Chapter 2

Free Fall and Harmonic Oscillators

2.1 Free Fall and Terminal Velocity

In this chapter we will study some common differential equations that appear in physics. We will begin with the simplest types of equations and standard techniques for solving them We will end this part of the discussion by returning to the problem of free fall with air resistance. We will then turn to the study of oscillations, which are modelled by second order differential equations.

Let us begin with a simple example from introductory physics. We recall that free fall is the vertical motion of an object under the force of gravity. It is experimentally determined that an object at some distance from the center of the earth falls at a constant acceleration in the absence of other forces, such as air resistance. This constant acceleration is denoted by $-g$, where g is called the acceleration due to gravity. The negative sign is an indication that up is positive.

We will be interested in determining the position, $y(t)$, of the body as a function of time. From the definition of free fall, we have

$$\ddot{y}(t) = -g. \tag{2.1}$$

Note that we will occasionally use a dot to indicate time differentiation.

This notation is standard in physics and we will begin to introduce you to this notation, though at times we might use the more familiar prime notation to indicate spatial differentiation, or general differentiation.

In Equation (2.1) we know g. It is a constant. Near the earth's surface it is about 9.81 m/s^2 or 32.2 ft/s^2. What we do not know is $y(t)$. This is our first differential equation. In fact it is natural to see differential equations appear in physics as Newton's Second Law, $F = ma$, plays an important role in classical physics. We will return to this point later.

So, how does one solve the differential equation in (2.1)? We can do so by using what we know about calculus. It might be easier to see how if we put in a particular number instead of g. You might still be getting used to the fact that some letters are used to represent constants. We will come back to the more general form after we see how to solve the differential equation.

Consider

$$\ddot{y}(t) = 5. \tag{2.2}$$

Recalling that the second derivative is just the derivative of a derivative, we can rewrite the equation as

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = 5. \tag{2.3}$$

This tells s that the derivative of dy/dt is 5. Can you think of a function whose derivative is 5? (Do not forget that the independent variable is t.) Yes, the derivative of $5t$ with respect to t is 5. Is this the only function whose derivative is 5? No! You can also differentiate $5t + 1$, $5t + \pi$, $5t - 6$, etc. In general, the derivative of $5t + C$ is 5.

So, our equation can be reduced to

$$\frac{dy}{dt} = 5t + C. \tag{2.4}$$

Now we ask if you know a function whose derivative is $5t + C$. Well, you might be able to do this one in your head, but we just need to recall the Fundamental Theorem of Calculus, which relates integrals and derivatives. Thus, we have

$$y(t) = \frac{5}{2}t^2 + Ct + D,$$

where D is a second integration constant.

This is a solution to the original equation. That means it is a function that when placed into the differential equation makes both sides of the equal sign the same. You can always check your answer by showing that it satisfies the equation. In this case we have

$$\ddot{y}(t) = \frac{d^2}{dt^2}(\frac{5}{2}t^2 + Ct + D) = \frac{d}{dt}(5t + C) = 5.$$

So, it is a solution.

We also see that there are two arbitrary constants, C and D. Picking any values for these gives a whole family of solutions. As we will see, our equation is a linear second order ordinary differential equation. We will see that the general solution of such an equation always has two arbitrary constants.

Let's return to the free fall problem. We solve it the same way. The only difference is that we can replace the constant 5 with the constant $-g$. So, we find that

$$\frac{dy}{dt} = -gt + C, \tag{2.5}$$

and

$$y(t) = -\frac{1}{2}gt^2 + Ct + D. \tag{2.6}$$

Once you get down the process, it only takes a line or two to solve.

There seems to be a problem. Imagine dropping a ball that then undergoes free fall. We just determined that there are an infinite number of solutions to where the ball is at any time! Well, that is not possible. Experience tells us that if you drop a ball you expect it to behave the same way every time. Or does it? Actually, you could drop the ball from anywhere. You could also toss it up or throw it down. So, there are many ways you can release the ball before it is in free fall. That is where the constants come in. They have physical meanings.

If you set $t = 0$ in the equation, then you have that $y(0) = D$. Thus, D gives the initial position of the ball. Typically, we denote initial values with a subscript. So, we will write $y(0) = y_0$. Thus, $D = y_0$.

That leaves us to determine C. It appears at first in Equation (2.5). Recall that $\frac{dy}{dt}$, the derivative of the position, is the vertical velocity, $v(t)$. It is positive when the ball moves upward. Now, denoting the initial velocity

$v(0) = v_0$, we see that Equation (2.5) becomes $\dot{y}(0) = C$. This implies that $C = v(0) = v_0$.

Putting this all together, we have the physical form of the solution for free fall as

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0. \tag{2.7}$$

Doesn't this equation look familiar? Now we see that our infinite family of solutions consists of free fall resulting from initially dropping a ball at position y_0 with initial velocity v_0. The conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$ are called the initial conditions. A solution of a differential equation satisfying a set of initial conditions is often called a particular solution.

So, we have solved the free fall equation. Along the way we have begun to see some of the features that will appear in the solutions of other problems that are modelled with differential equation. Throughout the book we will see several applications of differential equations. We will extend our analysis to higher dimensions, in which we case will be faced with so-called partial differential equations, which involve the partial derivatives of functions of more that one variable.

But are we done with free fall? Not at all! We can relax some of the conditions that we have imposed. We can add air resistance. We will visit this problem later in this chapter after introducing some more techniques.

Before we do that, we should also note that free fall at constant g only takes place near the surface of the Earth. What if a tile falls off the shuttle far from the surface? It will also fall to the earth. Actually, it may undergo projectile motion, which you may recall is a combination of horizontal motion and free fall.

To look at this problem we need to go to the origins of the acceleration due to gravity. This comes out of Newton's Law of Gravitation. Consider a mass m at some distance $h(t)$ from the surface of the (spherical) Earth. Letting M and R be the Earth's mass and radius, respectively, Newton's Law of Gravitation states that

$$\begin{aligned} ma &= F \\ m\frac{d^2h(t)}{dt^2} &= G\frac{mM}{(R+h(t))^2}. \end{aligned} \tag{2.8}$$

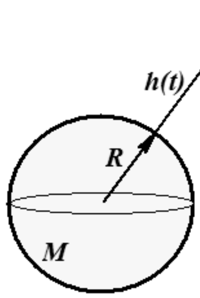

Figure 2.1: Free fall far from the Earth from a height $h(t)$ from the surface.

