mathbf{b}) - T(\mathbf{a})$, The Fundamental Theorem of Calculus for Vector Fields.

(c) $\oint_C (P\,dx + Q\,dy) = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$, Green's Theorem in the Plane.

(d) $\int_V \nabla \cdot \mathbf{F}\,dV = \oint_S \mathbf{F} \cdot d\mathbf{a}$, Gauss' Divergence Theorem.

(e) $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint_C \mathbf{F} \cdot d\mathbf{r}$, Stoke's Theorem.

The connections between these integral theorems are probably more easily seen by thinking in terms of fluids. Consider a fluid with mass density $\rho(x, y, z)$ and fluid velocity $\mathbf{v}(x, y, z, t)$. We define $(Q) = \rho\mathbf{v}$ as the mass flow rate. [Note the units are kg/m^2/s indicating the mass per area per time.]

Now consider the fluid flowing through an imaginary rectangular box. Let the fluid flow into the left face and out the right face. The rate at which the fluid mass flows through a face can be represented by $\mathbf{Q} \cdot d\boldsymbol{\sigma}$, where $d\boldsymbol{\sigma} = \mathbf{n} d\sigma$ represents the differential area element normal to the face. The rate of flow across the left face is

$$\mathbf{Q} \cdot d\boldsymbol{\sigma} = -Q_y \, dxdz\Big|_y$$

and that flowing across the right face is

$$\mathbf{Q} \cdot d\boldsymbol{\sigma} = Q_y \, dxdz\Big|_{y+dy}.$$

The net flow rate is the sum of these

$$Q_y dxdz\Big|_{y+dy} - Q_y dxdz\Big|_y = \frac{\partial Q_y}{\partial y} \, dxdydz.$$

A similar computation can be done for the other faces, leading to the result that the total rate of flow is $\nabla \cdot \mathbf{Q} \, d\tau$, where $d\tau = dxdydz$ is the volume element. So, the rate of flow per volume from the volume element gives

$$\nabla \cdot \mathbf{Q} = -\frac{\partial \rho}{\partial t}.$$

Note that if more fluid is flowing out the right face than is flowing into the left face, then the amount of fluid inside the region will decrease. That is why the right hand side of this equation has the negative sign.

Conservation of mass equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{Q} = 0.$$

If the fluid is incompressible, i.e., ρ =const., then $\nabla \cdot \mathbf{Q} = 0$, which implies $\nabla \cdot \mathbf{v} = 0$ assuming there are no sinks or sources. If there were a sink in the rectangular box, then there would be a loss of fluid not accounted for. Likewise, is a hose were inserted and fluid were supplied, then one would have a source.

If there are sinks, or sources, then the net mass due to these would contribute to an overall flow through the surrounding surface. This is captured by the equation

Gauss' Divergence Theorem

$$\underbrace{\int_V \nabla \cdot \mathbf{Q} \, d\tau}_{\text{Net mass due to sink/sources}} = \underbrace{\oint_S \mathbf{Q} \cdot \mathbf{n} \, d\sigma}_{\text{Net flow outward from } S} . \tag{8.49}$$

Dividing out the constant mass density, since $\mathbf{Q} = \rho\mathbf{v}$, this becomes

$$\boxed{\int_V \nabla \cdot \mathbf{v} \, d\tau = \oint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma.} \tag{8.50}$$

The surface integral on the right had side is called the flux of the vector field through surface S. This is nothing other than Gauss' Divergence Theorem.[1]

The unit normal can be written in terms of the direction cosines,

$$\mathbf{n} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k},$$

where the angles are the directions between $\mathbf{n}$ and the coordinate axes. For example, $\mathbf{n}\cdot\mathbf{i} = \cos\alpha$. For vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, we have

$$\begin{aligned}\int_S \mathbf{v}\cdot\mathbf{n}\, d\sigma &= \int_S (v_1\cos\alpha + v_2\cos\beta + v_3\cos\gamma)\, d\sigma \\ &= \int_S (u_1 dydz + u_2 dzdx + u_3 dxdy). \qquad (8.51)\end{aligned}$$

[1] We should note that the Divergence Theorem holds provided $\mathbf{v}$ is a continuous vector field and has continuous partial derivatives in a domain containing V. Also, $\mathbf{n}$ is the outward normal to the surface S.

Example 8.13. *Use the Divergence Theorem to compute*

$$\int_S (x^2 dydz + y^2 dzdx + z^2 dxdy)$$

for S the surface of the unit cube, $[0,1]\times[0,1]\times[0,1]$.

We first compute the divergence of the vector $\mathbf{v} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, *which we obtained from the coefficients in the given integral. Then*

$$\nabla\cdot\mathbf{v} = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 2(x+y+z).$$

Then,

$$\begin{aligned}\int_S (x^2 dydz + y^2 dzdx + z^2 dxdy) &= \int_V 2(x+y+z)\, d\tau \\ &= 2\int_0^1\int_0^1\int_0^1 (x+y+z)\, dxdydz \\ &= 2\int_0^1\int_0^1 (\frac{1}{2} + y + z)\, dydz \\ &= 2\int_0^1 (\frac{1}{2} + \frac{1}{2} + z)\, dz \\ &= 2(1+\frac{1}{2}) = 3. \qquad (8.52)\end{aligned}$$

The other integral theorem's are just a variation of the divergence theorem. For example, a two dimensional version of this obtained by considering a simply connected region, D, bounded by a simple closed curve, C. One could think of the laminar flow of a thin sheet of fluid. Then the total mass in contained in D and the net mass would be related to the next flow through the boundary, C. The integral theorem for this situation is given as

$$\int_D \nabla\cdot\mathbf{v}\, dA = \oint_C \mathbf{v}\cdot\mathbf{n}\, ds. \qquad (8.53)$$

The tangent vector to the curve at point $\mathbf{r}$ on the curve C, is

$$\frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}.$$

Therefore, the outward normal at that point is given by

$$\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

Letting $\mathbf{v} = Q(x,y)\mathbf{i} - P(x,y)\mathbf{j}$, the two dimensional version of the Divergence Theorem becomes

Green's Theorem in the Plane, which is a special case of Stoke's Theorem.

$$\boxed{\oint_C (P\,dx + Q\,dy) = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \qquad (8.54)}$$

This is just Green's Theorem in the Plane.

Example 8.14. *Evaluate* $\int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy$ *for* C *given by* $x^2 + 4y^2 = 4$.

Green's Theorem in the Plane gives

$$\begin{aligned} \int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy &= \int_S \left(\frac{\partial}{\partial x}(e^y + 6x) - \frac{\partial}{\partial y}(e^x - 3y)\right) dxdy \\ &= \int_S (6+3)\;dxdy \\ &= 9\int_S dxdy. \end{aligned} \qquad (8.55)$$

The integral that we need to compute is simply the area of the ellipse $x^2 + 4y^2 = 4$. *Recall that the area of an ellipse with semimajor axis* a *and semiminor axis* b *is* πab. *For this ellipse* $a = 2$ *and* $b = 1$. *So,*

$$\int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy = 18\pi.$$

We can obtain Stoke's Theorem by applying the Divergence Theorem to the vector $\mathbf{v} \times \mathbf{n}$.

$$\int_V \nabla \cdot (\mathbf{v} \times \mathbf{n})\,d\tau = \oint_S \mathbf{n}_s \cdot (\mathbf{v} \times \mathbf{n})\,d\sigma. \qquad (8.56)$$

Here $\mathbf{n}_s = \mathbf{u} \times \mathbf{n}$ where $\mathbf{u}$ is tangent to the curve C, and $\mathbf{n}$ is normal to the domain D. Noting that $(\mathbf{u} \times \mathbf{n}) \times (\mathbf{v} \times \mathbf{n}) = \mathbf{v} \cdot \mathbf{u}$ and $\nabla \cdot (\mathbf{v} \times \mathbf{n}) = \mathbf{n} \cdot \nabla \times \mathbf{v}$, then

$$\int_0^h \left(\int_D \mathbf{n} \cdot \nabla \times \mathbf{v}\,d\sigma\right) dh = \int_0^h \left(\oint_C \mathbf{v} \cdot \mathbf{u}\,ds\right) dh. \qquad (8.57)$$

Since h is arbitrary, then we obtain Stoke's Theorem:

Stoke's Theorem.

$$\boxed{\int_D \mathbf{n} \cdot \nabla \times \mathbf{v}\,d\sigma = \oint_C \mathbf{v} \cdot \mathbf{u}\,ds. \qquad (8.58)}$$

Example 8.15. *Evaluate* $\int_C (z\,dx + x\,dy + y\,dz)$ *for* C *the boundary of the triangle with vertices (1,0,0), (0,1,0), (0,0,1) using Stoke's Theorem.*

We first identify the vector $\mathbf{v} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$. *Then, we compute the curl,*

$$\begin{aligned}
\nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ z & x & y \end{vmatrix} \\
&= \mathbf{i} + \mathbf{j} + \mathbf{k}. \qquad (8.59)
\end{aligned}$$

Stoke's Theorem then gives

$$\int_C (z\,dx + x\,dy + y\,dz) = \int_D \mathbf{n} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\, d\sigma,$$

where $\mathbf{n}$ *is the outward normal to the surface of the triangle. For a surface defined by* $\phi(x, y, z)$ =*const, the normal is in the direction of* $\nabla\phi$. *In this case the triangle lives in the plane* $x + y + z = 1$. *Thus,* $\phi(x, y, z) = x + y + z$ *and* $\nabla\phi = \mathbf{i} + \mathbf{j} + \mathbf{k}$. *Thus,*

$$\int_C (z\,dx + x\,dy + y\,dz) = 3 \int_D d\sigma.$$

The remaining integral is just the area of the triangle. We can determine this area as follows. Imagine the vectors $\mathbf{a}$ *and* $\mathbf{b}$ *pointing from (1,0,0) to (0,1,0) and from (1,0,0) to (0,0,1), respectively. So,* $\mathbf{a} = \mathbf{j} - \mathbf{i}$ *and* $\mathbf{b} = \mathbf{k} - \mathbf{i}$.

These vectors are the sides of a parallelogram whose area is twice that of the triangle. The area of the parallelogram is given by $|\mathbf{a} \times \mathbf{b}|$. *The area of the triangle is thus*

$$\begin{aligned}
\int_D d\sigma &= \frac{1}{2}|\mathbf{a} \times \mathbf{b}| \\
&= \frac{1}{2}|(\mathbf{j} - \mathbf{i}) \times (\mathbf{k} - \mathbf{i})| \\
&= \frac{1}{2}|\mathbf{i} + \mathbf{j} + \mathbf{k}| = \frac{3}{2}. \qquad (8.60)
\end{aligned}$$

Finally, we have

$$\int_C (z\,dx + x\,dy + y\,dz) = \frac{9}{2}.$$

8.1.5 Vector Identities

In this section we will list some common vector identities and show how to prove a few of them. We will introduce two triple products and list first derivative and second derivative identities. These are useful in reducing some equations into simpler forms.

Proving these identities can be straight forward, though sometimes tedious in the more complicated cases. You should try to prove these yourself. Sometimes it is useful to write out the components on each side of the identity and see how one can fill in the needed arguments which would provide the proofs. We will provide a couple of examples of this process.

1. Triple Products

    (a) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$
    (b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

2. First Derivatives

    (a) $\nabla(fg) = f\nabla g + g\nabla f$
    (b) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
    (c) $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$
    (d) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
    (e) $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} - \mathbf{A} \times \nabla f$
    (f) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

3. Second Derivatives

    (a) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ div curl $= 0$.
    (b) $\nabla \times (\nabla f) = 0$ curl grad$= \mathbf{0}$.
    (c) $\nabla \cdot (\nabla f \times \nabla g) = 0$
    (d) $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$
    (e) $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
    (f) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

Example 8.16. *Prove* $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$.

In such problems one can write out the components on both sides of the identity. Using the determinant form of the triple scalar, the left hand side becomes

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \\ &= A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1). \end{aligned} \tag{8.61}$$

Similarly, the right hand side is given as

$$\begin{aligned} \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) &= \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= B_1(C_2A_3 - C_3A_2) - B_2(C_1A_3 - C_3A_1) + B_3(C_1A_2 - C_2A_1). \end{aligned} \tag{8.62}$$

We can rearrange this result by separating out the components of $\mathbf{A}$.

$$\begin{aligned} B_1(C_2A_3 &- C_3A_2) - B_2(C_1A_3 - C_3A_1) + B_3(C_1A_2 - C_2A_1) \\ &= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1). \end{aligned} \tag{8.63}$$

Upon inspection, we see that we obtain the same result as we had for the left hand side.

This problem can also be solved using the completely antisymmetric symbol, ϵ_{ijk}. Recall that the scalar triple product is given by

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \epsilon_{ijk} A_i B_j C_k.$$

(We have employed the Einstein summation convention.) Since $\epsilon_{ijk} = \epsilon_{jki}$, we have

$$\epsilon_{ijk} A_i B_j C_k = \epsilon_{jki} A_i B_j C_k = \epsilon_{jki} B_j C_k A_i.$$

But,

$$\epsilon_{jki} B_j C_k A_i = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}).$$

So, we have once again proven the identity. However, it took a little less work and an understanding of the antisymmetric symbol. Furthermore, you should note that this identity was proven earlier in the chapter.

Example 8.17. *Prove $\nabla(fg) = f\nabla g + g\nabla f$. In this problem we compte the gradient of fg. Then we note that each derivative is the derivative of a product and apply the Product Rule. Carefully writing out the terms, we obtain the desired result.*

$$\begin{aligned}
\nabla(fg) &= \frac{\partial fg}{\partial x}\mathbf{i} + \frac{\partial fg}{\partial y}\mathbf{j} + \frac{\partial fg}{\partial z}\mathbf{k} \\
&= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) g + f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) \\
&= f\nabla g + g\nabla f. \qquad (8.64)
\end{aligned}$$

8.2 Electromagnetic Waves

8.2.1 Maxwell's Equations

THERE ARE MANY APPLICATIONS leading to the equations in Table 8.1. One goal of this chapter is to derive the three dimensional wave equation for electromagnetic waves. This derivation was first carried out by James Clerk Maxwell in 1860. At the time much was known about the relationship between electric and magnetic fields through the work of of such people as Hans Christian Ørstead (1777-1851), Michael Faraday (1791-1867), and André-Marie Ampère. Maxwell provided a mathematical formalism for these relationships consisting of twenty partial differential equations in twenty unknowns. Later these equations were put into more compact notations, namely in terms of quaternions, only later to be cast in vector form.

Quaternions were introduced in 1843 by William Rowan Hamilton (1805-1865) as a four dimensional generalization of complex numbers.

In vector form, the original Maxwell's equations are given as

$$\nabla \cdot \mathbf{D} = \rho$$

$$\begin{aligned}
\nabla \times \mathbf{H} &= \mu_0 \mathbf{J}_{\text{tot}} \\
\mathbf{D} &= \epsilon \mathbf{E} \\
\mathbf{J} &= \sigma \mathbf{E} \\
\mathbf{J}_{\text{tot}} &= \mathbf{J} \frac{\partial \mathbf{D}}{\partial t} \\
\nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \\
\mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \\
\mu \mathbf{H} &= \nabla \times \mathbf{A}.
\end{aligned} \tag{8.65}$$

Note that Maxwell expressed the electric and magnetic fields in terms of the scalar and vector potentials, ϕ and $\mathbf{A}$, respectively, as defined in the last two equation. Here $\mathbf{H}$ is the magnetic field, $\mathbf{D}$ is the electric displacement field, $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the current density, ρ is the charge density, and σ is the conductivity.

This set of equations differs from what we typically present in physics courses. Several of these equations are defining quantities. While the potentials are part of a course in electrodynamics, they are not cast as the core set of equations now referred to as Maxwell's equations. Also, several equations are given as defining relations between the various variables, though they have some physical significance of their own, such as the continuity equation, given by $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$.

Furthermore, the distinction between the magnetic field strength, $\mathbf{H}$, and the magnetic flux density, $\mathbf{B}$, only becomes important in the presence of magnetic materials. Students are typically first introduced to $\mathbf{B}$ in introductory physics classes. In general, $\mathbf{B} = \mu \mathbf{H}$, where μ is the magnetic permeability of a material. In the absence of magnetic materials, $\mu = \mu_0$. In fact, in many applications of the propagation of electromagnetic waves, $\mu \approx \mu_0$.

These equations can be written in a more familiar form. The equations that we will refer to as Maxwell's equations from now on are

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \quad \text{(Gauss' Law)} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{(Faraday's Law)} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \text{(Maxwell-Ampère Law)}
\end{aligned} \tag{8.66}$$

We have noted the common names attributed to each law. There are corresponding integral forms of these laws, which are often presented in introductory physics class. The first law is Gauss' law. It allows one to determine the electric field due to specific charge distributions. The second law typically has no name attached to it, but in some cases is called Gauss' law for magnetic fields. It simply states that there are no free magnetic poles. The third law is Faraday's law, indicating that changing magnetic flux induces electric potential differences. Lastly, the fourth law is a modification of Ampere's law that states that electric currents produce magnetic fields.

It should be noted that the last term in the fourth equation was introduced by Maxwell. As we have seen, the divergence of the curl of any vector is zero,

The divergence of the curl of any vector is zero.

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0.$$

Computing the divergence of the curl of the electric field, we find from Maxwell's equations that

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{E}) &= -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} \\ &= -\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0. \end{aligned} \tag{8.67}$$

Thus, the relation works here.

However, before Maxwell, Ampère's law in differential form would have been written as

Ampère's law in differential form.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Computing the divergence of the curl of the magnetic field, we have

The introduction of the displacement current makes Maxwell's equations mathematically consistent.

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{B}) &= \mu_0 \nabla \cdot \mathbf{J} \\ &= -\mu_0 \frac{\partial \rho}{\partial t}. \end{aligned} \tag{8.68}$$

Here we made use of the continuity equation,

$$\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \nabla \cdot \mathbf{J} = 0.$$

As you can see, the vector identity, div curl $= 0$, does not work here! Maxwell argued that we need to account for a changing charge distribution. He introduced what he called the displacement current, $\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ into the Ampère Law. Now, we have

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{B}) &= \mu_0 \nabla \cdot \left(\mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= -\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} \\ &= -\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\epsilon_0} \right) = 0. \end{aligned} \tag{8.69}$$

So, Maxwell's introduction of the displacement current was not only physically important, it made the equations mathematically consistent.

8.2.2 *Electromagnetic Wave Equation*

We are now ready to derive the wave equation for electromagnetic waves. We will consider the case of free space in which there are no free charges or currents and the waves propagate in a vacuum. We then have Maxwell's equations in the form

Maxwell's equations in a vacuum.

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{8.70}$$

We will derive the wave equation for the electric field. You should confirm that a similar result can be obtained for the magnetic field. Consider the expression $\nabla \times (\nabla \times \mathbf{E})$. We note that the identities give

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

However, the divergence of $\mathbf{E}$ is zero, so we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}. \tag{8.71}$$

We can also use Faraday's Law on the right side of this equation to obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t} \right).$$

Interchanging the time and space derivatives, and using the Ampere-Maxwell Law, we find

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\ &= -\frac{\partial}{\partial t} \left(\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \end{aligned} \tag{8.72}$$

The three dimensional wave equations for electric and magnetic fields in a vacuum.

Combining the two expressions for $\nabla \times (\nabla \times \mathbf{E})$, we have the sought result:

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}.$$

This is the three dimensional equation for an oscillating electric field. A similar equation can be found for the magnetic field,

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla^2 \mathbf{B}.$$

Recalling that $\epsilon_0 = 8.85 \times 10^{-12}$ $\mathrm{C^2/Nm^2}$ and $\mu_0 = 4\pi \times 10^{-7}$ $\mathrm{N/A^2}$, one finds that $c = 3 \times 10^8$ m/s.

One can derive more general equations. For example, we could look for waves in what are called linear media. In this case one has $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. Here ϵ is called the electric permittivity and μ is the magnetic permeability of the material. Then, the wave speed in a vacuum, c, is replaced by the wave speed in the medium, v. It is given by

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}.$$

Here, n is the index of refraction, $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$. In many materials $\mu \approx \mu_0$. Introducing the dielectric constant, $\kappa = \frac{\epsilon}{\epsilon_0}$, one finds that $n \approx \sqrt{\kappa}$.

The wave equations lead to many of the properties of the electric and magnetic fields. We can also study systems in which these waves are confined, such as waveguides. In such cases we can impose boundary conditions and determine what modes are allowed to propagate within certain structures, such as optical fibers. However, these equation involve unknown vector fields. We have to solve for several inter-related component functions. In the next chapter we will look at simpler models in order to get some ideas as to how one can solve scalar wave equations in higher dimensions. However, we will first explore how the differential operators introduced in this chapter appear in different coordinate systems.

8.2.3 Potential Functions and Helmholtz's Theorem

Another application of the use of vector analysis for studying electromagnetism is that of potential theory. In this section we describe the use of a scalar potential, $\phi(\mathbf{r}, t)$ and a vector potential, $\mathbf{A}(\mathbf{r}, t)$ to solve problems in electromagnetic theory. Helmholtz's theorem says that a vector field is uniquely determined by knowing its divergence and its curl. Combining this result with the definitions of the electric and magnetic potentials, we will show that Maxwell's equations will the electric and magnetic fields can be found by simply solving a set of Poisson equations, $\nabla^2 u = f$, for the potential functions.

Hermann Ludwig Ferdinand von Helmholtz (1821-1894) made many contributions to physics. There are several theorems named after him.

A vector field is uniquely determined by knowing its divergence and its curl.

Electric and magnetic potentials.

In the case of static fields, we have from Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = 0.$$

We saw earlier in this chapter that the curl of a gradient is zero and the divergence of a curl is zero. This suggests that $\mathbf{E}$ is the gradient of a scalar function and $\mathbf{B}$ is the curl of a vector function:

$$\mathbf{E} = -\nabla\phi,$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

ϕ is called the electric potential and $\mathbf{A}$ is called the magnetic potential.

The remaining Maxwell equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Inserting the potential functions, we have

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}, \quad \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}.$$

Thus, ϕ satisfies a Poisson equation, which is a simple partial differential equation which can be solved using separation of variables, or other techniques.

The equation satisfied by the magnetic potential looks a little more complicated. However, we can use the identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

If $\nabla \cdot \mathbf{A} = 0$, then we find that

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

Thus, the components of the magnetic potential also satisfy Poisson equations!

It turns out that requiring $\nabla \cdot \mathbf{A} = 0$ is not as restrictive as one might first think. Potential functions are not unique. For example, adding a constant to a potential function will still give the same fields. For example

$$\nabla(\phi + c) = \nabla \phi = -\mathbf{E}.$$

This is not too alarming because it is the field that is physical and not the potential. In the case of the magnetic potential, adding the gradient of some field gives the same magnetic field, $\nabla \times (\mathbf{A} + \nabla \psi) = \nabla \times \mathbf{A} = \mathbf{B}$. So, we can choose ψ such that the new magnetic potential is divergenceless, $\nabla \cdot \mathbf{A} = 0$. A particular choice of the scalar and vector potentials is a called a gauge and the process is called fixing, or choosing, a gauge. The choice of $\nabla \cdot \mathbf{A} = 0$ is called the Coulomb gauge.

Coulomb gauge: $\nabla \cdot \mathbf{A} = 0$.

If the fields are dynamic, i.e., functions of time, then the magnetic potential also contributes to the electric field. In this case, we have

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t},$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Thus, two of Maxwell's equations are automatically satisfied,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

The other two equations become

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 \phi + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0},$$

and

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \Rightarrow$$
$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \frac{1}{c^2} \frac{\partial}{\partial t} \left(\nabla \phi + \frac{\partial \mathbf{A}}{\partial t} \right).$$

Rearranging, we have

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = -\mu_0 \mathbf{J}.$$

If we choose the Lorentz gauge, by requiring

Lorentz gauge: $\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$.

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t},$$

then

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = -\frac{\rho}{\epsilon_0},$$
$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu_0 \mathbf{J}.$$

In relativity, one defines the d'Alembertian by $\Box \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$. Then, the equations for the potentials become

$$\Box \phi = \frac{\rho}{\epsilon_0},$$

and

$$\Box \mathbf{A} = \mu_0 \mathbf{J}.$$

Thus, the potential satisfy nonhomogeneous wave equations, which can be solved with standard methods as one will see in a course in electrodynamics.

The above introduction of potentials to describe the electric and magnetic fields is a special case of Helmholtz's Theorem for vectors. This theorem states that "any sufficiently smooth, rapidly decaying vector field in three dimensions can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field."[2] This is known as the Helmholtz decomposition. Namely, given any nice vector field $\mathbf{v}$, we can write it as

[2] Wikipedia entry for the Helmholtz decomposition.

$$\mathbf{v} = \underbrace{-\nabla \phi}_{\text{irrotational}} + \underbrace{\nabla \times \mathbf{A}}_{\text{solenoidal}}.$$

Given

$$\nabla \cdot \mathbf{v} = \rho, \quad \nabla \times \mathbf{v} = \mathbf{F},$$

then one has

$$\nabla^2 \phi = \rho$$

and

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mathbf{F}.$$

Forcing $\nabla \cdot \mathbf{A} = 0$,

$$\nabla^2 \mathbf{A} = -\mathbf{F}.$$

Thus, one obtains Poisson equations for ϕ and $\mathbf{A}$. This is just repeating the above procedure which we had seen in the special case of static electromagnetic fields.

8.3 Curvilinear Coordinates

In order to study solutions of the wave equation, the heat equation, or even Schrödinger's equation in different geometries, we need to see how differential operators, such as the Laplacian, appear in these geometries. The most common coordinate systems arising in physics are polar coordinates, cylindrical coordinates, and spherical coordinates. These reflect the common geometrical symmetries often encountered in physics.

In such systems it is easier to describe boundary conditions and to make use of these symmetries. For example, specifying that the electric potential is 10.0 V on a spherical surface of radius one, we would say $\phi(x,y,z) = 10$ for $x^2 + y^2 + z^2 = 1$. However, if we use spherical coordinates, (r,θ,ϕ), then we would say $\phi(r,\theta,\phi) = 10$ for $r = 1$, or $\phi(1,\theta,\phi) = 10$. This is a much simpler representation of the boundary condition.

However, this simplicity in boundary conditions leads to a more complicated looking partial differential equation in spherical coordinates. In this section we will consider general coordinate systems and how the differential operators are written in the new coordinate systems. In the next chapter we will solve some of these new problems.

We begin by introducing the general coordinate transformations between Cartesian coordinates and the more general curvilinear coordinates. Let the Cartesian coordinates be designated by (x_1, x_2, x_3) and the new coordinates by (u_1, u_2, u_3). We will assume that these are related through the transformations

$$\begin{aligned} x_1 &= x_1(u_1,u_2,u_3), \\ x_2 &= x_2(u_1,u_2,u_3), \\ x_3 &= x_3(u_1,u_2,u_3). \end{aligned} \tag{8.73}$$

Thus, given the curvilinear coordinates (u_1, u_2, u_3) for a specific point in space, we can determine the Cartesian coordinates, (x_1, x_2, x_3), of that point. We will assume that we can invert this transformation: Given the Cartesian coordinates, one can determine the corresponding curvilinear coordinates.

Need to insert figures depicting this.

In the Cartesian system we can assign an orthogonal basis, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. As a particle traces out a path in space, one locates its position by the coordinates (x_1, x_2, x_3). Picking x_2 and x_3 constant, the particle lies on the curve $x_1 =$ value of the x_1 coordinate. This line lies in the direction of the basis vector $\mathbf{i}$. We can do the same with the other coordinates and essentially map out a grid in three dimensional space. All of the x_i-curves intersect at each point orthogonally and the basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ lie along the grid lines and are mutually orthogonal.

We would like to mimic this construction for general curvilinear coordinates. Requiring the orthogonality of the resulting basis vectors leads to orthogonal curvilinear coordinates.

As for the Cartesian case, we consider u_2 and u_3 constant. This leads to a curve parametrized by $u_1 : \mathbf{r} = x_1(u_1)\mathbf{i} + x_2(u_1)\mathbf{j} + x_3(u_1)\mathbf{k}$. We call this the u_1-curve. Similarly, when u_1 and u_3 are constant we obtain a u_2-curve and for u_1 and u_2 constant we obtain a u_3-curve. We will assume that these curves intersect such that each pair of curves intersect orthogonally. Furthermore, we will assume that the unit tangent vectors to these curves form a right handed system similar to the $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ systems for Cartesian coordinates. We will denote these as $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$.

We can quantify all of this. Consider the position vector as a function of the new coordinates,

$$\mathbf{r}(u_1, u_2, u_3) = x_1(u_1, u_2, u_3)\mathbf{i} + x_2(u_1, u_2, u_3)\mathbf{j} + x_3(u_1, u_2, u_3)\mathbf{k}.$$

Then the infinitesimal change in position is given by

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u_i} du_i.$$

We note that the vectors $\frac{\partial \mathbf{r}}{\partial u_i}$ are tangent to the u_i-curves. Thus, we define the unit tangent vectors

$$\hat{\mathbf{u}}_i = \frac{\frac{\partial \mathbf{r}}{\partial u_i}}{\left| \frac{\partial \mathbf{r}}{\partial u_i} \right|}.$$

Solving for the tangent vector, we have

$$\frac{\partial \mathbf{r}}{\partial u_i} = h_i \hat{\mathbf{u}}_i,$$

where

$$h_i \equiv \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|$$

are called the *scale factors* for the transformation.

The scale factors, $h_i \equiv \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|$.

Example 8.18. *Determine the scale factors for the polar coordinate transformation.*

Show an annotated polar plot here.

The transformation for polar coordinates is

$$x = r \cos\theta, \quad y = r \sin\theta.$$

Here we note that $x_1 = x$, $y_1 = y$, $u_1 = r$, and $u_2 = \theta$. The u_1-curves are curves with $\theta =$ const. Thus, these curves are radial lines. Similarly, the u_2-curves have $r =$ const. These curves are concentric circles about the origin.

