9
Oscillations in Higher Dimensions

"Equations of such complexity as are the equations of the gravitational field can be found only through the discovery of a logically simple mathematical condition that determines the equations completely or at least almost completely."

"What I have to say about this book can be found inside this book." Albert Einstein (1879-1955)

In this chapter we will explore several generic examples of the solution of initial-boundary value problems involving higher spatial dimensions. These are described by higher dimensional partial differential equations, such as the ones presented in Table 8.1 in the last chapter. We will solve these problems for different geometries, using rectangular, polar, cylindrical, or spherical coordinates.

We will solve these problems using the method of separation of variables, though there are other methods which we will not consider in this text. Using separation of variables will result in a system of ordinary differential equations for each problem. Adding the boundary conditions, we will need to solve a variety of eigenvalue problems. The product solutions that result will involve trigonometric or some of the special functions that we had encountered in Chapter 5.

As you go through the examples in this chapter, you will see some common features. For example, the two key equations that we have studied are the heat equation and the wave equation. For higher dimensional problems these take the form

$$u_t = k\nabla^2 u, \tag{9.1}$$
$$u_{tt} = c^2\nabla^2 u. \tag{9.2}$$

One can first separate out the time dependence. Let $u(\mathbf{r},t) = \phi(\mathbf{r})T(t)$. Inserting u into the heat and wave equations, we have

$$T'\phi = kT\nabla^2\phi, \tag{9.3}$$
$$T''\phi = c^2T\nabla^2\phi. \tag{9.4}$$

Separating out the time and space dependence, we find

$$\frac{1}{k}\frac{T'}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda, \tag{9.5}$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda. \tag{9.6}$$

Note that in each case we have that a function of time equals a function of the spatial variables. Thus, they must be constant functions. We set these equal to the constant $-\lambda$. The sign of λ is chosen because we expect decaying solutions in time for the heat equation and oscillations in time for the wave equation and will pick $\lambda > 0$.

First, we look at the time dependence. The respective set of equations for $T(t)$ are given by

$$T' = -\lambda k T, \tag{9.7}$$

$$T'' + c^2\lambda T = 0. \tag{9.8}$$

These are easily solved. We have

$$T(t) = T(0)e^{-\lambda k t}, \tag{9.9}$$

$$T(t) = a\cos\omega t + b\sin\omega t, \quad \omega = c\sqrt{\lambda}. \tag{9.10}$$

In both cases the spatial equation becomes

The Helmholtz equation.

$$\nabla^2\phi + \lambda\phi = 0. \tag{9.11}$$

This is called the Helmholtz equation. For one dimensional problems, which we have already solved, the Helmholtz equation takes the form $\phi'' + \lambda\phi = 0$. We had to impose the boundary conditions and found that there were a discrete set of eigenvalues, λ_n, and associated eigenfunctions, ϕ_n.

In higher dimensional problems we need to further separate out the spatial dependence. We will again use the boundary conditions and find the eigenvalues and eigenfunctions for the Helmholtz equation, though the eigenfunctions will be labeled with more than one index. The resulting boundary value problems are often second order ordinary differential equations, which can be set up as Sturm-Liouville problems. We know from Chapter 5 that such problems possess an orthogonal set of eigenfunctions. These can then be used to construct a general solution out of product solutions consisting of elementary or special functions, such as Legendre functions or Bessel functions.

We will begin our study of higher dimensional problems by considering the vibrations of two dimensional membranes. First we will solve the problem of a vibrating rectangular membrane and then we turn out attention to a vibrating circular membranes. The rest of the chapter will be devoted to the study of three dimensional problems possessing cylindrical or spherical symmetry.

9.1 *Vibrations of Rectangular Membranes*

Our first example will be the study of the vibrations of a rectangular membrane. You can think of this as a drum with a rectangular cross section as shown in Figure 9.1. We stretch the membrane over the drumhead and fasten the material to the boundary of the rectangle. The height of the vibrating membrane is described by its height from equilibrium, $u(x,y,t)$. This problem is a much simpler example of higher dimensional vibrations than that possessed by the oscillating electric and magnetic fields in the last chapter.

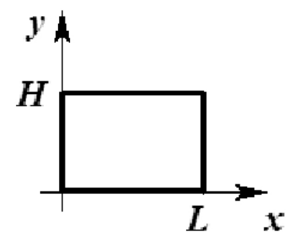

Figure 9.1: The rectangular membrane of length L and width H. There are fixed boundary conditions along the edges.

The problem is given by the two dimensional wave equation in Cartesian coordinates,

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{9.12}$$

a set of boundary conditions,

$$\begin{aligned} &u(0,y,t) = 0, \quad u(L,y,t) = 0, \quad t > 0, \quad 0 < y < H, \\ &u(x,0,t) = 0, \quad u(x,H,t) = 0, \quad t > 0, \quad 0 < x < L, \end{aligned} \tag{9.13}$$

and a pair of initial conditions (since the equation is second order in time),

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y). \tag{9.14}$$

The first step is to separate the variables: $u(x,y,t) = X(x)Y(y)T(t)$. Inserting the guess, $u(x,y,t)$ into the wave equation, we have

$$X(x)Y(y)T''(t) = c^2 \left(X''(x)Y(y)T(t) + X(x)Y''(y)T(t) \right).$$

Dividing by both $u(x,y,t)$ and c^2, we obtain

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{Function of } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{Function of } x \text{ and } y} = -\lambda. \tag{9.15}$$

We see that we have a function of t equals a function of x and y. Thus, both expressions are constant. We expect oscillations in time, so we chose the constant λ to be positive, $\lambda > 0$. (Note: As usual, the primes mean differentiation with respect to the specific dependent variable. So, there should be no ambiguity.)

These lead to two equations:

$$T'' + c^2\lambda T = 0, \tag{9.16}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda. \tag{9.17}$$

The first equation is easily solved. We have

$$T(t) = a \cos \omega t + b \sin \omega t, \tag{9.18}$$

where

$$\omega = c\sqrt{\lambda}. \tag{9.19}$$

This is the angular frequency in terms of the separation constant, or eigenvalue. It leads to the frequency of oscillations for the various harmonics of the vibrating membrane as

$$\nu = \frac{\omega}{2\pi} = \frac{c}{2\pi}\sqrt{\lambda}. \tag{9.20}$$

Once we know λ, we can compute these frequencies.

Now we solve the spatial equation. Again, we need to do a separation of variables. Rearranging the spatial equation, we have

$$\underbrace{\frac{X''}{X}}_{\text{Function of } x} = \underbrace{-\frac{Y''}{Y} - \lambda}_{\text{Function of } y} = -\mu. \tag{9.21}$$

Here we have a function of x equals a function of y. So, the two expressions are constant, which we indicate with a second separation constant, $-\mu < 0$. We pick the sign in this way because we expect oscillatory solutions for $X(x)$. This leads to two equations:

$$\begin{aligned} X'' + \mu X &= 0, \\ Y'' + (\lambda - \mu)Y &= 0. \end{aligned} \tag{9.22}$$

We now need to use the boundary conditions. We have $u(0, y, t) = 0$ for all $t > 0$ and $0 < y < H$. This implies that $X(0)Y(y)T(t) = 0$ for all t and y in the domain. This is only true if $X(0) = 0$. Similarly, from the other boundary conditions we find that $X(L) = 0$, $Y(0) = 0$, and $Y(H) = 0$. We note that homogeneous boundary conditions are important in carrying out this process. Nonhomogeneous boundary conditions could be imposed, but the techniques are a bit more complicated and we will not discuss these techniques here.

The boundary values problems we need to solve are:

$$\begin{aligned} X'' + \mu X = 0, &\quad X(0) = 0, X(L) = 0. \\ Y'' + (\lambda - \mu)Y = 0, &\quad Y(0) = 0, Y(H) = 0. \end{aligned} \tag{9.23}$$

We have seen the first of these problems before, except with a λ instead of a μ. The solutions of the eigenvalue problem are

$$X(x) = \sin \frac{n\pi x}{L}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The second equation is solved in the same way. The differences are that the "eigenvalue" is $\lambda - \mu$, the independent variable is y, and the interval is $[0, H]$. Thus, we can quickly write down the solutions of the eigenvalue problem as

$$Y(y) = \sin \frac{m\pi x}{H}, \quad \lambda - \mu = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, 3, \ldots.$$

We have successfully carried out the separation of variables for the wave equation for the vibrating rectangular membrane. The product solutions can be written as

$$u_{nm} = (a \cos \omega_{nm} t + b \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{9.24}$$

Recall that ω is given in terms of λ. We have that

The harmonics for the vibrating rectangular membrane are given by

$$\nu_{nm} = \frac{c}{2}\sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{H}\right)^2},$$

for $n, m = 1, 2, \ldots$.

$$\lambda_{mn} - \mu_n = \left(\frac{m\pi}{H}\right)^2$$

and

$$\mu_n = \left(\frac{n\pi}{L}\right)^2.$$

Therefore,

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2. \tag{9.25}$$

So,

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{9.26}$$

The most general solution can now be written as a linear combination of the product solutions and we can solve for the expansion coefficients that will lead to a solution satisfying he initial conditions. However, we will first concentrate on the two dimensional harmonics of this membrane.

For the vibrating string the nth harmonic corresponded to the function $\sin \frac{n\pi x}{L}$. The various harmonics corresponded to the pure tones supported by the string. These then lead to the corresponding frequencies that one would hear. The actual shapes of the harmonics could be sketched by locating the nodes, or places on the string that did not move.

In the same way, we can explore the shapes of the harmonics of the vibrating membrane. These are given by the spatial functions

$$\phi_{nm}(x, y) = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{9.27}$$

A discussion of the nodal lines.

Instead of nodes, we will look for the *nodal curves*, or *nodal lines*. These are the points (x, y) at which $\phi_{nm}(x, y) = 0$. Of course, these depend on the indices, n and m.

For example, when $n = 1$ and $m = 1$, we have

$$\sin \frac{\pi x}{L} \sin \frac{\pi y}{H} = 0.$$

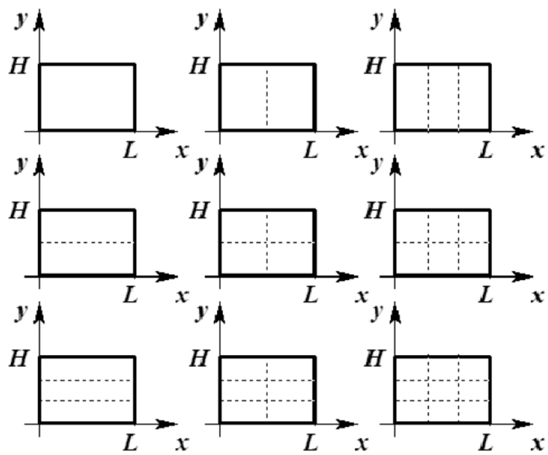

Figure 9.2: The first few modes of the vibrating rectangular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these the nodal lines to the 3D view in Figure 9.3

These are zero when either

$$\sin \frac{\pi x}{L} = 0, \quad \text{or} \quad \sin \frac{\pi y}{H} = 0.$$

Of course, this can only happen for $x = 0, L$ and $y = 0, H$. Thus, there are no interior nodal lines.

When $n = 2$ and $m = 1$, we have $y = 0, H$ and

$$\sin \frac{2\pi x}{L} = 0,$$

or, $x = 0, \frac{L}{2}, L$. Thus, there is one interior nodal line at $x = \frac{L}{2}$. These points stay fixed during the oscillation and all other points oscillate on either side of this line. A similar solution shape results for the (1,2)-mode; i.e., $n = 1$ and $m = 2$.

In Figure 9.2 we show the nodal lines for several modes for $n, m = 1, 2, 3$ The blocked regions appear to vibrate independently. A better view is the three dimensional view depicted in Figure 9.3 . The frequencies of vibration are easily computed using the formula for ω_{nm}.

For completeness, we now see how one satisfies the initial conditions. The general solution is given by a linear superposition of the product solutions. There are two indices to sum over. Thus, the general solution is

The general solution for the vibrating rectangular membrane.

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \tag{9.28}$$

where

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{9.29}$$

The first initial condition is $u(x, y, 0) = f(x, y)$. Setting $t = 0$ in the

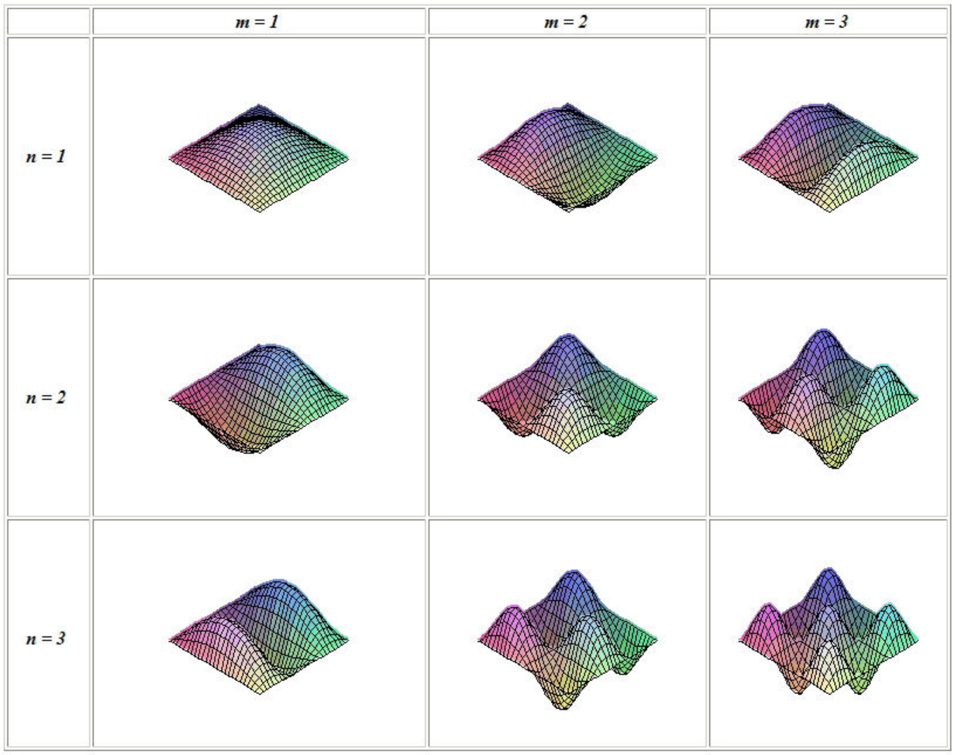

Figure 9.3: A three dimensional view of the vibrating rectangular membrane for the lowest modes. Compare these images with the nodal lines in Figure 9.2

general solution, we obtain

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{9.30}$$

This is a double Fourier sine series. The goal is to find the unknown coefficients a_{nm}. This can be done knowing what we already know about Fourier sine series. We can write the initial condition as the single sum

$$f(x,y) = \sum_{n=1}^{\infty} A_n(y) \sin \frac{n\pi x}{L}, \tag{9.31}$$

where

$$A_n(y) = \sum_{m=1}^{\infty} a_{nm} \sin \frac{m\pi y}{H}. \tag{9.32}$$

These are two Fourier sine series. Recalling that the coefficients of Fourier sine series can be computed as integrals, we have

$$\begin{aligned} A_n(y) &= \frac{2}{L} \int_0^L f(x,y) \sin \frac{n\pi x}{L}\, dx, \\ a_{nm} &= \frac{2}{H} \int_0^H A_n(y) \sin \frac{m\pi y}{H}\, dy. \end{aligned} \tag{9.33}$$

Inserting the integral for $A_n(y)$ into that for a_{nm}, we have an integral representation for the Fourier coefficients in the double Fourier sine series,

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dxdy. \tag{9.34}$$

The Fourier coefficients for the double Fourier sine series.