Thus, we arrive at a differential equation

$$\frac{d^2h(t)}{dt^2} = \frac{GM}{(R+h(t))^2}. \tag{2.9}$$

This equation is not as easy to solve. We will leave it as a homework exercise for the reader.

2.1.1 First Order Differential Equations

Before moving on, we first define an *n-th order ordinary differential equation*. This is is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x) = 0. \tag{2.10}$$

An *initial value problem* consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{2.11}$$

A *linear nth order differential equation* takes the form

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\ldots+a_1(x)y'(x)+a_0(x)y(x) = f(x). \tag{2.12}$$

If $f(x) \equiv 0$, then the equation is said to be *homogeneous*, otherwise it is *nonhomogeneous*.

We will return to these definitions as we explore a variety of examples. However, we will start with the simplest of ordinary differential equations.

Typically, the first differential equations encountered are first order equations. A *first order differential equation* takes the form

$$F(y', y, x) = 0. \tag{2.13}$$

There are two general forms for which one can formally obtain a solution. The first is the separable case and the second is a linear first order equation. We indicate that we can formally obtain solutions, as one can indicate the needed integration that leads to a solution. However, these integrals are not always reducible to elementary functions nor does one necessarily obtain explicit solutions when the integrals are doable.

A first order equation is *separable* if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \tag{2.14}$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be autonomous.

The *general solution* to equation (2.14) is obtained in terms of two integrals:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C, \tag{2.15}$$

where C is an integration constant. This yields a family of solutions to the differential equation corresponding to different values of C. If one can solve (2.15) for $y(x)$, then one obtains an *explicit solution*. Otherwise, one has a family of *implicit* solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a *particular solution*.

Example 1. $y' = 2xy$, $y(0) = 2$.

Applying (2.15), one has

$$\int \frac{dy}{y} = \int 2x\,dx + C.$$

Integrating yields

$$\ln|y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = e^{x^2+C} = Ae^{x^2}, \quad \text{where } A = e^C.$$

(Note that since C is arbitrary, then so is e^C. Thus, there is no loss in generality using A instead of e^C.)

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution is $y(x) = 2e^{x^2}$.

Example 2. $yy' = -x$.

Following the same procedure as in the last example, one obtains:

$$\int y\,dy = -\int x\,dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where} \quad A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$.

The second type of first order equation encountered is the *linear first order differential equation* in the form

$$y'(x) + p(x)y(x) = q(x). \tag{2.16}$$

In this case one seeks an *integrating factor*, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Multiplying the equation by μ, the resulting equation becomes

$$\frac{d}{dx}(\mu y) = \mu q. \tag{2.17}$$

The integrating factor that works is $\mu(x) = \exp(\int p(x)\,dx)$. The resulting equation is then easily integrated to obtain

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(\xi)q(\xi)\,d\xi + C\right]. \tag{2.18}$$

Example 3. $xy' + y = x, \quad x > 0 \; y(1) = 0.$

One first notes that this is a linear first order differential equation. Solving for y', one can see that it is not separable. However, it is not in the standard form. So, we first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \tag{2.19}$$

Next, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (2.19) by the $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, equation (2.17) becomes

$$(xy)' = x.$$

Integrating one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting this solution into the initial condition, $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is $y(x) = \frac{1}{2}(x - \frac{1}{x})$.

There are other first order equations that one can solve for closed form solutions. However, many equations are not solvable, or one is simply interested in the behavior of solutions. In such cases one turns to direction fields. We will return to a discussion of the qualitative behavior of differential equations later in the course.

2.1.2 Terminal Velocity

Now let's return to free fall. What if there is air resistance? We first need to model the air resistance. As an object falls faster and faster, the drag force becomes greater. So, this resistive force is a function of the velocity. There are a couple of standard models that people use to test this. The idea is to write $F = ma$ in the form

$$m\ddot{y} = -mg + f(v), \tag{2.20}$$

where $f(v)$ gives the resistive force and mg is the weight. Recall that this applies to free fall near the Earth's surface. Also, for it to be resistive, $f(v)$ should oppose the motion. If the body is falling, then $f(v)$ should be positive. If it is rising, then $f(v)$ would have to be negative to indicate the opposition to the motion.

On common determination derives from the drag force on an object moving through a fluid. This force is given by

$$f(v) = \frac{1}{2}CA\rho v^2, \tag{2.21}$$

where C is the drag coefficient, A is the cross sectional area and ρ is the fluid density. For laminar flow the drag coefficient is constant.

Unless you are into aerodynamics, you do not need to get into the details of the constants. So, it is best to absorb all of the constants into one to simplify the computation. So, we will write $f(v) = bv^2$. Our equation can then be rewritten as

$$\dot{v} = kv^2 - g, \tag{2.22}$$

where $k = b/m$. Note that this is a first order equation for $v(t)$. It is separable too!

Formally, we can separate the variables and integrate the time out to obtain

$$t + K = \int^v \frac{dz}{kz^2 - g}. \tag{2.23}$$

(Note: We used an integration constant of K since C is the drag coefficient in this problem.) If we can do the integral, then we have a solution for v. In fact, we can do this integral. You need to recall another common method of integration, which we have not reviewed yet. Do you remember Partial Fraction Decomposition? It involves factoring the denominator in our integral. Of course, this is ugly because our constants are represented by letters and are not specific numbers. Letting $\alpha^2 = g/k$, we can write the integrand as

$$\frac{1}{kz^2 - g} = \frac{1}{k}\frac{1}{z^2 - \alpha^2} = \frac{1}{2\alpha k}\left[\frac{1}{z - \alpha} - \frac{1}{z + \alpha}\right]. \tag{2.24}$$

Now, the integrand can be easily integrated giving

$$t + K = \frac{1}{2\alpha k} \ln \left| \frac{v-\alpha}{v+\alpha} \right| . \tag{2.25}$$

Solving for v, we have

$$v(t) = \frac{1 - Ae^{2\alpha kt}}{1 + Ae^{2\alpha kt}} \alpha, \tag{2.26}$$

where $A \equiv e^K$. A can be determined using the initial velocity.

There are other forms for the solution in terms of a tanh function, which the reader can determine as an exercise. One important conclusion is that for large times, the ratio in the solution approaches -1. Thus, $v \to -\alpha = -\sqrt{\frac{g}{k}}$. This means that the falling object will reach a terminal velocity.