The unit vectors are easily found. We will denote them by $\hat{\mathbf{u}}_r$ *and* $\hat{\mathbf{u}}_\theta$*. We can determine these unit be first computing* $\frac{\partial \mathbf{r}}{\partial u_i}$*. Let*

$$\mathbf{r} = x(r,\theta)\mathbf{i} + y(r,\theta)\mathbf{j} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.$$

Then,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial \theta} &= -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}. \end{aligned} \tag{8.74}$$

The first vector already is a unit vector. So,

$$\hat{\mathbf{u}}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}.$$

The second vector has length r since $|-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}| = r$*. Dividing* $\frac{\partial \mathbf{r}}{\partial \theta}$ *by* r*, we have*

$$\hat{\mathbf{u}}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}.$$

We can see these vectors are orthogonal and form a right hand system. That they form a right hand system can be seen by either drawing the vectors, or computing the cross product,

$$(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \times (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \mathbf{k}.$$

Since

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \hat{\mathbf{u}}_r, \\ \frac{\partial \mathbf{r}}{\partial \theta} &= r\hat{\mathbf{u}}_\theta, \end{aligned}$$

The scale factors are $h_r = 1$ *and* $h_\theta = r$*.*

We have determined that once we know the scale factors, we have that

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i.$$

The infinitesimal arclength is then given by

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = \sum_{i=1}^{3} h_i^2 du_i^2$$

when the system is orthogonal. Also, along the u_i-curves,

$$d\mathbf{r} = h_i du_i \hat{\mathbf{u}}_i, \quad \text{(no summation)}.$$

So, we consider at a given point (u_1, u_2, u_3) an infinitesimal parallelepiped of sides $h_i du_i$, $i = 1,2,3$. This infinitesimal parallelepiped has a volume of size

$$dV = \left|\frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3}\right| du_1 du_2 du_3.$$

The triple scalar product can be computed using determinants and the resulting determinant is call the Jacobian, and is given by

$$
\begin{aligned}
J &= \left| \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} \right| \\
&= \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| \\
&= \left| \begin{array}{ccc} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{array} \right| . \qquad (8.75)
\end{aligned}
$$

Therefore, the volume element can be written as

$$
dV = J\, du_1 du_2 du_3 = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} \right| du_1 du_2 du_3.
$$

Example 8.19. *Determine the volume element for cylindrical coordinates* (r, θ, z)*, given by*

$$
\begin{aligned}
x &= r \cos\theta, \qquad (8.76) \\
y &= r \sin\theta, \qquad (8.77) \\
z &= z. \qquad (8.78)
\end{aligned}
$$

Here, we have $(u_1, u_2, u_3) = (r, \theta, z)$*. Then, the Jacobian is given by*

$$
\begin{aligned}
J &= \left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| \\
&= \left| \begin{array}{ccc} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{array} \right| \\
&= \left| \begin{array}{ccc} \cos\theta & \sin\theta & 0 \\ -r \sin\theta & r \cos\theta & 0 \\ 0 & 0 & 1 \end{array} \right| \\
&= r \qquad (8.79)
\end{aligned}
$$

Thus, the volume element is given as

$$
dV = r dr d\theta dz.
$$

This result should be familiar from multivariate calculus.

Next we will derive the forms of the gradient, divergence, and curl in curvilinear coordinates. The results are given here for quick reference.

Gradient, divergence and curl in orthogonal curvilinear coordinates.

$$\begin{aligned}
\nabla \phi &= \sum_{i=1}^{3} \frac{\hat{\mathbf{u}}_i}{h_i} \frac{\partial \phi}{\partial u_i} \\
&= \frac{\hat{\mathbf{u}}_1}{h_1} \frac{\partial \phi}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2} \frac{\partial \phi}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3} \frac{\partial \phi}{\partial u_3}. && (8.80) \\
\nabla \cdot \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_1 h_3 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right). \\
& && (8.81) \\
\nabla \times \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}. && (8.82) \\
\nabla^2 \phi &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial u_2} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right) && (8.83) \\
& && (8.84)
\end{aligned}$$

We begin the derivations of these formulae by looking at the gradient, $\nabla \phi$, of the scalar function $\phi(u_1, u_2, u_3)$. We recall that the gradient operator appears in the differential change of a scalar function,

Derivation of the gradient form.

$$d\phi = \nabla \phi \cdot d\mathbf{r} = \sum_{i=1}^{3} \frac{\partial \phi}{\partial u_i} du_i.$$

Since

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i,$$

we also have that

$$d\phi = \nabla \phi \cdot d\mathbf{r} = \sum_{i=1}^{3} (\nabla \phi)_i \, h_i du_i.$$

Comparing these two expressions for $d\phi$, we determine that the components of the del operator can be written as

$$(\nabla \phi)_i = \frac{1}{h_i} \frac{\partial \phi}{\partial u_i}$$

and thus the gradient is given by

$$\nabla \phi = \frac{\hat{\mathbf{u}}_1}{h_1} \frac{\partial \phi}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2} \frac{\partial \phi}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3} \frac{\partial \phi}{\partial u_3}.$$

Next we compute the divergence,

Derivation of the divergence form.

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{3} \nabla \cdot (F_i \hat{\mathbf{u}}_i).$$

We can do this by computing the individual terms in the sum. We will compute $\nabla \cdot (F_1 \hat{\mathbf{u}}_1)$.

We first note that the gradients of the coordinate functions are found as $\nabla u_i = \frac{\hat{\mathbf{u}}_i}{h_i}$. (This results from a direct application of the gradient operator form just derived.) Then

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3}{h_2 h_3} = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}.$$

This gives

$$\begin{aligned} \nabla \cdot (F_1 \hat{\mathbf{u}}_1) &= \nabla \cdot (F_1 h_2 h_3 \nabla u_2 \times \nabla u_3) \\ &= \nabla (F_1 h_2 h_3) \cdot \nabla u_2 \times \nabla u_3 + F_1 h_2 h_2 \nabla \cdot (\nabla u_2 \times \nabla u_3). \end{aligned} \tag{8.85}$$

Here we used the vector identity

$$\nabla \cdot (f\mathbf{A}) = f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

The second term can be handled using the identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

where $\mathbf{A}$ and $\mathbf{B}$ are gradients. However, each term the curl of a gradient, which are identically zero! Or, you could just use the third identity in the previous list of second derivative identities,

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

Using the expression $\nabla u_2 \times \nabla u_3 = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}$ and the expression for the gradient operator in curvilinear coordinates, we have

$$\nabla \cdot (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_2 h_3) \cdot \frac{\hat{\mathbf{u}}_1}{h_2 h_3} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (F_1 h_2 h_3).$$

Similar computations can be done for the remaining components, leading to the sought expression for the divergence in curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_1 h_3 F_2) + \frac{\partial}{\partial u_2} (h_1 h_2 F_3) \right).$$

We now turn to the curl operator. In this case, we need to simplify

Derivation of the curl form.

$$\nabla \times \mathbf{F} = \sum_{i=1}^{3} \nabla \times (F_i \hat{\mathbf{u}}_i).$$

Using the identity

$$\nabla \times (f\mathbf{A}) = f \nabla \times \mathbf{A} - \mathbf{A} \times \nabla f,$$

we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla \times (F_1 h_1 \nabla u_1) \\ &= \nabla (F_1 h_1) \times \nabla u_1 + F_1 h_1 \nabla \times \nabla u_1. \end{aligned} \tag{8.86}$$

Again, the curl of the gradient vanishes, leaving

$$\nabla \times (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_1) \times \nabla u_1.$$

Since $\nabla u_1 = \frac{\hat{\mathbf{u}}_1}{h_1}$, we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla (F_1 h_1) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \left(\sum_{i=1}^{3} \frac{\hat{\mathbf{u}}_i}{h_i} \frac{\partial (F_1 h_1)}{\partial u_i} \right) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \frac{\hat{\mathbf{u}}_2}{h_3 h_1} \frac{\partial (F_1 h_1)}{\partial u_3} - \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \frac{\partial (F_1 h_1)}{\partial u_2}. \end{aligned} \tag{8.87}$$

The other terms can be handled in a similar manner. The overall result is that

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{\hat{\mathbf{u}}_1}{h_2 h_3} \left(\frac{\partial (h_3 F_3)}{\partial u_2} - \frac{\partial (h_2 F_2)}{\partial u_3} \right) + \frac{\hat{\mathbf{u}}_2}{h_1 h_3} \left(\frac{\partial (h_1 F_1)}{\partial u_3} - \frac{\partial (h_3 F_3)}{\partial u_1} \right) \\ &\quad + \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \left(\frac{\partial (h_2 F_2)}{\partial u_1} - \frac{\partial (h_1 F_1)}{\partial u_2} \right) \end{aligned} \tag{8.88}$$

This can be written more compactly as

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix} \tag{8.89}$$

Finally, we turn to the Laplacian. In the next chapter we will solve higher dimensional problems in various geometric settings such as the wave equation, the heat equation, and Laplace's equation. These all involve knowing how to write the Laplacian in different coordinate systems. Since $\nabla^2 \phi = \nabla \cdot \nabla \phi$, we need only combine the above results for the gradient and the divergence in curvilinear coordinates. This is straight forward and gives

$$\begin{aligned} \nabla^2 \phi &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial u_2} \right) \right. \\ &\quad \left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right). \end{aligned} \tag{8.90}$$

The results of rewriting the standard differential operators in cylindrical and spherical coordinates are shown in Problems 28 and 29. In particular, the Laplacians are given as

Cylindrical Coordinates:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{8.91}$$

Spherical Coordinates:

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{8.92}$$

These forms will be used in the next chapter for the solution of Laplace's equation, the heat equation, and the wave equation in these coordinate systems.

Problems

1. Compute $\mathbf{u} \times \mathbf{v}$ using the permutation symbol. Verify your answer by computing these products using traditional methods.

a. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$.

b. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$.

c. $\mathbf{u} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

2. Compute the following determinants using the permutation symbol. Verify your answer.

a. $\begin{vmatrix} 3 & 2 & 0 \\ 1 & 4 & -2 \\ -1 & 4 & 3 \end{vmatrix}$

b. $\begin{vmatrix} 1 & 2 & 2 \\ 4 & -6 & 3 \\ 2 & 3 & 1 \end{vmatrix}$

3. For the given expressions, write out all values for $i, j = 1, 2, 3$.

a. ϵ_{i2j}.

b. ϵ_{i13}.

c. $\epsilon_{ij1}\epsilon_{i32}$.

4. Show that

a. $\delta_{ii} = 3$.

b. $\delta_{ij}\epsilon_{ijk} = 0$

c. $\epsilon_{imn}\epsilon_{jmn} = 2\delta_{ij}$.

d. $\epsilon_{ijk}\epsilon_{ijk} = 6$.

5. Show that the vector $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ lies on the line of intersection of the two planes: (1) the plane containing $\mathbf{a}$ and $\mathbf{b}$ and (2) the plane containing $\mathbf{c}$ and $\mathbf{d}$.

6. Prove the following vector identities:

a. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$.

b. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{b} \times \mathbf{d})\mathbf{c} - (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\mathbf{d}$.

7. Use problem 6a to prove that $|\mathbf{a} \times \mathbf{b}| = ab \sin\theta$.

8. A particle moves on a straight line, $\mathbf{r} = t\mathbf{u}$, from the center of a disk. If the disk is rotating with angular velocity ω, then $\mathbf{u}$ rotates. Let $\mathbf{u} = (\cos\omega t)\mathbf{i} + (\sin\omega t)\mathbf{j}$.

a. Determine the velocity, $\mathbf{v}$.

b. Determine the acceleration, $\mathbf{a}$.

c. Describe the resulting acceleration terms identifying the centripetal acceleration and Coriolis acceleration.

9. Compute the gradient of the following:

a. $f(x,y) = x^2 - y^2$.

b. $f(x,y,z) = yz + xy + xz$.

c. $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.

d. $f(x,y,z) =$

10. Find the directional derivative of the given function at the indicated point in the given direction.

a. $f(x,y) = x^2 - y^2$, $(3,2)$, $\mathbf{u} = \mathbf{i} + \mathbf{j}$.

b. $f(x,y) = \frac{y}{x}$, $(2,1)$, $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

c. $f(x,y,z) = x^2 + y^2 + z^2$, $(1,0,2)$, $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$.

11. Zaphod Beeblebrox was in trouble after the infinite improbability drive caused the Heart of Gold, the spaceship Zaphod had stolen when he was President of the Galaxy, to appear between a small insignificant planet and its hot sun. The temperature of the ship's hull is given by $T(x,y,z) = e^{-k(x^2+y^2+z^2)}$ Nivleks. He is currently at $(1,1,1)$, in units of globs, and $k = 2$ globs^{-2}. (Check the *Hitchhikers Guide* for the current conversion of globs to kilometers and Nivleks to Kelvins.)

a. In what direction should he proceed so as to decrease the temperature the quickest?

b. If the Heart of Gold travels at e^6 globs per second, then how fast will the temperature decrease in the direction of fastest decline?

12. For the given vector field, find the divergence and curl of the field.

a. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.

b. $\mathbf{F} = \frac{y}{r}\mathbf{i} - \frac{x}{r}\mathbf{j}$, for $r = \sqrt{x^2 + y^2}$.

c. $\mathbf{F} = x^2y\mathbf{i} + z\mathbf{j} + xyz\mathbf{k}$.

13. Write the following using ϵ_{ijk} notation and simplify if possible.

a. $\mathbf{C} \times (\mathbf{A} \times (\mathbf{A} \times \mathbf{C}))$.

b. $\nabla \cdot (\nabla \times \mathbf{A})$.

c. $\nabla \times \nabla\phi$.

14. Prove the identities:

a. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

b. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

c. $\nabla r^n = nr^{n-2}\mathbf{r}, \quad n \geq 2$.

15. For $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$, simplify the following.

a. $\nabla \times (\mathbf{k} \times \mathbf{r})$.

b. $\nabla \cdot \left(\frac{\mathbf{r}}{r}\right)$.

c. $\nabla \times \left(\frac{\mathbf{r}}{r}\right)$.

d. $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right)$.

16. Newton's Law of Gravitation gives the gravitational force between two masses as

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r}.$$

a. Prove that $\mathbf{F}$ is irrotational.

b. Find a scalar potential for $\mathbf{F}$.

17. Consider an electric dipole moment $\mathbf{p}$ at the origin. It produces an electric potential of $\phi = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3}$ outside the dipole. Noting that $\mathbf{E} = -\nabla\phi$, find the electric field at $\mathbf{r}$.

18. In fluid dynamics the Euler equations govern inviscid fluid flow and provide quantitative statements on the conservation of mass, momentum and energy. The continuity equation is given by

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0,$$

where $\rho(x, y, z, t)$ is the mass density and $\mathbf{v}(x, y, z, t)$ is the fluid velocity. The momentum equations are given by

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla(\rho\mathbf{v}) = \mathbf{f} - \nabla p.$$

Here $p(x, y, z, t)$ is the pressure and $\mathbf{F}$ is the external force per volume.

a. Show that the continuity equation can be rewritten as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot (\mathbf{v}) + \mathbf{v} \cdot \nabla \rho = 0.$$

b. Prove the identity $\frac{1}{2}\nabla v^2 = \mathbf{v} \cdot \nabla \mathbf{v}$ for irrotational $\mathbf{v}$.

c. Assume that

- the external forces are conservative ($\mathbf{F} = -\nabla \phi$),
- the velocity field is irrotational ($\nabla \times \mathbf{v} = \mathbf{0}$).
- the fluid is incompressible ($\rho =$const), and
- the flow is steady, $\frac{\partial \mathbf{v}}{\partial t} = 0$.

Under these assumptions, prove Bernoulli's Principle:

$$\frac{1}{2}v^2 + \phi + \frac{p}{\rho} = \text{ const.}$$

19. Find the lengths of the following curves:

a. $y(x) = x$ for $x \in [0, 2]$.

b. $(x, y, z) = (t, \ln t, 2\sqrt{2}t)$ for $1 \leq t \leq 2$.

c. $y(x) = 2\cosh 3x$, $x \in [-2, 2]$. (Recall the hanging chain example from classical dynamics.)

20. Consider the integral $\int_C y^2\, dx - 2x^2\, dy$. Evaluate this integral for the following curves:

a. C is a straight line from (0,2) to (1,1).

b. C is the parabolic curve $y = x^2$ from (0,0) to (2,4).

c. C is the circular path from (1,0) to (0,1) in a clockwise direction.

21. Evaluate $\int_C (x^2 - 2xy + y^2)\, ds$ for the curve $x(t) = 2\cos t$, $y(t) = 2\sin t$, $0 \leq t \leq \pi$.

22. Prove that the magnetic flux density, $\mathbf{B}$, satisfies the wave equation.

23. Prove the identity

$$\int_C \phi \nabla \phi \cdot \mathbf{n}\, ds = \int_D (\phi \nabla^2 \phi + \nabla \cdot \nabla \phi)\, dA.$$

24. Compute the work done by the force $\mathbf{F} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$ in moving a particle counterclockwise around the boundary of the rectangle $R = [0, 3] \times [0, 5]$.

25. Compute the following integrals:

a. $\int_C (x^2 + y)\, dx + (3x + y^3)\, dy$ for C the ellipse $x^2 + 4y^2 = 4$.

b. $\int_S (x-y)\,dydz + (y^2+z^2)\,dzdx + (y-x^2)\,dxdy$ for S the positively oriented unit sphere.

c. $\int_C (y-z)\,dx + (3x+z)\,dy + (x+2y)\,dz$, where C is the curve of intersection between $z = 4 - x^2 - y^2$ and the plane $x + y + z = 0$.

d. $\int_C x^2y\,dx - xy^2\,dy$ for C a circle of radius 2 centered about the origin.

e. $\int_S x^2y\,dydz + 3y^2\,dzdx - 2xz^2\,dxdy$, where S is the surface of the cube $[-1,1]\times[-1,1]\times[-1,1]$.

26. Use Stoke's theorem to evaluate the integral

$$\int_C -y^3\,dx + x^3\,dy - z^3\,dz$$

for C the (positively oriented) curve of intersection between the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$.

27. Use Stoke's theorem to derive the integral form of Faraday's law,

$$\int_C \mathbf{E}\cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int\int_S \mathbf{H}\cdot d\mathbf{S}$$

from the differential form of Maxwell's equations.

28. For cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned} \tag{8.93}$$

find the scale factors and derive the following expressions:

Note that it is customary to write the basis as $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ instead of $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$.

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z. \tag{8.94}$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}. \tag{8.95}$$

$$\nabla\times\mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\mathbf{e}_z \tag{8.96}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{8.97}$$

29. For spherical coordinates,

$$\begin{aligned} x &= \rho\sin\theta\cos\phi, \\ y &= \rho\sin\theta\sin\phi, \\ z &= \rho\cos\theta. \end{aligned} \tag{8.98}$$

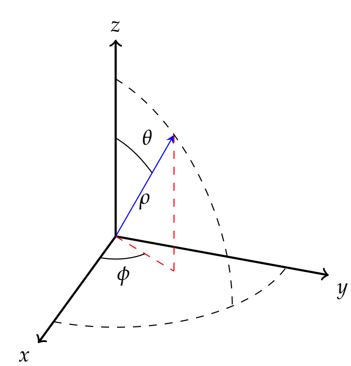


Figure 8.12: Definition of spherical coordinates for Problem 29.

find the scale factors and derive the following expressions:

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi. \tag{8.99}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2 F_\rho)}{\partial \rho} + \frac{1}{\rho \sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial \theta} + \frac{1}{\rho \sin\theta}\frac{\partial F_\phi}{\partial \phi}. \tag{8.100}$$

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{1}{\rho \sin\theta}\left(\frac{\partial(\sin\theta F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi}\right)\mathbf{e}_\rho + \frac{1}{\rho}\left(\frac{1}{\sin\theta}\frac{\partial F_\rho}{\partial \phi} - \frac{\partial(\rho F_\phi)}{\partial \rho}\right)\mathbf{e}_\theta \\ &\quad + \frac{1}{\rho}\left(\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}\right)\mathbf{e}_\phi \end{aligned} \tag{8.101}$$

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{8.102}$$

9
Oscillations in Higher Dimensions

"Equations of such complexity as are the equations of the gravitational field can be found only through the discovery of a logically simple mathematical condition that determines the equations completely or at least almost completely."

"What I have to say about this book can be found inside this book." Albert Einstein (1879-1955)

In this chapter we will explore several generic examples of the solution of initial-boundary value problems involving higher spatial dimensions. These are described by higher dimensional partial differential equations, such as the ones presented in Table 8.1 in the last chapter. We will solve these problems for different geometries, using rectangular, polar, cylindrical, or spherical coordinates.

We will solve these problems using the method of separation of variables, though there are other methods which we will not consider in this text. Using separation of variables will result in a system of ordinary differential equations for each problem. Adding the boundary conditions, we will need to solve a variety of eigenvalue problems. The product solutions that result will involve trigonometric or some of the special functions that we had encountered in Chapter 5.

As you go through the examples in this chapter, you will see some common features. For example, the two key equations that we have studied are the heat equation and the wave equation. For higher dimensional problems these take the form

$$u_t = k\nabla^2 u, \tag{9.1}$$

$$u_{tt} = c^2\nabla^2 u. \tag{9.2}$$

One can first separate out the time dependence. Let $u(\mathbf{r}, t) = \phi(\mathbf{r})T(t)$. Inserting u into the heat and wave equations, we have

$$T'\phi = kT\nabla^2\phi, \tag{9.3}$$

$$T''\phi = c^2T\nabla^2\phi. \tag{9.4}$$

Separating out the time and space dependence, we find

$$\frac{1}{k}\frac{T'}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda, \tag{9.5}$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda. \tag{9.6}$$

Note that in each case we have that a function of time equals a function of the spatial variables. Thus, they must be constant functions. We set these equal to the constant $-\lambda$. The sign of λ is chosen because we expect decaying solutions in time for the heat equation and oscillations in time for the wave equation and will pick $\lambda > 0$.

First, we look at the time dependence. The respective set of equations for $T(t)$ are given by

$$\begin{aligned} T' &= -\lambda k T, & (9.7)\\ T'' + c^2\lambda T &= 0. & (9.8) \end{aligned}$$

These are easily solved. We have

$$\begin{aligned} T(t) &= T(0)e^{-\lambda k t}, & (9.9)\\ T(t) &= a\cos\omega t + b\sin\omega t, \quad \omega = c\sqrt{\lambda}. & (9.10) \end{aligned}$$

In both cases the spatial equation becomes

The Helmholtz equation.

$$\nabla^2\phi + \lambda\phi = 0. \tag{9.11}$$

This is called the Helmholtz equation. For one dimensional problems, which we have already solved, the Helmholtz equation takes the form $\phi'' + \lambda\phi = 0$. We had to impose the boundary conditions and found that there were a discrete set of eigenvalues, λ_n, and associated eigenfunctions, ϕ_n.

In higher dimensional problems we need to further separate out the spatial dependence. We will again use the boundary conditions and find the eigenvalues and eigenfunctions for the Helmholtz equation, though the eigenfunctions will be labeled with more than one index. The resulting boundary value problems are often second order ordinary differential equations, which can be set up as Sturm-Liouville problems. We know from Chapter 5 that such problems possess an orthogonal set of eigenfunctions. These can then be used to construct a general solution out of product solutions consisting of elementary or special functions, such as Legendre functions or Bessel functions.

We will begin our study of higher dimensional problems by considering the vibrations of two dimensional membranes. First we will solve the problem of a vibrating rectangular membrane and then we turn out attention to a vibrating circular membranes. The rest of the chapter will be devoted to the study of three dimensional problems possessing cylindrical or spherical symmetry.

9.1 Vibrations of Rectangular Membranes

Our first example will be the study of the vibrations of a rectangular membrane. You can think of this as a drum with a rectangular cross section as shown in Figure 9.1. We stretch the membrane over the drumhead and fasten the material to the boundary of the rectangle. The height of the vibrating membrane is described by its height from equilibrium, $u(x,y,t)$. This problem is a much simpler example of higher dimensional vibrations than that possessed by the oscillating electric and magnetic fields in the last chapter.

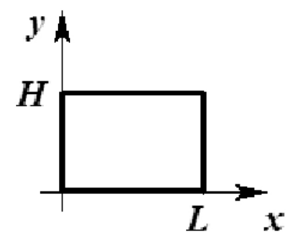


Figure 9.1: The rectangular membrane of length L and width H. There are fixed boundary conditions along the edges.

The problem is given by the two dimensional wave equation in Cartesian coordinates,

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{9.12}$$

a set of boundary conditions,

$$\begin{aligned} u(0,y,t) &= 0, \quad u(L,y,t) = 0, \quad t > 0, \quad 0 < y < H, \\ u(x,0,t) &= 0, \quad u(x,H,t) = 0, \quad t > 0, \quad 0 < x < L, \end{aligned} \tag{9.13}$$

and a pair of initial conditions (since the equation is second order in time),

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y). \tag{9.14}$$

The first step is to separate the variables: $u(x,y,t) = X(x)Y(y)T(t)$. Inserting the guess, $u(x,y,t)$ into the wave equation, we have

$$X(x)Y(y)T''(t) = c^2\left(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)\right).$$

Dividing by both $u(x,y,t)$ and c^2, we obtain

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{Function of } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{Function of } x \text{ and } y} = -\lambda. \tag{9.15}$$

We see that we have a function of t equals a function of x and y. Thus, both expressions are constant. We expect oscillations in time, so we chose the constant λ to be positive, $\lambda > 0$. (Note: As usual, the primes mean differentiation with respect to the specific dependent variable. So, there should be no ambiguity.)

These lead to two equations:

$$T'' + c^2\lambda T = 0, \tag{9.16}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda. \tag{9.17}$$

The first equation is easily solved. We have

$$T(t) = a\cos\omega t + b\sin\omega t, \tag{9.18}$$

where

$$\omega = c\sqrt{\lambda}. \tag{9.19}$$

This is the angular frequency in terms of the separation constant, or eigenvalue. It leads to the frequency of oscillations for the various harmonics of the vibrating membrane as

$$\nu = \frac{\omega}{2\pi} = \frac{c}{2\pi}\sqrt{\lambda}. \tag{9.20}$$

Once we know λ, we can compute these frequencies.

Now we solve the spatial equation. Again, we need to do a separation of variables. Rearranging the spatial equation, we have

$$\underbrace{\frac{X''}{X}}_{\text{Function of } x} = \underbrace{-\frac{Y''}{Y} - \lambda}_{\text{Function of } y} = -\mu. \tag{9.21}$$

Here we have a function of x equals a function of y. So, the two expressions are constant, which we indicate with a second separation constant, $-\mu < 0$. We pick the sign in this way because we expect oscillatory solutions for $X(x)$. This leads to two equations:

$$\begin{aligned} X'' + \mu X &= 0, \\ Y'' + (\lambda - \mu)Y &= 0. \end{aligned} \tag{9.22}$$

We now need to use the boundary conditions. We have $u(0, y, t) = 0$ for all $t > 0$ and $0 < y < H$. This implies that $X(0)Y(y)T(t) = 0$ for all t and y in the domain. This is only true if $X(0) = 0$. Similarly, from the other boundary conditions we find that $X(L) = 0$, $Y(0) = 0$, and $Y(H) = 0$. We note that homogeneous boundary conditions are important in carrying out this process. Nonhomogeneous boundary conditions could be imposed, but the techniques are a bit more complicated and we will not discuss these techniques here.

The boundary values problems we need to solve are:

$$\begin{aligned} X'' + \mu X = 0, &\quad X(0) = 0, X(L) = 0. \\ Y'' + (\lambda - \mu)Y = 0, &\quad Y(0) = 0, Y(H) = 0. \end{aligned} \tag{9.23}$$

We have seen the first of these problems before, except with a λ instead of a μ. The solutions of the eigenvalue problem are

$$X(x) = \sin\frac{n\pi x}{L}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The second equation is solved in the same way. The differences are that the "eigenvalue" is $\lambda - \mu$, the independent variable is y, and the interval is $[0, H]$. Thus, we can quickly write down the solutions of the eigenvalue problem as

$$Y(y) = \sin\frac{m\pi x}{H}, \quad \lambda - \mu = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, 3, \ldots.$$

We have successfully carried out the separation of variables for the wave equation for the vibrating rectangular membrane. The product solutions can be written as

$$u_{nm} = (a\cos\omega_{nm}t + b\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{9.24}$$

Recall that ω is given in terms of λ. We have that

The harmonics for the vibrating rectangular membrane are given by

$$\nu_{nm} = \frac{c}{2}\sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{H}\right)^2},$$

for $n, m = 1, 2, \ldots$.

$$\lambda_{mn} - \mu_n = \left(\frac{m\pi}{H}\right)^2$$

and

$$\mu_n = \left(\frac{n\pi}{L}\right)^2.$$

Therefore,

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2. \tag{9.25}$$

So,

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{9.26}$$

The most general solution can now be written as a linear combination of the product solutions and we can solve for the expansion coefficients that will lead to a solution satisfying he initial conditions. However, we will first concentrate on the two dimensional harmonics of this membrane.

For the vibrating string the nth harmonic corresponded to the function $\sin\frac{n\pi x}{L}$. The various harmonics corresponded to the pure tones supported by the string. These then lead to the corresponding frequencies that one would hear. The actual shapes of the harmonics could be sketched by locating the nodes, or places on the string that did not move.

In the same way, we can explore the shapes of the harmonics of the vibrating membrane. These are given by the spatial functions

$$\phi_{nm}(x, y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{9.27}$$

Instead of nodes, we will look for the *nodal curves*, or *nodal lines*. These are the points (x, y) at which $\phi_{nm}(x, y) = 0$. Of course, these depend on the indices, n and m.

A discussion of the nodal lines.

For example, when $n = 1$ and $m = 1$, we have

$$\sin\frac{\pi x}{L}\sin\frac{\pi y}{H} = 0.$$

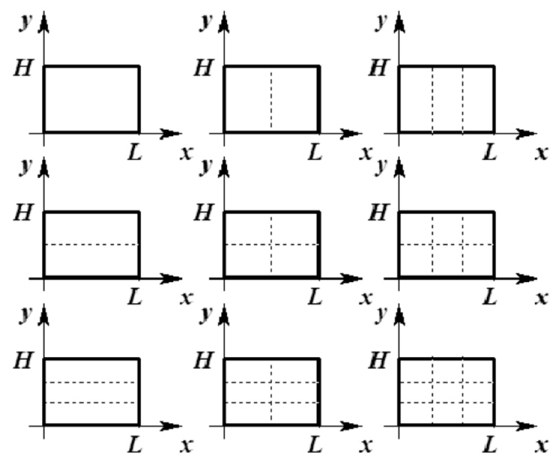


Figure 9.2: The first few modes of the vibrating rectangular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these the nodal lines to the 3D view in Figure 9.3

These are zero when either

$$\sin \frac{\pi x}{L} = 0, \quad \text{or} \quad \sin \frac{\pi y}{H} = 0.$$

Of course, this can only happen for $x = 0, L$ and $y = 0, H$. Thus, there are no interior nodal lines.