We can carry out the same process for satisfying the second initial condition, $u_t(x,y,0) = g(x,y)$ for the initial velocity of each point. Inserting this into the general solution, we have

$$g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\omega_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}. \tag{9.35}$$

Again, we have a double Fourier sine series. But, now we can write down Fourier coefficients quickly using the above expression for a_{nm}:

$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dxdy. \tag{9.36}$$

This completes the full solution of the vibrating rectangular membrane problem. Namely, we have obtained the solution

The full solution of the vibrating rectangular membrane.

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}, \tag{9.37}$$

where

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dxdy, \tag{9.38}$$

$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dxdy, \tag{9.39}$$

and the angular frequencies are given by

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{9.40}$$

9.2 *Vibrations of a Kettle Drum*

In this section we consider the vibrations of a circular membrane of radius a as shown in Figure 9.4. Again we are looking for the harmonics of the vibrating membrane, but with the membrane fixed around the circular boundary given by $x^2 + y^2 = a^2$. However, expressing the boundary condition in Cartesian coordinates is awkward. Namely, we can only write $u(x,y,t) = 0$ for $x^2 + y^2 = a^2$. It is more natural to use polar coordinates as indicated in Figure 9.4. Let the height of the membrane be given by $u = u(r,\theta,t)$ at time t and position (r,θ). Now the boundary condition is given as $u(a,\theta,t) = 0$ for all $t > 0$ and $\theta \in [0, 2\pi]$.

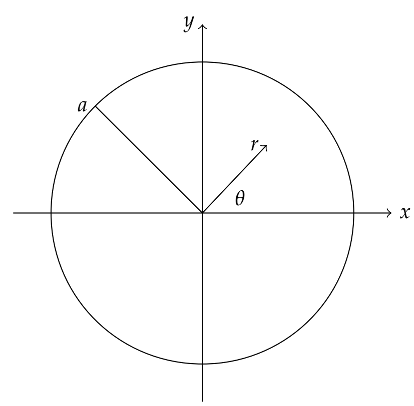

Figure 9.4: The circular membrane of radius a. A general point on the membrane is given by the distance from the center, r, and the angle, . There are fixed boundary conditions along the edge at $r = a$.

Before solving the initial-boundary value problem, we have to cast the full problem in polar coordinates. This means that we need to rewrite the Laplacian in r and θ. To do so would require that we know how to transform derivatives in x and y into derivatives with respect to r and θ. Using the results from Section 8.3 on curvilinear coordinates, we know that the Laplacian can be written in polar coordinates. In fact, we could use the results from Problem 28 for cylindrical coordinates for functions which are z-independent, $f = f(r,\theta)$. Then we would have

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

We can obtain this result using a more direct approach, namely applying the Chain Rule in higher dimensions. First recall the transformation between polar and Cartesian coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}.$$

Now, consider a function $f = f(x(r,\theta), y(r,\theta)) = g(r,\theta)$. (Technically, once we transform a given function of Cartesian coordinates we obtain a new function g of the polar coordinates. Many texts do not rigorously distinguish between the two functions.) Thinking of $x = x(r,\theta)$ and $y = y(r,\theta)$, we have from the chain rule for functions of two variables:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{\partial g}{\partial r}\frac{x}{r} - \frac{\partial g}{\partial \theta}\frac{y}{r^2} \\ &= \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}. \qquad (9.41)\end{aligned}$$

Here we have used

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r};$$

and

$$\frac{\partial \theta}{\partial x} = \frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right) = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{r^2}.$$

Similarly,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \frac{\partial g}{\partial r}\frac{y}{r} + \frac{\partial g}{\partial \theta}\frac{x}{r^2} \\ &= \sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}. \qquad (9.42)\end{aligned}$$

The 2D Laplacian can now be computed as

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right)+\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)\\
&= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial g}{\partial r}+\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial r}+\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&= \cos\theta\left(\cos\theta\frac{\partial^2 g}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial g}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}\right)\\
&\quad-\frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 g}{\partial\theta\partial r}-\frac{\sin\theta}{r}\frac{\partial^2 g}{\partial\theta^2}-\sin\theta\frac{\partial g}{\partial r}-\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\sin\theta\left(\sin\theta\frac{\partial^2 g}{\partial r^2}+\frac{\cos\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}-\frac{\cos\theta}{r^2}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 g}{\partial\theta\partial r}+\frac{\cos\theta}{r}\frac{\partial^2 g}{\partial\theta^2}+\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&= \frac{\partial^2 g}{\partial r^2}+\frac{1}{r}\frac{\partial g}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}. \qquad (9.43)
\end{aligned}$$

The last form often occurs in texts because it is in the form of a Sturm-Liouville operator. Also, it agrees with the result from using the Laplacian written in cylindrical coordinates as given in Problem 28.

Now that we have written the Laplacian in polar coordinates we can pose the problem of a vibrating circular membrane. It is given by a partial differential equation,[1]

$$u_{tt}=c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right], \qquad (9.44)$$

$$t>0,\quad 0<r<a,\quad -\pi<\theta<\pi,$$

the boundary condition,

$$u(a,\theta,t)=0,\quad t>0,\quad -\pi<\theta<\pi, \qquad (9.45)$$

and the initial conditions,

$$u(r,\theta,0)=f(r,\theta),\quad u_t(r,\theta,0)=g(r,\theta). \qquad (9.46)$$

Now we are ready to solve this problem using separation of variables. As before, we can separate out the time dependence. Let $u(r,\theta,t)=T(t)\phi(r,\theta)$. As usual, $T(t)$ can be written in terms of sines and cosines. This leads to the Helmholtz equation,

$$\nabla^2\phi+\lambda\phi=0.$$

[1] Here we state the problem of a vibrating circular membrane. We have chosen $-\pi<\theta<\pi$, but could have just as easily used $0<\theta<2\pi$. The symmetric interval about $\theta=0$ will make the use of boundary conditions simpler.

We now separate the Helmholtz equation by letting $\phi(r,\theta) = R(r)\Theta(\theta)$. This gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 R\Theta}{\partial\theta^2} + \lambda R\Theta = 0. \tag{9.47}$$

Dividing by $u = R\Theta$, as usual, leads to

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \lambda = 0. \tag{9.48}$$

The last term is a constant. The first term is a function of r. However, the middle term involves both r and θ. This can be remedied by multiplying the equation by r^2. Rearranging the resulting equation, we can separate out the θ-dependence from the radial dependence. Letting μ be the separation constant, we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu. \tag{9.49}$$

This gives us two ordinary differential equations:

$$\begin{aligned} \frac{d^2\Theta}{d\theta^2} + \mu\Theta &= 0, \\ r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - \mu)R &= 0. \end{aligned} \tag{9.50}$$

Let's consider the first of these equations. It should look familiar by now. For $\mu > 0$, the general solution is

$$\Theta(\theta) = a\cos\sqrt{\mu}\theta + b\sin\sqrt{\mu}\theta.$$

The next step typically is to apply the boundary conditions in θ. However, when we look at the given boundary conditions in the problem, we do not see anything involving θ. This is a case for which the boundary conditions that are needed are implied and not stated outright.

We can determine the hidden boundary conditions by making some observations. Let's consider the solution corresponding to the endpoints $\theta = \pm\pi$, noting that at these values for any $r < a$ we are at the same physical point. So, we would expect the solution to have the same value at $\theta = -\pi$ as it has at $\theta = \pi$. Namely, the solution is continuous at these physical points. Similarly, we expect the slope of the solution to be the same at these points. This tells us that

The boundary conditions in θ are periodic boundary conditions.

$$\Theta(\pi) = \Theta(-\pi) \quad \Theta'(\pi) = \Theta'(-\pi).$$

Such boundary conditions are called *periodic boundary conditions*.

Let's apply these conditions to the general solution for $\Theta(\theta)$. First, we set $\Theta(\pi) = \Theta(-\pi)$ and use the symmetries of the sine and cosine functions:

$$a\cos\sqrt{\mu}\pi + b\sin\sqrt{\mu}\pi = a\cos\sqrt{\mu}\pi - b\sin\sqrt{\mu}\pi.$$

This implies that

$$\sin\sqrt{\mu}\pi = 0.$$

This can only be true for $\sqrt{\mu} = m$, $m = 0, 1, 2, 3, \ldots$. Therefore, the eigenfunctions are given by

$$\Theta_m(\theta) = a\cos m\theta + b\sin m\theta, \quad m = 0, 1, 2, 3, \ldots.$$

For the other half of the periodic boundary conditions, $\Theta'(\pi) = \Theta'(-\pi)$, we have that

$$-am\sin m\pi + bm\cos m\pi = am\sin m\pi + bm\cos m\pi.$$

But, this gives no new information.

To summarize so far, we have found the general solutions to the temporal and angular equations. The product solutions will have various products of $\{\cos\omega t, \sin\omega t\}$ and $\{\cos m\theta, \sin m\theta\}_{m=0}^{\infty}$. We also know that $\mu = m^2$ and $\omega = c\sqrt{\lambda}$.

That leaves us with the radial equation. Inserting $\mu = m^2$, we have

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - m^2)R = 0. \tag{9.51}$$

A little rewriting,

$$r^2R''(r) + rR'(r) + (\lambda r^2 - m^2)R(r) = 0. \tag{9.52}$$

The reader should recognize this differential equation from Equation **??**. It is a Bessel equation with bounded solutions $R(r) = J_m(\sqrt{\lambda}r)$.

Recall there are two linearly independent solutions of this second order equation: $J_m(\sqrt{\lambda}r)$, the Bessel function of the first kind of order m, and $N_m(\sqrt{\lambda}r)$, the Bessel function of the second kind of order m. Plots of these functions are shown in Figures 5.8 and 5.9. Sometimes the N_m's are called Neumann functions. So, we have the general solution of the radial equation is

$$R(r) = c_1J_m(\sqrt{\lambda}r) + c_2N_m(\sqrt{\lambda}r).$$

Now we are ready to apply the boundary conditions to the radial factor in the product solutions. Looking at the original problem we find only one condition: $u(a, \theta, t) = 0$ for $t > 0$ and $-\pi < \ < \pi$. This implies that $R(0) = 0$. But where is the second condition?

This is another unstated boundary condition. Look again at the plots of the Bessel functions. Notice that the Neumann functions are not well behaved at the origin. Do you expect that the solution will become infinite at the center of the drum? No, the solutions should be finite at the center. So, this observation leads to the second boundary condition. Namely, $|R(0)| < \infty$. This implies that $c_2 = 0$.

Now we are left with

$$R(r) = J_m(\sqrt{\lambda} r).$$

We have set $c_1 = 1$ for simplicity. We can apply the vanishing condition at $r = a$. This gives

$$J_m(\sqrt{\lambda} a) = 0.$$

Looking again at the plots of $J_m(x)$, we see that there are an infinite number of zeros, but they are not as easy as π! In Table 9.1 we list the nth zeros of J_m, which were first seen in Table 5.3.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Table 9.1: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

Let's denote the nth zero of $J_m(x)$ by j_{mn}. Then the boundary condition tells us that

$$\sqrt{\lambda} a = j_{mn}.$$

This gives us the eigenvalue as

$$\lambda_{mn} = \left(\frac{j_{mn}}{a}\right)^2.$$

Thus, the radial function satisfying the boundary conditions is

$$R(r) = J_m\left(\frac{j_{mn}}{a} r\right).$$

We are finally ready to write out the product solutions for the vibrating circular membrane. They are given by

Product solutions for the vibrating circular membrane.

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m\left(\frac{j_{mn}}{a} r\right). \tag{9.53}$$

Here we have indicated choices with the braces, leading to four different types of product solutions. Also, $m = 0, 1, 2, \ldots$, and

$$\omega_{mn} = \frac{j_{mn}}{a} c.$$

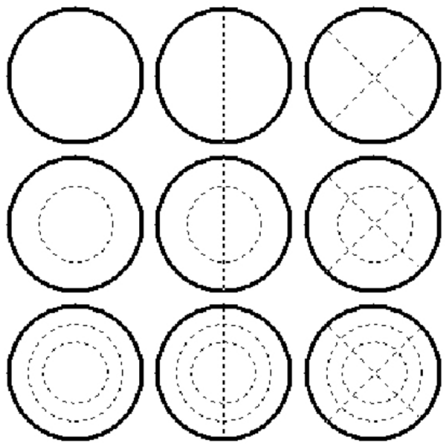

Figure 9.5: The first few modes of the vibrating circular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these nodal lines with the three dimensional images in Figure 9.6.

As with the rectangular membrane, we are interested in the shapes of the harmonics. So, we consider the spatial solution ($t = 0$)

$$\phi(r,\theta) = (\cos m\theta) J_m\left(\frac{j_{mn}}{a} r\right).$$

Including the solutions involving $\sin m\theta$ will only rotate these modes. The nodal curves are given by $\phi(r,\theta) = 0$. This can be satisfied if $\cos m\theta = 0$, or $J_m(\frac{j_{mn}}{a} r) = 0$. The various nodal curves which result are shown in Figure 9.5.

For the angular part, we easily see that the nodal curves are radial lines, $\theta =$const. For $m = 0$, there are no solutions, since $\cos m\theta = 1$ and $\sin m\theta = 1$ for $m = 0$. in Figure 9.5 this is seen by the absence of radial lines in the first column.

For $m = 1$, we have $\cos\theta = 0$. This implies that $\theta = \pm\frac{\pi}{2}$. These values give the vertical line as shown in the second column in Figure 9.5. For $m = 2$, $\cos 2\theta = 0$ implies that $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. This results in the two lines shown in the last column of Figure 9.5.

We can also consider the nodal curves defined by the Bessel functions. We seek values of r for which $\frac{j_{mn}}{a} r$ is a zero of the Bessel function and lies in the interval $[0, a]$. Thus, we have

$$\frac{j_{mn}}{a} r = j_{mj},$$

or

$$r = \frac{j_{mj}}{j_{mn}} a.$$

These will give circles of this radius with $j_{mj} \leq j_{mn}$, or $j \leq n$. The zeros can be found in Table 9.1. For $m = 0$ and $n = 1$, there is only one zero and $r = a$. In fact, for all $n = 1$ modes, there is only one zero and $r = a$. Thus, the first row in Figure 9.5 shows no interior nodal circles.