As a simple computation, we can determine the terminal velocity. We will take an 80 kg skydiver with a cross sectional area of about 0.093 m^2. (The skydiver is falling head first.) Assume that the air density is a constant 1.2 $\mathrm{kg/m}^3$ and the drag coefficient is $C = 2.0$. We first note that

$$v_{\text{terminal}} = -\sqrt{\frac{g}{k}} = -\sqrt{\frac{2mg}{CA\rho}}.$$

So,

$$v_{\text{terminal}} == -\sqrt{\frac{2(70)(9.8)}{(2.0)(0.093)(1.2)}} = 78m/s.$$

This is about 175 mph, which is slightly higher than the actual terminal velocity of a sky diver. One would need a more accurate determination of C.

2.2 The Simple Harmonic Oscillator

The next physical problem of interest is that of simple harmonic motion. Such motion comes up in many places in physics and provides a generic first approximation to models of oscillatory motion. This is the beginning of a major thread running throughout our course. You have seen simple harmonic motion in your introductory physics class. We will review SHM (or SHO in some texts) by looking at springs and pendula (the plural of

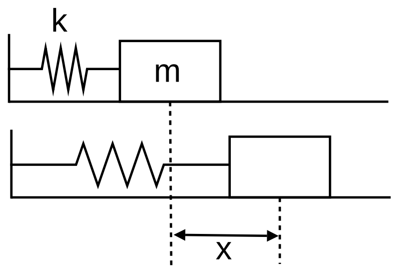

Figure 2.2: Spring-Mass system.

pendulum). We will use this as our jumping board into second order differential equation and later see how such oscillatory motion occurs in AC circuits.

2.2.1 Mass-Spring Systems

We begin with the case of a single block on a spring as shown in Figure 2.2. The net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant. Here x is the elongation, or displacement of the spring from equilibrium. When the displacement is positive, the spring force is negative and when the displacement is negative the spring force is positive. We have depicted a horizontal system sitting on a frictionless surface. A similar model can be provided for vertically oriented springs. However, you need to account for gravity to determine the location of equilibrium. Otherwise, the oscillatory motion about equilibrium is modelled the same.

From Newton's Second Law, $F = m\ddot{x}$, we obtain the equation for the motion of the mass on the spring:

$$m\ddot{x} + kx = 0.$$

We will later derive solutions of such equations in a methodical way. For now we note that two solutions of this equation are given by

$$x(t) = A\cos\omega t$$

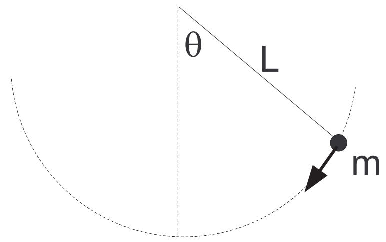

Figure 2.3: A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

$$x(t) = A \sin \omega t, \tag{2.27}$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

is the angular frequency, measured in rad/s. It is related to the frequency by

$$\omega = 2\pi f,$$

where f is measured in cycles per second, or Hertz. Furthermore, this is related to the period of oscillation, the time it takes the mass to go through one cycle:

$$T = 1/f.$$

Finally, A is called the amplitude of the oscillation.

2.2.2 The Simple Pendulum

The simple pendulum consists of a point mass m hanging on a string of length L from some support. [See Figure 2.3.] One pulls the mass back to some stating angle, θ_0, and releases it. The goal is to find the angular position as a function of time.

There are a couple of possible derivations. We could either use Newton's Second Law of Motion, $F = ma$, or its rotational analogue in terms of torque. We will use the former only to limit the amount of physics background needed.

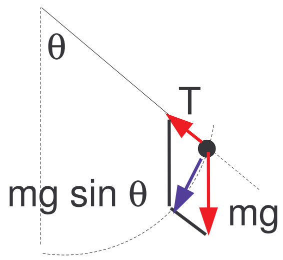

Figure 2.4: There are two forces acting on the mass, the weight mg and the tension T. The net force is found to be $F = mg\sin\theta$.

There are two forces acting on the point mass. The first is gravity. This points downward and has a magnitude of mg, where g is the standard symbol for the acceleration due to gravity. The other force is the tension in the string. In Figure 2.4 these forces and their sum are shown. The magnitude of the sum is easily found as $F = mg\sin\theta$ using the addition of these two vectors.

Now, Newton's Second Law of Motion tells us that the net force is the mass times the acceleration. So, we can write

$$m\ddot{x} = -mg\sin\theta.$$

Next, we need to relate x and θ. x is the distance traveled, which is the length of the arc traced out by our point mass. The arclength is related to the angle, provided the angle is measure in radians. Namely, $x = r\theta$ for $r = L$. Thus, we can write

$$mL\ddot{\theta} = -mg\sin\theta.$$

Cancelling the masses, this then gives us our nonlinear pendulum equation

$$L\ddot{\theta} + g\sin\theta = 0. \tag{2.28}$$

There are several variations of Equation (2.28) which will be used in this text. The first one is the linear pendulum. This is obtained by making a small angle approximation. For small angles we know that $\sin\theta \approx \theta$. Under this approximation (2.28) becomes

$$L\ddot{\theta} + g\theta = 0. \tag{2.29}$$

We note that this equation is of the same form as the mass-spring system. We define $\omega = \sqrt{g/L}$ and obtain the equation for simple harmonic motion,

$$\ddot{\theta} + \omega^2 \theta = 0.$$

2.3 Second Order Linear Differential Equations

In the last section we saw how second order differential equations naturally appear in the derivations for simple oscillating systems. In this section we will look at more general second order linear differential equations.

Second order differential equations are typically harder than first order. In most cases students are only exposed to *second order linear differential equations*. A general form is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.30}$$

One can rewrite this equation using operator terminology. Namely, we first define the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (2.30) becomes

$$Ly = f. \tag{2.31}$$

The solutions of linear differential equations are found by making use of the linearity of L. An operator L is said to be *linear* if it satisfies two properties:

1. $L(y_1 + y_2) = L(y_1) + L(y_2)$.
2. $L(ay) = aL(y)$ for a a constant.

One typically solves (2.30) by finding the general solution of the homogeneous problem, $Ly_h = 0$, and a particular solution of the nonhomogeneous problem, $Ly_p = f$. Then the general solution of (2.30) is simply given as $y = y + h + y_p$. This is found to be true using the linearity of L. Namely,

$$Ly = L(y_h + y_p) = Ly_h + Ly_p = 0 + f = f. \tag{2.32}$$

There are methods for finding a particular solution, $y_p(x)$, of the equation. These range from pure guessing to either using the Method of Undetermined Coefficients or the Method of Variation of Parameters.