When $n = 2$ and $m = 1$, we have $y = 0, H$ and

$$\sin \frac{2\pi x}{L} = 0,$$

or, $x = 0, \frac{L}{2}, L$. Thus, there is one interior nodal line at $x = \frac{L}{2}$. These points stay fixed during the oscillation and all other points oscillate on either side of this line. A similar solution shape results for the (1,2)-mode; i.e., $n = 1$ and $m = 2$.

In Figure 9.2 we show the nodal lines for several modes for $n, m = 1, 2, 3$ The blocked regions appear to vibrate independently. A better view is the three dimensional view depicted in Figure 9.3 . The frequencies of vibration are easily computed using the formula for ω_{nm}.

For completeness, we now see how one satisfies the initial conditions. The general solution is given by a linear superposition of the product solutions. There are two indices to sum over. Thus, the general solution is

The general solution for the vibrating rectangular membrane.

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \tag{9.28}$$

where

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{9.29}$$

The first initial condition is $u(x, y, 0) = f(x, y)$. Setting $t = 0$ in the

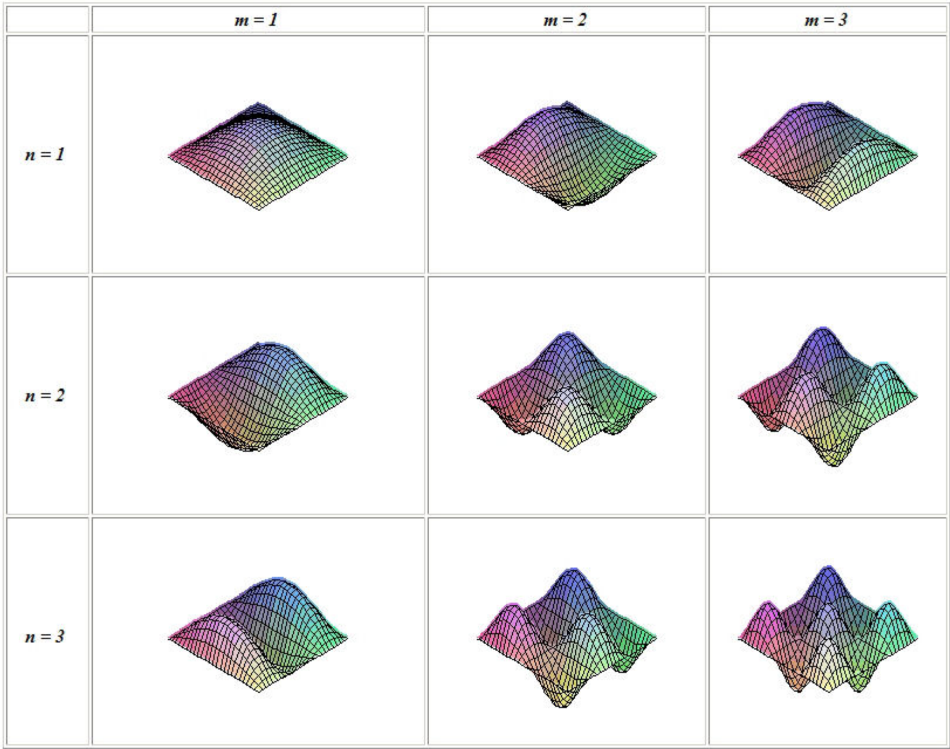


Figure 9.3: A three dimensional view of the vibrating rectangular membrane for the lowest modes. Compare these images with the nodal lines in Figure 9.2

general solution, we obtain

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}. \tag{9.30}$$

This is a double Fourier sine series. The goal is to find the unknown coefficients a_{nm}. This can be done knowing what we already know about Fourier sine series. We can write the initial condition as the single sum

$$f(x,y) = \sum_{n=1}^{\infty} A_n(y) \sin\frac{n\pi x}{L}, \tag{9.31}$$

where

$$A_n(y) = \sum_{m=1}^{\infty} a_{nm} \sin\frac{m\pi y}{H}. \tag{9.32}$$

These are two Fourier sine series. Recalling that the coefficients of Fourier sine series can be computed as integrals, we have

$$\begin{aligned} A_n(y) &= \frac{2}{L}\int_0^L f(x,y) \sin\frac{n\pi x}{L}\,dx, \\ a_{nm} &= \frac{2}{H}\int_0^H A_n(y) \sin\frac{m\pi y}{H}\,dy. \end{aligned} \tag{9.33}$$

Inserting the integral for $A_n(y)$ into that for a_{nm}, we have an integral representation for the Fourier coefficients in the double Fourier sine series,

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \, dxdy. \tag{9.34}$$

The Fourier coefficients for the double Fourier sine series.

We can carry out the same process for satisfying the second initial condition, $u_t(x,y,0) = g(x,y)$ for the initial velocity of each point. Inserting this into the general solution, we have

$$g(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \omega_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{9.35}$$

Again, we have a double Fourier sine series. But, now we can write down Fourier coefficients quickly using the above expression for a_{nm}:

$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \, dxdy. \tag{9.36}$$

This completes the full solution of the vibrating rectangular membrane problem. Namely, we have obtained the solution

The full solution of the vibrating rectangular membrane.

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \tag{9.37}$$

where

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \, dxdy, \tag{9.38}$$

$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \, dxdy, \tag{9.39}$$

and the angular frequencies are given by

$$\omega_{nm} = c \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{9.40}$$

9.2 *Vibrations of a Kettle Drum*

IN THIS SECTION we consider the vibrations of a circular membrane of radius a as shown in Figure 9.4. Again we are looking for the harmonics of the vibrating membrane, but with the membrane fixed around the circular boundary given by $x^2 + y^2 = a^2$. However, expressing the boundary condition in Cartesian coordinates is awkward. Namely, we can only write $u(x,y,t) = 0$ for $x^2 + y^2 = a^2$. It is more natural to use polar coordinates as indicated in Figure 9.4. Let the height of the membrane be given by $u = u(r,\theta,t)$ at time t and position (r,θ). Now the boundary condition is given as $u(a,\theta,t) = 0$ for all $t > 0$ and $\theta \in [0, 2\pi]$.

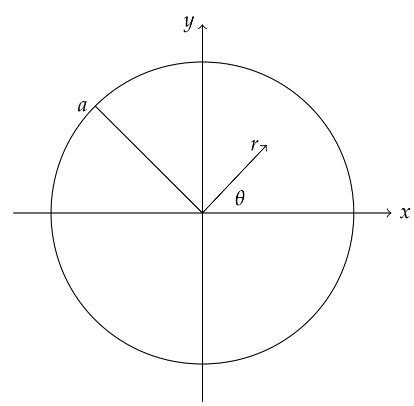


Figure 9.4: The circular membrane of radius a. A general point on the membrane is given by the distance from the center, r, and the angle, . There are fixed boundary conditions along the edge at $r = a$.

Before solving the initial-boundary value problem, we have to cast the full problem in polar coordinates. This means that we need to rewrite the Laplacian in r and θ. To do so would require that we know how to transform derivatives in x and y into derivatives with respect to r and θ. Using the results from Section 8.3 on curvilinear coordinates, we know that the Laplacian can be written in polar coordinates. In fact, we could use the results from Problem 28 for cylindrical coordinates for functions which are z-independent, $f = f(r, \theta)$. Then we would have

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

We can obtain this result using a more direct approach, namely applying the Chain Rule in higher dimensions. First recall the transformation between polar and Cartesian coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Now, consider a function $f = f(x(r,\theta), y(r,\theta)) = g(r,\theta)$. (Technically, once we transform a given function of Cartesian coordinates we obtain a new function g of the polar coordinates. Many texts do not rigorously distinguish between the two functions.) Thinking of $x = x(r,\theta)$ and $y = y(r,\theta)$, we have from the chain rule for functions of two variables:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{\partial g}{\partial r}\frac{x}{r} - \frac{\partial g}{\partial \theta}\frac{y}{r^2} \\ &= \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}. \end{aligned} \tag{9.41}$$

Here we have used

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r};$$

and

$$\frac{\partial \theta}{\partial x} = \frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right) = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{r^2}.$$

Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \frac{\partial g}{\partial r}\frac{y}{r} + \frac{\partial g}{\partial \theta}\frac{x}{r^2} \\ &= \sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}. \end{aligned} \tag{9.42}$$

The 2D Laplacian can now be computed as

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right)+\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)\\
&= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial g}{\partial r}+\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial r}+\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&= \cos\theta\left(\cos\theta\frac{\partial^2 g}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial g}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}\right)\\
&\quad-\frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 g}{\partial\theta\partial r}-\frac{\sin\theta}{r}\frac{\partial^2 g}{\partial\theta^2}-\sin\theta\frac{\partial g}{\partial r}-\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\sin\theta\left(\sin\theta\frac{\partial^2 g}{\partial r^2}+\frac{\cos\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}-\frac{\cos\theta}{r^2}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 g}{\partial\theta\partial r}+\frac{\cos\theta}{r}\frac{\partial^2 g}{\partial\theta^2}+\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&= \frac{\partial^2 g}{\partial r^2}+\frac{1}{r}\frac{\partial g}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}. \qquad (9.43)
\end{aligned}$$

The last form often occurs in texts because it is in the form of a Sturm-Liouville operator. Also, it agrees with the result from using the Laplacian written in cylindrical coordinates as given in Problem 28.

Now that we have written the Laplacian in polar coordinates we can pose the problem of a vibrating circular membrane. It is given by a partial differential equation,[1]

$$u_{tt}=c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right], \qquad (9.44)$$

$$t>0,\quad 0<r<a,\quad -\pi<\theta<\pi,$$

the boundary condition,

$$u(a,\theta,t)=0,\quad t>0,\quad -\pi<\theta<\pi, \qquad (9.45)$$

and the initial conditions,

$$u(r,\theta,0)=f(r,\theta),\quad u_t(r,\theta,0)=g(r,\theta). \qquad (9.46)$$

Now we are ready to solve this problem using separation of variables. As before, we can separate out the time dependence. Let $u(r,\theta,t)=T(t)\phi(r,\theta)$. As usual, $T(t)$ can be written in terms of sines and cosines. This leads to the Helmholtz equation,

$$\nabla^2\phi+\lambda\phi=0.$$

[1] Here we state the problem of a vibrating circular membrane. We have chosen $-\pi<\theta<\pi$, but could have just as easily used $0<\theta<2\pi$. The symmetric interval about $\theta=0$ will make the use of boundary conditions simpler.

We now separate the Helmholtz equation by letting $\phi(r,\theta) = R(r)\Theta(\theta)$. This gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 R\Theta}{\partial\theta^2} + \lambda R\Theta = 0. \tag{9.47}$$

Dividing by $u = R\Theta$, as usual, leads to

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \lambda = 0. \tag{9.48}$$

The last term is a constant. The first term is a function of r. However, the middle term involves both r and θ. This can be remedied by multiplying the equation by r^2. Rearranging the resulting equation, we can separate out the θ-dependence from the radial dependence. Letting μ be the separation constant, we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu. \tag{9.49}$$

This gives us two ordinary differential equations:

$$\begin{aligned} \frac{d^2\Theta}{d\theta^2} + \mu\Theta &= 0, \\ r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - \mu)R &= 0. \end{aligned} \tag{9.50}$$

Let's consider the first of these equations. It should look familiar by now. For $\mu > 0$, the general solution is

$$\Theta(\theta) = a\cos\sqrt{\mu}\theta + b\sin\sqrt{\mu}\theta.$$

The next step typically is to apply the boundary conditions in θ. However, when we look at the given boundary conditions in the problem, we do not see anything involving θ. This is a case for which the boundary conditions that are needed are implied and not stated outright.

We can determine the hidden boundary conditions by making some observations. Let's consider the solution corresponding to the endpoints $\theta = \pm\pi$, noting that at these values for any $r < a$ we are at the same physical point. So, we would expect the solution to have the same value at $\theta = -\pi$ as it has at $\theta = \pi$. Namely, the solution is continuous at these physical points. Similarly, we expect the slope of the solution to be the same at these points. This tells us that

The boundary conditions in θ are periodic boundary conditions.

$$\Theta(\pi) = \Theta(-\pi) \quad \Theta'(\pi) = \Theta'(-\pi).$$

Such boundary conditions are called *periodic boundary conditions*.

Let's apply these conditions to the general solution for $\Theta(\theta)$. First, we set $\Theta(\pi) = \Theta(-\pi)$ and use the symmetries of the sine and cosine functions:

$$a\cos\sqrt{\mu}\pi + b\sin\sqrt{\mu}\pi = a\cos\sqrt{\mu}\pi - b\sin\sqrt{\mu}\pi.$$

This implies that

$$\sin \sqrt{\mu}\pi = 0.$$

This can only be true for $\sqrt{\mu} = m$, $m = 0, 1, 2, 3, \ldots$. Therefore, the eigenfunctions are given by

$$\Theta_m(\theta) = a \cos m\theta + b \sin m\theta, \quad m = 0, 1, 2, 3, \ldots.$$

For the other half of the periodic boundary conditions, $\Theta'(\pi) = \Theta'(-\pi)$, we have that

$$-am \sin m\pi + bm \cos m\pi = am \sin m\pi + bm \cos m\pi.$$

But, this gives no new information.

To summarize so far, we have found the general solutions to the temporal and angular equations. The product solutions will have various products of $\{\cos \omega t, \sin \omega t\}$ and $\{\cos m\theta, \sin m\theta\}_{m=0}^{\infty}$. We also know that $\mu = m^2$ and $\omega = c\sqrt{\lambda}$.

That leaves us with the radial equation. Inserting $\mu = m^2$, we have

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - m^2)R = 0. \tag{9.51}$$

A little rewriting,

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - m^2)R(r) = 0. \tag{9.52}$$

The reader should recognize this differential equation from Equation **??**. It is a Bessel equation with bounded solutions $R(r) = J_m(\sqrt{\lambda}r)$.

Recall there are two linearly independent solutions of this second order equation: $J_m(\sqrt{\lambda}r)$, the Bessel function of the first kind of order m, and $N_m(\sqrt{\lambda}r)$, the Bessel function of the second kind of order m. Plots of these functions are shown in Figures 5.8 and 5.9. Sometimes the N_m's are called Neumann functions. So, we have the general solution of the radial equation is

$$R(r) = c_1 J_m(\sqrt{\lambda}r) + c_2 N_m(\sqrt{\lambda}r).$$

Now we are ready to apply the boundary conditions to the radial factor in the product solutions. Looking at the original problem we find only one condition: $u(a, \theta, t) = 0$ for $t > 0$ and $-\pi < \ < \pi$. This implies that $R(0) = 0$. But where is the second condition?

This is another unstated boundary condition. Look again at the plots of the Bessel functions. Notice that the Neumann functions are not well behaved at the origin. Do you expect that the solution will become infinite at the center of the drum? No, the solutions should be finite at the center. So, this observation leads to the second boundary condition. Namely, $|R(0)| < \infty$. This implies that $c_2 = 0$.

Now we are left with

$$R(r) = J_m(\sqrt{\lambda} r).$$

We have set $c_1 = 1$ for simplicity. We can apply the vanishing condition at $r = a$. This gives

$$J_m(\sqrt{\lambda} a) = 0.$$

Looking again at the plots of $J_m(x)$, we see that there are an infinite number of zeros, but they are not as easy as π! In Table 9.1 we list the nth zeros of J_m, which were first seen in Table 5.3.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Table 9.1: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

Let's denote the nth zero of $J_m(x)$ by j_{mn}. Then the boundary condition tells us that

$$\sqrt{\lambda} a = j_{mn}.$$

This gives us the eigenvalue as

$$\lambda_{mn} = \left(\frac{j_{mn}}{a}\right)^2.$$

Thus, the radial function satisfying the boundary conditions is

$$R(r) = J_m\left(\frac{j_{mn}}{a} r\right).$$

We are finally ready to write out the product solutions for the vibrating circular membrane. They are given by

Product solutions for the vibrating circular membrane.

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m\left(\frac{j_{mn}}{a} r\right). \tag{9.53}$$

Here we have indicated choices with the braces, leading to four different types of product solutions. Also, $m = 0, 1, 2, \ldots$, and

$$\omega_{mn} = \frac{j_{mn}}{a} c.$$

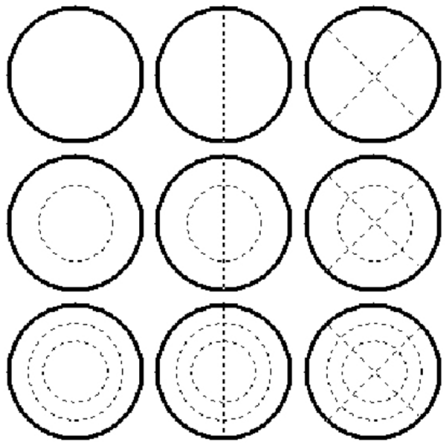


Figure 9.5: The first few modes of the vibrating circular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these nodal lines with the three dimensional images in Figure 9.6.

As with the rectangular membrane, we are interested in the shapes of the harmonics. So, we consider the spatial solution ($t = 0$)

$$\phi(r,\theta) = (\cos m\theta) J_m\left(\frac{j_{mn}}{a} r\right).$$

Including the solutions involving $\sin m\theta$ will only rotate these modes. The nodal curves are given by $\phi(r,\theta) = 0$. This can be satisfied if $\cos m\theta = 0$, or $J_m(\frac{j_{mn}}{a} r) = 0$. The various nodal curves which result are shown in Figure 9.5.

For the angular part, we easily see that the nodal curves are radial lines, $\theta =$const. For $m = 0$, there are no solutions, since $\cos m\theta = 1$ and $\sin m\theta = 1$ for $m = 0$. in Figure 9.5 this is seen by the absence of radial lines in the first column.

For $m = 1$, we have $\cos\theta = 0$. This implies that $\theta = \pm\frac{\pi}{2}$. These values give the vertical line as shown in the second column in Figure 9.5. For $m = 2$, $\cos 2\theta = 0$ implies that $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. This results in the two lines shown in the last column of Figure 9.5.

We can also consider the nodal curves defined by the Bessel functions. We seek values of r for which $\frac{j_{mn}}{a} r$ is a zero of the Bessel function and lies in the interval $[0, a]$. Thus, we have

$$\frac{j_{mn}}{a} r = j_{mj},$$

or

$$r = \frac{j_{mj}}{j_{mn}} a.$$

These will give circles of this radius with $j_{mj} \leq j_{mn}$, or $j \leq n$. The zeros can be found in Table 9.1. For $m = 0$ and $n = 1$, there is only one zero and $r = a$. In fact, for all $n = 1$ modes, there is only one zero and $r = a$. Thus, the first row in Figure 9.5 shows no interior nodal circles.

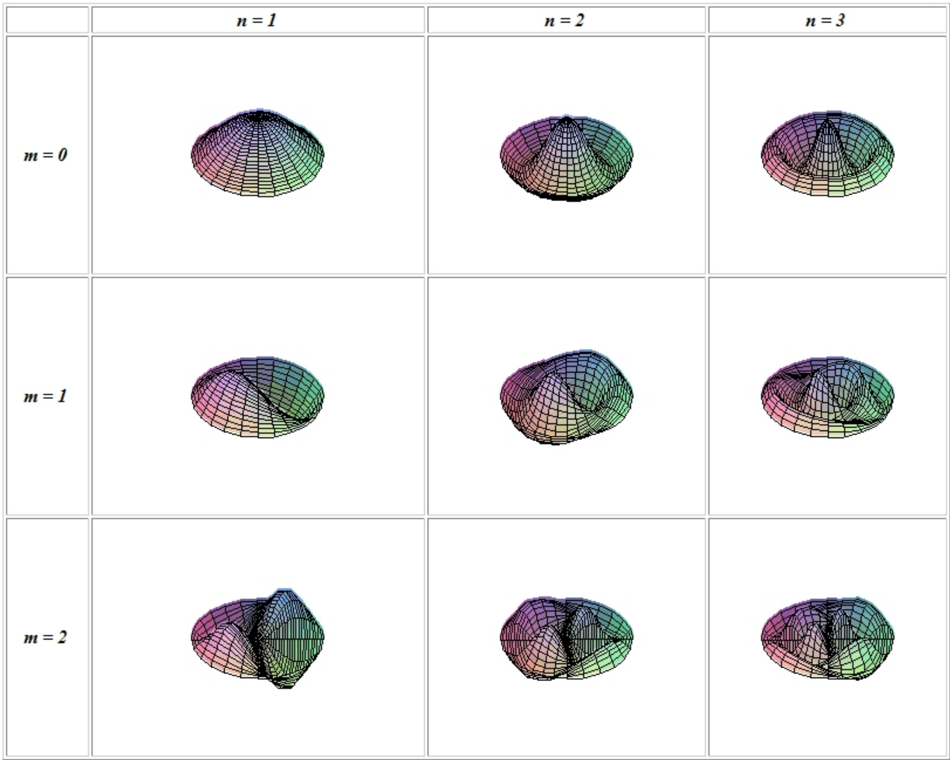


Figure 9.6: A three dimensional view of the vibrating circular membrane for the lowest modes. Compare these images with the nodal line plots in Figure 9.5.

For $n = 2$ modes, we have two circles, $r = a$ and $r = \frac{j_{m1}}{j_{m2}}$ as shown in the second row of Figure 9.5. For $m = 0$,

$$r = \frac{2.405}{5.520}a \approx 0.436a$$

for the inner circle. For $m = 1$,

$$r = \frac{3.832}{7.016}a \approx 0.546a,$$

and for $m = 2$,

$$r = \frac{5.135}{8.147}a \approx 0.630a.$$

For $n = 3$ we obtain circles of radii $r = a$,

$$r = \frac{j_{m1}}{j_{m3}}, \text{ and } r = \frac{j_{m2}}{j_{m3}}.$$

For $m = 0$,

$$r = a, \frac{5.520}{8.654}a \approx 0.638a, \frac{2.405}{8.654}a \approx 0.278a.$$

Similarly, for $m = 1$,

$$r = a, \frac{3.832}{10.173}0.377a \approx a, \frac{7.016}{10.173}a \approx 0.0.690a$$

and for $m = 2$,

$$r = a, \frac{5.135}{11.620}a \approx 0.442a, \frac{8.417}{11.620}a \approx 0.724a.$$

For a three dimensional view, one can look at Figure 9.6. Imagine that the various regions are oscillating independently and that the points on the nodal curves are not moving.

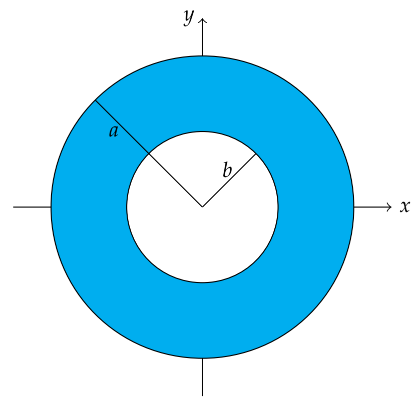


Figure 9.7: An annular membrane with radii a and $b > a$. There are fixed boundary conditions along the edges at $r = a$ and $r = b$.

Example 9.1. *Vibrating Annulus*

More complicated vibrations can be dreamt up for this geometry. We could consider an annulus in which the drum is formed from two concentric circular cylinders and the membrane is stretch between the two with an annular cross section as shown in Figure 9.7. The separation would follow as before except now the boundary conditions are that the membrane is fixed around the two circular boundaries. In this case we cannot toss out the Neumann functions because the origin is not part of the drum head.

With this in mind, we have that the product solutions take the form

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} R_m(r), \qquad (9.54)$$

where

$$R_m(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 N_m(\sqrt{\lambda} r)$$

and $\omega = c\sqrt{\lambda}$.

For this problem the radial boundary conditions are that the membrane is fixed at $r = a$ and $r = b$. Taking $b < a$, we then have to satisfy the conditions

$$\begin{aligned} R(a) &= c_1 J_m(\sqrt{\lambda} a) + c_2 N_m(\sqrt{\lambda} a) = 0, \\ R(b) &= c_1 J_m(\sqrt{\lambda} b) + c_2 N_m(\sqrt{\lambda} b) = 0. \end{aligned} \qquad (9.55)$$

This leads to two homogeneous equations for c_1 and c_2. The coefficient determinant of this system has to vanish if there are to be nontrivial solutions. This gives the eigenvalue equation λ :

$$J_m(\sqrt{\lambda} a) N_m(\sqrt{\lambda} b) - J_m(\sqrt{\lambda} b) N_m(\sqrt{\lambda} a) = 0.$$

This eigenvalue equation needs to be solved numerically. Choosing $a = 2$ and $b = 4$, we have for the first few modes

$$\begin{aligned} \sqrt{\lambda_{mn}} &\approx 1.562, 3.137, 4.709, \quad m = 0 \\ &\approx 1.598, 3.156, 4.722, \quad m = 1 \\ &\approx 1.703, 3.214, 4.761, \quad m = 2. \end{aligned} \qquad (9.56)$$

Note, since $\sqrt{\lambda_{mn}} = \frac{\omega_{mn}}{c}$, these numbers essentially give us the frequencies of oscillation.

For these particular roots, we then solve for c_1 and c_2 by setting $c_2 = -1$ and determining

$$c_1 = \frac{N_m(\sqrt{\lambda_{mn}} b)}{J_m(\sqrt{\lambda_{mn}} b)}$$

(This selection is not unique. We could replace the b's in c_1 with a's and that would work as well.) This leads to the basic modes of vibration,

$$R_{mn}(r)\Theta_m(\theta) = \cos m\theta \left(\frac{N_m(\sqrt{\lambda_{mn}}b)}{J_m(\sqrt{\lambda_{mn}}b)} J_m(\sqrt{\lambda_{mn}}r) - N_m(\sqrt{\lambda_{mn}}r) \right).$$

In Figure 9.8 we show various modes for the particular choice of membrane dimensions, $a = 2$ and $b = 4$.

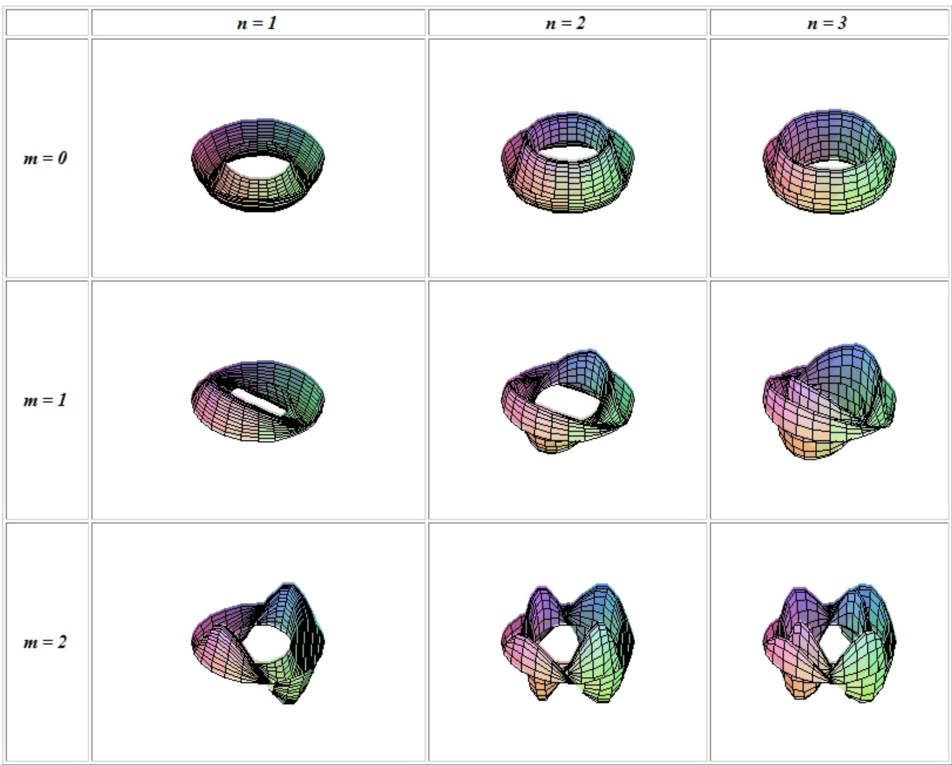


Figure 9.8: A three dimensional view of the vibrating annular membrane for the lowest modes.

9.3 Laplace's Equation in 2D

Another of the generic partial differential equations is Laplace's equation, $\nabla^2 u = 0$. This equation first appeared in the chapter on complex variables when we discussed harmonic functions. Another example is the electric potential for electrostatics. As we described in the last chapter, for static electromagnetic fields, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. Also, $\mathbf{E} = \nabla\phi$. In regions devoid of charge, we have $\nabla^2\phi = 0$.

Another example comes from studying temperature distributions. Consider a thin rectangular plate with the boundaries set at fixed temperatures. One can solve the heat equation. The solution is time dependent. However, if one wait a long time, the plate reaches thermal equilibrium. If the boundary temperature is zero, then the plate temperatures decays to zero. However, keeping the boundaries at a nonzero temperature, which means energies is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, which is $\nabla^2 u = 0$.

Finally, we could look at fluid flow. For an incompressible flow, $\nabla \cdot \mathbf{v} = 0$. If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$. We can introduce a velocity potential, $\mathbf{v} = \nabla\phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, once again we obtain Laplace's equation.

In this section we will look at a couple examples of Laplace's equation in two dimensions. The solutions could be examples of any of the above physical situations and can be determined appropriately.