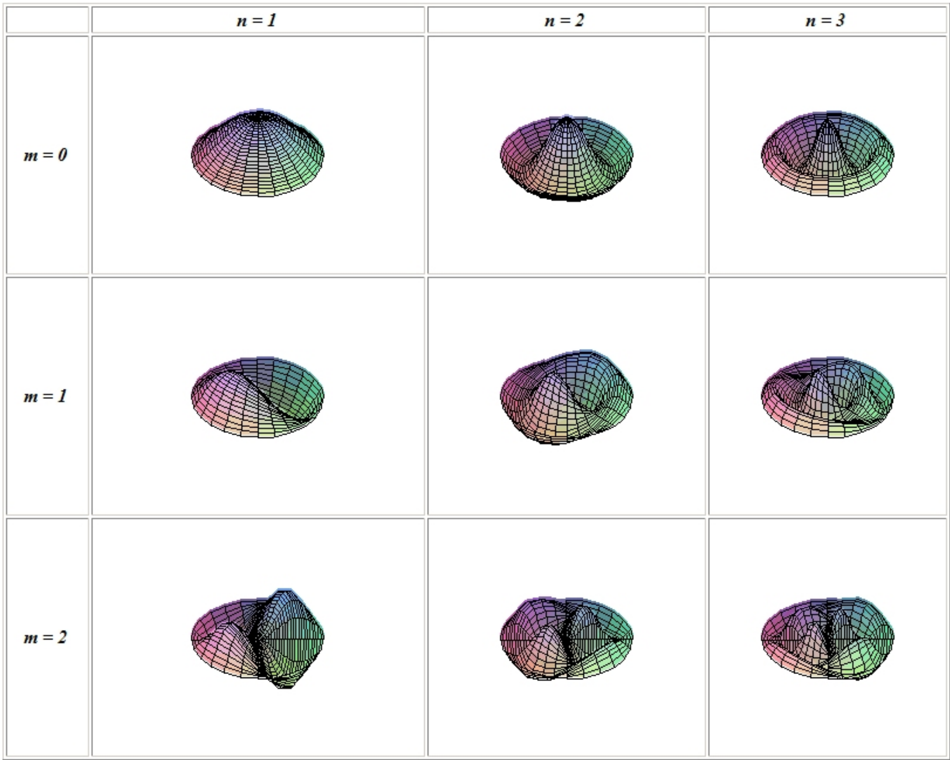

Figure 9.6: A three dimensional view of the vibrating circular membrane for the lowest modes. Compare these images with the nodal line plots in Figure 9.5.

For $n = 2$ modes, we have two circles, $r = a$ and $r = \frac{j_{m1}}{j_{m2}}$ as shown in the second row of Figure 9.5. For $m = 0$,

$$r = \frac{2.405}{5.520}a \approx 0.436a$$

for the inner circle. For $m = 1$,

$$r = \frac{3.832}{7.016}a \approx 0.546a,$$

and for $m = 2$,

$$r = \frac{5.135}{8.147}a \approx 0.630a.$$

For $n = 3$ we obtain circles of radii $r = a$,

$$r = \frac{j_{m1}}{j_{m3}}, \text{ and } r = \frac{j_{m2}}{j_{m3}}.$$

For $m = 0$,

$$r = a, \frac{5.520}{8.654}a \approx 0.638a, \frac{2.405}{8.654}a \approx 0.278a.$$

Similarly, for $m = 1$,

$$r = a, \frac{3.832}{10.173}0.377a \approx a, \frac{7.016}{10.173}a \approx 0.0.690a$$

and for $m = 2$,

$$r = a, \frac{5.135}{11.620}a \approx 0.442a, \frac{8.417}{11.620}a \approx 0.724a.$$

For a three dimensional view, one can look at Figure 9.6. Imagine that the various regions are oscillating independently and that the points on the nodal curves are not moving.

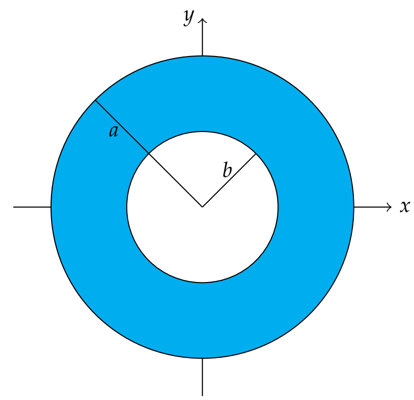

Figure 9.7: An annular membrane with radii a and $b > a$. There are fixed boundary conditions along the edges at $r = a$ and $r = b$.

Example 9.1. *Vibrating Annulus*

More complicated vibrations can be dreamt up for this geometry. We could consider an annulus in which the drum is formed from two concentric circular cylinders and the membrane is stretch between the two with an annular cross section as shown in Figure 9.7. The separation would follow as before except now the boundary conditions are that the membrane is fixed around the two circular boundaries. In this case we cannot toss out the Neumann functions because the origin is not part of the drum head.

With this in mind, we have that the product solutions take the form

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos\omega_{mn}t \\ \sin\omega_{mn}t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} R_m(r), \tag{9.54}$$

where

$$R_m(r) = c_1 J_m(\sqrt{\lambda}r) + c_2 N_m(\sqrt{\lambda}r)$$

and $\omega = c\sqrt{\lambda}$.

For this problem the radial boundary conditions are that the membrane is fixed at $r = a$ *and* $r = b$. *Taking* $b < a$, *we then have to satisfy the conditions*

$$\begin{aligned} R(a) &= c_1 J_m(\sqrt{\lambda}a) + c_2 N_m(\sqrt{\lambda}a) = 0, \\ R(b) &= c_1 J_m(\sqrt{\lambda}b) + c_2 N_m(\sqrt{\lambda}b) = 0. \end{aligned} \tag{9.55}$$

This leads to two homogeneous equations for c_1 *and* c_2. *The coefficient determinant of this system has to vanish if there are to be nontrivial solutions. This gives the eigenvalue equation* λ :

$$J_m(\sqrt{\lambda}a)N_m(\sqrt{\lambda}b) - J_m(\sqrt{\lambda}b)N_m(\sqrt{\lambda}a) = 0.$$

This eigenvalue equation needs to be solved numerically. Choosing $a = 2$ *and* $b = 4$, *we have for the first few modes*

$$\begin{aligned} \sqrt{\lambda_{mn}} &\approx 1.562, 3.137, 4.709, \quad m = 0 \\ &\approx 1.598, 3.156, 4.722, \quad m = 1 \\ &\approx 1.703, 3.214, 4.761, \quad m = 2. \end{aligned} \tag{9.56}$$

Note, since $\sqrt{\lambda_{mn}} = \frac{\omega_{mn}}{c}$, *these numbers essentially give us the frequencies of oscillation.*

For these particular roots, we then solve for c_1 *and* c_2 *by setting* $c_2 = -1$ *and determining*

$$c_1 = \frac{N_m(\sqrt{\lambda_{mn}}b)}{J_m(\sqrt{\lambda_{mn}}b)}$$

(This selection is not unique. We could replace the b's in c_1 with a's and that would work as well.) *This leads to the basic modes of vibration,*

$$R_{mn}(r)\Theta_m(\theta) = \cos m\theta \left(\frac{N_m(\sqrt{\lambda_{mn}}b)}{J_m(\sqrt{\lambda_{mn}}b)} J_m(\sqrt{\lambda_{mn}}r) - N_m(\sqrt{\lambda_{mn}}r) \right).$$

In Figure 9.8 we show various modes for the particular choice of membrane dimensions, $a = 2$ *and* $b = 4$.

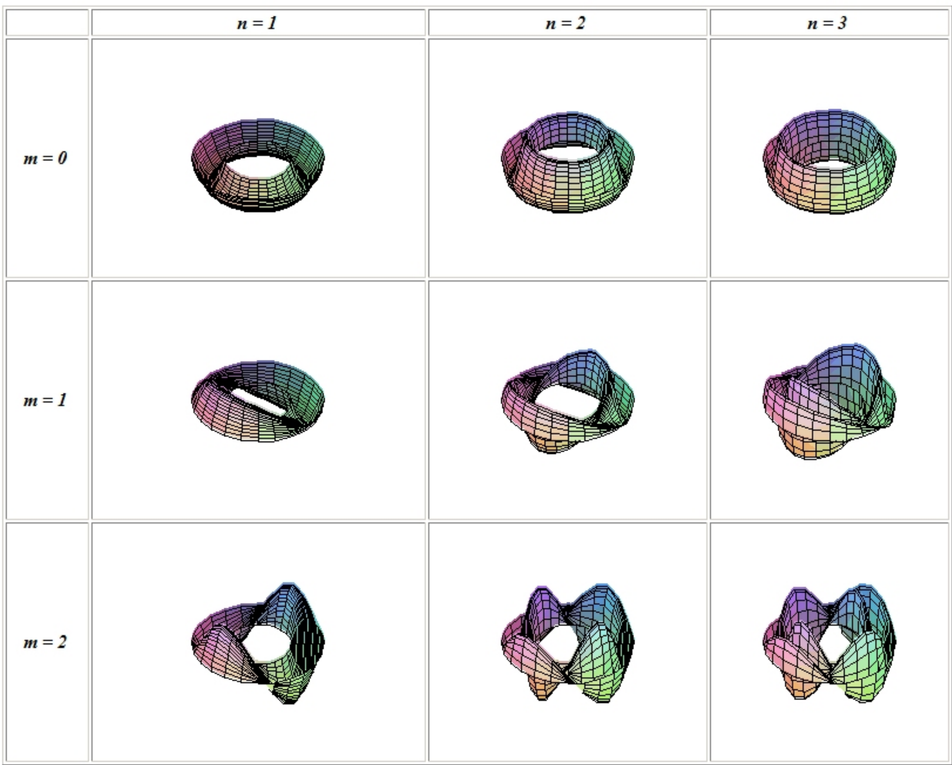

Figure 9.8: A three dimensional view of the vibrating annular membrane for the lowest modes.

9.3 Laplace's Equation in 2D

Another of the generic partial differential equations is Laplace's equation, $\nabla^2 u = 0$. This equation first appeared in the chapter on complex variables when we discussed harmonic functions. Another example is the electric potential for electrostatics. As we described in the last chapter, for static electromagnetic fields, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. Also, $\mathbf{E} = \nabla\phi$. In regions devoid of charge, we have $\nabla^2\phi = 0$.

Another example comes from studying temperature distributions. Consider a thin rectangular plate with the boundaries set at fixed temperatures. One can solve the heat equation. The solution is time dependent. However, if one wait a long time, the plate reaches thermal equilibrium. If the boundary temperature is zero, then the plate temperatures decays to zero. However, keeping the boundaries at a nonzero temperature, which means energies is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, which is $\nabla^2 u = 0$.

Finally, we could look at fluid flow. For an incompressible flow, $\nabla \cdot \mathbf{v} = 0$. If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$. We can introduce a velocity potential, $\mathbf{v} = \nabla\phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, once again we obtain Laplace's equation.

In this section we will look at a couple examples of Laplace's equation in two dimensions. The solutions could be examples of any of the above physical situations and can be determined appropriately.

Example 9.2. *Equilibrium Temperature Distribution for a Rectangular Plate*

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, 0 < y < H$$

with the boundary conditions

$$u(0, y) = 0, u(L, y) = 0, u(x, 0) = f(x), u(x, H) = 0.$$

The boundary conditions are shown in Figure 9.9

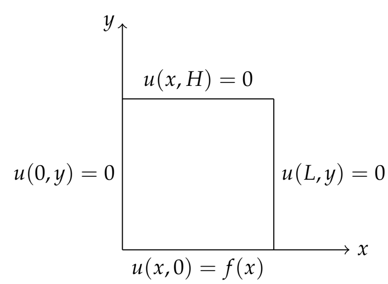

Figure 9.9: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

As usual, we solve this equation using the method of separation of variables. Let $u(x, y) = X(x)Y(y)$. Then Laplace's equation becomes

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda. \tag{9.57}$$

This leads to two differential equations,

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned} \tag{9.58}$$

We next turn to the boundary conditions. Since $u(0, y) = 0, u(L, y) = 0$, we have $X(0) = 0, X(L) = 0$. So, we have an eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0.$$

We can easily write down the solution to this problem,

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, \cdots.$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}.$$

The boundary condition $u(x,H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda}H} + c_2 e^{-\sqrt{\lambda}H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda}H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda}y} - c_1 e^{2\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \\ &= c_1 e^{2\sqrt{\lambda}H}\left(e^{-\sqrt{\lambda}H} e^{\sqrt{\lambda}y} - e^{\sqrt{\lambda}H} e^{-\sqrt{\lambda}y}\right) \\ &= -c_1 e^{2\sqrt{\lambda}H}\left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)}\right) \\ &= -2c_1 e^{2\sqrt{\lambda}H} \sinh\sqrt{\lambda}(H-y). \end{aligned} \tag{9.59}$$

Note: Having carried out this computation, we can now see that it would be better to guess at this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}(H-y)$. For $Y(0) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}y$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh\sqrt{\lambda}(H-y)$.

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that

$$Y_n(y) = \sinh\frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x,y) = \sin\frac{n\pi x}{L}\sinh\frac{n\pi(H-y)}{L}, \quad n = 1,2,\cdots.$$

These solutions satisfy the three homogeneous boundary conditions in the problem.

The remaining boundary condition, $u(x,0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first need to write the general solution, which is a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi(H-y)}{L}. \tag{9.60}$$

Now we apply the boundary condition to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh\frac{n\pi H}{L}\sin\frac{n\pi x}{L}. \tag{9.61}$$

Defining $b_n = a_n \sinh\frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{9.62}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. The Fourier coefficients are easily found as

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{9.63}$$

Finally, we have the solution to this problem,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \tag{9.64}$$

where

$$a_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{9.65}$$

Example 9.3. *Equilibrium Temperature Distribution for a Rectangular Plate for General Boundary Conditions*

Now we consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, 0 < y < H$$

with the non-zero boundary conditions on more than one side,

$$u(0,y) = g_1(y), u(L,y) = g_2(y), u(x,0) = f_1(x), u(x,H) = f_2(x).$$

The boundary conditions are shown in Figure 9.10

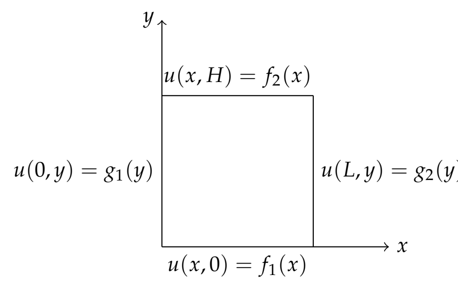

Figure 9.10: In this figure we show the domain and general boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

The problem with this example is that none of the boundary conditions are homogeneous, so we cannot specify the boundary conditions for the eigenvalue problems. However, we can express this problem as in terms of four problems with nonhomogeneous boundary conditions on only one side of the rectangle. In Figure 9.11 we show how the problem can be broken up into four separate problems. Since the boundary conditions and Laplace's equation are linear, the solution to the general problem is simply the sum of the solutions to these four problems.