Determining solutions to the homogeneous problem is not laways so easy. However, others have studied a variety of second order linear equations and have saved us the trouble in the case of differential equations that keep reappearing in applications. Again, linearity is useful.

If y_1 and y_2 are solutions of the homogeneous equation, then the linear combination $c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation. In fact, if y_1 and y_2 are *linearly independent*, namely,

$$c_1y_1 + c_2y_2 = 0 \Leftrightarrow c_1 = c_2 = 0,$$

then $c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem. Linear independence is established if the *Wronskian* of the solutions in not zero:

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0. \tag{2.33}$$

2.3.1 Constant Coefficient Equations

The simplest and most taught equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0. \tag{2.34}$$

Solutions to (2.34) are obtained by making a guess of $y(x) = e^{rx}$ and determining what possible values of r will yield a solution. Inserting this guess into (2.34) leads to the *characteristic equation*

$$ar^2 + br + c = 0. \tag{2.35}$$

The roots of this equation lead to three types of solution depending upon the nature of the roots.

1. Real, distinct roots r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$.

2. Real, equal roots $r_1 = r_2 = r = -\frac{b}{2a}$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses what is called the *Method of Reduction of Order.*[1] This gives the second solution as xe^{rx}. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x)e^{rx}$.

3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. (We will return to this identity later.) Namely, one has that $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$.

The solution of constant coefficient equations now follows easily. One solves the characteristic equation and then determines which case applies. Then one simply writes down the general solution. We will demonstrate this with a couple of examples. In the last section of this chapter we review the class of equations called the Cauchy-Euler equations. These equations occur often and follow a similar procedure. Students should be familiar with these kinds of equations as well.

Example 1. $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0$.

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be quickly written down:

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

[1] **The Method of Reduction of Order** We know that $y_1(x) = e^{rx}$ is a solution. We seek a second linearly independent solution of the form $y_2(x) = v(x)y_1(x)$. Inserting this into the ODE gives and equation for $v(x)$:

$$a(ve^{rx})'' + b(ve^{rx})' + cve^{rx} = 0,$$

or

$$[a(v'' + 2rv') + bv' + (ar^2 + br + c)v]e^{rx} = 0.$$

Since r satisfies the characteristic equation, the last term in the brackets vanishes. Cancelling out the exponential factor, which never vanishes, leaves a first order equation for v':

$$a(v')' + (2ar + b)v' = 0.$$

Since $r = -b/a$, we are left with $v'' = 0$. A solution of this equation is $v(x) = x$.

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have them with the information from the initial conditions. One needs

$$y'(x) = -2c_1e^{-2x} + 3c_2e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{2.36}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$. You should verify that this is indeed a solution.

Example 2. $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. Again, the solution is found as $y(x) = (c_1 + c_2x)e^{-3x}$.

Example 3. $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$ and the general solution consists purely of sinusoidal functions: $y(x) = c_1 \cos(2x) + c_2 \sin(2x)$.

Example 4. $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in the last example. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using variation of parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A \sin x$ and determine what A needs to be. (Recall, this is the Method of Undetermined Coefficients.) Inserting our guess in the equation gives $(-A + 4A) \sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \sin x$.

As we have seen, one of the most important applications of such equations is in the study of oscillations. Typical systems are a mass on a spring, or a simple pendulum. For a mass m on a spring with spring constant $k > 0$, one has from Hooke's law that the position as a function of time, $x(t)$, satisfies the equation

$$m\ddot{x} + kx = 0.$$

This constant coefficient equation has pure imaginary roots ($\alpha = 0$) and the solutions are pure sines and cosines. This is called simple harmonic motion. Adding a damping term and periodic forcing complicates the dynamics, but is nonetheless solvable. We will return to damped oscillations later and also investigate nonlinear oscillations.

2.4 LRC Circuits

Another typical problem often encountered in a first year physics class is that of an LRC series circuit. This circuit is pictured in Figure 2.5. The resistor is a circuit element satisfying Ohm's Law. The capacitor is a device that stores electrical energy and an inductor, or coil, store magnetic energy.

The physics for this problem stems from Kirchoff's Rules for circuits. Namely, the sum of the drops in electric potential are set equal to the rises in electric potential. The potential drops across each circuit element are given by

1. Resistor: $V = IR$.
2. Capacitor: $V = \frac{q}{C}$.
3. Inductor: $V = L\frac{dI}{dt}$.

Furthermore, we need to define the current as $I = \frac{dq}{dt}$. where q is the charge in the circuit. Adding these potential drops, we set them equal to the voltage supplied by the voltage source, $V(t)$. Thus, we obtain

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t).$$

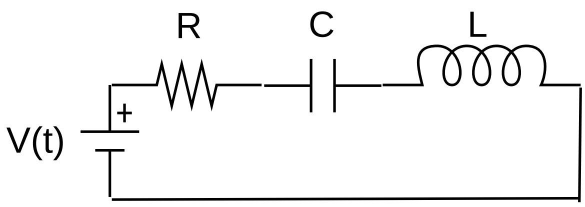

Figure 2.5: LRC Circuit.

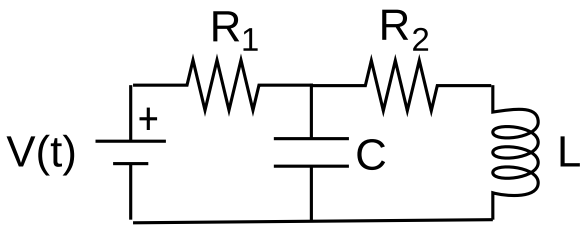

Figure 2.6: LRC Circuit.

Since both q and I are unknown, we can replace the current by its expression in terms of the charge to obtain

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t).$$

This is a second order equation for $q(t)$.

More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors. This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 2.6. This is not a problem that can be covered in the first year physics course. One can set up a system of second order equations and proceed to solve them.

2.4.1 Special Cases

In this section we will look at special cases that arise for the series LRC circuit equation. These include RC circuits, solvable by first order methods and LC circuits, leading to oscillatory behavior.

Case I. RC Circuits

We first consider the case of an RC circuit in which there is no inductor. Also, we will consider what happens when one charges a capacitor with a DC battery ($V(t) = V_0$) and when one discharges a charged capacitor ($V(t) = 0$).