Example 9.2. *Equilibrium Temperature Distribution for a Rectangular Plate*

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, 0 < y < H$$

with the boundary conditions

$$u(0,y) = 0, u(L,y) = 0, u(x,0) = f(x), u(x,H) = 0.$$

The boundary conditions are shown in Figure 9.9

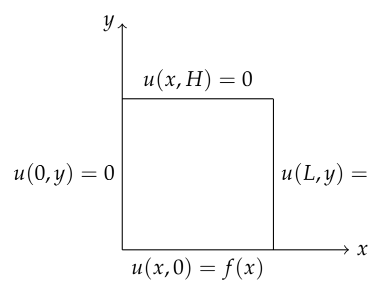


Figure 9.9: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

As usual, we solve this equation using the method of separation of variables. Let $u(x,y) = X(x)Y(y)$. Then Laplace's equation becomes

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda. \tag{9.57}$$

This leads to two differential equations,

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned} \tag{9.58}$$

We next turn to the boundary conditions. Since $u(0,y) = 0, u(L,y) = 0$, we have $X(0) = 0, X(L) = 0$. So, we have an eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0.$$

We can easily write down the solution to this problem,

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, \cdots.$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}.$$

The boundary condition $u(x,H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda}H} + c_2 e^{-\sqrt{\lambda}H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda}H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda}y} - c_1 e^{2\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \\ &= c_1 e^{2\sqrt{\lambda}H} \left(e^{-\sqrt{\lambda}H} e^{\sqrt{\lambda}y} - e^{\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \right) \\ &= -c_1 e^{2\sqrt{\lambda}H} \left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)} \right) \\ &= -2c_1 e^{2\sqrt{\lambda}H} \sinh \sqrt{\lambda}(H-y). \end{aligned} \tag{9.59}$$

Note: Having carried out this computation, we can now see that it would be better to guess at this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh \sqrt{\lambda}(H-y)$. For $Y(0) = 0$, one would guess a solution $Y(y) = \sinh \sqrt{\lambda}y$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh \sqrt{\lambda}(H-y)$.

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that

$$Y_n(y) = \sinh \frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x,y) = \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \quad n = 1, 2, \cdots.$$

These solutions satisfy the three homogeneous boundary conditions in the problem.

The remaining boundary condition, $u(x,0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first need to write the general solution, which is a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}. \tag{9.60}$$

Now we apply the boundary condition to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi H}{L} \sin \frac{n\pi x}{L}. \tag{9.61}$$

Defining $b_n = a_n \sinh \frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{9.62}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. The Fourier coefficients are easily found as

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{9.63}$$

Finally, we have the solution to this problem,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \tag{9.64}$$

where

$$a_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{9.65}$$

Example 9.3. *Equilibrium Temperature Distribution for a Rectangular Plate for General Boundary Conditions*

Now we consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, 0 < y < H$$

with the non-zero boundary conditions on more than one side,

$$u(0,y) = g_1(y), u(L,y) = g_2(y), u(x,0) = f_1(x), u(x,H) = f_2(x).$$

The boundary conditions are shown in Figure 9.10

The problem with this example is that none of the boundary conditions are homogeneous, so we cannot specify the boundary conditions for the eigenvalue problems. However, we can express this problem as in terms of four problems with nonhomogeneous boundary conditions on only one side of the rectangle. In Figure 9.11 we show how the problem can be broken up into four separate problems. Since the boundary conditions and Laplace's equation are linear, the solution to the general problem is simply the sum of the solutions to these four problems.

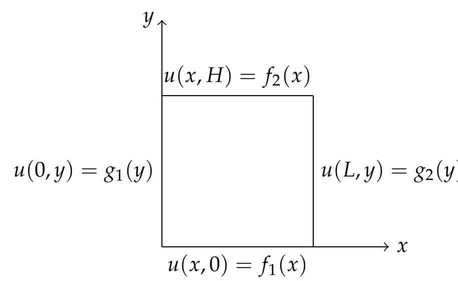


Figure 9.10: In this figure we show the domain and general boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

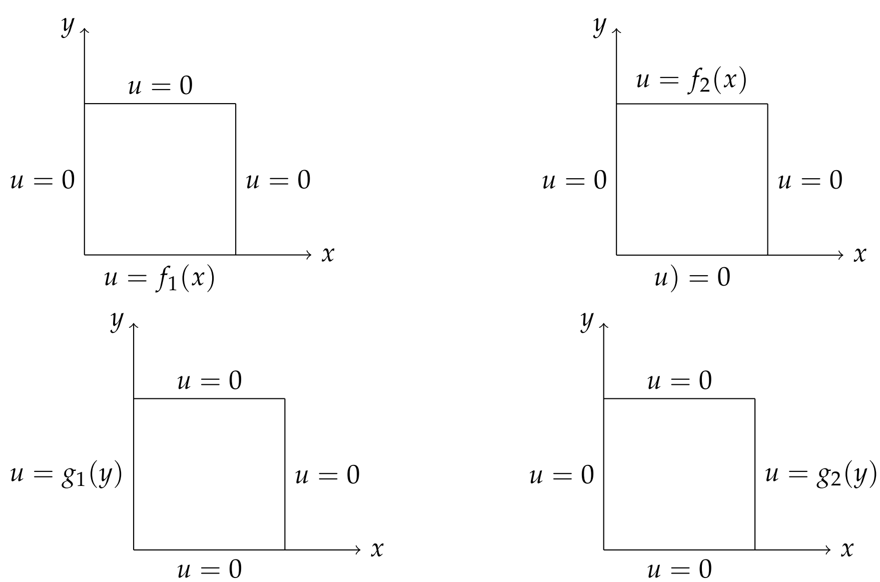


Figure 9.11: Breaking up the general boundary value problem for a rectangular plate.

We can solve each of the problems quickly, based on the solution obtained in the last example. The solution for boundary conditions

$$u(0,y) = 0, u(L,y) = 0, u(x,0) = f_1(x), u(x,H) = 0.$$

is the easiest to write down:

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi(H-y)}{L}. \tag{9.66}$$

where

$$a_n = \frac{2}{L\sinh\frac{n\pi H}{L}} \int_0^L f_1(x)\sin\frac{n\pi x}{L}\,dx. \tag{9.67}$$

For the boundary conditions

$$u(0,y) = 0, u(L,y) = 0, u(x,0) = 0, u(x,H) = f_2(x),$$

the boundary conditions for $X(x)$ *are* $X(0) = 0$ *and* $X(L) = 0$. *So, we get the same form for the eigenvalues and eigenfunctions as before:*

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,\cdots.$$

However, the remaining homogeneous boundary condition is now $Y(0) = 0$. *Recalling the equation satisfied by* $Y(y)$ *is*

$$Y'' - \lambda Y = 0,$$

we can write the general solution as

$$Y(y) = c_1 \cosh\sqrt{\lambda}y + c_2 \sinh\sqrt{\lambda}y.$$

Requiring $Y(0) = 0$, *we have* $c_1 = 0$, *or*

$$Y(y) = c_2 \sinh\sqrt{\lambda}y.$$

Then the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}. \tag{9.68}$$

We now force the nonhomogenous boundary condition, $u(x,H) = f_2(x)$,

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi H}{L}. \tag{9.69}$$

Once again we have a Fourier sine series. The Fourier coefficients are given by

$$b_n = \frac{2}{L\sinh\frac{n\pi H}{L}} \int_0^L f_2(x)\sin\frac{n\pi x}{L}\,dx. \tag{9.70}$$

Now we turn to the problem with the boundary conditions

$$u(0,y) = g_1(y), u(L,y) = 0, u(x,0) = 0, u(x,H) = 0.$$

In this case the pair of homogeneous boundary conditions $u(x,0) = 0, u(x,H) = 0$. *lead to solutions*

$$Y_n(y) = \sin\frac{n\pi y}{H}, \quad \lambda_n = -\left(\frac{n\pi}{H}\right)^2, n = 1,2\cdots.$$

The condition $u(L,0)=0$ gives $X(x)=\sinh\frac{n\pi(L-x)}{H}$. The general solution is

$$u(x,y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi(L-x)}{H}. \tag{9.71}$$

We now force the nonhomogenous boundary condition, $u(0,y)=g_1(y)$,

$$g_1(y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}. \tag{9.72}$$

The Fourier coefficients are given by

$$c_n=\frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H g_1(y)\sin\frac{n\pi y}{H}\,dy. \tag{9.73}$$

Finally, we can find the solution for

$$u(0,y)=0,u(L,y)=g_2(y),u(x,0)=0,u(x,H)=0.$$

Following the above analysis, we find the general solution

$$u(x,y)=\sum_{n=1}^{\infty}d_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi x}{H}. \tag{9.74}$$

We now force the nonhomogenous boundary condition, $u(L,y)=g_2(y)$,

$$g_2(y)=\sum_{n=1}^{\infty}d_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}. \tag{9.75}$$

The Fourier coefficients are given by

$$d_n=\frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H g_1(y)\sin\frac{n\pi y}{H}\,dy. \tag{9.76}$$

The solution to the general problem is given by the sum of these four solutions.

$$\begin{aligned}u(x,y) \quad = \quad & \sum_{n=1}^{\infty}\left[\left(a_n\sinh\frac{n\pi(H-y)}{L}+b_n\sinh\frac{n\pi y}{L}\right)\sin\frac{n\pi x}{L}\right.\\ & \left.+\left(c_n\sinh\frac{n\pi(L-x)}{H}+d_n\sinh\frac{n\pi x}{H}\right)\sin\frac{n\pi y}{H}\right],\end{aligned} \tag{9.77}$$

where the coefficients are given by the above Fourier integrals.

Example 9.4. *Laplace's Equation on a Disk*

We now turn to solving Laplace's equation on a disk of radius a as shown in Figure 9.12. Laplace's equation in polar coordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0,\quad 0<r<a,\quad -\pi<\theta<\pi. \tag{9.78}$$

The boundary conditions are given as

$$u(a, \theta) = f(\theta), \quad -\pi < \theta < \pi, \tag{9.79}$$

plus periodic boundary conditions in θ.

Separation of variable proceeds as usual. Let $u(r, \theta) = R(r)\Theta(\theta)$. *Then*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(R\Theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(R\Theta)}{\partial \theta^2} = 0, \tag{9.80}$$

or

$$\Theta\frac{1}{r}(rR')' + \frac{1}{r^2}R\Theta'' = 0. \tag{9.81}$$

Diving by $u(r, \theta) = R(r)\Theta(\theta)$ *and rearranging, we have*

$$\frac{r}{R}(rR')' = -\frac{\Theta''}{\theta} = \lambda. \tag{9.82}$$

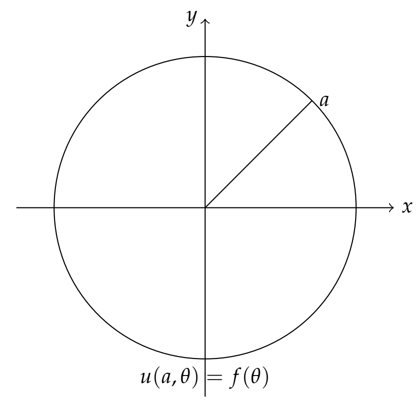


Figure 9.12: The circular plate of radius a with boundary condition along the edge at $r = a$.

Since this equation gives a function of r *equal to a function of* θ, *we set the equation equal to a constant. Thus, we have obtained two differential equations, which can be written as*

$$\begin{aligned} r(rR')' - \lambda R &= 0, && (9.83) \\ \Theta'' + \lambda\Theta &= 0. && (9.84) \end{aligned}$$

We can solve the second equation, using periodic boundary conditions. The reader should be able to confirm that

$$\Theta(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad \lambda = n^2, n = 0, 1, 2, \cdots$$

is the solution. Note that the $n = 0$ *case just leads to a constant solution.*

Inserting $\lambda = n^2$ *into the radial equation, we find*

$$r^2R'' + rR' - n^2R = 0.$$

This is a Cauchy-Euler type of ordinary differential equation. Recall that we solve such equations by guessing a solution of the form $R(r) = r^m$. *This leads to the characteristic equation* $m^2 - n^2 = 0$. *Therefore,* $m = \pm n$. *So,*

$$R(r) = c_1 r^n + c_2 r^{-n}.$$

Since we expect finite solutions at the origin, $r = 0$, *we can set* $c_2 = 0$. *Thus, the general solution is*

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n. \tag{9.85}$$

Note that we have taken the constant term out of the sum and put it into a familiar form.

Now were are ready to impose the remaining boundary condition, $u(a, \theta) = f(\theta)$. *This gives*

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) a^n. \tag{9.86}$$

This is a Fourier trigonometric series. The Fourier coefficients can be determined using the results from Chapter 4:

$$a_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta, \quad n = 0, 1, \cdots, \tag{9.87}$$

$$b_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta \quad n = 1, 2\cdots. \tag{9.88}$$

We can put the solution from the last example in a more compact form by inserting these coefficients into the general solution. Doing this, we have

$$\begin{aligned}
u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right) r^n \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi \\
&\quad + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\left[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\right]\left(\frac{r}{a}\right)^n f(\phi)\, d\phi \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n\right] f(\phi)\, d\phi.
\end{aligned} \tag{9.89}$$

The term in the brackets can be summed. We note that $\cos n(\theta - \phi) = Re(e^{in(\theta-\phi)})$. Then

$$\begin{aligned}
\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n &= Re\left(e^{i(\theta-\phi)}\left(\frac{r}{a}\right)^n\right) \\
&= Re\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n.
\end{aligned} \tag{9.90}$$

Therefore,

$$\sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n = Re\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n\right)$$

The right hand side of this equation is a geometric series with common ratio $\frac{r}{a}e^{i(\theta-\phi)}$. Since $\left|\frac{r}{a}e^{i(\theta-\phi)}\right| = \frac{r}{a} < 1$, the series converges. Summing the series, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n &= \frac{\frac{r}{a}e^{i(\theta-\phi)}}{1-\frac{r}{a}e^{i(\theta-\phi)}} \\
&= \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} \\
&= \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}}\,\frac{a - re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} \\
&= \frac{are^{-i(\theta-\phi)} - r^2}{a^2 + r^2 - 2ar\cos(\theta-\phi)}.
\end{aligned} \tag{9.91}$$

We have rewritten this sum so that we can easily take the real part,

$$Re\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n\right) = \frac{ar\cos(\theta-\phi) - r^2}{a^2 + r^2 - 2ar\cos(\theta-\phi)}$$

Therefore, the factor in the brackets under the integral in Equation (9.89) is

$$\begin{aligned}\frac{1}{2}+\sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^{n} &= \frac{1}{2}+\frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}\\ &= \frac{a^2-r^2}{2(a^2+r^2-2ar\cos(\theta-\phi))}. \qquad (9.92)\end{aligned}$$

Thus, we have shown that the solution of Laplace's equation on a disk of radius a with boundary condition $u(a,\theta)=f(\theta)$ can be written in the closed form

Poisson Integral Formula

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}f(\phi)\,d\phi. \qquad (9.93)$$

This result is called the Poisson Integral Formula and

$$K(\theta,\phi)=\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}$$

is called the Poisson kernel.

9.4 *Three Dimensional Cake Baking*

In the rest of the chapter we will extend our studies to three dimensions. In this section we will solve the heat equation as we look at examples of baking cakes. We consider cake batter, which is at room temperature of $T_i=80°$F. It is placed into an oven, also at a fixed temperature, $T_b=350°$F. For simplicity, we will assume that the thermal conductivity and cake density are constant. Of course, this is not quite true. However, it is an approximation which simplifies the model. We will consider two cases, one in which the cake is a rectangular solid ($0\le x\le W, 0\le y\le L, 0\le z\le H$), such as baking it in a $13''\times 9''\times 2''$ baking pan. The other case will lead to a cylindrical cake, such as you would obtain from a round cake pan.

This discussion of cake baking is adapted from R. Wilkinson's thesis work. That in turn was inspired by work done by Dr. Olszewski.

Assuming that the heat constant k is indeed constant and the temperature is given by $T(\mathbf{r},t)$, we begin with the heat equation in three dimensions,

$$\frac{\partial T}{\partial t}=k\nabla^2 T. \qquad (9.94)$$

We will need to specify initial and boundary conditions. Let T_i be the initial batter temperature, and write the initial condition as

$$T(x,y,z,0)=T_i.$$

We choose the boundary conditions to be fixed at the oven temperature T_b. However, these boundary conditions are not homogeneous and would lead to problems when carrying out separation of variables. This is easily remedied by subtracting the oven temperature from all temperatures involved and defining $u(x,y,z,t) = T(x,y,z,t) - T_b$. The heat equation then becomes

$$\frac{\partial u}{\partial t} = k\nabla^2 u \tag{9.95}$$

with initial condition

$$u(\mathbf{r},0) = T_i - T_b.$$

The boundary conditions are now that $u = 0$ on the boundary. We cannot be any more specific than this until we specify the geometry.

Example 9.5. *Temperature of a Rectangular Cake*

For this problem, we seek solutions of the heat equation plus the conditions

$$\begin{aligned} u(x,y,z,0) &= T_i - T_b, \\ u(0,y,z,t) = u(W,y,z,t) &= 0, \\ u(x,0,z,t) = u(x,L,z,t) &= 0, \\ u(x,y,0,t) = u(x,y,H,t) &= 0. \end{aligned}$$

Using the method of separation of variables, we seek solutions of the form

$$u(x,y,z,t) = X(x)Y(y)Z(z)G(t). \tag{9.96}$$

Substituting this form into the heat equation, we get

$$\frac{1}{k}\frac{G'}{G} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{9.97}$$

Setting these expressions equal to $-\lambda$, we get

$$\frac{1}{k}\frac{G'}{G} = -\lambda \quad \textit{and} \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda. \tag{9.98}$$

Therefore, the equation for $G(t)$ is given by

$$G' + k\lambda G = 0.$$

We further have to separate out the functions of x, y, and z. We anticipate that the homogeneous boundary conditions will lead to oscillatory solutions in these variables. Therefore, we expect separation of variable will lead to the eigenvalue problems

$$\begin{aligned} X'' + \mu^2 X &= 0, \quad X(0) = X(W) = 0, \\ Y'' + \nu^2 Y &= 0, \quad Y(0) = Y(L) = 0, \\ Z'' + \kappa^2 Z &= 0, \quad Z(0) = Z(H) = 0. \end{aligned} \tag{9.99}$$

Noting that

$$\frac{X''}{X} = -\mu^2, \quad \frac{Y''}{Y} = -\nu^2, \quad \frac{Z''}{Z} = -\kappa^2,$$

we have the relation $\lambda^2 = \mu^2 + \nu^2 + \kappa^2$.

From the boundary conditions, we get product solutions for $u(x, y, z, t)$ *in the form*

$$u_{mn\ell}(x, y, z, t) = \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z e^{-\lambda_{mn\ell} kt},$$

for

$$\lambda_{mnl} = \mu_m^2 + \nu_n^2 + \kappa_\ell^2 = \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{\ell\pi}{H}\right)^2, \quad m, n, \ell = 1, 2, \ldots.$$

The general solution is then

$$u(x, y, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_{mnl} \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z e^{-\lambda_{mn\ell} kt}, \quad (9.100)$$

where the $A_{mn\ell}$*'s are arbitrary constants.*

We can use the initial condition $u(x, y, z, 0) = T_i - T_b$ *to determine the* $A_{mn\ell}$*'s. We find*

$$T_i - T_b = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_{mnl} \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z. \quad (9.101)$$

This is a triple Fourier sine series. We can determine these coefficients in a manner similar to how we handled a double Fourier sine series earlier. Defining

$$b_m(y, z) = \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} A_{mnl} \sin \nu_n y \sin \kappa_\ell z,$$

we obtain a simple Fourier sine series:

$$T_i - T_b = \sum_{m=1}^{\infty} b_m(y, z) \sin \mu_m x. \quad (9.102)$$

The Fourier coefficients can then be found as

$$b_m(y, z) = \frac{2}{W} \int_0^W (T_i - T_b) \sin \mu_m x \, dx.$$

Using the same technique for the remaining sine series and noting that $T_i - T_b$ *is constant, we can compute the general coefficient* A_{mnl} *by carrying out the needed integrations:*

$$\begin{aligned}
A_{mnl} &= \frac{8}{WLH} \int_0^H \int_0^L \int_0^W (T_i - T_b) \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z \, dxdydz \\
&= (T_i - T_b) \frac{8}{\pi^3} \left[\frac{\cos\left(\frac{m\pi x}{W}\right)}{m}\right]_0^W \left[\frac{\cos\left(\frac{n\pi y}{L}\right)}{n}\right]_0^L \left[\frac{\cos\left(\frac{\ell\pi z}{H}\right)}{\ell}\right]_0^H \\
&= (T_i - T_b) \frac{8}{\pi^3} \left[\frac{\cos m\pi - 1}{m}\right] \left[\frac{\cos n\pi - 1}{n}\right] \left[\frac{\cos \ell\pi - 1}{\ell}\right] \\
&= (T_i - T_b) \frac{8}{\pi^3} \begin{cases} 0, & \text{for at least one } m, n, \ell \text{ even,} \\ \left[\frac{-2}{m}\right] \left[\frac{-2}{n}\right] \left[\frac{-2}{\ell}\right], & \text{for } m, n, \ell \text{ all odd.} \end{cases}
\end{aligned}$$

Since only the odd multiples yield non-zero $A_{mn\ell}$ we let $m = 2m' - 1$, $n = 2n' - 1$, and $\ell = 2\ell' - 1$. Thus

$$A_{mnl} = \frac{-64(T_i - T_b)}{(2m'-1)\,(2n'-1)\,(2\ell'-1)\,\pi^3}.$$

Substituting this result into general solution and dropping the primes, we find

$$u(x,y,z,t) = \frac{-64(T_i - T_b)}{\pi^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\sin \mu_m x \sin \nu_n y \sin \kappa_\ell z e^{-\lambda_{mn\ell} kt}}{(2m-1)(2n-1)(2\ell-1)},$$

where

$$\lambda_{mn\ell} = \left(\frac{(2m-1)\pi}{W}\right)^2 + \left(\frac{(2n-1)\pi}{L}\right)^2 + \left(\frac{(2\ell-1)\pi}{H}\right)^2$$

for $m, n, \ell = 1, 2, \ldots$.

Recalling $T(x,y,z,t) = u(x,y,z,t) - T_b$,

$$T(x,y,z,t) = T_b - \frac{64(T_i - T_b)}{\pi^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\sin \hat{\mu}_m x \sin \hat{\nu}_n y \sin \hat{\kappa}_\ell z e^{-\hat{\lambda}_{mn\ell} kt}}{(2m-1)(2n-1)(2\ell-1)}.$$

We show some temperature distributions in Figure 9.13. Vertical slices are taken at the positions and times indicated for a $13'' \times 9'' \times 2''$ cake. Obviously, this is not accurate because the cake consistency is changing and this will affect the parameter k. A more realistic model would be to allow $k = k(T(x,y,z,t))$. however, such problems are beyond the simple methods described in this book.

Example 9.6. *Circular Cakes*

In this case the geometry is cylindrical. Therefore, we need to express the boundary conditions and heat equation in cylindrical coordinates.

We assume $u(r,z,t) = T(r,z,t) - T_b$ is independent of θ due to symmetry. This gives the heat equation in cylindrical coordinates as

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \tag{9.103}$$

where $0 \le r \le a$ and $0 \le z \le Z$. The initial condition is

$$u(r,z,0) = T_i - T_b,$$

and the homogeneous boundary conditions are

$$u(a,z,t) = 0,$$
$$u(r,0,t) = u(r,Z,t) = 0.$$

Again, we seek solutions of the form $u(r,z,t) = R(r)H(z)G(t)$. Separation of variables leads to

$$\frac{1}{k}\frac{G'}{G} = \frac{1}{rR}\frac{d}{dr}\left(rR'\right) + \frac{H''}{H}. \tag{9.104}$$

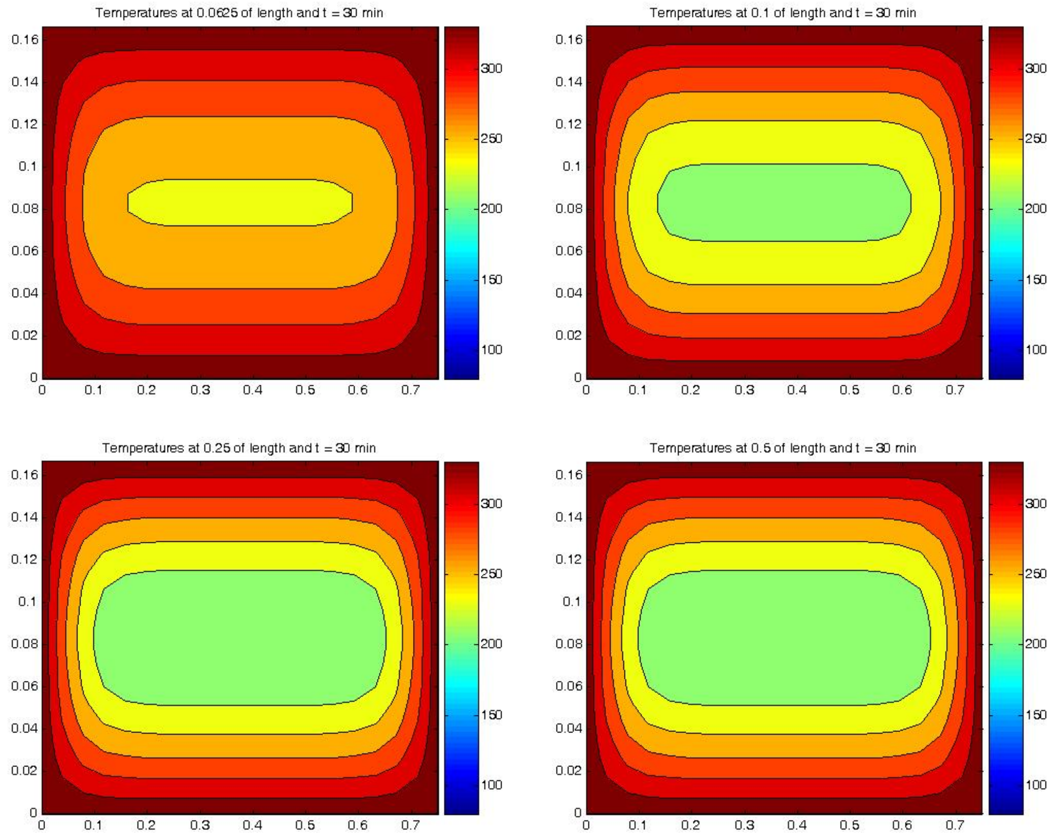


Figure 9.13: Temperature evolution for a $13'' \times 9'' \times 2''$ cake shown as vertical slices at the indicated length in feet.

Choosing λ as the separation constant, we get

$$G' - k\lambda G = 0, \tag{9.105}$$

and

$$\frac{1}{rR}\frac{d}{dr}\left(rR'\right) = -\frac{H''}{H} + \lambda. \tag{9.106}$$

Since negative eigenvalues yield the oscillatory solutions we expect, we continue as before by setting both sides of this equation equal to $-\mu^2$. After some rearrangement, we obtain the needed differential equations:

$$\frac{d}{dr}\left(rR'\right) + r\mu^2 R = 0 \tag{9.107}$$

and

$$H'' + \nu^2 H = 0. \tag{9.108}$$

Here $\lambda = -\left(\mu^2 + \nu^2\right)$.

We can easily write down the solutions

$$G(t) = Ae^{\lambda k t}$$

and

$$H_n(z) = \sin\left(\frac{n\pi z}{Z}\right), n = 1, 2, 3, \ldots,$$

where $\nu = \frac{n\pi}{Z}$. Recalling from the rectangular case that only odd terms arise in the Fourier sine series coefficients for the constant initial condition, we proceed by rewriting $H(z)$ as

$$H_n(z) = \sin\left(\frac{(2n-1)\pi z}{Z}\right), n = 1, 2, 3, \ldots \tag{9.109}$$

with $\nu = \frac{(2n-1)\pi}{Z}$.

The radial equation can be written in the form

$$r^2 R'' + rR' + r^2\mu^2 R = 0.$$

This is a Bessel equation of the first kind of order zero and the general solution is a linear combination of Bessel functions of the first and second kind,

$$R(r) = c_1 J_0(\mu r) + c_2 N_0(\mu r). \tag{9.110}$$

Since we wish to have $u(r, z, t)$ bounded at $r = 0$ and $N_0(\mu r)$ is not well behaved at $r = 0$, we set $c_2 = 0$. Up to a constant factor, the solution becomes

$$R(r) = J_0(\mu r). \tag{9.111}$$

The boundary condition $R(a) = 0$ gives $J_0(\mu a) = 0$ and thus $\mu_m = \frac{j_{0m}}{a}$, for $m = 1, 2, 3, \ldots$. Here the j_{0m}'s are the m^{th} roots of the zeroth-order Bessel function, $J_0(j_{0m}) = 0$, which are given in Table 9.1. This suggests that

$$R_m(r) = J_0\left(\frac{r}{a} j_{0m}\right), m = 1, 2, 3, \ldots. \tag{9.112}$$

Thus, we have found that the general solution is given as

$$u(r, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{(2n-1)\pi z}{Z}\right) J_0\left(\frac{r}{a} j_{0m}\right) e^{-\lambda_{nm} kt} \tag{9.113}$$

with

$$\lambda_{nm} = \left(\left(\frac{(2n-1)\pi}{Z}\right)^2 + \left(\frac{j_{0m}}{a}\right)^2\right),$$

for $n, m = 1, 2, 3, \ldots$.

