Mathematical Physics Review I

I. First Order Differential Equations

a. Separation of Variables

i. $\frac{dy}{dt} = f(t)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(t)\, dt$

ii. General Solutions – Implicit and Explicit

iii. Initial Value Problems – Particular Solutions

b. Linear Differential Equations

i. Find integrating factors and solve initial value problems

ii. $y' + a(x)y = f(x)$

$$\mu(x) = \exp \int^x a(x)\, dx \Rightarrow (\mu y)' = \mu f$$

$$y(x) = \frac{1}{\mu(x)} \left[\int^x \mu(t) f(t)\, dt + C \right]$$

II. Second Order Differential Equations

a. Homogeneous, Constant Coefficient Equations $ay'' + by' + cy = 0$

b. Solutions - $y(x) = e^{rx}$, $ar^2 + br + c = 0$.

i. Two, real distinct solutions $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

ii. One real solution $y = (c_1 + c_2 x)e^{rx}$

iii. Two complex conjugate solutions $y = (c_1 \cos \beta x + c_2 \sin \beta x)e^{\alpha x}$

c. Cauchy-Euler $ax^2 y'' + bxy' + cy = 0$ - Solve using $y(x) = x^r$

d. Nonhomogeneous Equations

i. First find solution to homogeneous problem

ii. Get Particular Solution

1. Method of Undetermined Coefficients
Also Applies to First Order! $y' + a(x)y = f(x)$.

$f(x)$	$y_p(x)$
$P_n(x) = a_n x^n + \cdots + a_1 x + a_0$	$Ax^n + \cdots + Bx + C$
$P_n(x)e^{ax}$	$(Ax^n + \cdots + Bx + C)e^{ax}$
$(P_n(x)\cos bx + Q_n(x)\sin bx)e^{ax}$	$[(Ax^n + \ldots)\cos bx + (Bx^n + \ldots)\sin bx]e^{ax}$

2. Method of Variation of Parameters

a. Determine Two Linearly Independent Solutions of Homogeneous Equation, $y_1(x), y_2(x)$

b. Solve System for c's and integrate

i. $c_1' y_1 + c_2' y_2 = 0,$

$c_1' y_1' + c_2' y_2' = f(x)/a(x)$

ii. or,

$$c_1 = -\int \frac{f y_2}{aW}\, dx,\ c_2 = \int \frac{f y_1}{aW}\, dx,\ W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

III. Oscillations - $x = x(t)$

a. Simple Harmonic Motion $x'' + \omega^2 x = 0,\ \omega = \sqrt{\frac{k}{m}}$

b. Damped Harmonic Motion $mx'' + bx' + kx = 0.$
Recognize solution behavior
underdamped, critically damped, overdamped.

c. Forced Harmonic Motion $mx'' + bx' + kx = F_0 \sin(\omega t + \phi).$

d. Models

i. Mass on Spring: $mx'' + bx' + kx = 0$

ii. Pendulum: $L\ddot{\theta} + g\theta = 0.$

iii. LRC: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$

IV. Linear Algebra

a. Vectors

i. Linear independence

ii. Bases, components

iii. Scalar Product $<\mathbf{u},\mathbf{v}> = \sum_{k=1}^{n} u_k v_k$

iv. Length $v = |v| = \sqrt{\sum_{k=1}^{n} v_k}$

v. Orthogonal, orthonormal basis $<\mathbf{e}_i, \mathbf{e}_j> = \delta_{ij}$

vi. Components w.r.t. basis $v_j = \frac{<\mathbf{a}_j, \mathbf{v}>}{<\mathbf{a}_j, \mathbf{a}_j>}$

b. Linear Transformations $L(a\mathbf{u} + b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$

c. Rotation Matrix – active vs passive, $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

d. Matrix Operations

i. Multiplication

ii. Transpose

iii. Inverse, Cofactors

iv. Determinant

v. Trace

vi. Cramer's Rule

e. Special Matrices – Identity, symmetric, etc.

V. Eigenvalue Problems

a. Eigenvalues and eigenvectors

b. Similarity transformations and Diagonalization, $\Lambda = S^{-1}AS$
columns of S = eigenvectors

c. Solution of Eigenvalue problems - $A\mathbf{v} = \lambda\mathbf{v}$

d. Application to Planar Systems $\frac{d\mathbf{x}}{dt} = A\mathbf{x} \Rightarrow \mathbf{x} = \mathbf{x}_0 e^{At}$ or $\mathbf{x}_i = e^{\lambda_i t}\mathbf{v}_i$

e. $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$

f. Application to conics $\mathbf{x}^T Q\mathbf{x} = D$, $\Lambda = R_\theta^{-1} A R_\theta$

VI. Methods of Integration

a. Substitution

b. Integration by parts $\int u\, dv = uv - \int v\, du$

c. Trigonometric Integrals $\int \sin^n x\, dx$, $\int \cos^n x\, dx$,

d. Differentiation of integrals with respect to parameters

VII. Integrals you should be able to do (or similar ones)

$\int x^n\, dx$	$\int \frac{1}{x}\, dx$	$\int e^{ax}\, dx$	$\int a^x\, dx$
$\int \sin ax\, dx$	$\int \cos ax\, dx$	$\int \sec^2 ax\, dx$	$\int \csc^2 ax\, dx$
$\int \sec x \tan x\, dx$	$\int \csc x \cot x\, dx$	$\int \sinh ax\, dx$	$\int \cosh ax\, dx$
$\int \tan ax\, dx$	$\int \cot ax\, dx$	$\int \frac{1}{x^2+a^2}\, dx$	$\int \frac{1}{\sqrt{a^2-x^2}}\, dx$
$\int \sec ax\, dx$	$\int \ln x\, dx$	$\int x^n e^{ax}\, dx$	$\int \frac{1}{x^2-a^2}\, dx$
$\int \sin^2 ax\, dx$	$\int \cos^2 ax\, dx$	$\int \sin ax \cos bx dx$	$\int \sin ax \sin bx dx$
$\int \frac{dx}{a+bx}$	$\int \frac{dx}{(x-a)(x-b)}$	$\int e^{ax} \cos bx\, dx$	$\int e^{ax} \sin bx\, dx$