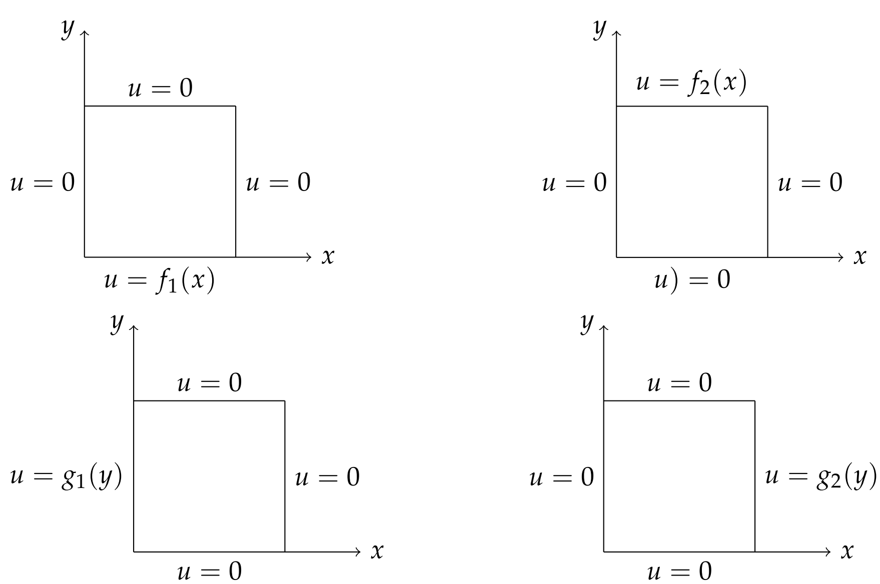

Figure 9.11: Breaking up the general boundary value problem for a rectangular plate.

We can solve each of the problems quickly, based on the solution obtained in the last example. The solution for boundary conditions

$$u(0,y) = 0, u(L,y) = 0, u(x,0) = f_1(x), u(x,H) = 0.$$

is the easiest to write down:

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi(H-y)}{L}. \tag{9.66}$$

where

$$a_n = \frac{2}{L\sinh\frac{n\pi H}{L}} \int_0^L f_1(x)\sin\frac{n\pi x}{L}\,dx. \tag{9.67}$$

For the boundary conditions

$$u(0,y) = 0, u(L,y) = 0, u(x,0) = 0, u(x,H) = f_2(x),$$

the boundary conditions for $X(x)$ are $X(0) = 0$ and $X(L) = 0$. So, we get the same form for the eigenvalues and eigenfunctions as before:

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, \cdots.$$

However, the remaining homogeneous boundary condition is now $Y(0) = 0$. Recalling the equation satisfied by $Y(y)$ is

$$Y'' - \lambda Y = 0,$$

we can write the general solution as

$$Y(y) = c_1 \cosh\sqrt{\lambda}y + c_2\sinh\sqrt{\lambda}y.$$

Requiring $Y(0) = 0$, we have $c_1 = 0$, or

$$Y(y) = c_2 \sinh\sqrt{\lambda}y.$$

Then the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}. \tag{9.68}$$

We now force the nonhomogenous boundary condition, $u(x,H) = f_2(x)$,

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi H}{L}. \tag{9.69}$$

Once again we have a Fourier sine series. The Fourier coefficients are given by

$$b_n = \frac{2}{L\sinh\frac{n\pi H}{L}} \int_0^L f_2(x)\sin\frac{n\pi x}{L}\,dx. \tag{9.70}$$

Now we turn to the problem with the boundary conditions

$$u(0,y) = g_1(y), u(L,y) = 0, u(x,0) = 0, u(x,H) = 0.$$

In this case the pair of homogeneous boundary conditions $u(x,0) = 0, u(x,H) = 0$. lead to solutions

$$Y_n(y) = \sin\frac{n\pi y}{H}, \quad \lambda_n = -\left(\frac{n\pi}{H}\right)^2, n = 1, 2\cdots.$$

The condition $u(L,0)=0$ gives $X(x)=\sinh\frac{n\pi(L-x)}{H}$. The general solution is

$$u(x,y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi(L-x)}{H}. \tag{9.71}$$

We now force the nonhomogenous boundary condition, $u(0,y)=g_1(y)$,

$$g_1(y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}. \tag{9.72}$$

The Fourier coefficients are given by

$$c_n=\frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H g_1(y)\sin\frac{n\pi y}{H}\,dy. \tag{9.73}$$

Finally, we can find the solution for

$$u(0,y)=0, u(L,y)=g_2(y), u(x,0)=0, u(x,H)=0.$$

Following the above analysis, we find the general solution

$$u(x,y)=\sum_{n=1}^{\infty}d_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi x}{H}. \tag{9.74}$$

We now force the nonhomogenous boundary condition, $u(L,y)=g_2(y)$,

$$g_2(y)=\sum_{n=1}^{\infty}d_n\sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}. \tag{9.75}$$

The Fourier coefficients are given by

$$d_n=\frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H g_1(y)\sin\frac{n\pi y}{H}\,dy. \tag{9.76}$$

The solution to the general problem is given by the sum of these four solutions.

$$\begin{aligned}u(x,y) &= \sum_{n=1}^{\infty}\left[\left(a_n\sinh\frac{n\pi(H-y)}{L}+b_n\sinh\frac{n\pi y}{L}\right)\sin\frac{n\pi x}{L}\right.\\ &\quad\left.+\left(c_n\sinh\frac{n\pi(L-x)}{H}+d_n\sinh\frac{n\pi x}{H}\right)\sin\frac{n\pi y}{H}\right],\end{aligned} \tag{9.77}$$

where the coefficients are given by the above Fourier integrals.

Example 9.4. *Laplace's Equation on a Disk*

We now turn to solving Laplace's equation on a disk of radius a as shown in Figure 9.12. Laplace's equation in polar coordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0,\quad 0<r<a,\quad -\pi<\theta<\pi. \tag{9.78}$$

The boundary conditions are given as

$$u(a,\theta) = f(\theta), \quad -\pi < \theta < \pi, \tag{9.79}$$

plus periodic boundary conditions in θ.

Separation of variable proceeds as usual. Let $u(r,\theta) = R(r)\Theta(\theta)$. *Then*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(R\Theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(R\Theta)}{\partial\theta^2} = 0, \tag{9.80}$$

or

$$\Theta\frac{1}{r}(rR')' + \frac{1}{r^2}R\Theta'' = 0. \tag{9.81}$$

Diving by $u(r,\theta) = R(r)\Theta(\theta)$ *and rearranging, we have*

$$\frac{r}{R}(rR')' = -\frac{\Theta''}{\theta} = \lambda. \tag{9.82}$$

Since this equation gives a function of r *equal to a function of* θ, *we set the equation equal to a constant. Thus, we have obtained two differential equations, which can be written as*

$$r(rR')' - \lambda R = 0, \tag{9.83}$$
$$\Theta'' + \lambda\Theta = 0. \tag{9.84}$$

We can solve the second equation, using periodic boundary conditions. The reader should be able to confirm that

$$\Theta(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad \lambda = n^2, n = 0, 1, 2, \cdots$$

is the solution. Note that the $n = 0$ *case just leads to a constant solution.*

Inserting $\lambda = n^2$ *into the radial equation, we find*

$$r^2R'' + rR' - n^2R = 0.$$

This is a Cauchy-Euler type of ordinary differential equation. Recall that we solve such equations by guessing a solution of the form $R(r) = r^m$. *This leads to the characteristic equation* $m^2 - n^2 = 0$. *Therefore,* $m = \pm n$. *So,*

$$R(r) = c_1 r^n + c_2 r^{-n}.$$

Since we expect finite solutions at the origin, $r = 0$, *we can set* $c_2 = 0$. *Thus, the general solution is*

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\, r^n. \tag{9.85}$$

Note that we have taken the constant term out of the sum and put it into a familiar form.

Now were are ready to impose the remaining boundary condition, $u(a,\theta) = f(\theta)$. *This gives*

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\, a^n. \tag{9.86}$$

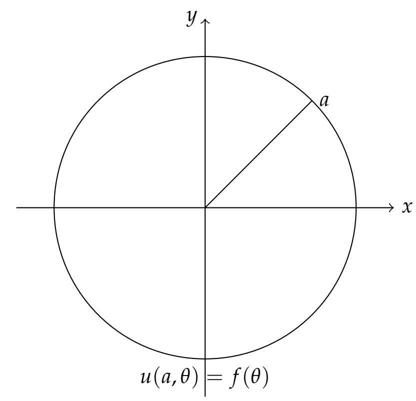

Figure 9.12: The circular plate of radius a with boundary condition along the edge at $r = a$.

This is a Fourier trigonometric series. The Fourier coefficients can be determined using the results from Chapter 4:

$$a_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta, \quad n = 0, 1, \cdots, \tag{9.87}$$

$$b_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta \quad n = 1, 2\cdots. \tag{9.88}$$

We can put the solution from the last example in a more compact form by inserting these coefficients into the general solution. Doing this, we have

$$\begin{aligned} u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos n\theta + b_n\sin n\theta\right) r^n \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi \\ &\quad + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\left[\cos n\phi\cos n\theta + \sin n\phi\sin n\theta\right]\left(\frac{r}{a}\right)^n f(\phi)\, d\phi \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n\right] f(\phi)\, d\phi. \end{aligned} \tag{9.89}$$

The term in the brackets can be summed. We note that $\cos n(\theta - \phi) = Re(e^{in(\theta-\phi)})$. Then

$$\begin{aligned} \cos n(\theta-\phi)\left(\frac{r}{a}\right)^n &= Re\left(e^{i(\theta-\phi)}\left(\frac{r}{a}\right)^n\right) \\ &= Re\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n. \end{aligned} \tag{9.90}$$

Therefore,

$$\sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n = Re\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n\right)$$

The right hand side of this equation is a geometric series with common ratio $\frac{r}{a}e^{i(\theta-\phi)}$. Since $\left|\frac{r}{a}e^{i(\theta-\phi)}\right| = \frac{r}{a} < 1$, the series converges. Summing the series, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n &= \frac{\frac{r}{a}e^{i(\theta-\phi)}}{1-\frac{r}{a}e^{i(\theta-\phi)}} \\ &= \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}} \\ &= \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}\frac{a-re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}} \\ &= \frac{are^{-i(\theta-\phi)}-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}. \end{aligned} \tag{9.91}$$

We have rewritten this sum so that we can easily take the real part,

$$Re\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n\right) = \frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}$$

Therefore, the factor in the brackets under the integral in Equation (9.89) is

$$\begin{aligned}\frac{1}{2}+\sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n &= \frac{1}{2}+\frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)} \\ &= \frac{a^2-r^2}{2(a^2+r^2-2ar\cos(\theta-\phi))}. \quad (9.92)\end{aligned}$$

Thus, we have shown that the solution of Laplace's equation on a disk of radius a with boundary condition $u(a,\theta)=f(\theta)$ can be written in the closed form

Poisson Integral Formula

$$u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}f(\phi)\,d\phi. \quad (9.93)$$

This result is called the Poisson Integral Formula and

$$K(\theta,\phi)=\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}$$

is called the Poisson kernel.

9.4 *Three Dimensional Cake Baking*

IN THE REST OF THE CHAPTER we will extend our studies to three dimensions. In this section we will solve the heat equation as we look at examples of baking cakes. We consider cake batter, which is at room temperature of $T_i = 80^\circ$F. It is placed into an oven, also at a fixed temperature, $T_b = 350^\circ$F. For simplicity, we will assume that the thermal conductivity and cake density are constant. Of course, this is not quite true. However, it is an approximation which simplifies the model. We will consider two cases, one in which the cake is a rectangular solid ($0 \le x \le W, 0 \le y \le L, 0 \le z \le H$), such as baking it in a $13'' \times 9'' \times 2''$ baking pan. The other case will lead to a cylindrical cake, such as you would obtain from a round cake pan.

This discussion of cake baking is adapted from R. Wilkinson's thesis work. That in turn was inspired by work done by Dr. Olszewski.

Assuming that the heat constant k is indeed constant and the temperature is given by $T(\mathbf{r},t)$, we begin with the heat equation in three dimensions,

$$\frac{\partial T}{\partial t}=k\nabla^2 T. \quad (9.94)$$

We will need to specify initial and boundary conditions. Let T_i be the initial batter temperature, and write the initial condition as

$$T(x,y,z,0)=T_i.$$

We choose the boundary conditions to be fixed at the oven temperature T_b. However, these boundary conditions are not homogeneous and would lead to problems when carrying out separation of variables. This is easily remedied by subtracting the oven temperature from all temperatures involved and defining $u(x,y,z,t) = T(x,y,z,t) - T_b$. The heat equation then becomes

$$\frac{\partial u}{\partial t} = k\nabla^2 u \tag{9.95}$$

with initial condition

$$u(\mathbf{r},0) = T_i - T_b.$$

The boundary conditions are now that $u = 0$ on the boundary. We cannot be any more specific than this until we specify the geometry.

Example 9.5. *Temperature of a Rectangular Cake*

For this problem, we seek solutions of the heat equation plus the conditions

$$\begin{aligned} u(x,y,z,0) &= T_i - T_b, \\ u(0,y,z,t) = u(W,y,z,t) &= 0, \\ u(x,0,z,t) = u(x,L,z,t) &= 0, \\ u(x,y,0,t) = u(x,y,H,t) &= 0. \end{aligned}$$

Using the method of separation of variables, we seek solutions of the form

$$u(x,y,z,t) = X(x)Y(y)Z(z)G(t). \tag{9.96}$$

Substituting this form into the heat equation, we get

$$\frac{1}{k}\frac{G'}{G} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{9.97}$$

Setting these expressions equal to $-\lambda$, we get

$$\frac{1}{k}\frac{G'}{G} = -\lambda \quad \text{and} \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda. \tag{9.98}$$

Therefore, the equation for $G(t)$ is given by

$$G' + k\lambda G = 0.$$

We further have to separate out the functions of x, y, and z. We anticipate that the homogeneous boundary conditions will lead to oscillatory solutions in these variables. Therefore, we expect separation of variable will lead to the eigenvalue problems

$$\begin{aligned} X'' + \mu^2 X &= 0, \quad X(0) = X(W) = 0, \\ Y'' + \nu^2 Y &= 0, \quad Y(0) = Y(L) = 0, \\ Z'' + \kappa^2 Z &= 0, \quad Z(0) = Z(H) = 0. \end{aligned} \tag{9.99}$$

Noting that

$$\frac{X''}{X} = -\mu^2, \quad \frac{Y''}{Y} = -\nu^2, \quad \frac{Z''}{Z} = -\kappa^2,$$

we have the relation $\lambda^2 = \mu^2 + \nu^2 + \kappa^2$.