Using the constant initial condition to find the A_{nm}'s, we have

$$T_i - T_b = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left[\frac{(2n-1)\pi z}{Z}\right] J_0\left(\frac{r}{a} j_{0m}\right).$$

If we let $b_n(r) = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right)$, we have

$$T_i - T_b = \sum_{n=1}^{\infty} b_n(r) \sin\left(\frac{(2n-1)\pi z}{Z}\right).$$

As seen previously, this is a Fourier sine series and the Fourier coefficients are given by

$$\begin{aligned} b_n(r) &= \frac{2}{Z}\int_0^Z (T_i - T_b)\sin\left(\frac{(2n-1)\pi z}{Z}\right)dz \\ &= \frac{2(T_i-T_b)}{Z}\left[-\frac{Z}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi z}{Z}\right)\right]_0^Z \\ &= \frac{4(T_i-T_b)}{(2n-1)\pi}. \end{aligned}$$

Then, we have

$$b_n(r) = \frac{4(T_i-T_b)}{(2n-1)\pi} = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right).$$

This is a Fourier-Bessel series. Given $b_n(r) = \frac{4(T_i-T_b)}{(2n-1)\pi}$, we seek to find the Fourier coefficients A_{nm}. Recall from Chapter 5 that the Fourier-Bessel series is given by

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}\frac{x}{a}), \tag{9.114}$$

where the Fourier-Bessel coefficients are found as

$$c_n = \frac{2}{a^2\left[J_{p+1}(j_{pn})\right]^2}\int_0^a x f(x) J_p(j_{pn}\frac{x}{a})\,dx. \tag{9.115}$$

For this problem, we have

$$A_{nm} = \frac{2}{a^2 J_1^2(j_{0m})}\frac{4(T_i-T_b)}{(2n-1)\pi}\int_0^a J_0(\mu_m r) r\,dr. \tag{9.116}$$

In order to evaluate $\int_0^a J_0(\mu_k r) r\,dr$, we let $y = \mu_k r$ and get

$$\begin{aligned} \int_0^a J_0(\mu_k r) r dr &= \int_0^{\mu_k a} J_0(y)\frac{y}{\mu_k}\frac{dy}{\mu_k} \\ &= \frac{1}{\mu_k^2}\int_0^{\mu_k a} J_0(y) y\,dy \\ &= \frac{1}{\mu_k^2}\int_0^{\mu_k a}\frac{d}{dy}\left(y J_1(y)\right)\,dy \\ &= \frac{1}{\mu_k^2}(\mu_k a) J_1(\mu_k a) = \frac{a^2}{j_{0k}} J_1(j_{0k}). \end{aligned} \tag{9.117}$$

Here we have made use of the identity $\frac{d}{dx}\left(x J_1(x)\right) = J_0(x)$.

Substituting the result of this integral computation into the expression for A_{nm}, we find

$$A_{nm} = \frac{8(T_i-T_b)}{(2n-1)\pi}\frac{1}{j_{0m}J_1(j_{0m})}.$$

Substituting A_{nm} into the original expression for $u(r,z,t)$, gives

$$u(r,z,t) = \frac{8(T_i-T_b)}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{(2n-1)\pi z}{Z}\right)}{(2n-1)}\frac{J_0(\frac{r}{a} j_{0m}) e^{\lambda_{nm} D t}}{j_{0m} J_1(j_{0m})}.$$

Therefore, $T(r,z,t)$ can be found as

$$T(r,z,t) = T_b + \frac{8(T_i - T_b)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin\left(\frac{(2n-1)\pi z}{Z}\right)}{(2n-1)} \frac{J_0(\frac{r}{a} j_{0m}) e^{\lambda_{nm} kt}}{j_{0m} J_1(j_{0m})}.$$

We have therefore found the general solution for the three-dimensional heat equation in cylindrical coordinates with constant diffusivity. Similar to the solutions shown in Figure 9.13 of the previous section, we show in Figure 9.14 the temperature temperature evolution throughout a standard 9″ round cake pan.

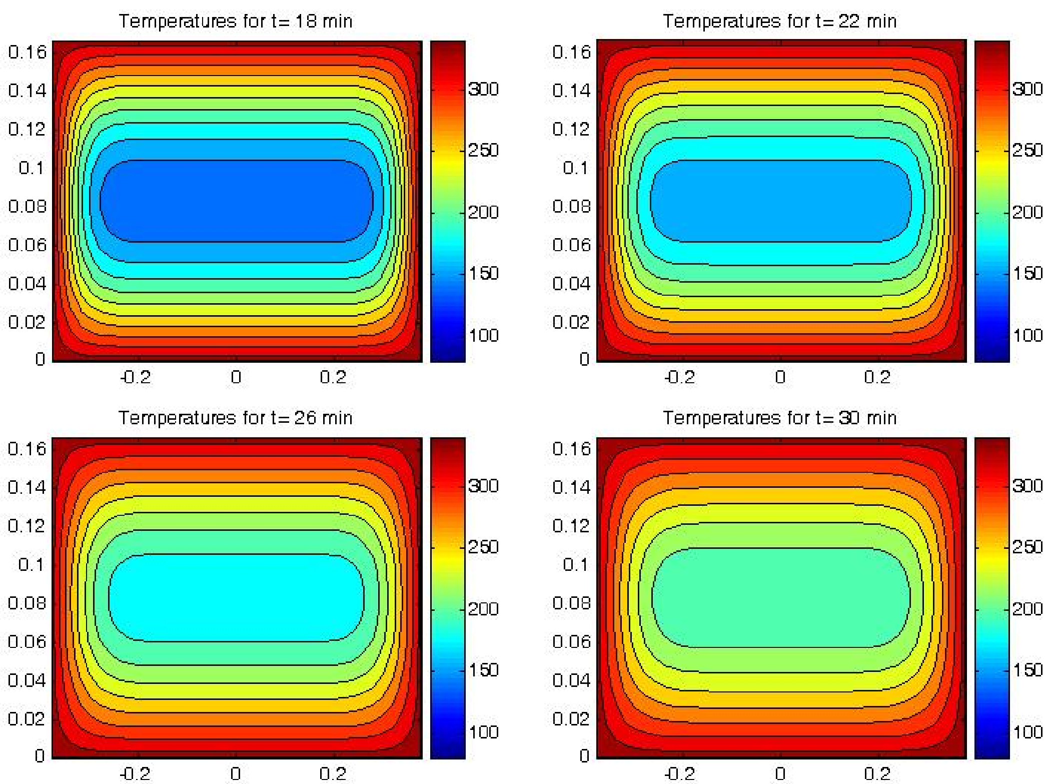


Figure 9.14: Temperature evolution for a standard 9″ cake shown as vertical slices through the center.

9.5 *Laplace's Equation and Spherical Symmetry*

We have seen that Laplace's equation, $\nabla^2 u = 0$, arises in electrostatics as an equation for electric potential outside a charge distribution and it occurs as the equation governing equilibrium temperature distributions. As we had seen in the last chapter, Laplace's equation generally occurs in the study of potential theory, which also includes the study of gravitational and fluid potentials. The equation is named after Pierre-Simon Laplace (1749-1827) who had studied the properties of this equation. solutions of Laplace's equation are called harmonic functions.

Laplace's equation in spherical coordinates is given by[2]

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0, \qquad (9.118)$$

where $u=u(\rho,\theta,\phi)$.

We seek solutions of this equation inside a sphere of radius r subject to the boundary condition $u(r,\theta,\phi)=g(\theta,\phi)$ as shown in Figure 9.15.

As before, we perform a separation of variables by seeking product solutions of the form $u(\rho,\theta,\phi)=R(\rho)\Theta(\theta)\Phi(\phi)$. Inserting this form into the Laplace equation, we obtain

[2] The Laplacian in spherical coordinates is given in Problem 29 in Chapter 8.

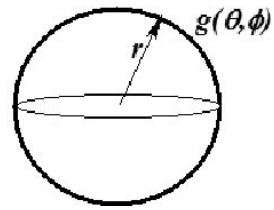


Figure 9.15: A sphere of radius r with the boundary condition $u(r,\theta,\phi)=g(\theta,\phi)$.

$$\frac{\Theta\Phi}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{R\Phi}{\rho^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{R\Theta}{\rho^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0. \qquad (9.119)$$

Multiplying this equation by ρ^2 and dividing by $R\Theta\Phi$, yields

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=0. \qquad (9.120)$$

Note that the first term is the only term depending upon ρ. Thus, we can separate out the radial part. However, there is still more work to do on the other two terms, which give the angular dependence. Thus, we have

$$-\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda, \qquad (9.121)$$

where we have introduced the first separation constant. This leads to two equations:

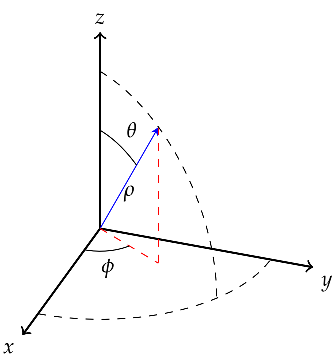


Figure 9.16: Definition of spherical coordinates (ρ,θ,ϕ). Note that there are different conventions for labeling spherical coordinates. This labeling is used often in physics.

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)-\lambda R=0 \qquad (9.122)$$

and

$$\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda. \qquad (9.123)$$

Equation (9.123) is a key equation which occurs when studying problems possessing spherical symmetry. It is an eigenvalue problem for $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$, $LY=-\lambda Y$, where

$$L=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The eigenfunctions of this operator are referred to as spherical harmonics.

The final separation can be performed by multiplying the last equation by $\sin^2\theta$, rearranging the terms, and introducing a second separation constant:

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\lambda\sin^2\theta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\mu. \qquad (9.124)$$

From this expression we can determine the differential equations satisfied by $\Theta(\theta)$ and $\Phi(\phi)$:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+(\lambda\sin^2\theta-\mu)\Theta=0, \qquad (9.125)$$

and
$$\frac{d^2\Phi}{d\phi^2} + \mu\Phi = 0. \tag{9.126}$$

We now have three ordinary differential equations to solve. These are the radial equation (9.122) and the two angular equations (9.125)-(9.126). We note that all three are in Sturm-Liouville form. We will solve each eigenvalue problem subject to appropriate boundary conditions.

The simplest of these differential equations is the one for $\Phi(\phi)$, Equation (9.126). We have seen equations of this form many times and the general solution is a linear combination of sines and cosines. As argues in such problems, we have to impose periodic boundary conditions. For example, we expect that

$$u(\rho,\theta,0) = u(\rho,\theta,2\pi), \quad u_\phi(\rho,\theta,0) = u_\phi(\rho,\theta,2\pi).$$

Since these conidtions hold for all ρ and θ, we must require that

$$\Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi).$$

As we have seen before, the eigenfunctions and eigenvalues are then found as

$$\Phi(\phi) = \{\cos m\phi, \sin m\phi\}, \quad \mu = m^2, \quad m = 0, 1, \ldots. \tag{9.127}$$

Next we turn to solving equation, (9.126). We first transform this equation in order to identify the solutions. Let $x = \cos\theta$. Then the derivatives with respect to θ transform as

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = \sin\theta\frac{d}{dx}.$$

Letting $y(x) = \Theta(\theta)$ and noting that $\sin^2\theta = 1 - x^2$, Equation (9.126) becomes
$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0. \tag{9.128}$$
We further note that $x \in [-1, 1]$, as can be easily confirmed by the reader.

This is a Sturm-Liouville eigenvalue problem. The solutions consist of a set of orthogonal eigenfunctions. For the special case that $m = 0$ Equation (9.128) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \lambda y = 0. \tag{9.129}$$

In a course in differential equations one learns to seek solutions of this equation in the form
$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

This leads to the recursion relation

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n.$$

Setting $n = 0$ and seeking a series solution, one finds that the resulting series does not converge for $x = \pm 1$. This is remedied by choosing $\lambda = \ell(\ell+1)$ for $\ell = 0, 1, \ldots$, leading to the differential equation

$$\frac{d}{dx}\left((1 - x^2)\frac{dy}{dx}\right) + \ell(\ell+1)y = 0. \tag{9.130}$$

We saw this equation in Chapter 5. The solutions of this Legendre differential equation are the Legendre polynomials, denoted by $P_\ell(x)$.

For the more general case, $m \neq 0$, the differential equation (9.128) with $\lambda = \ell(\ell+1)$ becomes

Associated Legendre Functions

$$\frac{d}{dx}\left((1 - x^2)\frac{dy}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1 - x^2}\right) y = 0. \tag{9.131}$$

The solutions of this equation are called the associated Legendre functions. The two linearly independent solutions are denoted by $P_\ell^m(x)$ and $Q_\ell^m(x)$. The latter functions are not well behaved at $x = \pm 1$, corresponding to the north and south poles of the original problem. So, we can throw out these solutions, leaving

$$\Theta(\theta) = P_\ell^m(\cos\theta)$$

as the needed solutions. In Table 9.2 we list a few of these.

	$P_n^m(x)$	$P_n^m(\cos\theta)$
$P_1^0(x)$	x	$\cos\theta$
$P_1^1(x)$	$(1 - x^2)^{\frac{1}{2}}$	$\sin\theta$
$P_2^0(x)$	$\frac{1}{2}(3x^2 - 1)$	$\frac{1}{2}(\cos^2\theta - 1)$
$P_2^1(x)$	$3x(1 - x^2)^{\frac{1}{2}}$	$3\cos\theta\sin\theta$
$P_2^2(x)$	$3(1 - x^2)$	$3\sin^2\theta$
$P_3^0(x)$	$\frac{1}{2}(5x^3 - 32x)$	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$
$P_3^1(x)$	$\frac{3}{2}(5x^2 - 1)(1 - x^2)^{\frac{1}{2}}$	$\frac{3}{2}(5\cos^2\theta - 1)\sin\theta$
$P_3^2(x)$	$15x(1 - x^2)$	$15\cos\theta\sin^2\theta$
$P_3^3(x)$	$15(1 - x^2)^{\frac{3}{2}}$	$15\sin^3\theta$

Table 9.2: Associated Legendre Functions, $P_n^m(x)$.

The associated Legendre functions are related to the Legendre polynomials by[3]

[3] Some definitions do not include the $(-1)^m$ factor.

$$P_\ell^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x), \tag{9.132}$$

for $\ell = 0, 1, 2, , \ldots$ and $m = 0, 1, \ldots, \ell$. We further note that $P_\ell^0(x) = P_\ell(x)$, as one can see in the table. Since $P_\ell(x)$ is a polynomial of degree

ℓ, then for $m > \ell$, $\frac{d^m}{dx^m}P_\ell(x) = 0$ and $P_\ell^m(x) = 0$. Furthermore, since the differential equation only depends on m^2, $P_\ell^{-m}(x)$ is proportional to $P_\ell^m(x)$. One normalization is given by

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x).$$

The associated Legendre functions also satisfy the orthogonality condition

$$\int_{-1}^{1} P_\ell^m(x) P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}. \tag{9.133}$$

The last differential equation we need to solve is the radial equation. With $\lambda = \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$, the radial equation (9.122) can be written as

$$\rho^2 R'' + 2\rho R' - \ell(\ell+1)R = 0. \tag{9.134}$$

The radial equation is a Cauchy-Euler type of equation. So, we can guess the form of the solution to be $R(\rho) = \rho^s$, where s is a yet to be determined constant. Inserting this guess, we obtain the characteristic equation

$$s(s+1) = \ell(\ell+1).$$

Solving for s, we have

$$s = \ell, -(\ell+1).$$

Thus, the general solution of the radial equation is

$$R(\rho) = a\rho^\ell + b\rho^{-(\ell+1)}. \tag{9.135}$$

We would normally apply boundary conditions at this point. Recall that we gave that for $\rho = r$, $u(r, \theta, \phi) = g(\theta, \phi)$. This is not a homogeneous boundary condition, so we will need to hold off using it until we have the general solution to the three dimensional problem. However, we do have a hidden condition. Since we are interested in solutions inside the sphere, we need to consider what happens at $\rho = 0$. Note that $\rho^{-(\ell+1)}$ is not defined at the origin. Since the solution is expected to be bounded at the origin, we can set $b = 0$. So, in the current problem we have established that

$$R(\rho) = a\rho^\ell.$$

We have carried out the full separation of Laplace's equation in spherical coordinates. The product solutions consist of the forms

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta)\cos m\phi$$

and

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta)\sin m\phi$$

for $\ell = 0, 1, 2, \ldots$ and $m = 0, \pm 1, , \ldots, \pm \ell$. These solutions can be combined to give a complex representation of the product solutions as

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta) e^{im\phi}.$$

The general solution is then given as a linear combination of these product solutions. As there are two indices, we have a double sum:[4]

$$u(\rho, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \rho^\ell P_\ell^m(\cos\theta) e^{im\phi}. \tag{9.136}$$

[4] While this appears to be a complex-valued solution, it can be rewritten as a sum over real functions. The inner sum contains terms for both $m = k$ and $m = -k$. Adding these contributions, we have that

$$a_{\ell k} \rho^\ell P_\ell^k(\cos\theta) e^{ik\phi} + a_{\ell(-k)} \rho^\ell P_\ell^{-k}(\cos\theta) e^{-ik\phi}$$

can be rewritten as

$$(A_{\ell k} \cos k\phi + B_{\ell k} \sin k\phi) \rho^\ell P_\ell^k(\cos\theta).$$

$Y_{\ell m}(\theta, \phi)$, are the spherical harmonics. Spherical harmonics are important in applications from atomic electron configurations to gravitational fields, planetary magnetic fields, and the cosmic microwave background radiation.

The solutions of the angular parts of the problem are often combined into one function of two variables, as problems with spherical symmetry arise often, leaving the main differences between such problems confined to the radial equation. These functions are referred to as spherical harmonics, $Y_{\ell m}(\theta, \phi)$, which are defined with a special normalization as

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m e^{im\phi}. \tag{9.137}$$

These satisfy the simple orthogonality relation

$$\int_0^\pi \int_0^{2\pi} Y_{\ell m}(\theta, \phi) Y^*_{\ell' m'}(\theta, \phi) \sin\theta \, d\phi \, d\theta = \delta_{\ell \ell'} \delta_{m m'}.$$

As noted in an earlier side note, the spherical harmonics are eigenfunctions of the eigenvalue problem $LY = -\lambda Y$, where

$$L = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}.$$

This operator appears in many problems in which there is spherical symmetry, such as obtaining the solution of Schrödinger's equation for the hydrogen atom as we will see later. Therefore, it is customary to plot spherical harmonics. Because the $Y_{\ell m}$'s are complex functions, one typically plots either the real part or the modulus squared. One rendition of $|Y_{\ell m}(\theta, \phi)|^2$ is shown in Figure 9.17.

We could also look for the nodal curves of the spherical harmonics like we had for vibrating membranes. Such surface plots on a sphere are shown in Figure 9.18. The colors provide for the amplitude of the $|Y_{\ell m}(\theta, \phi)|^2$. We can match these with the shapes in Figure 9.17 by coloring the plots with some of the same colors. This is shown in Figure 9.19. However, by plotting just the sign of the spherical harmonics, as in Figure 9.20, we can pick out the nodal curves much easier.

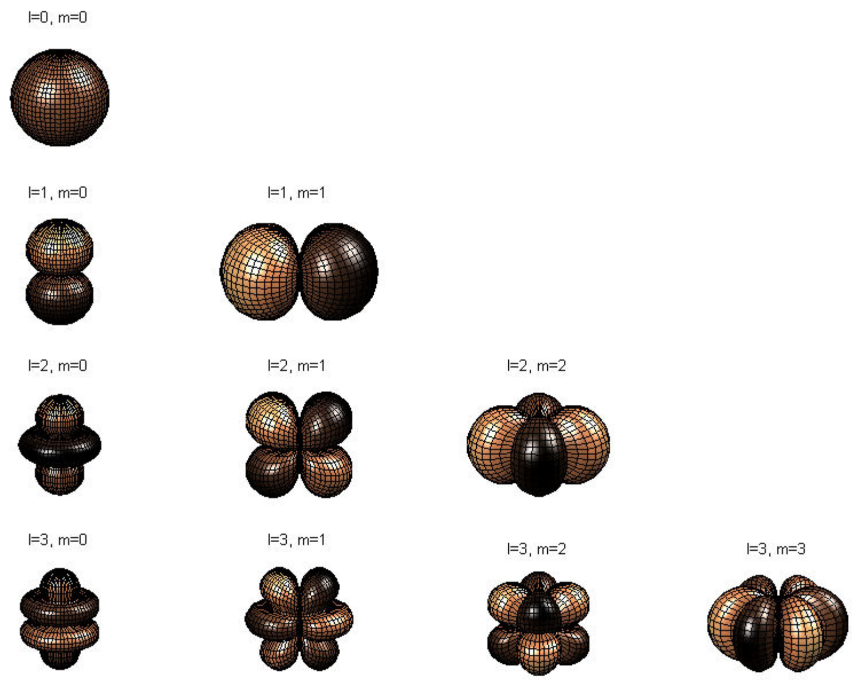


Figure 9.17: The first few spherical harmonics, $|Y_{\ell m}(\theta, \phi)|^2$

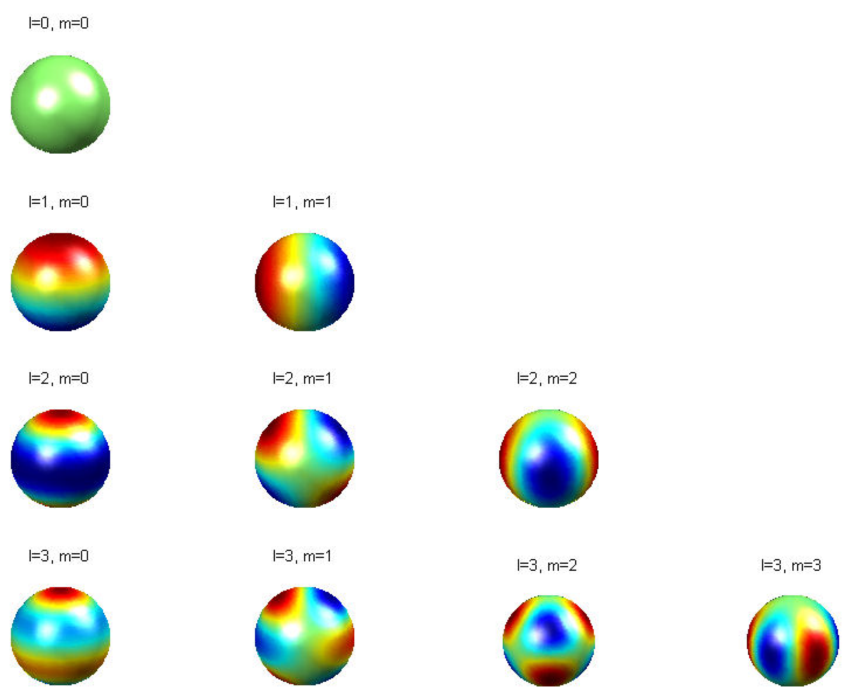


Figure 9.18: Spherical harmonic contours for $|Y_{\ell m}(\theta, \phi)|^2$.

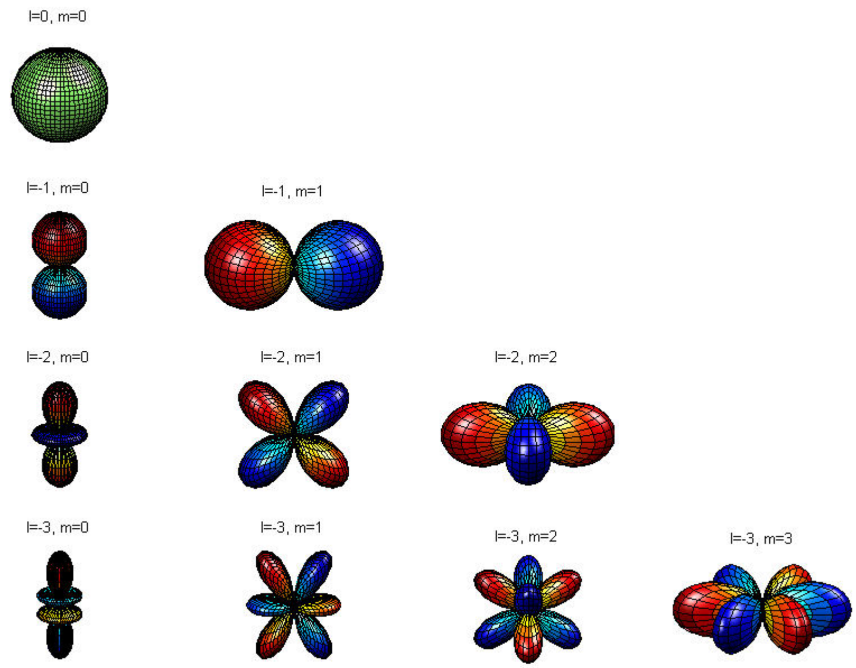


Figure 9.19: The first few spherical harmonics, $|Y_{\ell m}(\theta, \phi)|^2$

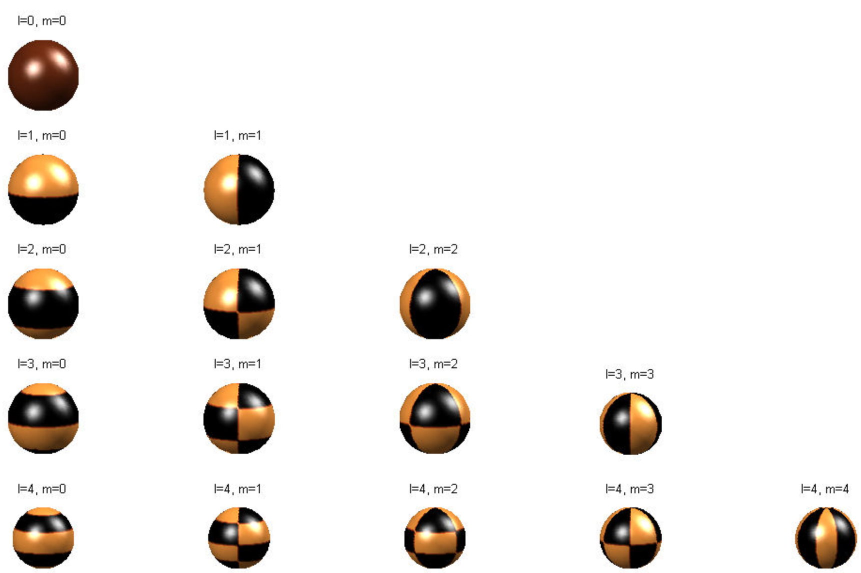


Figure 9.20: In these figures we show the nodal curves of $|Y_{\ell m}(\theta, \phi)|^2$

Spherical, or surface, harmonics can be further grouped into zonal, sectoral, and tesseral harmonics. Zonal harmonics correspond to the $m = 0$ modes. In this case, one seeks nodal curves for which $P_\ell(\cos\theta) = 0$. These lead to constant θ values such that $\cos\theta$ is a zero of the Legendre polynomial, $P_\ell(x)$. These correspond to the first column in Figure 9.20. Since $P_\ell(x)$ is a polynomial of degree ℓ, the zonal harmonics consist of ℓ latitudinal circles.

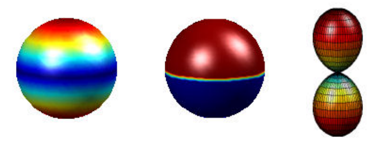


Figure 9.21: Zonal harmonics, $\ell = 1$, $m = 0$.

Sectoral, or meridional, harmonics result for the case that $m = \pm\ell$. For this case, we note that $P_\ell^{\pm\ell}(x) \propto (1-x^2)^{m/2}$. This vanishes for $x = \pm 1$, or $\theta = 0, \pi$. Therefore, the spherical harmonics can only produce nodal curves for $e^{im\phi} = 0$. Thus, one obtains the meridians corresponding to solutions of $A\cos m\phi + B\sin m\phi = 0$. Such solutions are constant values of ϕ. These can be seen in Figure 9.20 in the top diagonal and can be described as m circles passing through the poles, or longitudinal circles.

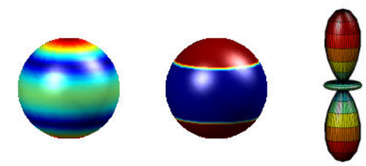


Figure 9.22: Zonal harmonics, $\ell = 2$, $m = 0$.

Tesseral harmonics are all of the rest, which typically look like a checker board glued to the surface of a sphere. Examples can be seen in the pictures of nodal curves, such as Figure 9.20. Looking in Figure 9.20 along the diagonals going downward from left to right, one can see the same number of latitudinal circles. In fact, there are $\ell - m$ latitudinal nodal curves in these figures

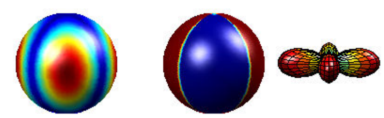


Figure 9.23: Sectoral harmonics, $\ell = 2$, $m = 2$.

In summary, the spherical harmonics have several representations, as show in Figures 9.17-9.20. Note that there are ℓ nodal lines, m meridional curves, and $\ell - m$ horizontal curves in these figures. The plots in Figures 9.17 and 9.19 are the typical plots shown in physics for discussion of the wavefunctions of the hydrogen atom. Those in 9.18 are useful for describing gravitational or electric potential functions, temperature distributions, or wave modes on a spherical surface. The relationships between these pictures and the nodal curves can be better understood by comparing respective plots. Several modes were separated out in Figures 9.21-9.26 to make this comparison easier.

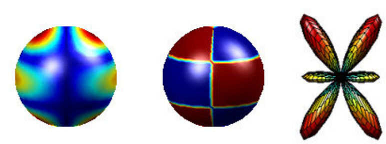


Figure 9.24: Tesseral harmonics, $\ell = 3$, $m = 1$.

Example 9.7. *Laplace's Equation with Azimuthal Symmetry*

As a simple example we consider the solution of Laplace's equation in which there is azimuthal symmetry. Let

$$u(r,\theta,\phi) = g(\theta) = 1 - \cos 2\theta.$$

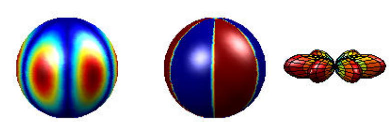


Figure 9.25: Sectoral harmonics, $\ell = 3$, $m = 3$.

This function is zero at the poles and has a maximum at the equator. So, this could be a crude model of the temperature distribution of the Earth with zero temperature at the poles and a maximum near the equator.

In problems in which there is no ϕ-dependence, only the $m = 0$ term of the general solution survives. Thus, we have that

$$u(\rho,\theta,\phi) = \sum_{\ell=0}^{\infty} a_\ell \rho^\ell P_\ell(\cos\theta). \tag{9.138}$$

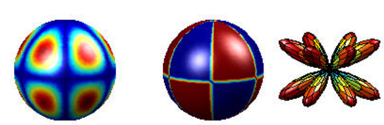


Figure 9.26: Tesseral harmonics, $\ell = 4$, $m = 3$.