For charging a capacitor, we have the initial value problem

$$R\frac{dq}{dt} + \frac{q}{C} = V_0, \quad q(0) = 0. \tag{2.37}$$

This equation is an example of a linear first order equation for $q(t)$. However, we can also rewrite it and solve it as a separable equation, since V_0 is a constant. We will do the former only as another example of finding the integrating factor.

We first write the equation in standard form:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}. \tag{2.38}$$

The integrating factor is then

$$\mu(t) = e^{\int \frac{dt}{RC}} = e^{t/RC}.$$

Thus,

$$\frac{d}{dt}\left(qe^{t/RC}\right) = \frac{V_0}{R}e^{t/RC}. \tag{2.39}$$

Integrating, we have

$$qe^{t/RC} = \frac{V_0}{R}\int e^{t/RC} = \frac{V_0}{C}\int e^{t/RC} + K. \tag{2.40}$$

Note that we introduced the integration constant, K. Now divide out the exponential to get the general solution:

$$q = \frac{V_0}{C} + Ke^{-t/RC}. \tag{2.41}$$

(If we had forgotten the K, we would not have gotten a correct solution for the differential equation.)

Next, we use the initial condition to get our particular solution. Namely, setting $t = 0$, we have that

$$0 = q(0) = \frac{V_0}{C} + K.$$

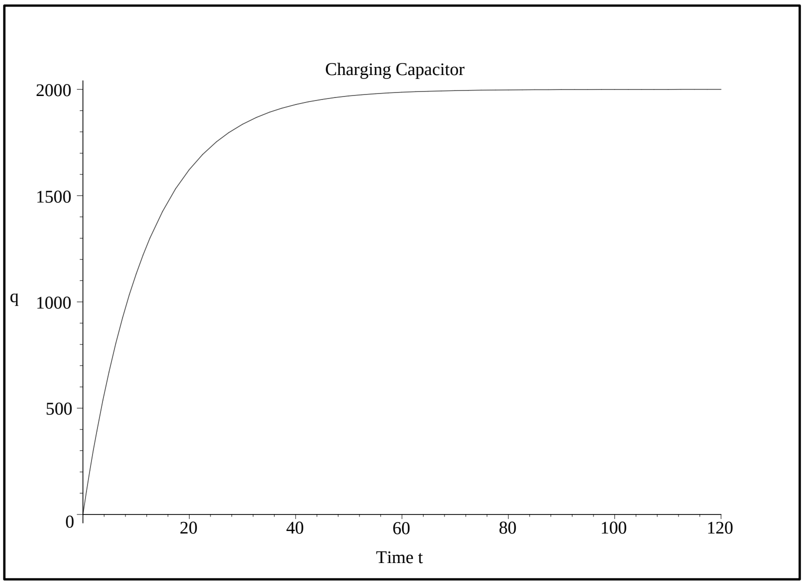

Figure 2.7: The charge as a function of time for a charging capacitor with $R = 2.00$ kΩ, $C = 6.00$ mF, and $V_0 = 12$ V.

So, $K = -\frac{V_0}{C}$. Inserting this into our solution, we have

$$q(t) = \frac{V_0}{C}(1 - e^{-t/RC}). \tag{2.42}$$

Now we can study the behavior of this solution. For large times the second term goes to zero. Thus, the capacitor charges up, asymptotically, to the final value of $q_0 = \frac{V_0}{C}$. This is what we expect, because the current is no longer flowing over R and this just gives the relation between the potential difference across the capacitor plates when a charge of q_0 is established on the plates.

Let's put in some values for the parameters. We let $R = 2.00$ kΩ, $C = 6.00$ mF, and $V_0 = 12$ V. A plot of the solution is given in Figure 2.7. We see that the charge builds up to the value of $V_0/C = 2000$ C. If we use a smaller resistance, $R = 200\ \Omega$, we see in Figure 2.8 that the capacitor charges to the same value, but much faster.

The rate at which a capacitor charges, or discharges, is governed by the time constant, $\tau = RC$. This is the constant factor in the exponential. The larger it is, the slower the exponential term decays.

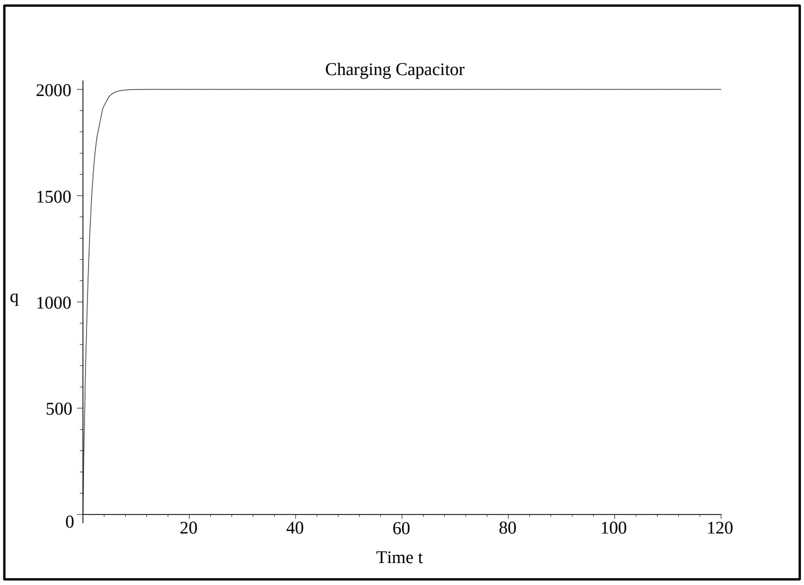

Figure 2.8: The charge as a function of time for a charging capacitor with $R = 200\ \Omega$, $C = 6.00$ mF, and $V_0 = 12$ V.

If we set $t = \tau$, we find that

$$q(\tau) = \frac{V_0}{C}(1 - e^{-1}) = (1 - 0.3678794412\ldots)q_0 \approx 0.63q_0.$$

Thus, at time $t = \tau$, the capacitor has almost charged to two thirds of its final value. For the first set of parameters, $\tau = 12$s. For the second set, $\tau = 1.2$s.