From the boundary conditions, we get product solutions for $u(x,y,z,t)$ *in the form*

$$u_{mn\ell}(x,y,z,t) = \sin\mu_m x \sin\nu_n y \sin\kappa_\ell z e^{-\lambda_{mn\ell}kt},$$

for

$$\lambda_{mnl} = \mu_m^2 + \nu_n^2 + \kappa_\ell^2 = \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{\ell\pi}{H}\right)^2, \quad m,n,\ell = 1,2,\ldots.$$

The general solution is then

$$u(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl}\sin\mu_m x \sin\nu_n y \sin\kappa_\ell z e^{-\lambda_{mn\ell}kt}, \quad (9.100)$$

where the $A_{mn\ell}$*'s are arbitrary constants.*

We can use the initial condition $u(x,y,z,0) = T_i - T_b$ *to determine the* $A_{mn\ell}$*'s. We find*

$$T_i - T_b = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl}\sin\mu_m x \sin\nu_n y \sin\kappa_\ell z. \quad (9.101)$$

This is a triple Fourier sine series. We can determine these coefficients in a manner similar to how we handled a double Fourier sine series earlier. Defining

$$b_m(y,z) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl}\sin\nu_n y \sin\kappa_\ell z,$$

we obtain a simple Fourier sine series:

$$T_i - T_b = \sum_{m=1}^{\infty} b_m(y,z)\sin\mu_m x. \quad (9.102)$$

The Fourier coefficients can then be found as

$$b_m(y,z) = \frac{2}{W}\int_0^W (T_i - T_b)\sin\mu_m x\, dx.$$

Using the same technique for the remaining sine series and noting that $T_i - T_b$ *is constant, we can compute the general coefficient* A_{mnl} *by carrying out the needed integrations:*

$$\begin{aligned}
A_{mnl} &= \frac{8}{WLH}\int_0^H\int_0^L\int_0^W (T_i - T_b)\sin\mu_m x \sin\nu_n y \sin\kappa_\ell z\, dxdydz \\
&= (T_i - T_b)\frac{8}{\pi^3}\left[\frac{\cos\left(\frac{m\pi x}{W}\right)}{m}\right]_0^W \left[\frac{\cos\left(\frac{n\pi y}{L}\right)}{n}\right]_0^L \left[\frac{\cos\left(\frac{\ell\pi z}{H}\right)}{\ell}\right]_0^H \\
&= (T_i - T_b)\frac{8}{\pi^3}\left[\frac{\cos m\pi - 1}{m}\right]\left[\frac{\cos n\pi - 1}{n}\right]\left[\frac{\cos\ell\pi - 1}{\ell}\right] \\
&= (T_i - T_b)\frac{8}{\pi^3}\begin{cases} 0, & \text{for at least one } m,n,\ell \text{ even,} \\ \left[\frac{-2}{m}\right]\left[\frac{-2}{n}\right]\left[\frac{-2}{\ell}\right], & \text{for } m,n,\ell \text{ all odd.} \end{cases}
\end{aligned}$$

Since only the odd multiples yield non-zero $A_{mn\ell}$ we let $m = 2m' - 1$, $n = 2n' - 1$, and $\ell = 2\ell' - 1$. Thus

$$A_{mnl} = \frac{-64(T_i - T_b)}{(2m'-1)\,(2n'-1)\,(2\ell'-1)\,\pi^3}.$$

Substituting this result into general solution and dropping the primes, we find

$$u(x,y,z,t) = \frac{-64(T_i - T_b)}{\pi^3} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{\sin\mu_m x \sin\nu_n y \sin\kappa_\ell z e^{-\lambda_{mn\ell}kt}}{(2m-1)(2n-1)(2\ell-1)},$$

where

$$\lambda_{mn\ell} = \left(\frac{(2m-1)\pi}{W}\right)^2 + \left(\frac{(2n-1)\pi}{L}\right)^2 + \left(\frac{(2\ell-1)\pi}{H}\right)^2$$

for $m, n, \ell = 1, 2, \ldots$.

Recalling $T(x,y,z,t) = u(x,y,z,t) - T_b$,

$$T(x,y,z,t) = T_b - \frac{64(T_i - T_b)}{\pi^3} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{\sin\hat{\mu}_m x \sin\hat{\nu}_n y \sin\hat{\kappa}_\ell z e^{-\hat{\lambda}_{mn\ell}kt}}{(2m-1)(2n-1)(2\ell-1)}.$$

We show some temperature distributions in Figure 9.13. Vertical slices are taken at the positions and times indicated for a $13'' \times 9'' \times 2''$ cake. Obviously, this is not accurate because the cake consistency is changing and this will affect the parameter k. A more realistic model would be to allow $k = k(T(x,y,z,t))$. however, such problems are beyond the simple methods described in this book.

Example 9.6. *Circular Cakes*

In this case the geometry is cylindrical. Therefore, we need to express the boundary conditions and heat equation in cylindrical coordinates.

We assume $u(r,z,t) = T(r,z,t) - T_b$ is independent of θ due to symmetry. This gives the heat equation in cylindrical coordinates as

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \tag{9.103}$$

where $0 \le r \le a$ and $0 \le z \le Z$. The initial condition is

$$u(r,z,0) = T_i - T_b,$$

and the homogeneous boundary conditions are

$$u(a,z,t) = 0,$$
$$u(r,0,t) = u(r,Z,t) = 0.$$

Again, we seek solutions of the form $u(r,z,t) = R(r)H(z)G(t)$. Separation of variables leads to

$$\frac{1}{k}\frac{G'}{G} = \frac{1}{rR}\frac{d}{dr}\left(rR'\right) + \frac{H''}{H}. \tag{9.104}$$

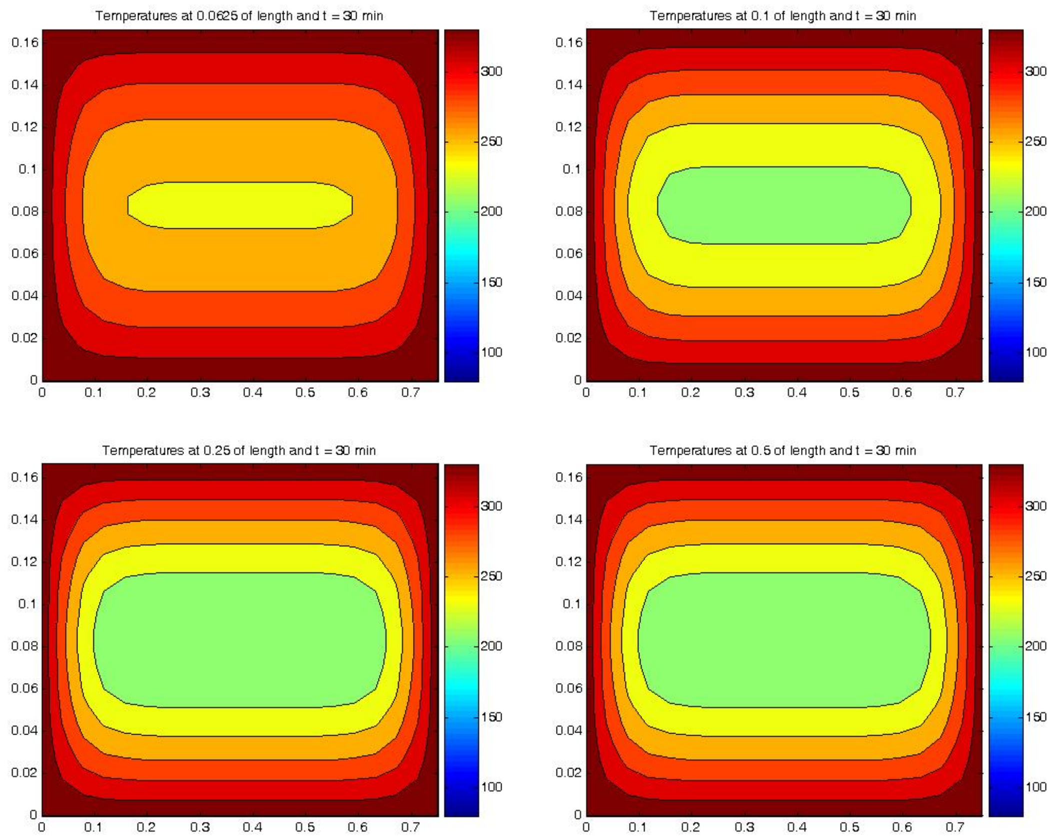

Figure 9.13: Temperature evolution for a $13'' \times 9'' \times 2''$ cake shown as vertical slices at the indicated length in feet.

Choosing λ as the separation constant, we get

$$G' - k\lambda G = 0, \tag{9.105}$$

and

$$\frac{1}{rR}\frac{d}{dr}\left(rR'\right) = -\frac{H''}{H} + \lambda. \tag{9.106}$$

Since negative eigenvalues yield the oscillatory solutions we expect, we continue as before by setting both sides of this equation equal to $-\mu^2$. After some rearrangement, we obtain the needed differential equations:

$$\frac{d}{dr}\left(rR'\right) + r\mu^2 R = 0 \tag{9.107}$$

and

$$H'' + \nu^2 H = 0. \tag{9.108}$$

Here $\lambda = -\left(\mu^2 + \nu^2\right)$.

We can easily write down the solutions

$$G(t) = Ae^{\lambda kt}$$

and

$$H_n(z) = \sin\left(\frac{n\pi z}{Z}\right), n = 1, 2, 3, \ldots,$$

where $\nu = \frac{n\pi}{Z}$. Recalling from the rectangular case that only odd terms arise in the Fourier sine series coefficients for the constant initial condition, we proceed by rewriting $H(z)$ as

$$H_n(z) = \sin\left(\frac{(2n-1)\,\pi z}{Z}\right), n = 1, 2, 3, \ldots \tag{9.109}$$

with $\nu = \frac{(2n-1)\pi}{Z}$.

The radial equation can be written in the form

$$r^2 R'' + rR' + r^2\mu^2 R = 0.$$

This is a Bessel equation of the first kind of order zero and the general solution is a linear combination of Bessel functions of the first and second kind,

$$R(r) = c_1 J_0(\mu r) + c_2 N_0(\mu r). \tag{9.110}$$

Since we wish to have $u(r,z,t)$ bounded at $r = 0$ and $N_0(\mu r)$ is not well behaved at $r = 0$, we set $c_2 = 0$. Up to a constant factor, the solution becomes

$$R(r) = J_0(\mu r). \tag{9.111}$$

The boundary condition $R(a) = 0$ gives $J_0(\mu a) = 0$ and thus $\mu_m = \frac{j_{0m}}{a}$, for $m = 1, 2, 3, \ldots$. Here the j_{0m}'s are the m^{th} roots of the zeroth-order Bessel function, $J_0(j_{0m}) = 0$, which are given in Table 9.1. This suggests that

$$R_m(r) = J_0\left(\frac{r}{a} j_{0m}\right), m = 1, 2, 3, \ldots. \tag{9.112}$$

Thus, we have found that the general solution is given as

$$u(r,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{(2n-1)\,\pi z}{Z}\right) J_0\left(\frac{r}{a} j_{0m}\right) e^{-\lambda_{nm} kt} \tag{9.113}$$

with

$$\lambda_{nm} = \left(\left(\frac{(2n-1)\pi}{Z}\right)^2 + \left(\frac{j_{0m}}{a}\right)^2\right),$$

for $n, m = 1, 2, 3, \ldots$.

Using the constant initial condition to find the A_{nm}'s, we have

$$T_i - T_b = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left[\frac{(2n-1)\pi z}{Z}\right] J_0\left(\frac{r}{a} j_{0m}\right).$$

If we let $b_n(r) = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right)$, we have

$$T_i - T_b = \sum_{n=1}^{\infty} b_n(r) \sin\left(\frac{(2n-1)\pi z}{Z}\right).$$

As seen previously, this is a Fourier sine series and the Fourier coefficients are given by

$$\begin{aligned} b_n(r) &= \frac{2}{Z}\int_0^Z (T_i - T_b)\sin\left(\frac{(2n-1)\,\pi z}{Z}\right)dz \\ &= \frac{2(T_i-T_b)}{Z}\left[-\frac{Z}{(2n-1)\,\pi}\cos\left(\frac{(2n-1)\,\pi z}{Z}\right)\right]_0^Z \\ &= \frac{4(T_i-T_b)}{(2n-1)\,\pi}. \end{aligned}$$

Then, we have

$$b_n(r) = \frac{4(T_i-T_b)}{(2n-1)\,\pi} = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right).$$

This is a Fourier-Bessel series. Given $b_n(r) = \frac{4(T_i-T_b)}{(2n-1)\pi}$, we seek to find the Fourier coefficients A_{nm}. Recall from Chapter 5 that the Fourier-Bessel series is given by

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}\frac{x}{a}), \tag{9.114}$$

where the Fourier-Bessel coefficients are found as

$$c_n = \frac{2}{a^2\left[J_{p+1}(j_{pn})\right]^2}\int_0^a x f(x) J_p(j_{pn}\frac{x}{a})\,dx. \tag{9.115}$$

For this problem, we have

$$A_{nm} = \frac{2}{a^2 J_1^2(j_{0m})}\frac{4(T_i-T_b)}{(2n-1)\,\pi}\int_0^a J_0(\mu_m r) r\,dr. \tag{9.116}$$

In order to evaluate $\int_0^a J_0(\mu_k r) r\,dr$, we let $y = \mu_k r$ and get

$$\begin{aligned} \int_0^a J_0(\mu_k r) r dr &= \int_0^{\mu_k a} J_0(y)\frac{y}{\mu_k}\frac{dy}{\mu_k} \\ &= \frac{1}{\mu_k^2}\int_0^{\mu_k a} J_0(y) y\,dy \\ &= \frac{1}{\mu_k^2}\int_0^{\mu_k a}\frac{d}{dy}\left(y J_1(y)\right)\,dy \\ &= \frac{1}{\mu_k^2}(\mu_k a) J_1(\mu_k a) = \frac{a^2}{j_{0k}} J_1(j_{0k}). \end{aligned} \tag{9.117}$$

Here we have made use of the identity $\frac{d}{dx}\left(x J_1(x)\right) = J_0(x)$.

Substituting the result of this integral computation into the expression for A_{nm}, we find

$$A_{nm} = \frac{8(T_i-T_b)}{(2n-1)\pi}\frac{1}{j_{0m} J_1(j_{0m})}.$$

Substituting A_{nm} into the original expression for $u(r,z,t)$, gives

$$u(r,z,t) = \frac{8(T_i-T_b)}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{(2n-1)\pi z}{Z}\right)}{(2n-1)}\frac{J_0(\frac{r}{a} j_{0m}) e^{\lambda_{nm} D t}}{j_{0m} J_1(j_{0m})}.$$

Therefore, $T(r,z,t)$ can be found as

$$T(r,z,t) = T_b + \frac{8(T_i - T_b)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin\left(\frac{(2n-1)\pi z}{Z}\right)}{(2n-1)} \frac{J_0(\frac{r}{a} j_{0m}) e^{\lambda_{nm} kt}}{j_{0m} J_1(j_{0m})}.$$

We have therefore found the general solution for the three-dimensional heat equation in cylindrical coordinates with constant diffusivity. Similar to the solutions shown in Figure 9.13 of the previous section, we show in Figure 9.14 the temperature temperature evolution throughout a standard 9″ round cake pan.