Here we have used the fact that $P_\ell^0(x) = P_\ell(x)$. *We just need to determine the unknown expansion coefficients,* a_ℓ. *Imposing the boundary condition at* $\rho = r$, *we are lead to*

$$g(\theta) = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta). \tag{9.139}$$

This is a Fourier-Legendre series representation of $g(\theta)$. *Since the Legendre polynomials are an orthogonal set of eigenfunctions, we can extract the coefficients. In Chapter 5 we had proven that*

$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = \int_{-1}^1 P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}.$$

So, multiplying the expression for $g(\theta)$ *by* $P_m(\cos\theta)\sin\theta$ *and integrating, we obtain the expansion coefficients:*

$$a_\ell = \frac{2\ell+1}{2r^\ell}\int_0^\pi g(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta. \tag{9.140}$$

Sometimes it is easier to rewrite $g(\theta)$ *as a polynomial in* $\cos\theta$ *and avoid the integration. For this example we see that*

$$\begin{aligned} g(\theta) &= 1-\cos 2\theta \\ &= 2\sin^2\theta \\ &= 2-2\cos^2\theta. \end{aligned} \tag{9.141}$$

Thus, setting $x = \cos\theta$, *we have* $g(\theta) = 2-2x^2$. *We seek the form*

$$g(\theta) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x),$$

where $P_0(x) = 1$, $P_1(x) = x$, *and* $P_2(x) = \frac{1}{2}(3x^2-1)$. *Since* $g(\theta) = 2-2x^2$ *does not have any x terms, we know that* $c_1 = 0$. *So,*

$$2-2x^2 = c_0(1) + c_2\frac{1}{2}(3x^2-1) = c_0 - \frac{1}{2}c_2 + \frac{3}{2}c_2x^2.$$

By observation we have $c_2 = -\frac{4}{3}$ *and thus,* $c_0 = 2 + \frac{1}{2}c_2 = \frac{4}{3}$. *This gives the sought expansion for* $g(\theta)$:

$$g(\theta) = \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}P_2(\cos\theta). \tag{9.142}$$

Therefore, the nonzero coefficients in the general solution become

$$a_0 = \frac{4}{3}, \quad a_2 = \frac{4}{3}\frac{1}{r^2},$$

and the rest of the coefficients are zero. Inserting these into the general solution, we have

$$\begin{aligned} u(\rho,\theta,\phi) &= \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}\left(\frac{\rho}{r}\right)^2 P_2(\cos\theta) \\ &= \frac{4}{3} - \frac{2}{3}\left(\frac{\rho}{r}\right)^2(3\cos^2\theta - 1). \end{aligned} \tag{9.143}$$

9.6 Schrödinger Equation in Spherical Coordinates

Another important eigenvalue problem in physics is the Schrödinger equation. The time-dependent Schrödinger equation is given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \tag{9.144}$$

Here $\Psi(\mathbf{r},t)$ is the wave function, which is determines the quantum state of a particle of mass m subject to a (time independent) potential, $V(\mathbf{r})$. $\hbar = \frac{h}{2\pi}$, where h is Planck's constant. The probability of finding the particle in an infinitesimal volume, dV, is given by $|\Psi(\mathbf{r},t)|^2\,dV$, assuming the wave function is normalized,

$$\int_{\text{all space}} |\Psi(\mathbf{r},t)|^2\,dV = 1.$$

One can separate out the time dependence by assuming a special form, $\Psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-iEt/\hbar}$, where E is the energy of the particular stationary state solution, or product solution. Inserting this form into the time-dependent equation, one finds that $\psi(\mathbf{r})$ satisfies

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi. \tag{9.145}$$

Assuming that the potential depends only on distance from the origin, $V = V(\rho)$, we can further separate out the radial part of this solution using spherical coordinates. Recall that the Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{9.146}$$

Then, the time-independent Schrödinger equation can be written as

$$\begin{aligned} -&\frac{\hbar^2}{2m}\left[\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] \\ =& \; [E - V(\rho)]\psi. \end{aligned} \tag{9.147}$$

Let's continue with the separation of variables. Assuming that the wave function takes the form $\psi(\rho,\theta,\phi) = R(\rho)Y(\theta,\phi)$, we obtain

$$\begin{aligned} -&\frac{\hbar^2}{2m}\left[\frac{Y}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{R}{\rho^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right) + \frac{R}{\rho^2\sin^2\theta}\frac{d^2Y}{d\phi^2}\right] \\ =& \; RY[E - V(\rho)]\psi. \end{aligned} \tag{9.148}$$

Now dividing by $\psi = RY$, multiplying by $-\frac{2m\rho^2}{\hbar^2}$, and rearranging, we have

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \frac{2m\rho^2}{\hbar^2}\left[V(\rho) - E\right] = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2Y}{d\phi^2}\right].$$

We have a function of ρ equal to a function of the angular variables. So, we set each side equal to a constant. We will judiciously set the constant equal to $\ell(\ell+1)$. The resulting equations are

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)-\frac{2m\rho^2}{\hbar^2}\left[V(\rho)-E\right]R=\ell(\ell+1)R, \tag{9.149}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}=-\ell(\ell+1)Y. \tag{9.150}$$

The second of these equations should look familiar from the last section. This is the equation for spherical harmonics,

$$Y_{\ell m}(\theta,\phi)=\sqrt{\frac{2\ell+1}{2}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m e^{im\phi}. \tag{9.151}$$

So, any further analysis of the problem depends upon the choice of potential, $V(\rho)$, and the solution of the radial equation. For this, we turn to the determination of the wave function for an electron in orbit about a proton.

Example 9.8. *The Hydrogen Atom - $\ell=0$ States*

Historically, the first test of the Schrödinger equation was the determination of the energy levels in a hydrogen atom. This is modeled by an electron orbiting a proton. The potential energy is provided by the Coulomb potential,

$$V(\rho)=-\frac{e^2}{4\pi\epsilon_0\rho}.$$

Thus, the radial equation becomes

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{2m\rho^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0\rho}+E\right]R=\ell(\ell+1)R. \tag{9.152}$$

Before looking for solutions, we need to simplify the equation by absorbing some of the constants. One way to do this is to make an appropriate change of variables. Let $\rho=ar$. Then by the Chain Rule we have

$$\frac{d}{d\rho}=\frac{dr}{d\rho}\frac{d}{dr}=\frac{1}{a}\frac{d}{dr}.$$

Under this transformation, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right)+\frac{2ma^2r^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 ar}+E\right]u=\ell(\ell+1)u, \tag{9.153}$$

where $u(r)=R(\rho)$. Expanding the second term,

$$\frac{2ma^2r^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 ar}+E\right]u=\left[\frac{mae^2}{2\pi\epsilon_0\hbar^2}r+\frac{2mEa^2}{\hbar^2}r^2\right]u,$$

we see that we can define

$$a = \frac{2\pi\epsilon_0\hbar^2}{me^2}, \tag{9.154}$$

$$\begin{aligned} \epsilon &= -\frac{2mEa^2}{\hbar^2} \\ &= -\frac{2(2\pi\epsilon_0)^2\hbar^2}{me^4}E. \end{aligned} \tag{9.155}$$

Using these constants, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + ru - \ell(\ell+1)u = \epsilon r^2 u. \tag{9.156}$$

Expanding the derivative and dividing by r^2,

$$u'' + \frac{2}{r}u' + \frac{1}{r}u - \frac{\ell(\ell+1)}{r^2}u = \epsilon u. \tag{9.157}$$

The first two terms in this differential equation came from the Laplacian. The third term came from the Coulomb potential. The next term can be thought to contribute to the potential and is attributed to angular momentum. Thus, ℓ *is called the angular momentum quantum number. This is an eigenvalue problem for the radial eigenfunctions* $u(r)$ *and energy eigenvalues* ϵ.

The solutions of this equation are determined in a quantum mechanics course. In order to get a feeling for the solutions, we will consider the zero angular momentum case, $\ell = 0$:

$$u'' + \frac{2}{r}u' + \frac{1}{r}u = \epsilon u. \tag{9.158}$$

Even this equation is one we have not encountered in this book. Let's see if we can find some of the solutions.

First, we consider the behavior of the solutions for large r. *For large* r *the second and third terms on the left hand side of the equation are negligible. So, we have the approximate equation*

$$u'' - \epsilon u = 0. \tag{9.159}$$

The solutions thus behave like $u(r) = e^{\pm\sqrt{\epsilon}r}$. *For bounded solutions, we choose the decaying solution.*

This suggests that solutions take the form $u(r) = v(r)e^{-\sqrt{\epsilon}r}$ *for some unknown function,* $v(r)$. *Inserting this guess into Equation (9.158), gives an equation for* $v(r)$:

$$rv'' + 2\left(1 - \sqrt{\epsilon}r\right)v' + (1 - 2\sqrt{\epsilon})v = 0. \tag{9.160}$$

Next we seek a series solution to this equation. Let

$$v(r) = \sum_{k=0}^{\infty} c_k r^k.$$

Inserting this series into Equation (9.160), we have

$$\sum_{k=1}^{\infty}[k(k-1)+2k]c_k r^{k-1} + \sum_{k=1}^{\infty}[1-2\sqrt{\epsilon}(k+1)]c_k r^k = 0.$$

We can re-index the dummy variable in each sum. Let $k = m$ in the first sum and $k = m-1$ in the second sum. We then find that

$$\sum_{k=1}^{\infty}[m(m+1)c_m + [1-2m\sqrt{\epsilon}]c_{m-1}]r^{m-1} = 0.$$

Since this has to hold for all $m \geq 1$,

$$c_m = \frac{2m\sqrt{\epsilon}-1}{m(m+1)}c_{m-1}.$$

Further analysis indicates that the resulting series leads to unbounded solutions unless the series terminates. This is only possible if the numerator, $2m\sqrt{\epsilon}-1$, vanishes for $m = n$, $n = 1, 2\ldots$. Thus,

$$\epsilon = \frac{1}{4n^2}.$$

Since ϵ is related to the energy eigenvalue, E, we have

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2 n^2}.$$

Inserting the values for the constants, this gives

$$E_n = -\frac{13.6\ eV}{n^2}.$$

This is the well known set of energy levels for the hydrogen atom.

The corresponding eigenfunctions are polynomials, since the infinite series was forced to terminate. We could obtain these polynomials by iterating the recursion equation for the c_m's. However, we will instead rewrite the radial equation (9.160).

Let $x = 2\sqrt{\epsilon}r$ and define $y(x) = v(r)$. Then

$$\frac{d}{dr} = 2\sqrt{\epsilon}\frac{d}{dx}.$$

This gives

$$2\sqrt{\epsilon}xy'' + (2-x)2\sqrt{\epsilon}y' + (1-2\sqrt{\epsilon})y = 0.$$

Rearranging, we have

$$xy'' + (2-x)y' + \frac{1}{2\sqrt{\epsilon}}(1-2\sqrt{\epsilon})y = 0.$$

Noting that $2\sqrt{\epsilon} = \frac{1}{n}$, this becomes

$$xy'' + (2-x)y' + (n-1)y = 0. \tag{9.161}$$

The resulting equation is well known. It takes the form

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{9.162}$$

Solutions of this equation are the associated Laguerre polynomials. The solutions are denoted by $L_n^\alpha(x)$. They can be defined in terms of the Laguerre polynomials,

$$L_n(x) = e^x \left(\frac{d}{dx}\right)^n (e^{-x}x^n).$$

The associated Laguerre polynomials are named after the French mathematician Edmond Laguerre (1834-1886).

The associated Laguerre polynomials are defined as

$$L_{n-m}^m(x) = (-1)^m \left(\frac{d}{dx}\right)^m L_n(x).$$

Note: The Laguerre polynomials were first encountered in Problem 2 in Chapter 5 as an example of a classical orthogonal polynomial defined on $[0, \infty)$ with weight $w(x) = e^{-x}$. Some of these polynomials are listed in Table 9.3.

Comparing Equation (9.161) with Equation (9.162), we find that $y(x) = L_{n-1}^1(x)$.

	$L_n^m(x)$
$L_0^0(x)$	1
$L_1^0(x)$	$1-x$
$L_2^0(x)$	x^2-4x+2
$L_0^1(x)$	1
$L_1^1(x)$	$4-2x$
$L_2^1(x)$	$3x^2-18x+18$
$L_0^2(x)$	2
$L_1^2(x)$	$-6x+18$
$L_2^2(x)$	$12x^2-96x+144$
$L_0^3(x)$	6
$L_1^3(x)$	$-24x+96$
$L_2^3(x)$	$60x^2-600x+1200$

Table 9.3: Associated Laguerre Functions, $L_n^m(x)$.

In summary, we have made the following transformations:

1. $R(\rho) = u(r)$, $\rho = ar$.
2. $u(r) = v(r)e^{-\sqrt{\epsilon}r}$.
3. $v(r) = y(x) = L_{n-1}^1(x)$, $x = 2\sqrt{\epsilon}r$.

In most derivation in quantum mechanics $a = \frac{a_0}{2}$. where $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ is the Bohr radius and $a_0 = 5.2917 \times 10^{-11}$m.

Therefore,

$$R(\rho) = e^{-\sqrt{\epsilon}\rho/a} L_{n-1}^1(2\sqrt{\epsilon}\rho/a).$$

However, we also found that $2\sqrt{\epsilon} = 1/n$. So,

$$R(\rho) = e^{-\rho/2na} L_{n-1}^1(\rho/na).$$

For the general case, for all $\ell \geq 0$, we need to solve the differential equation

$$u'' + \frac{2}{r}u' + \frac{1}{r}u - \frac{\ell(\ell+1)}{r^2}u = \epsilon u. \tag{9.163}$$

Instead of letting $u(r) = v(r)e^{-\sqrt{\epsilon}r}$, we let

$$u(r) = v(r)r^{\ell}e^{-\sqrt{\epsilon}r}.$$

This lead to the differential equation

$$rv'' + 2(\ell + 1 - \sqrt{\epsilon}r)v' + (1 - 2(\ell+1)\sqrt{\epsilon})v = 0. \tag{9.164}$$

as before, we let $x = 2\sqrt{\epsilon}r$ to obtain

$$xy'' + 2\left[\ell + 1 - \frac{x}{2}\right]v' + \left[\frac{1}{2\sqrt{\epsilon}} - \ell(\ell+1)\right]v = 0.$$

Noting that $2\sqrt{\epsilon} = 1/n$, we have

$$xy'' + 2\left[2(\ell+1) - x\right]v' + (n - \ell(\ell+1))v = 0.$$

We see that this is once again in the form of the associate Laguerre equation and the solutions are

$$y(x) = L_{n-\ell-1}^{2\ell+1}(x).$$

So, the solution to the radial equation for the hydrogen atom is given by

$$\begin{aligned} R(\rho) &= r^{\ell}e^{-\sqrt{\epsilon}r}L_{n-\ell-1}^{2\ell+1}(2\sqrt{\epsilon}r) \\ &= \left(\frac{\rho}{2na}\right)^{\ell} e^{-\rho/2na}L_{n-\ell-1}^{2\ell+1}\left(\frac{\rho}{na}\right). \end{aligned} \tag{9.165}$$

Interpretations of these solutions will be left for your quantum mechanics course.

Problems

1. Consider Laplace's equation on the unit square, $u_{xx} + u_{yy} = 0$, $0 \leq x, y \leq 1$. Let $u(0, y) = 0, u(1, y) = 0$ for $0 < y < 1$ and $u_y(x, 0) = 0$ for $0 < y < 1$. Carry out the needed separation of variables and write down the product solutions satisfying these boundary conditions.

2. Consider a cylinder of height H and radius a.

a. Write down LaplaceŠs Equation for this cylinder in cylindrical coordinates.

b. Carry out the separation of variables and obtain the three ordinary differential equations that result from this problem.

c. What kind of boundary conditions could be satisfied in this problem in the independent variables?

3. Consider a square drum of side s and a circular drum of radius a.

a. Rank the modes corresponding to the first 6 frequencies for each.

b. Write each frequency (in Hz) in terms of the fundamental (i.e., the lowest frequency.)

c. What would the lengths of the sides of the square drum have to be to have the same fundamental frequency? (Assume that $c = 1.0$ for each one.)

4. A copper cube 10.0 cm on a side is heated to 100° C. The block is placed on a surface that is kept at 0° C. The sides of the block are insulated, so the normal derivatives on the sides are zero. Heat flows from the top of the block to the air governed by the gradient $u_z = -10^\circ\mathrm{C/m}$. Determine the temperature of the block at its center after 1.0 minutes. Note that the thermal diffusivity is given by $k = \frac{K}{\rho c_p}$, where K is the thermal conductivity, ρ is the density, and c_p is the specific heat capacity.

5. Consider a spherical balloon of radius a. Small deformations on the surface can produce waves on the balloon's surface.

a. Write the wave equation in spherical polar coordinates. (Note: ρ is constant!)

b. Carry out a separation of variables and find the product solutions for this problem.

c. Describe the nodal curves for the first six modes.

d. For each mode determine the frequency of oscillation in Hz assuming $c = 1.0$ m/s.

6. Consider a circular cylinder of radius $R = 4.00$ cm and height $H = 20.0$ cm which obeys the steady state heat equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz}.$$

Find the temperature distribution, $u(r, z)$, given that $u(r, 0) = 0$, $u(r, 20) = 20$, and heat is lost through the sides due to Newton's Law of Cooling

$$[u_r + hu]_{r=4} = 0,$$

for $h = 1.0\ \mathrm{cm}^{-1}$.

A

Review of Sequences and Infinite Series

"Once you eliminate the impossible, whatever remains, no matter how improbable, must be the truth." Sherlock Holmes (by Sir Arthur Conan Doyle, 1859-1930)

In this chapter we will review and extend some of the concepts and definitions related to infinite series that you might have seen previously in your calculus class ([1], [2], [3]). Working with infinite series can be a little tricky and we need to understand some of the basics before moving on to the study of series of trigonometric functions in the next chapter.

The material in this chapter is a review of material covered in a standard course in calculus with some additional notions from advanced calculus. It is provided as a review before encountering the notion of Fourier series and their convergence as seen in the next chapter.

[1]

[2]

[3]

For example, one can show that the infinite series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges to $\ln 2$. However, the terms can be rearranged to give

$$1 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{11} - \frac{1}{6} + \frac{1}{13}\right) + \cdots = \frac{3}{2}\ln 2.$$

In fact, other rearrangements can be made to give any desired sum!

As we will see, $\ln(1+x) = x - \frac{x}{2} + \frac{x}{3} - \ldots$. So, inserting $x = 1$ yields the first result - at least formally! It was shown in Cowen, Davidson and Kaufman (in *The American Mathematical Monthly*, Vol. 87, No. 10. (Dec., 1980), pp. 817-819) that expressions like

$$\begin{aligned} f(x) &= \frac{1}{2}\left[\ln\frac{1+x}{1-x} + \ln(1-x^4)\right] \\ &= \frac{1}{2}\ln\left[(1+x)^2(1+x^2)\right] \end{aligned}$$

lead to alternate sums of the rearrangement of the alternating harmonic series.

Other problems with infinite series can occur. Try to sum the following infinite series to find that

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^2} \sim 0.937548\ldots.$$

A sum of even as many as 10^7 terms only gives convergence to four or five decimal places.

The series

$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \cdots, \quad x > 0$$

diverges for all x. So, you might think this divergent series is useless. However, truncation of this divergent series leads to an approximation of the integral

$$\int_0^{\infty} \frac{e^{-t}}{x+t}\, dt, \quad x > 0.$$

So, can we make sense out of any of these, or other manipulations, of infinite series? We will not answer these questions now, but we will go back and review what you have seen in your calculus classes.

A.1 Sequences of Real Numbers

WE FIRST BEGIN with the definitions for sequences and series of numbers.

Definition A.1. A *sequence* is a function whose domain is the set of positive integers, $a(n)$, $n \in N$ $[N = \{1, 2, \ldots .\}]$.

Examples are

1. $a(n) = n$ yields the sequence $\{1, 2, 3, 4, 5, \ldots\}$
2. $a(n) = 3n$ yields the sequence $\{3, 6, 9, 12, \ldots\}$

However, one typically uses subscript notation and not functional notation: $a_n = a(n)$. We then call a_n the nth term of the sequence. Furthermore, we will denote sequences by $\{a_n\}_{n=1}^{\infty}$. Sometimes we will only give the nth term of the sequence and will assume that $n \in N$ unless otherwise noted.

Another way to define a particular sequence is recursively.

Definition A.2. A *recursive sequence* is defined in two steps:

1. The value of first term (or first few terms) is given.
2. A rule, or recursion formula, to determine later terms from earlier ones is given.

Example A.1. *A typical example is given by the Fibonacci*[4] *sequence. It can be defined by the recursion formula $a_{n+1} = a_n + a_{n-1}$, $n \geq 2$ and the starting values of $a_1 = 0$ and $a_1 = 1$. The resulting sequence is $\{a_n\}_{n=1}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$. Writing the general expression for the nth term is possible, but it is not as simply stated. Recursive definitions are often useful in doing computations for large values of n.*

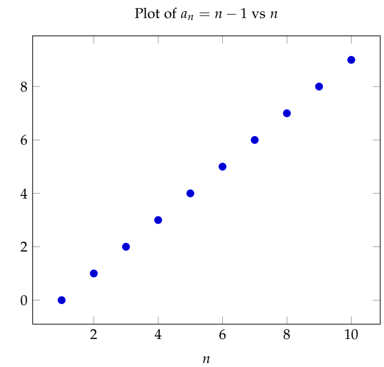


Figure A.1: Plot of $a_n = n - 1$ for $n = 1 \ldots 10$.

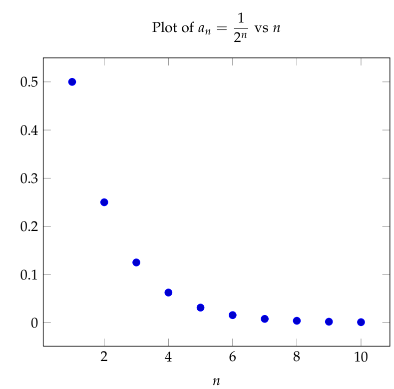


Figure A.2: Plot of $a_n = \dfrac{1}{2^n}$ for $n = 1 \ldots 10$.

[4] Leonardo Pisano Fibonacci (c.1170-c.1250) is best known for this sequence of numbers. This sequence is the solution of a problem in one of his books: *A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive* http://www-history.mcs.st-and.ac.uk

A.2 Convergence of Sequences

NEXT WE ARE INTERESTED in the behavior of the sequence as n gets large. For the sequence defined by $a_n = n - 1$, we find the behavior as shown in Figure A.1. Notice that as n gets large, a_n also gets large. This sequence is said to be divergent.

On the other hand, the sequence defined by $a_n = \frac{1}{2^n}$ approaches a limit as n gets large. This is depicted in Figure A.2. Another related series, $a_n = \frac{(-1)^n}{2^n}$, is shown in Figure A.3. This sequence is the alternating sequence $\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\}$.

Definition A.3. The sequence a_n *converges* to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \Rightarrow |a - L| < \epsilon.$$

If no such number exists, then the sequence is said to *diverge*.

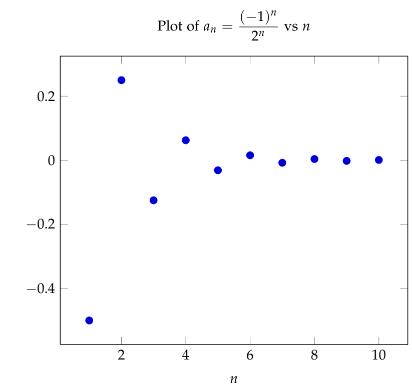


Figure A.3: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$.

In Figures A.4-A.5 we see what this means. For the sequence given by $a_n = \frac{(-1)^n}{2^n}$, we see that the terms approach $L = 0$. Given an $\epsilon > 0$, we ask for what value of N the nth terms $(n > N)$ lie in the interval $[L - \epsilon, L + \epsilon]$. In these figures this interval is depicted by a horizontal band. We see that for convergence, sooner, or later, the tail of the sequence ends up entirely within this band.

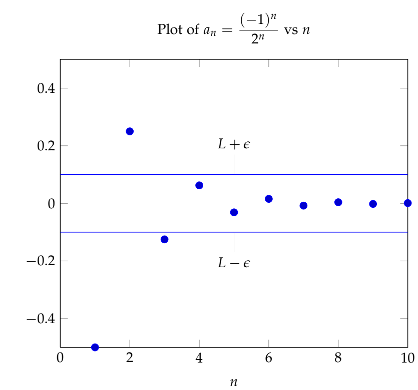


Figure A.4: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$. Picking $\epsilon = 0.1$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 3$.

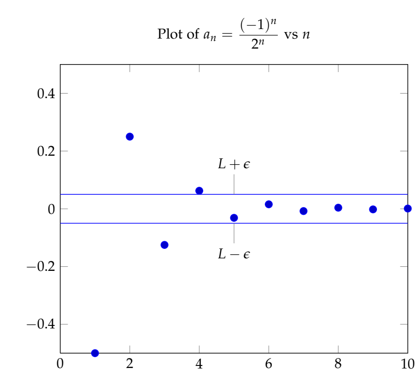


Figure A.5: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$. Picking $\epsilon = 0.05$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 4$.

If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit L, then we write either $a_n \to L$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = L$. For example, we have already seen in Figure A.3 that $\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$.

A.3 Limit Theorems

Once we have defined the notion of convergence of a sequence to some limit, then we can investigate the properties of the limits of sequences. Here we list a few general limit theorems and some special limits, which arise often.

Limit Theorem

Theorem A.1. *Consider two convergent sequences $\{a_n\}$ and $\{b_n\}$ and a number k. Assume that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then we have*

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B.$
2. $\lim_{n\to\infty}(kb_n) = kB.$
3. $\lim_{n\to\infty}(a_n b_n) = AB.$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}, \quad B \neq 0.$

Some special limits are given next. These are generally first encountered in a second course in calculus.

Special Limits

Theorem A.2. *The following are special cases:*

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$
2. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$
3. $\lim_{n\to\infty} x^{\frac{1}{n}} = 1, \quad x > 0.$
4. $\lim_{n\to\infty} x^n = 0, \quad |x| < 1.$
5. $\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x.$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$

The proofs generally are straight forward and found in beginning calculus texts. For example, one can prove the first limit by first realizing that $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x}$. This limit is indeterminate as $x \to \infty$ in its current form since the numerator and the denominator get large for large x. In such cases one employs L'Hopital's Rule: One computes

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

The second limit in Theorem A.2 can be proven by first looking at

$$\lim_{n\to\infty} \ln n^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Now, if $\lim_{n\to\infty} \ln f(n) = 0$, then $\lim_{n\to\infty} f(n) = e^0 = 1$. Thus proving the second limit.[5]

The third limit can be done similarly. The reader is left to confirm the other limits. We finish this section with a few selected examples.

L'Hopital's Rule is used often in computing limits. We recall this powerful rule here as a reference for the reader.

Theorem A.3. *Let c be a finite number or $c = \infty$. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

If $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

[5] We should note that we are assuming something about limits of composite functions. Let a and b be real numbers. Suppose f and g are continuous functions, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to b} g(x) = b$, and $g(b) = a$. Then, $\lim_{x\to b} f(g(x)) = f(\lim_{x\to b} g(x)) = f(g(b)) = f(a)$.

Example A.2. $\lim_{n\to\infty} \frac{n^2+2n+3}{n^3+n}$

Divide the numerator and denominator by n^2. Then

$$\lim_{n\to\infty} \frac{n^2 + 2n + 3}{n^3 + n} = \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{3}{n^2}}{n + \frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Another approach to this example is to consider the behavior of the numerator and denominator as $n \to \infty$. As n gets large, the numerator behaves like n^2, since $2n + 3$ becomes negligible for large enough n. Similarly, the denominator behaves like n^3 for large n. Thus,

$$\lim_{n\to\infty} \frac{n^2 + 2n + 3}{n^3 + n} = \lim_{n\to\infty} \frac{n^2}{n^3} = 0.$$

Example A.3. $\lim_{n\to\infty} \frac{\ln n^2}{n}$

Rewriting $\frac{\ln n^2}{n} = \frac{2 \ln n}{n}$, we find from identity 1 of the Theorem A.2 that

$$\lim_{n\to\infty} \frac{\ln n^2}{n} = 2 \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Example A.4. $\lim_{n\to\infty}(n^2)^{\frac{1}{n}}$

To compute this limit, we rewrite

$$\lim_{n\to\infty}(n^2)^{\frac{1}{n}} = \lim_{n\to\infty}(n)^{\frac{1}{n}}(n)^{\frac{1}{n}} = 1,$$

using identity 2 of the Theorem A.2.

Example A.5. $\lim_{n\to\infty}(\frac{n-2}{n})^n$

This limit can be written as

$$\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{(-2)}{n}\right)^n = e^{-2}.$$

Here we used identity 5 of the Theorem A.2.

A.4 Infinite Series

In this section we investigate the meaning of infinite series, which are infinite sums of the form

$$a_1 + a_2 + a_2 + \ldots. \tag{A.1}$$

There is story that's described in E.T. Bell's "Men of Mathematics" about Carl Friedrich Gauß (1777-1855). Gauß' third grade teacher needed to occupy the students, so she asked the class to sum the first 100 integers thinking that this would occupy the students for a while. However, Gauß was able to do so in practically no time. He recognized the sum could be written as $(1+100)+(2+99)+\ldots(50+51) = 50(101)$. $\sum_{k=1}^n k = \frac{n(n+1)}{2}$. This is an example of an arithmetic progression which is a finite sum of terms.

A typical example is the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots. \tag{A.2}$$

How would one evaluate this sum? We begin by just adding the terms. For example,

$$\begin{aligned} 1+\frac{1}{2} &= \frac{3}{2}, \\ 1+\frac{1}{2}+\frac{1}{4} &= \frac{7}{4}, \\ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} &= \frac{15}{8}, \\ 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} &= \frac{31}{16}, \end{aligned} \tag{A.3}$$

etc. The values tend to a limit. We can see this graphically in Figure A.6.

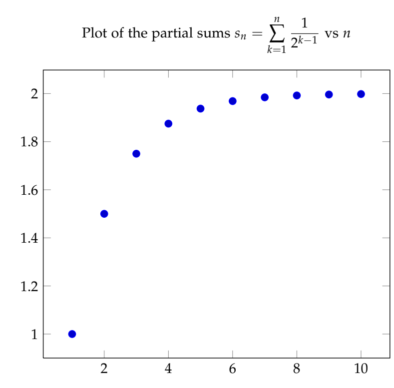


Figure A.6: Plot of $s_n = \sum_{k=1}^n \frac{1}{2^{k-1}}$ for $n = 1\ldots10$.

