Mathematical Physics Review I

I. First Order Differential Equations
   a. Separation of Variables
      i. \( \frac{dy}{dt} = f(t)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(t) \, dt \)
   ii. General Solutions – Implicit and Explicit
   iii. Initial Value Problems – Particular Solutions
   b. Linear Differential Equations
      i. Find integrating factors and solve initial value problems
      ii. \( y' + a(x)y = f(x) \)
         \[ \mu(x) = \exp \left( \int a(x) \, dx \right) \Rightarrow \left( \mu y \right)' = \mu f \]
         \[ y(x) = \frac{1}{\mu(x)} \left[ \int \mu(t) f(t) \, dt + C \right] \]

II. Second Order Differential Equations
   a. Homogeneous, Constant Coefficient Equations \( ay'' + by' + cy = 0 \)
   b. Solutions - \( y(x) = e^x, \ ax^2 + bx + c = 0 \)
      i. Two, real distinct solutions \( y = c_1 e^{x} + c_2 e^{2x} \)
      ii. One real solution \( y = (c_1 + c_2 x)e^x \)
      iii. Two complex conjugate solutions \( y = (c_1 \cos bx + c_2 \sin bx)e^{ax} \)
   c. Cauchy-Euler \( ax^2 y'' + by' + cy = 0 \) - Solve using \( y(x) = x^r \)
   d. Nonhomogeneous Equations
      i. First find solution to homogeneous problem
      ii. Get Particular Solution
   1. Method of Undetermined Coefficients
      Also Applies to First Order! \( y' + a(x)y = f(x) \).

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( y_p(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_n(x) = a_n x^n + \cdots + a_1 x + a_0 )</td>
<td>( Ax^n + \cdots + Bx + C )</td>
</tr>
<tr>
<td>( P_n(x)e^{ax} )</td>
<td>( (Ax^n + \cdots + Bx + C)e^{ax} )</td>
</tr>
<tr>
<td>( (P_n(x) \cos bx + Q_n(x) \sin bx)e^{ax} )</td>
<td>( [(Ax^n + \cdots) \cos bx + (Bx^n + \cdots) \sin bx]e^{ax} )</td>
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2. Method of Variation of Parameters
   a. Determine Two Linearly Independent Solutions of Homogeneous Equation, \( y_1(x), y_2(x) \)
   b. Solve System for \( c \)'s and integrate
      i. \( c_1'y_1 + c_2'y_2 = 0, \)
      \[ c_1'y_1 + c_2'y_2' = f(x)/a(x) \]
      ii. or,
      \[ c_1 = -\int \frac{f y_2}{aW} \, dx, \quad c_2 = \int \frac{f y_1}{aW} \, dx, \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \]
III. Oscillations - \( x = x(t) \)

a. Simple Harmonic Motion \( x'' + \omega^2 x = 0, \omega = \sqrt{\frac{k}{m}} \)

b. Damped Harmonic Motion \( m x'' + b x' + k x = 0 \).
   Recognize solution behavior
   underdamped, critically damped, overdamped.

c. Forced Harmonic Motion \( m x'' + b x' + k x = F_0 \sin(\omega t + \phi) \).

d. Models
   i. Mass on Spring: \( m x'' + b x' + k x = 0 \)
   ii. Pendulum: \( L \ddot{\theta} + g \theta = 0 \).
   iii. LRC: \( L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V(t) \)

IV. Linear Algebra

a. Vectors
   i. Linear independence
   ii. Bases, components
   iii. Scalar Product \( < \mathbf{u}, \mathbf{v}> = \sum_{k=1}^{n} u_k v_k \)
   iv. Length \( v = |v| = \sqrt{\sum_{k=1}^{n} v_k} \)
   v. Orthogonal, orthonormal basis \( < \mathbf{e}_i, \mathbf{e}_j > = \delta_{ij} \)
   vi. Components w.r.t. basis \( v_j = \frac{< \mathbf{a}_j, \mathbf{v}>}{< \mathbf{a}_j, \mathbf{a}_j >} \)

b. Linear Transformations \( L(\mathbf{a} \mathbf{u} + b \mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v}) \)

c. Rotation Matrix – active vs passive, \( R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \)

d. Matrix Operations
   i. Multiplication
   ii. Transpose
   iii. Inverse, Cofactors
   iv. Determinant
   v. Trace
   vi. Cramer’s Rule

e. Special Matrices – Identity, symmetric, etc.

V. Eigenvalue Problems

a. Eigenvalues and eigenvectors
b. Similarity transformations and Diagonalization, \( \Lambda = S^{-1} A S \)
   columns of \( S = \) eigenvectors
c. Solution of Eigenvalue problems - \( A \mathbf{v} = \lambda \mathbf{v} \)

d. Application to Planar Systems \( \frac{d\mathbf{x}}{dt} = A \mathbf{x} \Rightarrow \mathbf{x} = x_0 e^{\lambda t} \) or \( \mathbf{x}_i = e^{\lambda t} \mathbf{v}_i \)

e. \( e^t = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots \)
VI. Methods of Integration
a. Substitution
b. Integration by parts $\int u\, dv = uv - \int v\, du$
c. Trigonometric Integrals $\int \sin^n x\, dx, \int \cos^n x\, dx$
d. Differentiation of integrals with respect to parameters

VII. Integrals you should be able to do (or similar ones)

<table>
<thead>
<tr>
<th>$\int x^n, dx$</th>
<th>$\int \frac{1}{x}, dx$</th>
<th>$\int e^{ax}, dx$</th>
<th>$\int a^x, dx$</th>
</tr>
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<tr>
<td>$\int \sin ax, dx$</td>
<td>$\int \cos ax, dx$</td>
<td>$\int \sec^2 ax, dx$</td>
<td>$\int \csc^2 ax, dx$</td>
</tr>
<tr>
<td>$\int \sec x\tan x, dx$</td>
<td>$\int \csc x\cot x, dx$</td>
<td>$\int \sinh ax, dx$</td>
<td>$\int \cosh ax, dx$</td>
</tr>
<tr>
<td>$\int \tan ax, dx$</td>
<td>$\int \cot ax, dx$</td>
<td>$\int \frac{1}{x^2 + a^2}, dx$</td>
<td>$\int \frac{1}{\sqrt{a^2 - x^2}}, dx$</td>
</tr>
<tr>
<td>$\int \sec ax, dx$</td>
<td>$\int \ln x, dx$</td>
<td>$\int x^a e^{ax}, dx$</td>
<td>$\int \frac{1}{x^2 - a^2}, dx$</td>
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<tr>
<td>$\int \sin^2 ax, dx$</td>
<td>$\int \cos^2 ax, dx$</td>
<td>$\int \sin ax\cos bx, dx$</td>
<td>$\int \sin ax\sin bx, dx$</td>
</tr>
<tr>
<td>$\int \frac{dx}{a + bx}$</td>
<td>$\int \frac{dx}{(x-a)(x-b)}$</td>
<td>$\int e^{ax}\cos bx, dx$</td>
<td>$\int e^{ax}\sin bx, dx$</td>
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