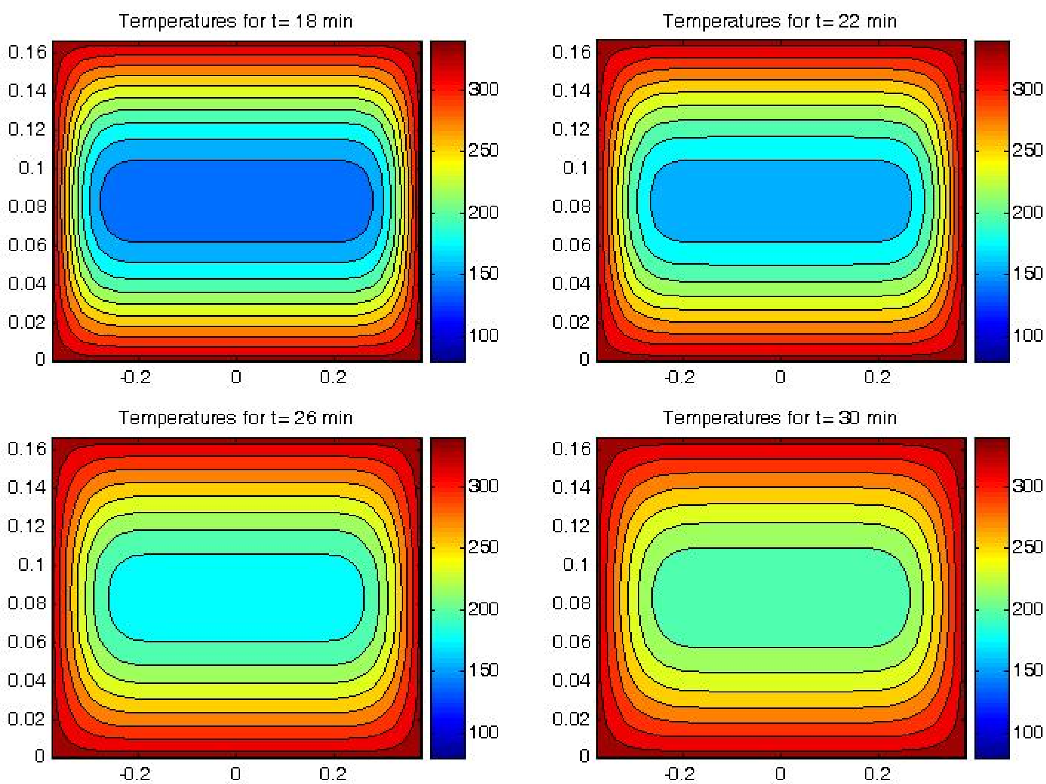

Figure 9.14: Temperature evolution for a standard 9″ cake shown as vertical slices through the center.

9.5 *Laplace's Equation and Spherical Symmetry*

WE HAVE SEEN THAT LAPLACE'S EQUATION, $\nabla^2 u = 0$, arises in electrostatics as an equation for electric potential outside a charge distribution and it occurs as the equation governing equilibrium temperature distributions. As we had seen in the last chapter, Laplace's equation generally occurs in the study of potential theory, which also includes the study of gravitational and fluid potentials. The equation is named after Pierre-Simon Laplace (1749-1827) who had studied the properties of this equation. solutions of Laplace's equation are called harmonic functions.

Laplace's equation in spherical coordinates is given by[2]

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0, \quad (9.118)$$

where $u = u(\rho, \theta, \phi)$.

[2] The Laplacian in spherical coordinates is given in Problem 29 in Chapter 8.

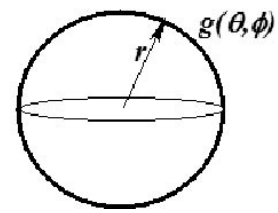

Figure 9.15: A sphere of radius r with the boundary condition $u(r, \theta, \phi) = g(\theta, \phi)$.

We seek solutions of this equation inside a sphere of radius r subject to the boundary condition $u(r, \theta, \phi) = g(\theta, \phi)$ as shown in Figure 9.15.

As before, we perform a separation of variables by seeking product solutions of the form $u(\rho, \theta, \phi) = R(\rho)\Theta(\theta)\Phi(\phi)$. Inserting this form into the Laplace equation, we obtain

$$\frac{\Theta\Phi}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{R\Phi}{\rho^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{R\Theta}{\rho^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0. \quad (9.119)$$

Multiplying this equation by ρ^2 and dividing by $R\Theta\Phi$, yields

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)+\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=0. \quad (9.120)$$

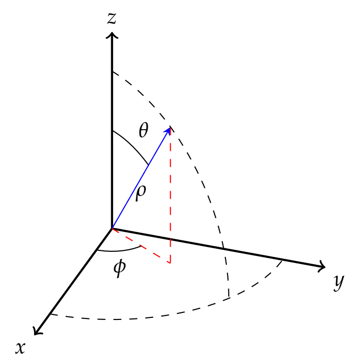

Figure 9.16: Definition of spherical coordinates (ρ, θ, ϕ). Note that there are different conventions for labeling spherical coordinates. This labeling is used often in physics.

Note that the first term is the only term depending upon ρ. Thus, we can separate out the radial part. However, there is still more work to do on the other two terms, which give the angular dependence. Thus, we have

$$-\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)=\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda, \quad (9.121)$$

where we have introduced the first separation constant. This leads to two equations:

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)-\lambda R=0 \quad (9.122)$$

and

$$\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda. \quad (9.123)$$

Equation (9.123) is a key equation which occurs when studying problems possessing spherical symmetry. It is an eigenvalue problem for $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$, $LY = -\lambda Y$, where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The eigenfunctions of this operator are referred to as spherical harmonics.

The final separation can be performed by multiplying the last equation by $\sin^2\theta$, rearranging the terms, and introducing a second separation constant:

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\lambda\sin^2\theta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\mu. \quad (9.124)$$

From this expression we can determine the differential equations satisfied by $\Theta(\theta)$ and $\Phi(\phi)$:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+(\lambda\sin^2\theta-\mu)\Theta=0, \quad (9.125)$$

and

$$\frac{d^2\Phi}{d\phi^2} + \mu\Phi = 0. \tag{9.126}$$

We now have three ordinary differential equations to solve. These are the radial equation (9.122) and the two angular equations (9.125)-(9.126). We note that all three are in Sturm-Liouville form. We will solve each eigenvalue problem subject to appropriate boundary conditions.

The simplest of these differential equations is the one for $\Phi(\phi)$, Equation (9.126). We have seen equations of this form many times and the general solution is a linear combination of sines and cosines. As argues in such problems, we have to impose periodic boundary conditions. For example, we expect that

$$u(\rho,\theta,0) = u(\rho,\theta,2\pi), \quad u_\phi(\rho,\theta,0) = u_\phi(\rho,\theta,2\pi).$$

Since these conidtions hold for all ρ and θ, we must require that

$$\Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi).$$

As we have seen before, the eigenfunctions and eigenvalues are then found as

$$\Phi(\phi) = \{\cos m\phi, \sin m\phi\}, \quad \mu = m^2, \quad m = 0, 1, \ldots. \tag{9.127}$$

Next we turn to solving equation, (9.126). We first transform this equation in order to identify the solutions. Let $x = \cos\theta$. Then the derivatives with respect to θ transform as

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = \sin\theta\frac{d}{dx}.$$

Letting $y(x) = \Theta(\theta)$ and noting that $\sin^2\theta = 1 - x^2$, Equation (9.126) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0. \tag{9.128}$$

We further note that $x \in [-1, 1]$, as can be easily confirmed by the reader.

This is a Sturm-Liouville eigenvalue problem. The solutions consist of a set of orthogonal eigenfunctions. For the special case that $m = 0$ Equation (9.128) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \lambda y = 0. \tag{9.129}$$

In a course in differential equations one learns to seek solutions of this equation in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

This leads to the recursion relation

$$a_{n+2} = \frac{n(n+1)-\lambda}{(n+2)(n+1)} a_n.$$

Setting $n = 0$ and seeking a series solution, one finds that the resulting series does not converge for $x = \pm 1$. This is remedied by choosing $\lambda = \ell(\ell+1)$ for $\ell = 0, 1, \ldots$, leading to the differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \ell(\ell+1)y = 0. \tag{9.130}$$

We saw this equation in Chapter 5. The solutions of this Legendre differential equation are the Legendre polynomials, denoted by $P_\ell(x)$.

For the more general case, $m \neq 0$, the differential equation (9.128) with $\lambda = \ell(\ell+1)$ becomes

Associated Legendre Functions

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)y = 0. \tag{9.131}$$

The solutions of this equation are called the associated Legendre functions. The two linearly independent solutions are denoted by $P_\ell^m(x)$ and $Q_\ell^m(x)$. The latter functions are not well behaved at $x = \pm 1$, corresponding to the north and south poles of the original problem. So, we can throw out these solutions, leaving

$$\Theta(\theta) = P_\ell^m(\cos\theta)$$

as the needed solutions. In Table 9.2 we list a few of these.

	$P_n^m(x)$	$P_n^m(\cos\theta)$
$P_1^0(x)$	x	$\cos\theta$
$P_1^1(x)$	$(1-x^2)^{\frac{1}{2}}$	$\sin\theta$
$P_2^0(x)$	$\frac{1}{2}(3x^2-1)$	$\frac{1}{2}(\cos^2\theta-1)$
$P_2^1(x)$	$3x(1-x^2)^{\frac{1}{2}}$	$3\cos\theta\sin\theta$
$P_2^2(x)$	$3(1-x^2)$	$3\sin^2\theta$
$P_3^0(x)$	$\frac{1}{2}(5x^3-32x)$	$\frac{1}{2}(5\cos^3\theta-3\cos\theta)$
$P_3^1(x)$	$\frac{3}{2}(5x^2-1)(1-x^2)^{\frac{1}{2}}$	$\frac{3}{2}(5\cos^2\theta-1)\sin\theta$
$P_3^2(x)$	$15x(1-x^2)$	$15\cos\theta\sin^2\theta$
$P_3^3(x)$	$15(1-x^2)^{\frac{3}{2}}$	$15\sin^3\theta$

Table 9.2: Associated Legendre Functions, $P_n^m(x)$.

The associated Legendre functions are related to the Legendre polynomials by[3]

$$P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x), \tag{9.132}$$

for $\ell = 0, 1, 2, , \ldots$ and $m = 0, 1, \ldots, \ell$. We further note that $P_\ell^0(x) = P_\ell(x)$, as one can see in the table. Since $P_\ell(x)$ is a polynomial of degree

[3] Some definitions do not include the $(-1)^m$ factor.

ℓ, then for $m > \ell$, $\frac{d^m}{dx^m}P_\ell(x) = 0$ and $P_\ell^m(x) = 0$. Furthermore, since the differential equation only depends on m^2, $P_\ell^{-m}(x)$ is proportional to $P_\ell^m(x)$. One normalization is given by

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x).$$

The associated Legendre functions also satisfy the orthogonality condition

$$\int_{-1}^{1} P_\ell^m(x) P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}. \tag{9.133}$$

The last differential equation we need to solve is the radial equation. With $\lambda = \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$, the radial equation (9.122) can be written as

$$\rho^2 R'' + 2\rho R' - \ell(\ell+1)R = 0. \tag{9.134}$$

The radial equation is a Cauchy-Euler type of equation. So, we can guess the form of the solution to be $R(\rho) = \rho^s$, where s is a yet to be determined constant. Inserting this guess, we obtain the characteristic equation

$$s(s+1) = \ell(\ell+1).$$

Solving for s, we have

$$s = \ell, -(\ell+1).$$

Thus, the general solution of the radial equation is

$$R(\rho) = a\rho^\ell + b\rho^{-(\ell+1)}. \tag{9.135}$$

We would normally apply boundary conditions at this point. Recall that we gave that for $\rho = r$, $u(r,\theta,\phi) = g(\theta,\phi)$. This is not a homogeneous boundary condition, so we will need to hold off using it until we have the general solution to the three dimensional problem. However, we do have a hidden condition. Since we are interested in solutions inside the sphere, we need to consider what happens at $\rho = 0$. Note that $\rho^{-(\ell+1)}$ is not defined at the origin. Since the solution is expected to be bounded at the origin, we can set $b = 0$. So, in the current problem we have established that

$$R(\rho) = a\rho^\ell.$$

We have carried out the full separation of Laplace's equation in spherical coordinates. The product solutions consist of the forms

$$u(\rho,\theta,\phi) = \rho^\ell P_\ell^m(\cos\theta)\cos m\phi$$

and

$$u(\rho,\theta,\phi) = \rho^\ell P_\ell^m(\cos\theta)\sin m\phi$$

for $\ell = 0, 1, 2, \ldots$ and $m = 0, \pm 1, , \ldots, \pm \ell$. These solutions can be combined to give a complex representation of the product solutions as

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta) e^{im\phi}.$$

The general solution is then given as a linear combination of these product solutions. As there are two indices, we have a double sum:[4]

$$u(\rho, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \rho^\ell P_\ell^m(\cos\theta) e^{im\phi}. \tag{9.136}$$

[4] While this appears to be a complex-valued solution, it can be rewritten as a sum over real functions. The inner sum contains terms for both $m = k$ and $m = -k$. Adding these contributions, we have that

$$a_{\ell k} \rho^\ell P_\ell^k(\cos\theta) e^{ik\phi} + a_{\ell(-k)} \rho^\ell P_\ell^{-k}(\cos\theta) e^{-ik\phi}$$

can be rewritten as

$$(A_{\ell k} \cos k\phi + B_{\ell k} \sin k\phi) \rho^\ell P_\ell^k(\cos\theta).$$

$Y_{\ell m}(\theta, \phi)$, are the spherical harmonics. Spherical harmonics are important in applications from atomic electron configurations to gravitational fields, planetary magnetic fields, and the cosmic microwave background radiation.

The solutions of the angular parts of the problem are often combined into one function of two variables, as problems with spherical symmetry arise often, leaving the main differences between such problems confined to the radial equation. These functions are referred to as spherical harmonics, $Y_{\ell m}(\theta, \phi)$, which are defined with a special normalization as

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_\ell^m e^{im\phi}. \tag{9.137}$$

These satisfy the simple orthogonality relation

$$\int_0^\pi \int_0^{2\pi} Y_{\ell m}(\theta, \phi) Y^*_{\ell' m'}(\theta, \phi) \sin\theta \, d\phi \, d\theta = \delta_{\ell\ell'} \delta_{mm'}.$$

As noted in an earlier side note, the spherical harmonics are eigenfunctions of the eigenvalue problem $LY = -\lambda Y$, where

$$L = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}.$$

This operator appears in many problems in which there is spherical symmetry, such as obtaining the solution of Schrödinger's equation for the hydrogen atom as we will see later. Therefore, it is customary to plot spherical harmonics. Because the $Y_{\ell m}$'s are complex functions, one typically plots either the real part or the modulus squared. One rendition of $|Y_{\ell m}(\theta, \phi)|^2$ is shown in Figure 9.17.