In general, we want to make sense out of Equation (A.1). As with the example, we look at a sequence of partial sums . Thus, we consider the sums

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ s_4 &= a_1 + a_2 + a_3 + a_4, \end{aligned} \tag{A.4}$$

etc. In general, we define the nth partial sum as

$$s_n = a_1 + a_2 + \ldots + a_n.$$

If the infinite series (A.1) is to make any sense, then the sequence of partial sums should converge to some limit. We define this limit to be the sum of the infinite series, $S = \lim_{n\to\infty} s_n$.

Definition A.4. If the sequence of partial sums converges to the limit L as n gets large, then the infinite series is said to have the sum L.

We will use the compact summation notation

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots .$$

Here n will be referred to as the index and it may start at values other than $n = 1$.

A.5 Convergence Tests

GIVEN A GENERAL INFINITE SERIES, it would be nice to know if it converges, or not. Often, we are only interested in the convergence and not the actual sum as it is often difficult to determine the sum even if the series converges. In this section we will review some of the standard tests for convergence, which you should have seen in Calculus II.

First, we have the nth term divergence test. This is motivated by two examples:

1. $\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \ldots .$
2. $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots .$

In the first example it is easy to see that each term is getting larger and larger, and thus the partial sums will grow without bound. In the second case, each term is bigger than one. Thus, the series will be bigger than adding the same number of ones as there are terms in the sum. Obviously, this series will also diverge.

The nth Term Divergence Test.

This leads to the nth Term Divergence Test:

Theorem A.4. *If* $\lim a_n \neq 0$ *or if this limit does not exist, then* $\sum_n a_n$ *diverges.*

This theorem does not imply that just because the terms are getting smaller, the series will converge. Otherwise, we would not need any other convergence theorems.

For the next theorems, we will assume that the series has nonnegative terms.

1. **Comparison Test**

The Comparison Test.

The series $\sum a_n$ converges if there is a convergent series $\sum c_n$ such that $a_n \leq c_n$ for all $n > N$ for some N. The series $\sum a_n$ diverges if there is a divergent series $\sum d_n$ such that $d_n \leq a_n$ for all $n > N$ for some N.

This is easily seen. In the first case, we have

$$a_n \leq c_n, \forall n > N.$$

Summing both sides of the inequality, we have

$$\sum_n a_n \leq \sum_n c_n.$$

If $\sum c_n$ converges, $\sum c_n < \infty$, the $\sum a_n$ converges as well. A similar argument applies for the divergent series case.

For this test one has to dream up a second series for comparison. Typically, this requires some experience with convergent series. Often it is better to use other tests first if possible.

Example A.6. $\sum_{n=0}^{\infty} \frac{1}{3^n}$

We already know that $\sum_{n=0}^{\infty} \frac{1}{2^n}$ *converges. So, we compare these two series. In the above notation, we have* $a_n = \frac{1}{3^n}$ *and* $c_n = \frac{1}{2^n}$. *Since* $\frac{1}{2^n} \leq \frac{1}{3^n}$ *for* $n \geq 0$ *and* $\sum_{n=0}^{\infty} \frac{1}{2^n}$ *converges, then* $\sum_{n=0}^{\infty} \frac{1}{3^n}$ *converges by the Comparison Test.*

2. **Limit Comparison Test**

The Limit comparison Test.

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite, then $\sum a_n$ and $\sum b_n$ converge together or diverge together.

Example A.7. $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

In order to establish the convergence, or divergence, of this series, we look to see how the terms, $a_n = \frac{2n+1}{(n+1)^2}$, *behave for large* n. *As* n *gets large, the numerator behaves like* $2n$ *and the denominator behaves like* n^2. *So,* a_n *behaves like* $\frac{2n}{n^2} = \frac{2}{n}$. *The factor of* 2 *does not really matter. So, will compare the infinite series* $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ *with* $\sum_{n=1}^{\infty} \frac{1}{n}$. *Then,* $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{(n+1)^2} = 2$. *Thus, these two series both converge, or both diverge. If we knew the behavior of the second series, then we could draw a conclusion. Using the next test, we will prove that* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges, therefore* $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ *diverges by the Limit Comparison Test. Another example of this test is given in Example A.9.*

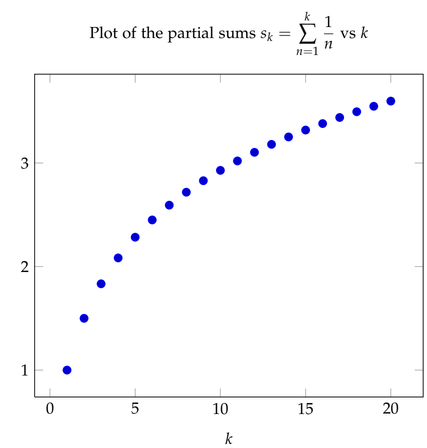


Figure A.7: Plot of the partial sums, $s_k = \sum_{n=1}^{k} \frac{1}{n}$, for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

3. **Integral Test**

The Integral Test.

Consider the infinite series $\sum_{n=1}^{\infty} a_n$, where $a_n = f(n)$. Then, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge. Here we mean that the integral converges or diverges as an improper integral.

Example A.8. *The harmonic series:* $\sum_{n=1}^{\infty} \frac{1}{n}$

We are interested in the convergence or divergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ *which we saw in the Limit Comparison Test example. This infinite series is famous and is called the harmonic series. The plot of the partial sums is given in Figure A.7. It appears that the series could possibly converge or diverge. It is hard to tell graphically.*

In this case we can use the Integral Test. In Figure A.8 we plot $f(x) = \frac{1}{x}$ *and at each integer* n *we plot a box from* n *to* $n+1$ *of height* $\frac{1}{n}$. *We can see from the figure that the total area of the boxes is greater than the area under the curve. Since the area of each box is* $\frac{1}{n}$, *then we have that*

$$\int_1^{\infty} \frac{dx}{x} < \sum_{n=1}^{\infty} \frac{1}{n}.$$

But, we can compute the integral.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{x \to \infty} (\ln x) = \infty.$$

Thus, the integral diverges. However, the infinite series is larger than this! So, the harmonic series diverges by the Integral Test.

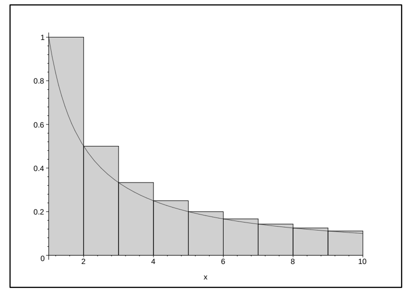


Figure A.8: Plot of $f(x) = x$ and boxes of height $\frac{1}{n}$ and width 1.

The Integral Test provides us with the convergence behavior for a class of infinite series called a p-series . These series are of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Recalling that the improper integrals $\int_1^{\infty} \frac{dx}{x^p}$ converge for $p > 1$ and diverge otherwise, we have the p-test :

p-series and p-test.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1$$

and diverges otherwise.

Example A.9. $\sum_{n=1}^{\infty} \frac{n+1}{n^3-2}$.

We first note that as n *gets large, the general term behaves like* $\frac{1}{n^2}$ *since the numerator behaves like* n *and the denominator behaves like* n^3. *So, we expect that this series behaves like the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *Thus, by the Limit Comparison Test,*

$$\lim_{n \to \infty} \frac{n+1}{n^3-2}(n^2) = 1.$$

These series both converge, or both diverge. However, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges by the* p*-test since* $p = 2$. *Therefore, the original series converges.*

4. **Ratio Test**

The Ratio Test.

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$. Then the behavior of the infinite series can be determined from the conditions

$$\begin{aligned} \rho < 1, &\quad \text{converges} \\ \rho > 1, &\quad \text{diverges} \end{aligned}$$

Example A.10. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$.

We compute

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(n+1)^{10}}{n^{10}} \frac{10^n}{10^{n+1}} \\
&= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} \frac{1}{10} \\
&= \frac{1}{10} < 1.
\end{aligned} \tag{A.5}$$

Therefore, the series is said to converge by the Ratio Test.

Example A.11. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

In this case we make use of the fact that[6] $(n+1)! = (n+1)n!$. *We compute*

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1
\end{aligned} \tag{A.6}$$

This series also converges by the Ratio Test.

[6] Note that the Ratio Test works when factorials are involved because using $(n+1)! = (n+1)n!$ helps to reduce the needed ratios into something manageable.

5. ***n*th Root Test**

The nth Root Test.

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n\to\infty} {a_n}^{1/n}$. Then the behavior of the infinite series can be determined using

$$\begin{aligned}
\rho < 1, &\quad \text{converges} \\
\rho > 1, &\quad \text{diverges}
\end{aligned}$$

Example A.12. $\sum_{n=0}^{\infty} e^{-n}$.

We use the nth Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{-1} = e^{-1} < 1$. *Thus, this series converges by the nth Root Test.*[7]

[7] Note that the Root Test works when there are no factorials and simple powers are involved. In such cases special limit rules help in the evaluation.

Example A.13. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$.

This series also converges by the nth Root Test.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n^n}{2^{n^2}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{2^n} = 0 < 1.$$

We next turn to series which have both positive and negative terms. We can toss out the signs by taking absolute values of each of the terms. We note that since $a_n \le |a_n|$, we have

$$-\sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} |a_n|.$$

If the sum $\sum_{n=1}^{\infty}|a_n|$converges, then the original series converges. This type of convergence is useful, because we can use the previous tests to establish convergence of such series. Thus, we say that a series *converges absolutely* if $\sum_{n=1}^{\infty}|a_n|$ converges. If a series converges, but does not converge absolutely, then it is said to *converge conditionally.*

Conditional and absolute convergence.

Example A.14. $\sum_{n=1}^{\infty}\frac{\cos \pi n}{n^2}$. *This series converges absolutely because* $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ *is a p-series with* $p = 2$.

Finally, there is one last test that we recall from your introductory calculus class. We consider the alternating series, given by $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. The convergence of an alternating series is determined from **Leibniz's Theorem**[8] .

Convergence of alternating series.

[8] Gottfried Wilhelm Leibniz (1646-1716) developed calculus independently of Sir Isaac Newton (1643-1727).

Theorem A.5. *The series* $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *converges if*

1. a_n*'s are positive.*
2. $a_n \geq a_{n+1}$ *for all* n.
3. $a_n \to 0$.

The first condition guarantees that we have alternating signs in the series. The next condition says that the magnitude if the terms gets smaller and the last condition imposes further that the terms approach zero.

Example A.15. *The alternating harmonic series:* $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$.

First of all, this series is an alternating series. The a_n's in Leibniz's Theorem are given by $a_n = \frac{1}{n}$. Condition 2 for this case is

$$\frac{1}{n} \geq \frac{1}{n+1}.$$

This is certainly true, as condition 2 just means that the terms are not getting bigger as n increases. Finally, condition 3 says that the terms are in fact going to zero as n increases. This is true in this example. Therefore, the the alternating harmonic series converges by Leibniz's Theorem. Note: The alternating harmonic series converges conditionally, since $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ gives the (divergent) harmonic series. So, the alternating harmonic series does not converge absolutely.

Example A.16. $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}$ *also passes the conditions of Leibniz's Theorem. It should be clear that the terms of this alternating series are getting smaller and approach zero. Furthermore, this series converges absolutely!*

A.6 Sequences of Functions

OUR IMMEDIATE GOAL is to prepare for studying Fourier series, which are series whose terms are functions. So, in this section we begin to discuss series of functions and the convergence of such series. Once more we will need to resort to the convergence of the sequence of partial sums. This means we really need to start with sequences of functions.

Definition A.5. A *sequence of functions* is simply a set of functions $f_n(x)$, $n = 1, 2, \ldots$ defined on a common domain D. A frequently used example will be the sequence of functions $\{1, x, x^2, \ldots\}$, $x \in [-1, 1]$.

Evaluating each sequences of functions at a given value of x, we obtain a sequence of real numbers. As before, we can ask if this sequence converges. Doing this for each point in the domain D, we then ask if the resulting collection of limits defines a function on D. More formally, this leads us to the idea of pointwise convergence.

Definition A.6. A sequence of functions f_n *converges pointwise* on D to a limit g if

Pointwise convergence.

$$\lim_{n\to\infty} f_n(x) = g(x)$$

for each $x \in D$. More formally, we write that

$$\lim_{n\to\infty} f_n = g \text{ (pointwise on } D)$$

if given $x \in D$ and $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N.$$

Example A.17. *Consider the sequence of functions*

$$f_n(x) = \frac{1}{1+nx}, \quad |x| < \infty, \quad n = 1, 2, 3, \ldots.$$

The limits depends on the value of x. We consider two cases, $x = 0$ and $x \neq 0$.

1. $x = 0$. *Here* $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1 = 1$.
2. $x \neq 0$. *Here* $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = 0$.

Therefore, we can say that $f_n \to g$ pointwise for $|x| < \infty$, where

$$g(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{A.7}$$

We also note that in Definition A.6 N generally depends on both x and ϵ.

Example A.18. *We consider the functions $f_n(x) = x^n, x \in [0,1], n = 1, 2, \ldots$. We recall that the definition for pointwise convergence suggests that for each x we seek an N such that $|f_n(x) - g(x)| < \epsilon, \forall n \geq N$. This is not at first easy to see. So, we will provide some simple examples showing how N can depend on both x and ϵ.*

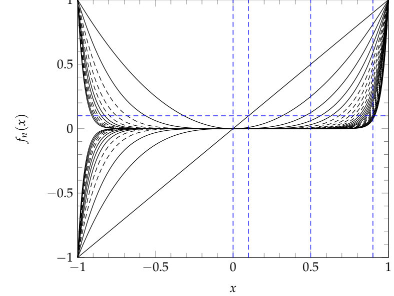


Figure A.9: Plot of $f_n(x) = x^n$ showing how N depends on $x = 0, 0.1, 0.5, 0.9$ (the vertical lines) and $\epsilon = 0.1$ (the horizontal line). Look at the intersection of a given vertical line with the horizontal line and determine N from the number of curves not under the intersection point.

1. *$x = 0$. Here we have $f_n(0) = 0$ for all n. So, given $\epsilon > 0$ we seek an N such that $|f_n(0) - 0| < \epsilon, \forall n \geq N$. Inserting $f_n(0) = 0$, we have $0 < \epsilon$. Since this is true for all n, we can pick $N = 1$.*

2. *$x = \frac{1}{2}$. In this case we have $f_n(\frac{1}{2}) = \frac{1}{2^n}$, for $n = 1, 2, \ldots$. As n gets large, $f_n \to 0$. So, given $\epsilon > 0$, we seek N such that $|\frac{1}{2^n} - 0| < \epsilon, \quad \forall n \geq N$. This means that $\frac{1}{2^n} < \epsilon$. Solving the inequality for n, we have $n > -\frac{\ln \epsilon}{\ln 2}$ We choose $N \geq -\frac{\ln \epsilon}{\ln 2}$. Thus, our choice of N depends on ϵ. For, $\epsilon = 0.1$, this gives*

$$N \geq -\frac{\ln 0.1}{\ln 2} = \frac{\ln 10}{\ln 2} \approx 3.32.$$

So, we pick $N = 4$ and we have $n > N = 4$.

3. *$x = \frac{1}{10}$. This can be examined like the last example. We have $f_n(\frac{1}{10}) = \frac{1}{10^n}$, for $n = 1, 2, \ldots$. This leads to $N \geq -\frac{\ln \epsilon}{\ln 10}$ For $\epsilon = 0.1$, this gives $N \geq 1$, or $n > 1$.*

4. *$x = \frac{9}{10}$. This can be examined like the last two examples. We have $f_n(\frac{9}{10}) = \left(\frac{9}{10}\right)^n$, for $n = 1, 2, \ldots$. So given an $\epsilon > 0$, we seek an N such that $\left(\frac{9}{10}\right)^n < \epsilon$ for all $n > N$. Therefore,*

$$n > N \geq \frac{\ln \epsilon}{\ln\left(\frac{9}{10}\right)}.$$

For $\epsilon = 0.1$, we have $N \geq 21.85$, or $n > N = 22$.

So, for these cases, we have shown that N can depend on both x and ϵ. These cases are shown in Figure A.9.

There are other questions that can be asked about sequences of functions. Let the sequence of functions f_n be continuous on D. If the sequence of functions converges pointwise to g on D then we can ask the following.

1. Is g continuous on D?

2. If each f_n is integrable on $[a, b]$, then does

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b g(x)\, dx?$$

3. If each f_n is differentiable at c, then does

$$\lim_{n \to \infty} f_n'(c) = g'(c)?$$

It turns out that pointwise convergence is not enough to provide an affirmative answer to any of these questions. Though we will not prove it here, what we will need is uniform convergence.

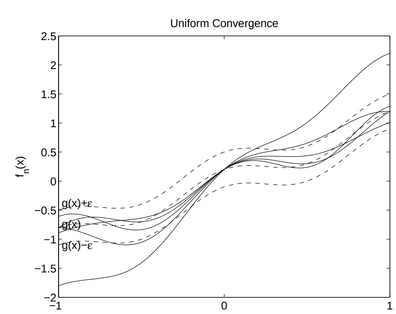


Figure A.10: For uniform convergence, as n gets large, $f_n(x)$ lies in the band $g(x) - \epsilon, g(x) - \epsilon$.

Definition A.7. Consider a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ on D. Let $g(x)$ be defined for $x \in D$. Then the sequence *converges uniformly* on D, or

$$\lim_{n\to\infty} f_n = g \text{ uniformly on } D,$$

if given $\epsilon > 0$, there exists an N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N \text{ and } \forall x \in D.$$

This definition almost looks like the definition for pointwise convergence. However, the seemingly subtle difference lies in the fact that N does not depend upon x. The sought N works for all x in the domain. As seen in Figure A.10 as n gets large, $f_n(x)$ lies in the band $(g(x) - \epsilon, g(x) + \epsilon)$.

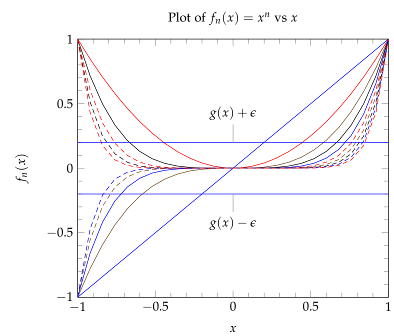


Figure A.11: Plot of $f_n(x) = x^n$ on [-1,1] for $n = 1 \ldots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

Example A.19. $f_n(x) = x^n$*, for* $x \in [0,1]$.
Note that in this case as n gets large, $f_n(x)$ *does not lie in the band* $(g(x) - \epsilon, g(x) + \epsilon)$*. This is displayed in Figure A.11.*

Example A.20. $f_n(x) = \cos(nx)/n^2$ *on* $[-1,1]$.
For this example we plot the first several members of the sequence in Figure A.12. We can see that eventually $(n \geq N)$ *members of this sequence do lie inside a band of width* ϵ *about the limit* $g(x) \equiv 0$ *for all values of* x*. Thus, this sequence of functions will converge uniformly to the limit.*

Finally, we should note that if a sequence of functions is uniformly convergent then it converges pointwise. However, the examples should bear out that the converse is not true.

A.7 Infinite Series of Functions

We now turn our attention to infinite series of functions, which will form the basis of our study of Fourier series. An infinite *series of functions* is given by $\sum_{n=1}^{\infty} f_n(x), \quad x \in D$. Using powers of x again, an example would be $\sum_{n=1}^{\infty} x^n, \quad x \in [-1,1]$. In order to investigate the convergence of this series; i.e., we would substitute values for x and determine if the resulting series of numbers converges. This means that we would need to consider the Nth partial sums

$$s_N(x) = \sum_{n=1}^{N} f_n(x).$$

Does this sequence of functions converge? We begin to answer this question by defining pointwise and uniform convergence.

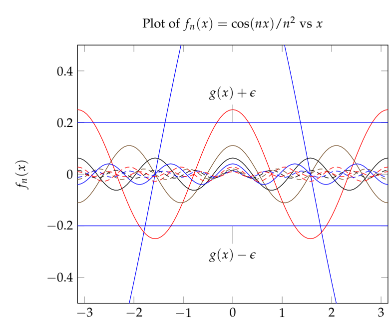


Figure A.12: Plot of $f_n(x) = \cos(nx)/n^2$ on $[-\pi, \pi]$ for $n = 1 \ldots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

Pointwise convergence.

Definition A.8. $\sum f_j(x)$ *converges pointwise* to $f(x)$ on D if given $x \in D$, and $\epsilon > 0$, there exists and N such that

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$.

Definition A.9. $\sum f_j(x)$ *converges uniformly* to $f(x)$ on D given $\epsilon > 0$, there exists and N such that

Uniform convergence.

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$ and all $x \in D$.

Again, we state without proof the following:

1. Uniform convergence implies pointwise convergence.
2. If f_n is continuous on D, and $\sum_n^\infty f_n$ converges uniformly to f on D, then f is continuous on D.
3. If f_n is continuous on $[a, b] \subset D$, $\sum_n^\infty f_n$ converges uniformly on D, and $\int_a^b f_n(x)\,dx$ exists, then
$$\sum_n^\infty \int_a^b f_n(x)\,dx = \int_a^b \sum_n^\infty f_n(x)\,dx = \int_a^b g(x)\,dx.$$
4. If f_n' is continuous on $[a, b] \subset D$, $\sum_n^\infty f_n$ converges pointwise to g on D, and $\sum_n^\infty f_n'$ converges uniformly on D, then $\sum_n^\infty f_n'(x) = \frac{d}{dx}(\sum_n^\infty f_n(x)) = g'(x)$ for $x \in (a, b)$.

Uniform convergence give nice properties under some additional conditions, such as being able to integrate, or differentiate, term by term.

Since uniform convergence of series gives so much, like term by term integration and differentiation, we would like to be able to recognize when we have a uniformly convergent series. One test for such convergence is the **Weierstraß M-Test**[9].

[9] Karl Theodor Wilhelm Weierstraß (1815-1897) was a German mathematician who may be thought of as the father of analysis.

Theorem A.6. Weierstraß M-Test *Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions on D. If $|f_n(x)| \le M_n$, for $x \in D$ and $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty f_n$ converges uniformly of D.*

Proof. First, we note that for $x \in D$,

$$\sum_{n=1}^\infty |f_n(x)| \le \sum_{n=1}^\infty M_n.$$

Thus, since by the assumption that $\sum_{n=1}^\infty M_n$ converges, we have that $\sum_{n=1}^\infty f_n$ converges absolutely on D. Therefore, $\sum_{n=1}^\infty f_n$ converges pointwise on D. So, let $\sum_{n=1}^\infty f_n = g$.

We now want to prove that this convergence is in fact uniform. Given $\epsilon > 0$, we need to find an N such that

$$|g(x) - \sum_{j=1}^n f_j(x)| < \epsilon$$

if $n \geq N$ for all $x \in D$.

So, for any $x \in D$,

$$\begin{aligned}
|g(x) - \sum_{j=1}^{n} f_j(x)| &= |\sum_{j=1}^{\infty} f_j(x) - \sum_{j=1}^{n} f_j(x)| \\
&= |\sum_{j=n+1}^{\infty} f_j(x)| \\
&\leq \sum_{j=n+1}^{\infty} |f_j(x)|, \quad \text{by the triangle inequality} \\
&\leq \sum_{j=n+1}^{\infty} M_j. \qquad \text{(A.8)}
\end{aligned}$$

Now, the sum over the M_j's is convergent, so we can choose N such that

$$\sum_{j=n+1}^{\infty} M_j < \epsilon, \quad n \geq N.$$

Then, we have from above that

$$|g(x) - \sum_{j=1}^{n} f_j(x)| \leq \sum_{j=n+1}^{\infty} M_j < \epsilon$$

for all $n \geq N$ and $x \in D$. Thus, $\sum f_j \to g$ uniformly on D. □

We now given an example of how to use the Weierstraß M-Test.

Example A.21. *We consider the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ defined on $[-\pi, \pi]$. Each term is bounded by $\left|\frac{\cos nx}{n^2}\right| = \frac{1}{n^2} \equiv M_n$. We know that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Thus, we can conclude that the original series converges uniformly, as it satisfies the conditions of the Weierstraß M-Test.*

A.8 Power Series

A TYPICAL EXAMPLE OF A SERIES of functions that the student has encountered in previous courses is the power series. Examples of such series were provided by Taylor and Maclaurin series.[10]

[10] Actually, what are now known as Taylor and Maclaurin series were known long before they were named. James Gregory (1638-1675) has been recognized for discovering Taylor series, which were later named after Brook Taylor (1685-1731) . Similarly, Colin Maclaurin (1698-1746) did not actually discover Maclaurin series, but because of his particular use of them.

Definition A.10. A *power series* expansion about $x = a$ with coefficient sequence c_n is given by $\sum_{n=0}^{\infty} c_n (x-a)^n$.

For now we will consider all constants to be real numbers with x in some subset of the set of real numbers.

An example of such a power series is the following expansion about $x = 0$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots . \qquad \text{(A.9)}$$

We would like to make sense of such expansions. For what values of x will this infinite series converge? Until now we did not pay much attention to which infinite series might converge. However, this particular series is already familiar to us. It is a geometric series. Note that each term is gotten from the previous one through multiplication by $r = x$. The first term is $a = 1$. So, from Equation (1.74), we have the sum of the series is given by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

In this case we see that the sum, when it exists, is a simple function. In fact, when x is small, we can use this infinite series to provide approximations to the function $(1-x)^{-1}$. If x is small enough, we can write

$$(1-x)^{-1} \approx 1 + x.$$

In Figure A.13 we see that for small values of x these functions do agree.

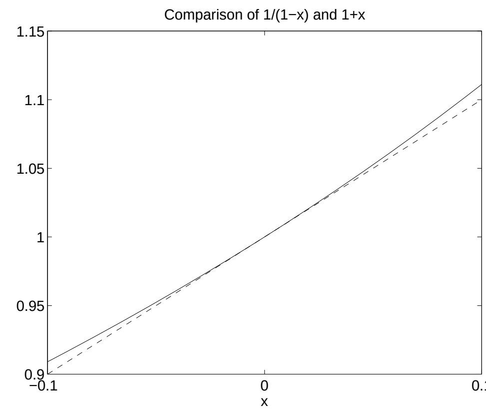


Figure A.13: Comparison of $\frac{1}{1-x}$ (solid) to $1 + x$ (dashed) for $x \in [-0.1, 0.1]$.

Of course, if we want better agreement, we select more terms. In Figure A.14 we see what happens when we do so. The agreement is much better. But extending the interval, we see in Figure A.15 shows that keeping only quadratic terms may not be good enough. Keeping the cubic terms gives better agreement over the interval.

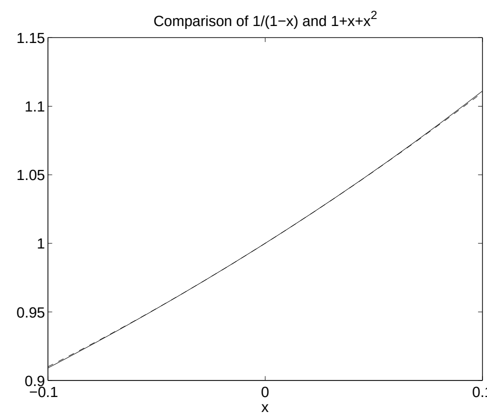


Figure A.14: Comparison of $\frac{1}{1-x}$ (solid) to $1 + x + x^2$ (dashed) for $x \in [-0.1, 0.1]$.

Finally, in Figure A.16 we show the sum of the first 21 terms over the entire interval $[-1, 1]$. Note that there are problems with approximations near the endpoints of the interval, $x = \pm 1$.

Such polynomial approximations are called *Taylor polynomials*. Thus, $T_3(x) = 1 + x + x^2 + x^3$ is the third order Taylor polynomial approximation of $f(x) = \frac{1}{1-x}$.

With this example we have seen how useful a series representation might be for a given function. However, the series representation was a simple geometric series, which we already knew how to sum. Is there a way to begin with a function and then find its series representation? Once we have such a representation, will the series converge to the function with which we started? For what values of x will it converge? These questions can be answered by recalling the definitions of Taylor and Maclaurin series.

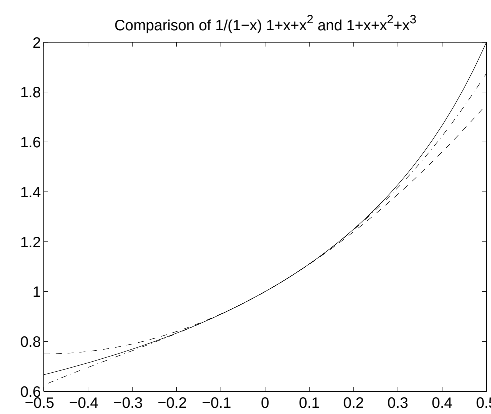


Figure A.15: Comparison of $\frac{1}{1-x}$ (solid) to $1 + x + x^2$ (dashed) and $1 + x + x^2 + x^3$ (dash-dot) for $x \in [-0.5, 0.5]$.

Taylor series expansion.

Definition A.11. A *Taylor series* expansion of $f(x)$ about $x = a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{A.10}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{A.11}$$

Note that we use $\sim$ to indicate that we have yet to determine when the series may converge to the given function. A special class of series are those Taylor series for which the expansion is about $x = 0$.

Definition A.12. A *Maclaurin series* expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x = 0$, or

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \tag{A.12}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{A.13}$$

Maclaurin series expansion.

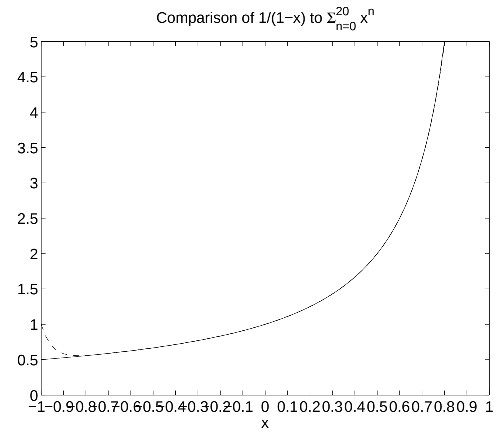


Figure A.16: Comparison of $\frac{1}{1-x}$ (solid) to $\sum_{n=0}^{\infty} x^n$ for $x \in [-1, 1]$.