Now, let's assume the capacitor is charged with charge $\pm q_0$ on its plates. If we disconnect the battery and reconnect the wires to complete the circuit, the charge will then move off the plates, discharging the capacitor. The relevant form of our initial value problem becomes

$$R\frac{dq}{dt} + \frac{q}{C} = 0, \quad q(0) = q_0. \tag{2.43}$$

This equation is simpler to solve. Rearranging, we have

$$\frac{dq}{dt} = -\frac{q}{RC}. \tag{2.44}$$

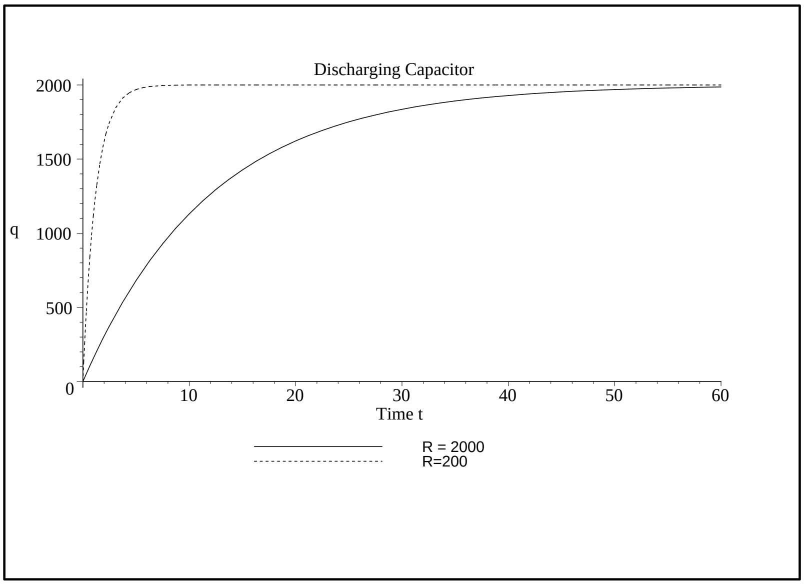

Figure 2.9: The charge as a function of time for a discharging capacitor with $R = 2.00$ kΩ or $R = 200$ Ω, and $C = 6.00$ mF, and $q_0 = 2000$ C.

This is a simple exponential decay problem, which you can solve using separation of variables. However, by now you should know how to immediately write down the solution to such problems of the form $y' = ky$. The solution is

$$q(t) = q_0 e^{-t/\tau}, \quad \tau = RC.$$

We see that the charge decays exponentially. In principle, the capacitor never fully discharges. That is why you are often instructed to place a shunt across a discharged capacitor to fully discharge it.

In Figure 2.9 we show the discharging of our two previous RC circuits. Once again, $\tau = RC$ determines the behavior. At $t = \tau$ we have

$$q(\tau) = q_0 e^{-1} = (0.3678794412\ldots)q_0 \approx 0.37 q_0.$$

So, at this time the capacitor only has about a third of its original value.

Case II. LC Circuits

Another simple result comes from studying *LC* circuits. We will now connect a charged capacitor to an inductor. In this case, we consider the initial value problem

$$L\ddot{q} + \frac{1}{C}q = 0, \quad q(0) = q_0, \dot{q}(0) = I(0) = 0. \tag{2.45}$$

Dividing out the inductance, we have

$$\ddot{q} + \frac{1}{LC}q = 0. \tag{2.46}$$

This equation is a second order, constant coefficient equation. It is of the same form as the ones for simple harmonic motion of a mass on a spring or the linear pendulum. So, we expect oscillatory behavior. The characteristic equation is

$$r^2 + \frac{1}{LC} = 0.$$

The solutions are

$$r_{1,2} = \pm\frac{i}{\sqrt{LC}}.$$

Thus, the solution of (2.46) is of the form

$$q(t) = c_1\cos(\omega t) + c_2\sin(\omega t), \quad \omega = (LC)^{-1/2}. \tag{2.47}$$

Inserting the initial conditions yields

$$q(t) = q_0\cos(\omega t). \tag{2.48}$$

The oscillations that result are understandable. As the charge leaves the plates, the changing current induces a changing magnetic field in the inductor. The stored electrical energy in the capacitor changes to stored magnetic energy in the inductor. However, the process continues until the plates are charged with opposite polarity and then the process begins in reverse. The charged capacitor then discharges and the capacitor eventually returns to its original state and the whole system repeats this over and over.

The frequency of this simple harmonic motion is easily found. It is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}. \tag{2.49}$$

This is called the tuning frequency because of its role in tuning circuits.

Of course, this is an ideal situation. There is always resistance in the circuit, even if only a small amount from the wires. So, we really need to account for resistance, or even add a resistor. This leads to a slightly more complicated system in which damping will be present.

2.5 Damped Oscillations

As we have indicated, simple harmonic motion is an ideal situation. In real systems we often have to contend with some energy loss in the system. This leads to the damping of our oscillations. This energy loss could be in the spring, in the way a pendulum is attached to its support, or in the resistance to the flow of current in an LC circuit. The simplest models of resistance are the addition of a term in first derivative of the dependent variable. Thus, our three main examples with damping added look like:

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{2.50}$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \tag{2.51}$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0. \tag{2.52}$$

These are all examples of the general constant coefficient equation

$$ay''(x) + by'(x) + cy(x) = 0. \tag{2.53}$$

We have seen that solutions are obtained by looking at the characteristic equation $ar^2 + br + c = 0$. This leads to three different behaviors depending on the discriminant in the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.54}$$

We will consider the example of the damped spring. Then we have

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}. \tag{2.55}$$

For $b > 0$, there are three types of damping.

I. Overdamped, $b^2 > 4mk$

In this case we obtain two real root. Since this is Case I for constant coefficient equations, we have that

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

We note that $b^2 - 4mk < b^2$. Thus, the roots are both negative. So, both terms in the solution exponentially decay. The damping is so strong that there is no oscillation in the system.

II. Critically Damped, $b^2 = 4mk$

In this case we obtain one real root. This is Case II for constant coefficient equations and the solution is given by

$$x(t) = (c_1 + c_2 t)e^{rt},$$

where $r = -b/2m$. Once again, the solution decays exponentially. The damping is just strong enough to hinder any oscillation. If it were any weaker the discriminant would be negative and we would need the third case.

III. Underdamped, $b^2 < 4mk$

In this case we have complex conjugate roots. We can write $\alpha = -b/2m$ and $\beta = \sqrt{4mk - b^2}/2m$. Then the solution is

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

These solutions exhibit oscillations due to the trigonometric functions, but we see that the amplitude may decay in time due the the overall factor of $e^{\alpha t}$ when $\alpha < 0$.. Consider the case that the initial conditions give $c_1 = A$ and $c_2 = 0$. (When is this?) Then, the solution, $x(t) = Ae^{\alpha t} \cos \beta t$, looks like the plot in Figure 2.10.