We could also look for the nodal curves of the spherical harmonics like we had for vibrating membranes. Such surface plots on a sphere are shown in Figure 9.18. The colors provide for the amplitude of the $|Y_{\ell m}(\theta, \phi)|^2$. We can match these with the shapes in Figure 9.17 by coloring the plots with some of the same colors. This is shown in Figure 9.19. However, by plotting just the sign of the spherical harmonics, as in Figure 9.20, we can pick out the nodal curves much easier.

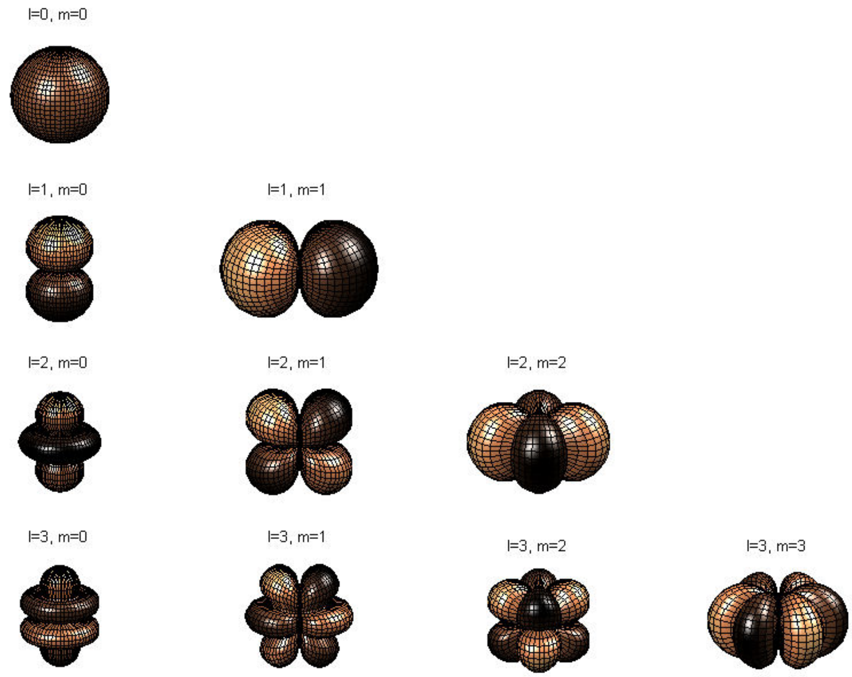

Figure 9.17: The first few spherical harmonics, $|Y_{\ell m}(\theta, \phi)|^2$

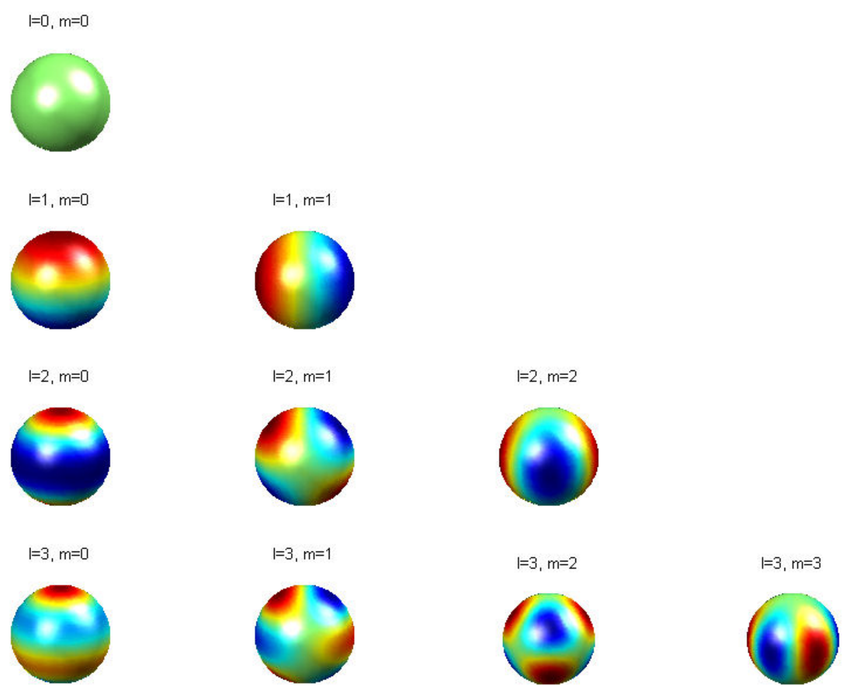

Figure 9.18: Spherical harmonic contours for $|Y_{\ell m}(\theta, \phi)|^2$.

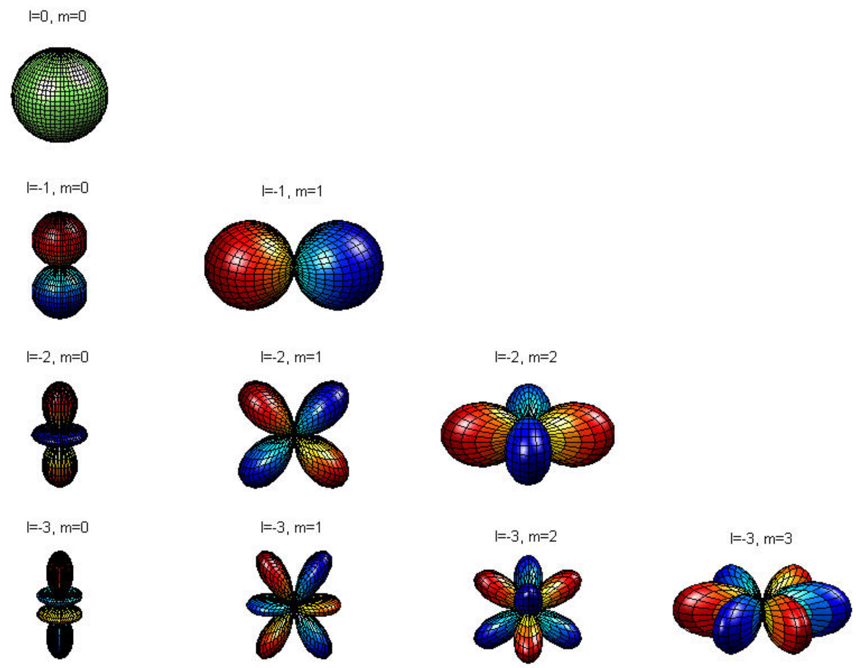

Figure 9.19: The first few spherical harmonics, $|Y_{\ell m}(\theta, \phi)|^2$

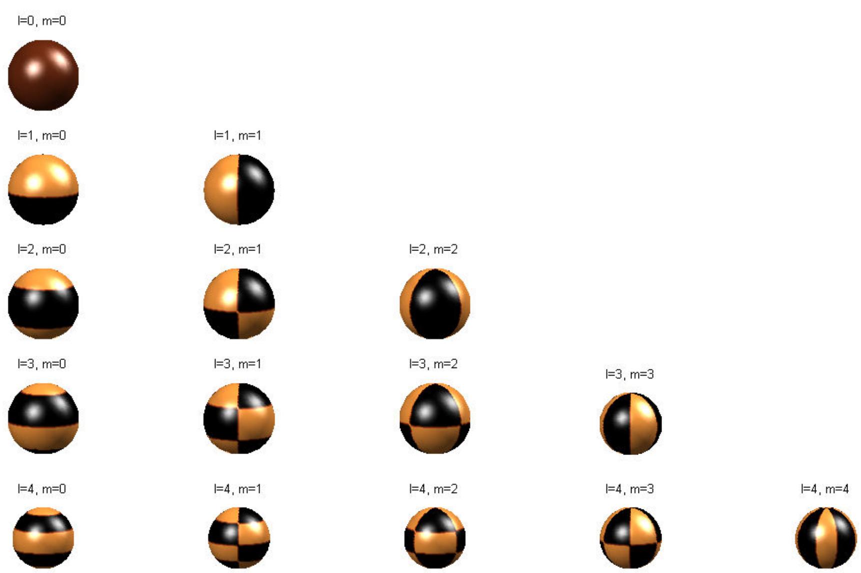

Figure 9.20: In these figures we show the nodal curves of $|Y_{\ell m}(\theta, \phi)|^2$

Spherical, or surface, harmonics can be further grouped into zonal, sectoral, and tesseral harmonics. Zonal harmonics correspond to the $m = 0$ modes. In this case, one seeks nodal curves for which $P_\ell(\cos\theta) = 0$. These lead to constant θ values such that $\cos\theta$ is a zero of the Legendre polynomial, $P_\ell(x)$. These correspond to the first column in Figure 9.20. Since $P_\ell(x)$ is a polynomial of degree ℓ, the zonal harmonics consist of ℓ latitudinal circles.

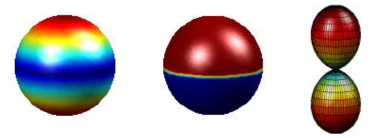

Figure 9.21: Zonal harmonics, $\ell = 1$, $m = 0$.

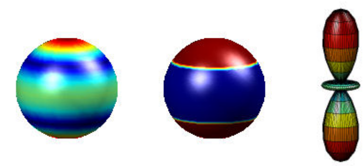

Figure 9.22: Zonal harmonics, $\ell = 2$, $m = 0$.

Sectoral, or meridional, harmonics result for the case that $m = \pm\ell$. For this case, we note that $P_\ell^{\pm\ell}(x) \propto (1-x^2)^{m/2}$. This vanishes for $x = \pm 1$, or $\theta = 0, \pi$. Therefore, the spherical harmonics can only produce nodal curves for $e^{im\phi} = 0$. Thus, one obtains the meridians corresponding to solutions of $A\cos m\phi + B\sin m\phi = 0$. Such solutions are constant values of ϕ. These can be seen in Figure 9.20 in the top diagonal and can be described as m circles passing through the poles, or longitudinal circles.

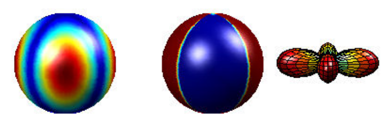

Figure 9.23: Sectoral harmonics, $\ell = 2$, $m = 2$.

Tesseral harmonics are all of the rest, which typically look like a checker board glued to the surface of a sphere. Examples can be seen in the pictures of nodal curves, such as Figure 9.20. Looking in Figure 9.20 along the diagonals going downward from left to right, one can see the same number of latitudinal circles. In fact, there are $\ell - m$ latitudinal nodal curves in these figures

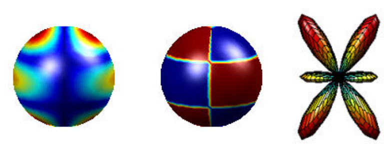

Figure 9.24: Tesseral harmonics, $\ell = 3$, $m = 1$.

In summary, the spherical harmonics have several representations, as show in Figures 9.17-9.20. Note that there are ℓ nodal lines, m meridional curves, and $\ell - m$ horizontal curves in these figures. The plots in Figures 9.17 and 9.19 are the typical plots shown in physics for discussion of the wavefunctions of the hydrogen atom. Those in 9.18 are useful for describing gravitational or electric potential functions, temperature distributions, or wave modes on a spherical surface. The relationships between these pictures and the nodal curves can be better understood by comparing respective plots. Several modes were separated out in Figures 9.21-9.26 to make this comparison easier.

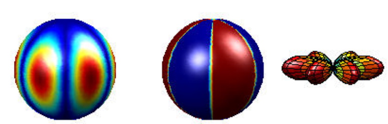

Figure 9.25: Sectoral harmonics, $\ell = 3$, $m = 3$.

Example 9.7. *Laplace's Equation with Azimuthal Symmetry*

As a simple example we consider the solution of Laplace's equation in which there is azimuthal symmetry. Let

$$u(r,\theta,\phi) = g(\theta) = 1 - \cos 2\theta.$$

This function is zero at the poles and has a maximum at the equator. So, this could be a crude model of the temperature distribution of the Earth with zero temperature at the poles and a maximum near the equator.

In problems in which there is no ϕ-dependence, only the $m = 0$ term of the general solution survives. Thus, we have that

$$u(\rho,\theta,\phi) = \sum_{\ell=0}^{\infty} a_\ell \rho^\ell P_\ell(\cos\theta). \tag{9.138}$$

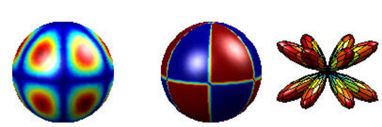

Figure 9.26: Tesseral harmonics, $\ell = 4$, $m = 3$.

Here we have used the fact that $P_\ell^0(x) = P_\ell(x)$. *We just need to determine the unknown expansion coefficients,* a_ℓ. *Imposing the boundary condition at* $\rho = r$, *we are lead to*

$$g(\theta) = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta). \tag{9.139}$$

This is a Fourier-Legendre series representation of $g(\theta)$. *Since the Legendre polynomials are an orthogonal set of eigenfunctions, we can extract the coefficients. In Chapter 5 we had proven that*

$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = \int_{-1}^1 P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}.$$

So, multiplying the expression for $g(\theta)$ *by* $P_m(\cos\theta)\sin\theta$ *and integrating, we obtain the expansion coefficients:*

$$a_\ell = \frac{2\ell+1}{2r^\ell}\int_0^\pi g(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta. \tag{9.140}$$

Sometimes it is easier to rewrite $g(\theta)$ *as a polynomial in* $\cos\theta$ *and avoid the integration. For this example we see that*

$$\begin{aligned} g(\theta) &= 1-\cos 2\theta \\ &= 2\sin^2\theta \\ &= 2-2\cos^2\theta. \end{aligned} \tag{9.141}$$

Thus, setting $x = \cos\theta$, *we have* $g(\theta) = 2-2x^2$. *We seek the form*

$$g(\theta) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x),$$

where $P_0(x) = 1$, $P_1(x) = x$, *and* $P_2(x) = \frac{1}{2}(3x^2-1)$. *Since* $g(\theta) = 2-2x^2$ *does not have any x terms, we know that* $c_1 = 0$. *So,*

$$2-2x^2 = c_0(1) + c_2\frac{1}{2}(3x^2-1) = c_0 - \frac{1}{2}c_2 + \frac{3}{2}c_2x^2.$$

By observation we have $c_2 = -\frac{4}{3}$ *and thus,* $c_0 = 2+\frac{1}{2}c_2 = \frac{4}{3}$. *This gives the sought expansion for* $g(\theta)$:

$$g(\theta) = \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}P_2(\cos\theta). \tag{9.142}$$

Therefore, the nonzero coefficients in the general solution become

$$a_0 = \frac{4}{3}, \quad a_2 = \frac{4}{3}\frac{1}{r^2},$$

and the rest of the coefficients are zero. Inserting these into the general solution, we have

$$\begin{aligned} u(\rho,\theta,\phi) &= \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}\left(\frac{\rho}{r}\right)^2 P_2(\cos\theta) \\ &= \frac{4}{3} - \frac{2}{3}\left(\frac{\rho}{r}\right)^2(3\cos^2\theta-1). \end{aligned} \tag{9.143}$$