Example A.22. *Expand $f(x) = e^x$ about $x = 0$.*

We begin by creating a table. In order to compute the expansion coefficients, c_n, we will need to perform repeated differentiations of $f(x)$. So, we provide a table for these derivatives. Then we only need to evaluate the second column at $x = 0$ and divide by $n!$.

n	$f^{(n)}(x)$	c_n
0	e^x	$\frac{e^0}{0!} = 1$
1	e^x	$\frac{e^0}{1!} = 1$
2	e^x	$\frac{e^0}{2!} = \frac{1}{2!}$
3	e^x	$\frac{e^0}{3!} = \frac{1}{3!}$

Next, one looks at the last column and tries to determine some pattern so as to write down the general term of the series. If there is only a need to get a polynomial approximation, then the first few terms may be sufficient.

In this case, we have that the pattern is obvious: $c_n = \frac{1}{n!}$. So,

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example A.23. *Expand $f(x) = e^x$ about $x = 1$.*

Here we seek an expansion of the form $e^x \sim \sum_{n=0}^{\infty} c_n (x-1)^n$. We could create a table like the last example. In fact, the last column would have values of the form $\frac{e}{n!}$. (You should confirm this.) However, we could make use of the Maclaurin series expansion for e^x and get the result quicker. Note that $e^x = e^{x-1+1} = ee^{x-1}$. Now, apply the known expansion for e^x :

$$e^x \sim e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \dots\right) = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}.$$

Example A.24. *Expand* $f(x) = \frac{1}{1-x}$ *about* $x = 0$.

This is the example with which we started our discussion. We set up a table again. We see from the last column that we get back our geometric series (A.9).

n	$f^{(n)}(x)$	c_n
0	$\frac{1}{1-x}$	$\frac{1}{0!} = 1$
1	$\frac{1}{(1-x)^2}$	$\frac{1}{1!} = 1$
2	$\frac{2(1)}{(1-x)^3}$	$\frac{2!}{2!} = 1$
3	$\frac{3(2)(1)}{(1-x)^4}$	$\frac{3!}{3!} = 1$

So, we have found

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n.$$

We can replace $\sim$ by equality if we can determine the range of x-values for which the resulting infinite series converges. We will investigate such convergence shortly.

Series expansions for many elementary functions arise in a variety of applications. Some common expansions are provided below.

Series Expansions You Should Know

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{(A.14)}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{(A.15)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{(A.16)}$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{(A.17)}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{(A.18)}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \quad \text{(A.19)}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-x)^n \quad \text{(A.20)}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{(A.21)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{(A.22)}$$

What is still left to be determined is for what values do such power series converge. The first five of the above expansions converge for all reals, but the others only converge for $|x| < 1$.

We consider the convergence of $\sum_{n=0}^{\infty} c_n (x-a)^n$. For $x = a$ the series obviously converges. Will it converge for other points? One can prove

Theorem A.7. *If $\sum_{n=0}^{\infty} c_n (b-a)^n$ converges for $b \neq a$, then $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely for all x satisfying $|x-a| < |b-a|$.*

This leads to three possibilities

1. $\sum_{n=0}^{\infty} c_n (x-a)^n$ may only converge at $x = a$.
2. $\sum_{n=0}^{\infty} c_n (x-a)^n$ may converge for all real numbers.
3. $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Interval and radius of convergence.

The number R is called the *radius of convergence* of the power series and $(a-R, a+R)$ is called the *interval of convergence.* Convergence at the endpoints of this interval has to be tested for each power series.

In order to determine the interval of convergence, one needs only note that when a power series converges, it does so absolutely. So, we need only test the convergence of $\sum_{n=0}^{\infty} |c_n (x-a)^n| = \sum_{n=0}^{\infty} |c_n||x -$

$a|^n$. This is easily done using either the ratio test or the nth root test. We first identify the nonnegative terms $a_n = |c_n||x-a|^n$, using the notation from Section A.4. Then we apply one of our convergence tests.

For example, the nth Root Test gives the convergence condition

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{|c_n|}|x-a| < 1.$$

Thus,

$$|x-a| < \left(\lim_{n\to\infty} \sqrt[n]{|c_n|}\right)^{-1} \equiv R.$$

This, R is the radius of convergence.

Similarly, we can apply the Ratio Test.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}|x-a| < 1.$$

Again, we rewrite this result to determine the radius of convergence:

$$|x-a| < \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)^{-1} \equiv R.$$

Example A.25. $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$.

Since there is a factorial, we will use the Ratio Test with $a = 0$..

$$\rho = \lim_{n\to\infty} \frac{|n!|}{|(n+1)!|}|x| = \lim_{n\to\infty} \frac{1}{n+1}|x| = 0.$$

Since $\rho = 0$, it is independent of $|x|$ and thus the series converges for all x. We also can say that the radius of convergence is infinite.

Example A.26. $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$.

In this example we will use the nth Root Test with $a = 0$.

$$\rho = \lim_{n\to\infty} \sqrt[n]{1}|x| = |x| < 1.$$

Thus, we find that we have absolute convergence for $|x| < 1$. Setting $x = 1$ or $x = -1$, we find that the resulting series do not converge. So, the endpoints are not included in the complete interval of convergence.

In this example we could have also used the Ratio Test. Thus,

$$\rho = \lim_{n\to\infty} \frac{1}{1}|x| = |x| < 1.$$

We have obtained the same result as when we used the nth Root Test.

Example A.27. $\sum_{n=1}^\infty \frac{3^n(x-2)^n}{n}$.

In this example, we have an expansion about $x = 2$. Using the nth Root Test we find that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n}}|x-2| = 3|x-2| < 1.$$

Solving for $|x-2|$ in this inequality, we find $|x-2| < \frac{1}{3}$. Thus, the radius of convergence is $R = \frac{1}{3}$ and the interval of convergence is $\left(2-\frac{1}{3}, 2+\frac{1}{3}\right) = \left(\frac{5}{3}, \frac{7}{3}\right)$.

As for the endpoints, we need to first test at $x = \frac{7}{3}$. The resulting series is $\sum_{n=1}^{\infty} \frac{3^n(\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, and thus it does not converge. Inserting $x = \frac{5}{3}$ we get the alternating harmonic series, which does converge. So, we have convergence on $[\frac{5}{3}, \frac{7}{3})$. However, it is only conditionally convergent at the left endpoint, $x = \frac{5}{3}$.

Example A.28. *Find an expansion of $f(x) = \frac{1}{x+2}$ about $x = 1$.*

Instead of explicitly computing the Taylor series expansion for this function, we can make use of an already known function. We first write $f(x)$ as a function of $x-1$, since we are expanding about $x = 1$. This is easily done by noting that $\frac{1}{x+2} = \frac{1}{(x-1)+3}$. Factoring out a 3, we can rewrite this as a sum of a geometric series. Namely, we use the expansion for

$$\begin{aligned} g(z) &= \frac{1}{1+z} \\ &= 1 - z + z^2 - z^3 + \ldots. \end{aligned} \tag{A.23}$$

and then we rewrite $f(x)$ as

$$\begin{aligned} f(x) &= \frac{1}{x+2} \\ &= \frac{1}{(x-1)+3} \\ &= \frac{1}{3[1+\frac{1}{3}(x-1)]} \\ &= \frac{1}{3}\frac{1}{1+\frac{1}{3}(x-1)}. \end{aligned} \tag{A.24}$$

Note that $f(x) = \frac{1}{3}g(\frac{1}{3}(x-1))$ for $g(z) = \frac{1}{1+z}$ So, the expansion becomes

$$f(x) = \frac{1}{3}\left[1 - \frac{1}{3}(x-1) + \left(\frac{1}{3}(x-1)\right)^2 - \left(\frac{1}{3}(x-1)\right)^3 + \ldots\right].$$

This can further be simplified as

$$f(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \ldots.$$

Convergence is easily established. The expansion for $g(z)$ converges for $|z| < 1$. So, the expansion for $f(x)$ converges for $|-\frac{1}{3}(x-1)| < 1$. This implies that $|x-1| < 3$. Putting this inequality in interval notation, we have that the power series converges absolutely for $x \in (-2, 4)$. Inserting the endpoints, one can show that the series diverges for both $x = -2$ and $x = 4$. You should verify this!

As a final application, we can derive Euler's Formula ,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, is an important formula and will be used throughout the text.

where $i = \sqrt{-1}$. We naively use the expansion for e^x with $x = i\theta$. This leads us to

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots.$$

Next we note that each term has a power of i. The sequence of powers of i is given as $\{1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \dots\}$. See the pattern? We conclude that

$$i^n = i^r, \text{ where } r = \text{ remainder after dividing } n \text{ by } 4.$$

This gives

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right).$$

We recognize the expansions in the parentheses as those for the cosine and sine functions. Thus, we end with Euler's Formula.

We further derive relations from this result, which will be important for our next studies. From Euler's formula we have that for integer n:

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

We also have

$$e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n.$$

Equating these two expressions, we are led to de Moivre's Formula, named after Abraham de Moivre (1667-1754),

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{A.25}$$

This formula is useful for deriving identities relating powers of sines or cosines to simple functions. For example, if we take $n = 2$ in Equation (A.25), we find

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.$$

Here we see elegant proofs of well known trigonometric identities. We will later make extensive use of these identities. Namely, you should know:

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta, && \text{(A.26)} \\ \sin 2\theta &= 2\sin\theta\cos\theta, && \text{(A.27)} \\ \cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta), && \text{(A.28)} \\ \sin^2\theta &= \frac{1}{2}(1 - \cos 2\theta). && \text{(A.29)} \end{aligned}$$

Looking at the real and imaginary parts of this result leads to the well known double angle identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Replacing $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$ leads to the half angle formulae:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

We can also use Euler's Formula to write sines and cosines in terms of complex exponentials. We first note that due to the fact that the cosine is an even function and the sine is an odd function, we have

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Combining this with Euler's Formula, we have that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Trigonometric functions can be written in terms of complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We finally note that there is a simple relationship between hyperbolic functions and trigonometric functions. Recall that

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

If we let $x = i\theta$, then we have that $\cosh(i\theta) = \cos\theta$ and $\cos(ix) = \cosh x$. Similarly, we can show that $\sinh(i\theta) = i\sin\theta$ and $\sin(ix) = -i\sinh x$.

Hyperbolic functions and trigonometric functions are intimately related.

$$\cos(ix) = \cosh x,$$
$$\sin(ix) = -i\sinh x.$$

The binomial expansion is a special series expansion used to approximate expressions of the form $(a+b)^p$ for $b \ll a$, or $(1+x)^p$ for $|x| \ll 1$.

ONE SERIES EXPANSION which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$ in powers of a and b. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p. While the binomial expansion can be obtained using Taylor series, we will provide a more interesting derivation here to show that

$$(a+b)^p = \sum^{\infty} C_p^r a^{n-r} b^r, \tag{A.30}$$

where the C_p^r are called the *binomial coefficients*.

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
&\cdots
\end{aligned} \tag{A.31}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $(k+1)$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not yet know the numerical coefficient in the expansion.

Let's list the coefficients for the above expansions.

$$\begin{array}{lccccccccc} n=0: & & & & & 1 & & & & \\ n=1: & & & & 1 & & 1 & & & \\ n=2: & & & 1 & & 2 & & 1 & & \\ n=3: & & 1 & & 3 & & 3 & & 1 & \\ n=4: & 1 & & 4 & & 6 & & 4 & & 1 \end{array} \tag{A.32}$$

This pattern is the famous Pascal's triangle.[11] There are many interesting features of this triangle. But we will first ask how each row can be generated.

[11] Pascal's triangle is named after Blaise Pascal (1623-1662). While such configurations of number were known earlier in history, Pascal published them and applied them to probability theory.

Pascal's triangle has many unusual properties and a variety of uses:

- Horizontal rows add to powers of 2.
- The horizontal rows are powers of 11 (1, 11, 121, 1331, etc.).
- Adding any two successive numbers in the diagonal 1-3-6-10-15-21-28... results in a perfect square.
- When the first number to the right of the 1 in any row is a prime number, all numbers in that row are divisible by that prime number.
- Sums along certain diagonals leads to the Fibonacci sequence.

We see that each row begins and ends with a one. The second term and next to last term have a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have for rows $n=2$ and $n=3$ that $1+2=3$ and $2+1=3:$

$$\begin{array}{lccccccccc} n=2: & & 1 & & & 2 & & & 1 & \\ & & & \searrow & \swarrow & & \searrow & \swarrow & & \\ n=3: & 1 & & & 3 & & 3 & & & 1 \end{array} \tag{A.33}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{lccccccccccccc} n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\ n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \tag{A.34}$$

So, we use the numbers in row $n=4$ to generate entries in row $n=5:$ $1+4=5$, $4+6=10$. We then use row $n=5$ to get row $n=6$, etc.

Of course, it would take a while to compute each row up to the desired n. Fortunately, there is a simple expression for computing a specific coefficient. Consider the kth term in the expansion of $(a+b)^n$. Let $r=k-1$. Then this term is of the form $C_r^n a^{n-r}b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the binomial coefficients have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \left(\begin{array}{c} n \\ r \end{array} \right).$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count the number of ways that we can arrange r products of b with $n-r$ products of a. There are n slots to place the b's. For example, the $r=2$ case for $n=4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*. Thus, it is natural to use this notation.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) a^{n-r} b^r. \tag{A.35}$$

Now consider the geometric series $1 + x + x^2 + \ldots$. We have seen that such a series converges for $|x| < 1$, giving

$$1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$.

This is again a binomial to a power, but the power is not an integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (A.35).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$(1+x)^p = \sum_{r=0}^{\infty} \left(\begin{array}{c} p \\ r \end{array} \right) x^r. \tag{A.36}$$

However, we quickly run into problems with this form. Consider the coefficient for $r=1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\left(\begin{array}{c} -1 \\ 1 \end{array} \right) = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion.

We can write the general coefficient as

$$\begin{aligned}\begin{pmatrix} p \\ r \end{pmatrix} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \tag{A.37}$$

With this in mind we now state the theorem:

General Binomial Expansion

The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (A.35). For p real, we have the following binomial series:

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r, \quad |x| < 1. \tag{A.38}$$

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2} x^2 + O(x^3). \tag{A.39}$$

Example A.29. *Approximate* $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ *for* $v \ll c$. *This can be rewritten as*

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}.$$

Using the binomial expansion for $p = -1/2$, *we have*

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) = 1 + \frac{v^2}{2c^2}.$$

The factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is important in special relativity. Namely, this is the factor relating differences in time and length measurements by observers moving relative inertial frames. For celestial speeds, this is an appropriate approximation.

Example A.30. *Small differences in large numbers.*

As an example, we could compute $f(R,h) = \sqrt{R^2+h^2} - R$ *for* $R = 6378.164$ *km and* $h = 1.0$ *m. Inserting these values into a scientific calculator, one finds that*

$$f(6378164, 1) = \sqrt{6378164^2 + 1} - 6378164 = 1 \times 10^{-7}\ m.$$

In some calculators one might obtain 0, in other calculators, or computer algebra systems like Maple, one might obtain other answers. What answer do you get and how accurate is your answer?

The problem with this computation is that $R \gg h$. *Therefore, the computation of* $f(R,h)$ *depends on how many digits the computing device can handle.*

The best way to get an answer is to use the binomial approximation. Writing $x = \frac{h}{R}$, *we have*

$$\begin{aligned} f(R,h) &= \sqrt{R^2+h^2} - R \\ &= R\sqrt{1+x^2} - R \\ &\simeq R\left[1+\frac{1}{2}x^2\right] - R \\ &= \frac{1}{2}Rx^2 \\ &= \frac{1}{2}\frac{h}{R^2} = 7.83926 \times 10^{-8}\ m. \end{aligned} \tag{A.40}$$

Of course, you should verify how many digits should be kept in reporting the result.

In the next examples, we show how computations taking a more general form can be handled. Such general computations appear in proofs involving general expansions without specific numerical values given.

Example A.31. *Obtain an approximation to* $(a+b)^p$ *when* a *is much larger than* b, *denoted by* $a \gg b$.

If we neglect b *then* $(a+b)^p \simeq a^p$. *How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion. Namely, what is the power of* b/a *of the first neglected term in this expansion?*

In order to do this we first divide out a *as*

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p.$$

Now we have a small parameter, $\frac{b}{a}$. *According to what we have seen earlier, we can use the binomial expansion to write*

$$\left(1+\frac{b}{a}\right)^n = \sum_{r=0}^{\infty}\begin{pmatrix} p \\ r \end{pmatrix}\left(\frac{b}{a}\right)^r. \tag{A.41}$$

Thus, we have a sum of terms involving powers of $\frac{b}{a}$. *Since* $a \gg b$, *most of these terms can be neglected. So, we can write*

$$\left(1+\frac{b}{a}\right)^p = 1 + p\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

Here we used $O()$, *big-Oh notation, to indicate the size of the first neglected term. (This notation is formally defined in another section.)*

Summarizing, this then gives

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p$$

$$
\begin{aligned}
&= a^p\left(1+p\frac{b}{a}+O\left(\left(\frac{b}{a}\right)^2\right)\right) \\
&= a^p+pa^p\frac{b}{a}+a^pO\left(\left(\frac{b}{a}\right)^2\right). \qquad \text{(A.42)}
\end{aligned}
$$

Therefore, we can approximate $(a+b)^p \simeq a^p + pba^{p-1}$*, with an error on the order of* b^2a^{p-2}*. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that* $(a+b)^p \simeq a^p$*, but it is not typically good enough in applications because the error in this case is of the order* ba^{p-1}*.*

Example A.32. *Approximate* $f(x) = (a+x)^p - a^p$ *for* $x \ll a$*.*

In an earlier example we computed $f(R,h) = \sqrt{R^2+h^2} - R$ *for* $R = 6378.164$ *km and* $h = 1.0$ *m. We can make use of the binomial expansion to determine the behavior of similar functions in the form* $f(x) = (a+x)^p - a^p$*. Inserting the binomial expression into* $f(x)$*, we have as* $\frac{x}{a} \to 0$ *that*

$$
\begin{aligned}
f(x) &= (a+x)^p - a^p \\
&= a^p\left[\left(1+\frac{x}{a}\right)^p - 1\right] \\
&= a^p\left[\frac{px}{a} + O\left(\left(\frac{x}{a}\right)^2\right)\right] \\
&= O\left(\frac{x}{a}\right) \quad \text{as } \frac{x}{a} \to 0. \qquad \text{(A.43)}
\end{aligned}
$$

This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as

$$
(a+x)^p - a^p = a^{p-1}px + O\left(\left(\frac{x}{a}\right)^2\right) \quad \text{as } \frac{x}{a} \to 0.
$$

We could use this approximation to answer the original question by letting $a = R^2$*,* $x = 1$ *and* $p = \frac{1}{2}$*. Then, our approximation would be of order*

$$
O\left(\left(\frac{x}{a}\right)^2\right) = O\left(\left(\frac{1}{6378164^2}\right)^2\right) \sim 2.4 \times 10^{-14}.
$$

Thus, we have

$$
\sqrt{6378164^2+1} - 6378164 \approx a^{p-1}px
$$

where

$$
a^{p-1}px = (6378164^2)^{-1/2}(0.5)1 = 7.83926 \times 10^{-8}.
$$

This is the same result we had obtained before.

A.9 The Order of Sequences and Functions

Often we are interested in comparing the rates of convergence of sequences or asymptotic behavior of functions. This is useful in approximation theory as we had seen in the last section. We begin with the comparison of sequences and introduce *big-Oh* notation. We will then extend this to functions of continuous variables.

Definition A.13. Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Then if there are numbers N and K (independent of N) such that

$$\left|\frac{a_n}{b_n}\right| < K \quad \text{whenever} \quad n > N,$$

then we say that a_n is *of the order of* b_n. We write this as

$$a_n = O(b_n) \quad \text{as} \quad n \to \infty$$

and say a_n is "big O" of b_n.

Example A.33. *Consider the sequences given by* $a_n = \frac{2n+1}{3n^2+2}$ *and* $b_n = \frac{1}{n}$.

In this case we consider the ratio,

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{\frac{2n+1}{3n^2+2}}{\frac{1}{n}}\right| = \left|\frac{2n^2+n}{3n^2+2}\right|.$$

We want to find a bound on the last expression as n gets large. We divide the numerator and denominator by n^2 *and find that*

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{2+1/n}{3+2/n^2}\right| = \frac{2}{3}\left|\frac{1+1/2n}{1+2/3n^2}\right|.$$

The last expresssion is largest for $n = 1$. *This gives*

$$\left|\frac{a_n}{b_n}\right| = \frac{2}{3}\left|\frac{1+1/2n}{1+2/3n^2}\right| \le \frac{2}{3}\left|\frac{1+1/2}{1+2/3}\right| = \frac{9}{10}.$$

Thus, for $n > 1$, *we have that*

$$\left|\frac{a_n}{b_n}\right| \le \frac{9}{10} < 1 \equiv K.$$

We then conclude from Definition A.13 that

$$a_n = O(b_n) = O\left(\frac{1}{n}\right).$$

In practice one is often given a sequence like a_n, but the second simpler sequence needs to be found by looking at the large n behavior of a_n.

Referring to the last example, we are given $a_n = \frac{2n+1}{3n^2+2}$. We look at the large n behavior. The numerator behaves like $2n$ and the denominator behaves like $3n^2$. Thus, $a_n = \frac{2n+1}{3n^2+2} \sim \frac{2n}{3n^2} = \frac{2}{3n}$ for large n. Therefore, we say that $a_n = O(\frac{1}{n})$ for large n. Note that we are only interested in the n-dependence and not the multiplicative constant since $\frac{1}{n}$ and $\frac{2}{3n}$ have the same growth rate.

In a similar way, we can compare functions. We modify our definition of big-Oh for functions of a continuous variable.

Definition A.14. $f(x)$ is *of the order of* $g(x)$, or $f(x) = O(g(x))$ as $x \to x_0$ if

$$\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| < K$$

for some K independent of x_0.

Example A.34. *Show that*

$$\cos x - 1 + \frac{x^2}{2} = O(x^4) \quad \textit{as } x \to 0.$$

This should be apparent from the Taylor series expansion for $\cos x$,

$$\cos x = 1 - \frac{x^2}{2} + O(x^4) \textit{ as } x \to 0.$$

However, we will show that $\cos x - 1 + \frac{x^2}{2}$ *is of the order of* $O(x^4)$ *using the above definition.*

We need to compute

$$\lim_{x \to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right|.$$

The numerator and denominator separately go to zero, so we have an indeterminate form. This suggests that we need to apply L'Hopital's Rule. In fact, we apply it several times to find that

$$\begin{aligned} \lim_{x \to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| &= \lim_{x \to 0} \left| \frac{-\sin x + x}{4x^3} \right| \\ &= \lim_{x \to 0} \left| \frac{-\cos x + 1}{12x^2} \right| \\ &= \lim_{x \to 0} \left| \frac{\sin x}{24x} \right| = \frac{1}{24}. \end{aligned}$$

Thus, for any number $K > \frac{1}{24}$, *we have that*

$$\lim_{x \to 0} \left| \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \right| < K.$$

We conclude that

$$\cos x - 1 + \frac{x^2}{2} = O(x^4) \quad \textit{as } x \to 0.$$

Example A.35. *Determine the order of* $f(x) = (x^3 - x)^{1/3} - x$ *as* $x \to \infty$. *We can use a binomial expansion to write the first term in powers of* x. *However, since* $x \to \infty$, *we want to write* $f(x)$ *in powers of* $\frac{1}{x}$, *so that we can neglect higher order powers. We can do this by first factoring out the* x^3 :

$$(x^3 - x)^{1/3} - x = x\left(1 - \frac{1}{x^2}\right)^{1/3} - x$$

$$
\begin{aligned}
&= x\left(1-\frac{1}{3x^2}+O\left(\frac{1}{x^4}\right)\right)-x \\
&= -\frac{1}{3x}+O\left(\frac{1}{x^3}\right). \qquad \text{(A.44)}
\end{aligned}
$$

Now we can see from the first term on the right that $(x^3-x)^{1/3}-x = O\left(\frac{1}{x}\right)$ *as* $x \to \infty$.

Problems

1. For those sequences that converge, find the limit $\lim_{n\to\infty} a_n$.

a. $a_n = \frac{n^2+1}{n^3+1}$.

b. $a_n = \frac{3n+1}{n+2}$.

c. $a_n = \left(\frac{3}{n}\right)^{1/n}$.

d. $a_n = \frac{2n^2+4n^3}{n^3+5\sqrt{2+n^6}}$.

e. $a_n = n\ln\left(1+\frac{1}{n}\right)$.

f. $a_n = n\sin\left(\frac{1}{n}\right)$.

g. $a_n = \frac{(2n+3)!}{(n+1)!}$.

2. Find the sum for each of the series:

a. $\sum_{n=0}^{\infty} \frac{(-1)^n 3}{4^n}$.

b. $\sum_{n=2}^{\infty} \frac{2}{5^n}$.

c. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n}+\frac{1}{3^n}\right)$.

d. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$.

3. Determine if the following converge, or diverge, using one of the convergence tests. If the series converges, is it absolute or conditional?

a. $\sum_{n=1}^{\infty} \frac{n+4}{2n^3+1}$.

b. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

c. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

d. $\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{2n^2-3}$.

e. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

f. $\sum_{n=1}^{\infty} \frac{100^n}{n^{200}}$.

g. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+3}$.

h. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{5n}}{n+1}$.

4. Do the following:

a. Compute: $\lim_{n\to\infty} n \ln\left(1-\frac{3}{n}\right)$.

b. Use L'Hopital's Rule to evaluate $L = \lim_{x\to\infty}\left(1-\frac{4}{x}\right)^x$. Hint: Consider $\ln L$.

c. Determine the convergence of $\sum_{n=1}^{\infty}\left(\frac{n}{3n+2}\right)^{n^2}$.

d. Sum the series $\sum_{n=1}^{\infty}\left[\tan^{-1} n - \tan^{-1}(n+1)\right]$ by first writing the Nth partial sum and then computing $\lim_{N\to\infty} s_N$.

5. Consider the sum $\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)}$.

a. Use an appropriate convergence test to show that this series converges.

b. Verify that

$$\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}-\frac{n}{n+1}\right).$$

c. Find the nth partial sum of the series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}-\frac{n}{n+1}\right)$ and use it to determine the sum of the resulting *telescoping* series.

6. Recall that the alternating harmonic series converges conditionally.

a. From the Taylor series expansion for $f(x) = \ln(1+x)$, inserting $x = 1$ gives the alternating harmonic series. What is the sum of the alternating harmonic series?

Since the alternating harmonic series does not converge absolutely, then a rearrangement of the terms in the series will result in series whose sums vary. One such rearrangement in alternating p positive terms and n negative terms leads to the following sum[12]:

[12] This is discussed by Lawrence H. Riddle in the *Kenyon Math. Quarterly*, 1(2), 6-21.

$$\begin{aligned}\frac{1}{2}\ln\frac{4p}{n} &= \underbrace{\left(1+\frac{1}{3}+\cdots+\frac{1}{2p-1}\right)}_{p \text{ terms}} - \underbrace{\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)}_{n \text{ terms}} \\ &\quad + \underbrace{\left(\frac{1}{2p+1}+\cdots+\frac{1}{4p-1}\right)}_{p \text{ terms}} - \underbrace{\left(\frac{1}{2n+2}+\cdots+\frac{1}{4n}\right)}_{n \text{ terms}} + \cdots .\end{aligned}$$

Find rearrangements of the alternating harmonic series to give the following sums; i.e., determine p and n for the given expression and write down the above series explicitly; i.e, determine p and n leading to the following sums.

b. $\frac{5}{2}\ln 2$.

c. $\ln 8$.

d. 0.

e. A sum that is close to π.

7. Determine the radius and interval of convergence of the following infinite series:

a. $\sum_{n=1}^{\infty}(-1)^n \frac{(x-1)^n}{n}$.

b. $\sum_{n=1}^{\infty} \frac{x^n}{2^n n!}$.

c. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{5}\right)^n$

d. $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{\sqrt{n}}$.

8. Find the Taylor series centered at $x = a$ and its corresponding radius of convergence for the given function. In most cases, you need not employ the direct method of computation of the Taylor coefficients.

a. $f(x) = \sinh x$, $a = 0$.

b. $f(x) = \sqrt{1+x}$, $a = 0$.

c. $f(x) = xe^x$, $a = 1$.

d. $f(x) = \frac{x-1}{2+x}$, $a = 1$.

9. Test for pointwise and uniform convergence on the given set. [The Weierstraß M-Test might be helpful.]

a. $f(x) = \sum_{n=1}^{\infty} \frac{\ln nx}{n^2}$, $x \in [1, 2]$.

b. $f(x) = \sum_{n=1}^{\infty} \frac{1}{3^n} \cos \frac{x}{2^n}$ on R.

10. Consider Gregory's expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

a. Derive Gregory's expansion by using the definition

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2},$$

expanding the integrand in a Maclaurin series, and integrating the resulting series term by term.

b. From this result, derive Gregory's series for π by inserting an appropriate value for x in the series expansion for $\tan^{-1} x$.

11. Use deMoivre's Theorem to write $\sin^3 \theta$ in terms of $\sin \theta$ and $\sin 3\theta$. Hint: Focus on the imaginary part of $e^{3i\theta}$.

12. Evaluate the following expressions at the given point. Use your calculator and your computer (such as Maple). Then use series expansions to find an approximation to the value of the expression to as many places as you trust.

a. $\frac{1}{\sqrt{1+x^3}} - \cos x^2$ at $x = 0.015$.

b. $\ln \sqrt{\frac{1+x}{1-x}} - \tan x$ at $x = 0.0015$.

c. $f(x) = \frac{1}{\sqrt{1+2x^2}} - 1 + x^2$ at $x = 5.00 \times 10^{-3}$.

d. $f(R, h) = R - \sqrt{R^2 + h^2}$ for $R = 1.374 \times 10^3$ km and $h = 1.00$ m.

e. $f(x) = 1 - \frac{1}{\sqrt{1-x}}$ for $x = 2.5 \times 10^{-13}$.

13. Determine the order, $O(x^p)$, of the following functions. You may need to use series expansions in powers of x when $x \to 0$, or series expansions in powers of $1/x$ when $x \to \infty$.

a. $\sqrt{x(1-x)}$ as $x \to 0$.

b. $\frac{x^{5/4}}{1-\cos x}$ as $x \to 0$.

c. $\frac{x}{x^2-1}$ as $x \to \infty$.

d. $\sqrt{x^2 + x} - x$ as $x \to \infty$.

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