2.6 Forced Oscillations

All of the systems presented at the beginning of the last section exhibit the same general behavior when a damping term is present. An additional term can be added that can cause even more complicated behavior. In the

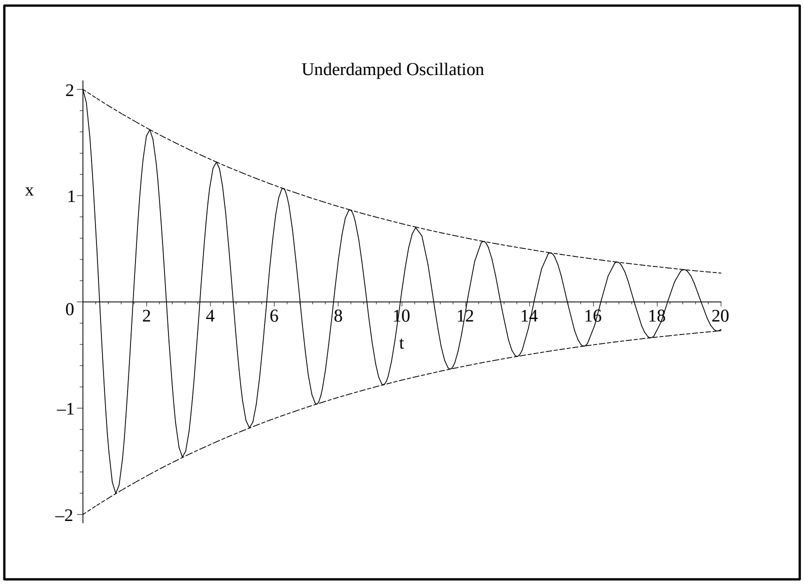

Figure 2.10: A plot of underdamped oscillation given by $x(t) = 2e^{0.1t} \cos 3t$. The dashed lines are given by $x(t) = \pm 2e^{0.1t}$, indicating the bounds on the amplitude of the motion.

case of LRC circuits, we have seen that the voltage source makes the system nonhomogeneous. It provides what is called a source term. Such terms can also arise in the mass-spring and pendulum systems. One can drive such systems by periodically pushing the mass, or having the entire system moved, or impacted by an outside force. Such systems are called forced, or driven.

Typical systems in physics can be modeled by nonhomogenous second order equations. Thus, we want to find solutions of equations of the form

$$Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.56}$$

Earlier we saw that the solution of equations are found in two steps.

1. First you solve the homogeneous equation for a general solution of $Ly_h = 0$, $y_h(x)$.

2. Then, you obtain a particular solution of the nonhomogeneous equation, $y_p(x)$.

To date, we only know how to solve constant coefficient, homogeneous equations. So, by adding a nonhomogeneous to such equations we need to figure out what to do with the extra term. In other words, how does one find the particular solution?

You could guess a solution, but that is not usually possible without a little bit of experience. So we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically developed the particular solution. We will come back to this method the Method of Variation of Parameters, later in this section.

2.6.1 Method of Undetermined Coefficients

Let's solve a simple differential equation highlighting how we can handle nonhomogeneous equations. Consider the equation

$$y'' + 2y' - 3y = 4. \tag{2.57}$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \tag{2.58}$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (2.57). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x, then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (2.57), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining our two solutions, we have the general solution to the original nonhomogeneous equation (2.57). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine our arbitrary constants.

What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \tag{2.59}$$

The only thing that would change is our particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (2.59), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (2.59)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \tag{2.60}$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \tag{2.61}$$

$f(x)$	Guess
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$
ae^{bx}	Ae^{bx}
$a\cos\omega x + b\sin\omega x$	$A\cos\omega x + B\sin\omega x$

So, our particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 2.6.1. More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have your solution. This solution is then added to the general solution of the homogeneous differential equation.

As a final example, let's consider the equation

$$y'' + 2y' - 3y = 2e^{-3x}. \tag{2.62}$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A, disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^x and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If our driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$. We compute the derivative of our guess, $y_p' = A(1-3x)e^{-3x}$ and $y_p'' = A(9x-6)e^{-3x}$. Inserting these into the equation, we obtain

$$[(9x-6) + 2(1-3x) - 3x]Ae^{-3x} = 2e^{-3x},$$

or

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$.

2.6.2 Method of Variation of Parameters

A more systematic way to find particular solutions is through the use of the Method of Variation of Parameters. The derivation is a little messy and the solution is sometimes messy, but the application of the method is straight forward if you can do the required integrals. We will first derive the needed equations and then do some examples.

We begin with the nonhomogeneous equation. Let's assume it is of the standard form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \tag{2.63}$$

We know that the solution of the homogeneous equation can be written in terms of two linearly independent solutions, which we will call $y_1(x)$ and $y_2(x)$:

$$y_h(x) = c_1y_1(x) + c_2y_2(x).$$

If one replaces the constants with functions, then you now longer have a solution to the homogeneous equation. Is it possible that you could stumble across the right functions with which to replace the constants and somehow end up with $f(x)$ when inserted into the left side of the differential equation? It turns out that you can.

So, let's assume that the constants are replaced with two unknown functions, which we will call $c_1(x)$ and $c_2(x)$. This change of the parameters is where the name of the method derives. Thus, we are assuming that a particular solution takes the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \tag{2.64}$$

If this is to be a solution, then insertion into the differential equation should make it true. To do this we will first need to compute some derivatives.

The first derivative is given by

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{2.65}$$

Next we will need the second derivative. But, this will give use eight terms. So, we will first make an assumption. Let's assume that the last two terms add to zero:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{2.66}$$

It turns out that we will get the same results in the end if we did not assume this. The important thing is that it works!