9.6 Schrödinger Equation in Spherical Coordinates

Another important eigenvalue problem in physics is the Schrödinger equation. The time-dependent Schrödinger equation is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \tag{9.144}$$

Here $\Psi(\mathbf{r}, t)$ is the wave function, which is determines the quantum state of a particle of mass m subject to a (time independent) potential, $V(\mathbf{r})$. $\hbar = \frac{h}{2\pi}$, where h is Planck's constant. The probability of finding the particle in an infinitesimal volume, dV, is given by $|\Psi(\mathbf{r}, t)|^2\, dV$, assuming the wave function is normalized,

$$\int_{\text{all space}} |\Psi(\mathbf{r}, t)|^2\, dV = 1.$$

One can separate out the time dependence by assuming a special form, $\Psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-iEt/\hbar}$, where E is the energy of the particular stationary state solution, or product solution. Inserting this form into the time-dependent equation, one finds that $\psi(\mathbf{r})$ satisfies

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi. \tag{9.145}$$

Assuming that the potential depends only on distance from the origin, $V = V(\rho)$, we can further separate out the radial part of this solution using spherical coordinates. Recall that the Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{9.146}$$

Then, the time-independent Schrödinger equation can be written as

$$\begin{aligned} & -\frac{\hbar^2}{2m}\left[\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] \\ & = [E - V(\rho)]\psi. \end{aligned} \tag{9.147}$$

Let's continue with the separation of variables. Assuming that the wave function takes the form $\psi(\rho, \theta, \phi) = R(\rho)Y(\theta, \phi)$, we obtain

$$\begin{aligned} & -\frac{\hbar^2}{2m}\left[\frac{Y}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{R}{\rho^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right) + \frac{R}{\rho^2\sin^2\theta}\frac{d^2Y}{d\phi^2}\right] \\ & = RY[E - V(\rho)]\psi. \end{aligned} \tag{9.148}$$

Now dividing by $\psi = RY$, multiplying by $-\frac{2m\rho^2}{\hbar^2}$, and rearranging, we have

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \frac{2m\rho^2}{\hbar^2}[V(\rho) - E] = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dY}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2Y}{d\phi^2}\right].$$

We have a function of ρ equal to a function of the angular variables. So, we set each side equal to a constant. We will judiciously set the constant equal to $\ell(\ell+1)$. The resulting equations are

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \frac{2m\rho^2}{\hbar^2}\left[V(\rho)-E\right]R = \ell(\ell+1)R, \tag{9.149}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)Y. \tag{9.150}$$

The second of these equations should look familiar from the last section. This is the equation for spherical harmonics,

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{2}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m e^{im\phi}. \tag{9.151}$$

So, any further analysis of the problem depends upon the choice of potential, $V(\rho)$, and the solution of the radial equation. For this, we turn to the determination of the wave function for an electron in orbit about a proton.

Example 9.8. *The Hydrogen Atom - $\ell = 0$ States*

Historically, the first test of the Schrödinger equation was the determination of the energy levels in a hydrogen atom. This is modeled by an electron orbiting a proton. The potential energy is provided by the Coulomb potential,

$$V(\rho) = -\frac{e^2}{4\pi\epsilon_0\rho}.$$

Thus, the radial equation becomes

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{2m\rho^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0\rho}+E\right]R = \ell(\ell+1)R. \tag{9.152}$$

Before looking for solutions, we need to simplify the equation by absorbing some of the constants. One way to do this is to make an appropriate change of variables. Let $\rho = ar$. Then by the Chain Rule we have

$$\frac{d}{d\rho} = \frac{dr}{d\rho}\frac{d}{dr} = \frac{1}{a}\frac{d}{dr}.$$

Under this transformation, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \frac{2ma^2r^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 ar}+E\right]u = \ell(\ell+1)u, \tag{9.153}$$

where $u(r) = R(\rho)$. Expanding the second term,

$$\frac{2ma^2r^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 ar}+E\right]u = \left[\frac{mae^2}{2\pi\epsilon_0\hbar^2}r + \frac{2mEa^2}{\hbar^2}r^2\right]u,$$

we see that we can define

$$a = \frac{2\pi\epsilon_0\hbar^2}{me^2}, \tag{9.154}$$

$$\begin{aligned}\epsilon &= -\frac{2mEa^2}{\hbar^2} \\ &= -\frac{2(2\pi\epsilon_0)^2\hbar^2}{me^4}E.\end{aligned} \tag{9.155}$$

Using these constants, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + ru - \ell(\ell+1)u = \epsilon r^2 u. \tag{9.156}$$

Expanding the derivative and dividing by r^2,

$$u'' + \frac{2}{r}u' + \frac{1}{r}u - \frac{\ell(\ell+1)}{r^2}u = \epsilon u. \tag{9.157}$$

The first two terms in this differential equation came from the Laplacian. The third term came from the Coulomb potential. The next term can be thought to contribute to the potential and is attributed to angular momentum. Thus, ℓ is called the angular momentum quantum number. This is an eigenvalue problem for the radial eigenfunctions $u(r)$ and energy eigenvalues ϵ.

The solutions of this equation are determined in a quantum mechanics course. In order to get a feeling for the solutions, we will consider the zero angular momentum case, $\ell = 0$:

$$u'' + \frac{2}{r}u' + \frac{1}{r}u = \epsilon u. \tag{9.158}$$

Even this equation is one we have not encountered in this book. Let's see if we can find some of the solutions.

First, we consider the behavior of the solutions for large r. For large r the second and third terms on the left hand side of the equation are negligible. So, we have the approximate equation

$$u'' - \epsilon u = 0. \tag{9.159}$$

The solutions thus behave like $u(r) = e^{\pm\sqrt{\epsilon}r}$. For bounded solutions, we choose the decaying solution.

This suggests that solutions take the form $u(r) = v(r)e^{-\sqrt{\epsilon}r}$ for some unknown function, $v(r)$. Inserting this guess into Equation (9.158), gives an equation for $v(r)$:

$$rv'' + 2\left(1 - \sqrt{\epsilon}r\right)v' + (1 - 2\sqrt{\epsilon})v = 0. \tag{9.160}$$

Next we seek a series solution to this equation. Let

$$v(r) = \sum_{k=0}^{\infty} c_k r^k.$$

Inserting this series into Equation (9.160), we have

$$\sum_{k=1}^{\infty}[k(k-1)+2k]c_k r^{k-1} + \sum_{k=1}^{\infty}[1-2\sqrt{\epsilon}(k+1)]c_k r^k = 0.$$

We can re-index the dummy variable in each sum. Let $k = m$ in the first sum and $k = m-1$ in the second sum. We then find that

$$\sum_{k=1}^{\infty}[m(m+1)c_m + [1-2m\sqrt{\epsilon}]c_{m-1}]r^{m-1} = 0.$$

Since this has to hold for all $m \geq 1$,

$$c_m = \frac{2m\sqrt{\epsilon}-1}{m(m+1)}c_{m-1}.$$

Further analysis indicates that the resulting series leads to unbounded solutions unless the series terminates. This is only possible if the numerator, $2m\sqrt{\epsilon}-1$, vanishes for $m = n$, $n = 1, 2 \ldots$. Thus,

$$\epsilon = \frac{1}{4n^2}.$$

Since ϵ is related to the energy eigenvalue, E, we have

$$E_n = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2 n^2}.$$

Inserting the values for the constants, this gives

$$E_n = -\frac{13.6\ eV}{n^2}.$$

This is the well known set of energy levels for the hydrogen atom.

The corresponding eigenfunctions are polynomials, since the infinite series was forced to terminate. We could obtain these polynomials by iterating the recursion equation for the c_m's. However, we will instead rewrite the radial equation (9.160).

Let $x = 2\sqrt{\epsilon}r$ and define $y(x) = v(r)$. Then

$$\frac{d}{dr} = 2\sqrt{\epsilon}\frac{d}{dx}.$$

This gives

$$2\sqrt{\epsilon}xy'' + (2-x)2\sqrt{\epsilon}y' + (1-2\sqrt{\epsilon})y = 0.$$

Rearranging, we have

$$xy'' + (2-x)y' + \frac{1}{2\sqrt{\epsilon}}(1-2\sqrt{\epsilon})y = 0.$$

Noting that $2\sqrt{\epsilon} = \frac{1}{n}$, this becomes

$$xy'' + (2-x)y' + (n-1)y = 0. \tag{9.161}$$

The resulting equation is well known. It takes the form

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{9.162}$$

Solutions of this equation are the associated Laguerre polynomials. The solutions are denoted by $L_n^\alpha(x)$. They can be defined in terms of the Laguerre polynomials,

The associated Laguerre polynomials are named after the French mathematician Edmond Laguerre (1834-1886).

$$L_n(x) = e^x \left(\frac{d}{dx}\right)^n (e^{-x}x^n).$$

The associated Laguerre polynomials are defined as

$$L_{n-m}^m(x) = (-1)^m \left(\frac{d}{dx}\right)^m L_n(x).$$

Note: The Laguerre polynomials were first encountered in Problem 2 in Chapter 5 as an example of a classical orthogonal polynomial defined on $[0, \infty)$ with weight $w(x) = e^{-x}$. Some of these polynomials are listed in Table 9.3.

Comparing Equation (9.161) with Equation (9.162), we find that $y(x) = L_{n-1}^1(x)$.

	$L_n^m(x)$
$L_0^0(x)$	1
$L_1^0(x)$	$1 - x$
$L_2^0(x)$	$x^2 - 4x + 2$
$L_0^1(x)$	1
$L_1^1(x)$	$4 - 2x$
$L_2^1(x)$	$3x^2 - 18x + 18$
$L_0^2(x)$	2
$L_1^2(x)$	$-6x + 18$
$L_2^2(x)$	$12x^2 - 96x + 144$
$L_0^3(x)$	6
$L_1^3(x)$	$-24x + 96$
$L_2^3(x)$	$60x^2 - 600x + 1200$

Table 9.3: Associated Laguerre Functions, $L_n^m(x)$.

In summary, we have made the following transformations:

1. $R(\rho) = u(r)$, $\rho = ar$.
2. $u(r) = v(r)e^{-\sqrt{\epsilon}r}$.
3. $v(r) = y(x) = L_{n-1}^1(x)$, $x = 2\sqrt{\epsilon}r$.

In most derivation in quantum mechanics $a = \frac{a_0}{2}$. where $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ is the Bohr radius and $a_0 = 5.2917 \times 10^{-11}$m.

Therefore,

$$R(\rho) = e^{-\sqrt{\epsilon}\rho/a} L_{n-1}^1(2\sqrt{\epsilon}\rho/a).$$

However, we also found that $2\sqrt{\epsilon} = 1/n$. So,

$$R(\rho) = e^{-\rho/2na} L_{n-1}^1(\rho/na).$$

For the general case, for all $\ell \geq 0$, we need to solve the differential equation

$$u'' + \frac{2}{r}u' + \frac{1}{r}u - \frac{\ell(\ell+1)}{r^2}u = \epsilon u. \qquad (9.163)$$

Instead of letting $u(r) = v(r)e^{-\sqrt{\epsilon}r}$, we let

$$u(r) = v(r)r^{\ell}e^{-\sqrt{\epsilon}r}.$$

This lead to the differential equation

$$rv'' + 2(\ell+1-\sqrt{\epsilon}r)v' + (1-2(\ell+1)\sqrt{\epsilon})v = 0. \qquad (9.164)$$

as before, we let $x = 2\sqrt{\epsilon}r$ to obtain

$$xy'' + 2\left[\ell+1-\frac{x}{2}\right]v' + \left[\frac{1}{2\sqrt{\epsilon}} - \ell(\ell+1)\right]v = 0.$$

Noting that $2\sqrt{\epsilon} = 1/n$, we have

$$xy'' + 2\left[2(\ell+1) - x\right]v' + (n - \ell(\ell+1))v = 0.$$

We see that this is once again in the form of the associate Laguerre equation and the solutions are

$$y(x) = L_{n-\ell-1}^{2\ell+1}(x).$$

So, the solution to the radial equation for the hydrogen atom is given by

$$\begin{aligned} R(\rho) &= r^{\ell}e^{-\sqrt{\epsilon}r}L_{n-\ell-1}^{2\ell+1}(2\sqrt{\epsilon}r) \\ &= \left(\frac{\rho}{2na}\right)^{\ell} e^{-\rho/2na}L_{n-\ell-1}^{2\ell+1}\left(\frac{\rho}{na}\right). \end{aligned} \qquad (9.165)$$

Interpretations of these solutions will be left for your quantum mechanics course.

Problems

1. Consider Laplace's equation on the unit square, $u_{xx} + u_{yy} = 0$, $0 \leq x, y \leq 1$. Let $u(0,y) = 0, u(1,y) = 0$ for $0 < y < 1$ and $u_y(x,0) = 0$ for $0 < y < 1$. Carry out the needed separation of variables and write down the product solutions satisfying these boundary conditions.

2. Consider a cylinder of height H and radius a.

a. Write down LaplaceŠs Equation for this cylinder in cylindrical coordinates.

b. Carry out the separation of variables and obtain the three ordinary differential equations that result from this problem.

c. What kind of boundary conditions could be satisfied in this problem in the independent variables?

3. Consider a square drum of side s and a circular drum of radius a.

a. Rank the modes corresponding to the first 6 frequencies for each.

b. Write each frequency (in Hz) in terms of the fundamental (i.e., the lowest frequency.)

c. What would the lengths of the sides of the square drum have to be to have the same fundamental frequency? (Assume that $c = 1.0$ for each one.)

4. A copper cube 10.0 cm on a side is heated to 100° C. The block is placed on a surface that is kept at 0° C. The sides of the block are insulated, so the normal derivatives on the sides are zero. Heat flows from the top of the block to the air governed by the gradient $u_z = -10^\circ\text{C/m}$. Determine the temperature of the block at its center after 1.0 minutes. Note that the thermal diffusivity is given by $k = \frac{K}{\rho c_p}$, where K is the thermal conductivity, ρ is the density, and c_p is the specific heat capacity.

5. Consider a spherical balloon of radius a. Small deformations on the surface can produce waves on the balloon's surface.

a. Write the wave equation in spherical polar coordinates. (Note: ρ is constant!)

b. Carry out a separation of variables and find the product solutions for this problem.

c. Describe the nodal curves for the first six modes.

d. For each mode determine the frequency of oscillation in Hz assuming $c = 1.0$ m/s.

6. Consider a circular cylinder of radius $R = 4.00$ cm and height $H = 20.0$ cm which obeys the steady state heat equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz}.$$

Find the temperature distribution, $u(r,z)$, given that $u(r,0) = 0$, $u(r,20) = 20$, and heat is lost through the sides due to Newton's Law of Cooling

$$[u_r + hu]_{r=4} = 0,$$

for $h = 1.0\ \text{cm}^{-1}$.