Mathematical Physics Review II
Chapters 4,5,6

1. 1D PDEs
   a. Heat Equation, $u_t = ku_{xx}$
   b. Wave Equation, $u_{tt} = c^2 u_{xx}$
   c. Boundary Value Problems, Types of Boundary Conditions
   d. Method of Separation of Variables, $u(x,t) = X(x)T(t)$
   e. Product solutions, General Solutions
   f. Eigenvalue Problems: Eigenvalues and Eigenfunctions
2. Special Formulae – Know Trigonometric Identities!
   a. Orthogonality Relations and Trigonometric Identities
      i. $\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$
      ii. $\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$
      iii. For example: $\begin{aligned} 2\cos nx \cos mx &= \cos(n+m)x + \cos(n-m)x \\ 2\sin nx \sin mx &= \cos(n-m)x - \cos(n+m)x \end{aligned}$
      iv. Hyperbolic function Identities, $\begin{aligned} \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \end{aligned}$
3. Fourier Series on particular intervals
   a. Trigonometric
      i. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$
      ii. $\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} \, dx \\ b_n &= \frac{1}{L} f(x) \int_{-L}^{L} \sin\frac{n\pi x}{L} \, dx \end{aligned}$
   b. Sine
      i. $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
      ii. $b_n = \frac{2}{L} f(x) \int_{0}^{L} \sin\frac{n\pi x}{L} \, dx$
   c. Cosine
      i. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$
      ii. $a_n = \frac{2}{L}\int_{0}^{L} f(x) \cos\frac{n\pi x}{L} \, dx$
   d. Even and Odd Periodic Extensions of Functions
   e. Know your integrals! Ex $\int_{a}^{b} x^k \cos(\frac{n\pi x}{L}) \, dx$, etc.
   f. What is Gibbs Phenomenon and when do you expect it?

4. Infinite Dimensional Spaces
    a. Inner Product $< f,g >= \int_a^b f(x)g(x)\sigma(x)\,dx$
    b. Generalized Fourier Coefficients: $f(x) \sim \sum_n c_n \phi_n(x),\ c_n = \frac{\langle \phi_n, f\rangle}{\langle \phi_n, \phi_n\rangle}$
    c. Gram Schmidt Orthogonalization, $\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2}\mathbf{e}_j,\ n \ge 2$
    d. Legendre Polynomials – use of Rodrigues formula, three term recursion formula, generating function, normalization - $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$
    e. Gamma Function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$,

       $\Gamma(1) = 1,\ \Gamma(x+1) = x\Gamma(x),\ \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$
    f. Bessel Functions - General behavior
    g. Sturm-Liouville Operator – put ODE in SL form $\frac{d}{dx}\left(p\frac{d}{dx}\right) + q$
    h. Properties – real eigenvalues, orthogonal – self adjoint $< u, Lv >=< Lu, v >$
    i. Lagrange & Green identities: $< u, Lv > - < Lu, v >= \left(p(uv' - vu')\right)'$
5. Complex Numbers
    a. Know how to use polar forms $z = re^{i\theta}$, $x = r\cos\theta,\ y = r\sin\theta$ and

       $r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}$
    b. $e^{i\pi} = -1, e^{2\pi i k} = 1$ for k an integer
    c. Complex Modulus and complex conjugate
    d. Use of Euler's Formula and DeMoivre's Theorem
        i. $e^{i\theta} = \cos\theta + i\sin\theta$
        ii. $\left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta$
    e. nth roots $z^{1/n} = r^{1/n}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n})$ for $k = 0,1,\ldots,n-1$
6. Complex Functions
    a. Determine real and imaginary parts of functions: $f(z) = u(x,y) + iv(x,y)$
    b. Logarithms
7. Differentiation
    a. Compute Derivative $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$
    b. Differentiability and CR Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$
    c. Harmonic Functions: CR => $\nabla^2 u = 0$ and harmonic conjugates
    d. Holomorphic, Analytic, Entire, ….
8. Integration

a. Complex Path Integrals – parametrized over line segment, arcs, etc.

$$\int_C f(z)\,dz = \int_a^b f(x(t)+iy(t))\left(\frac{dx}{dt}+i\frac{dy}{dt}\right)dt$$

b. Path Independence, When can one deform contours?

c. Cauchy's Theorem $\oint_C f(z)\,dz = 0$ if $f(z)$ is differentiable

d. Cauchy Integral Formula $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$

e. Singularities – removable, poles, essential

f. Computing Residues

  i. $\text{Res}[f(z); z = z_0] = \lim_{z\to z_0}(z-z_0)f(z)$ - simple poles

  ii. $\text{Res}[f(z); z = z_0] = \lim_{z\to z_0}\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}[(z-z_0)^k f(z)]$ - poles of order k

g. Residue Theorem $\int_C f(z)\,dz = 2\pi i \sum_{\text{Poles inside C}}$ Residues

h. $\cos\theta = \frac{z+z^{-1}}{2}$, $\sin\theta = \frac{z-z^{-1}}{2i}$

i. Going from integrals over R to complex integrals

9. Series Expansions

a. Power series, Laurent series, Taylor series

b. Circle of convergence

c. Using geometric series

  i. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z|<1,$

  ii. $\frac{1}{1-z} = \frac{-1}{z}\times\frac{1}{1-\frac{1}{z}} = -\sum_{n=1}^{\infty} z^{-n}, |z|>1$