So, we now have the first derivative as

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x). \tag{2.67}$$

The second derivative is then only four terms:

$$y_p'(x) = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x). \tag{2.68}$$

Now that we have the derivatives, we can insert our guess into the differential equation. Thus, we have

$$\begin{aligned} f(x) &= a(x)(c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x)) \\ &\quad +b(x)(c_1(x)y_1'(x) + c_2(x)y_2'(x)) \\ &\quad +c(x)(c_1(x)y_1(x) + c_2(x)y_2(x)). \end{aligned} \tag{2.69}$$

Regrouping the terms, we obtain

$$\begin{aligned} f(x) &= c_1(x)(a(x)y_1''(x) + b(x)y_1'(x) + c(x)y_1(x)) \\ &\quad c_2(x)(a(x)y_2''(x) + b(x)y_2'(x) + c(x)y_2(x)) \\ &\quad +a(x)(c_1'(x)y_1'(x) + c_2'(x)y_2'(x)). \end{aligned} \tag{2.70}$$

Note that the first two rows vanish since y_1 and y_2 are solutions of the homogeneous problem. This leaves the equation

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = \frac{f(x)}{a(x)}. \tag{2.71}$$

In summary, we have assumed a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

This is only possible if the unknown functions $c_1(x)$ and $c_2(x)$ satisfy the system of equations

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{a(x)}. \end{aligned} \tag{2.72}$$

It is standard to solve this system for the derivatives of the unknown functions and then present the integrated forms. However, one could just start from here.

Example Consider the problem: $y'' - y = e^{2x}$. We want the general solution of this nonhomogeneous problem.

The general solution to the homogeneous problem $y_h'' - y_h = 0$ is

$$y_h(x) = c_1e^x + c_2e^{-x}.$$

In order to use the Method of Variation of Parameters, we seek a solution of the form

$$y_p(x) = c_1(x)e^x + c_2(x)e^{-x}.$$

We find the unknown functions by solving the system in (2.72), which in this case becomes

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= e^{2x}. \end{aligned} \tag{2.73}$$

Adding these equations we find that

$$2c_1'e^x = e^{2x} \to c_1' = \frac{1}{2}e^x.$$

Solving for $c_1(x)$ we find

$$c_1(x) = \frac{1}{2}\int e^x\,dx = \frac{1}{2}e^x.$$

Subtracting the equations in the system yields

$$2c_2'e^{-x} = -e^{2x} \rightarrow c_2' = -\frac{1}{2}e^{3x}.$$

Thus,

$$c_2(x) = -\frac{1}{2}\int e^{3x}\,dx = -\frac{1}{6}e^{3x}.$$

The particular solution is found by inserting these results into y_p:

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (\frac{1}{2}e^x)e^x + (-\frac{1}{6}e^{3x})e^{-x} \\ &= \frac{1}{3}e^{2x}. \end{aligned} \tag{2.74}$$

Thus, we have the general solution of the nonhomogeneous problem as

$$y(x) = c_1e^x + c_2e^{-x} + \frac{1}{3}e^{2x}.$$

Example Now consider the problem: $y'' + 4y = \sin x$.

The solution to the homogeneous problem is

$$y_h(x) = c_1\cos 2x + c_2\sin 2x. \tag{2.75}$$

We now seek a particular solution of the form

$$y_h(x) = c_1(x)\cos 2x + c_2(x)\sin 2x.$$

We let $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, $a(x) = 1$, $f(x) = \sin x$ in system (2.72):

$$\begin{aligned} c_1'(x)\cos 2x + c_2'(x)\sin 2x &= 0 \\ -2c_1'(x)\sin 2x + 2c_2'(x)\cos 2x &= \sin x. \end{aligned} \tag{2.76}$$

Now, use your favorite method for solving a system of two equations and two unknowns. In this case, we can multiply the first equation by $2\sin 2x$ and the second equation by $\cos 2x$. Adding the resulting equations will eliminate the c_1' terms. Thus, we have

$$c_2'(x) = \frac{1}{2}\sin x \cos 2x = \frac{1}{2}(2\cos^2 x - 1)\sin x.$$

Inserting this into the first equation of the system, we have

$$c_1'(x) = -c_2'(x)\frac{\sin 2x}{\cos 2x} = -\frac{1}{2}\sin x \sin 2x = -\sin^2 x \cos x.$$

These can easily be solved:

$$c_2(x) = \frac{1}{2}\int (2\cos^2 x - 1)\sin x\, dx = \frac{1}{2}(\cos x - \frac{2}{3}\cos^3 x).$$

$$c_1(x) = -\int \sin^x \cos x\, dx = -\frac{1}{3}\sin^3 x.$$

The final step in getting the particular solution is to insert these functions into $y_p(x)$. This gives

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= (-\frac{1}{3}\sin^3 x)\cos 2x + (\frac{1}{2}\cos x - \frac{1}{3}\cos^3 x)\sin x \\ &= \frac{1}{3}\sin x. \end{aligned} \tag{2.77}$$

So, the general solution is

$$y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\sin x. \tag{2.78}$$

2.7 Numerical Solutions of ODEs

So far we have seen some of the standard methods for solving first and second order differential equations. However, we have had to restrict ourselves to very special cases in order to get nice analytical solutions to our initial value problems. While these are not the only equations for which we can get exact results (see Section 2.10 for another common class of second order differential equations), there are many cases in which exact solutions are not possible. In such cases we have to rely on approximation techniques, including the numerical solution of the equation at hand.

The use of numerical methods to obtain approximate solutions of differential equations and systems of differential equations has been known for some time. However, with the advent of powerful computers and desktop computers, we can now solve many of these problems with relative ease. The simple ideas used to solve first order differential equations can be extended to the solutions of more complicated systems of partial differential equations, such as the large scale problems of modelling ocean dynamics, weather systems and even cosmological problems stemming from general relativity.

In this section we will look at the simplest method for solving first order equations, Euler's Method. While it is not the most efficient method, it does provide us with a picture of how one proceeds and can be improved by introducing better techniques, which are typically covered in a numerical analysis text.

Let's consider the class of first order initial value problems of the form

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{2.79}$$

We are interested in finding the solution $y(x)$ of this equation which passes through the initial point (x_0, y_0) in the xy-plane for values of x in the interval $[a, b]$, where $a = x_0$. We will seek approximations of the solution at N points, labeled x_n for $n = 1, \ldots, N$. For equally spaced points we have $\Delta x = x_1 - x_0 = x_2 - x_1$, etc. Then, $x_n = x_0 + n\Delta x$. In Figure 2.11 we show three such points on the x-axis.

We will develop a simple numerical method, called Euler's Method. We rely on Figure 2.11 to do this. As already noted, we first break the interval of interest into N subintervals with $N + 1$ points x_n. We already know a point on the solution $(x_0, y(x_0)) = (x_0, y_0)$. How do we find the solution for other x values?

We first note that the differential equation gives us the slope of the tangent line at $(x, y(x))$ of our solution $y(x)$. The slope is $f(x, y(x))$. Referring to Figure 2.11, we see the tangent line drawn at (x_0, y_0). We

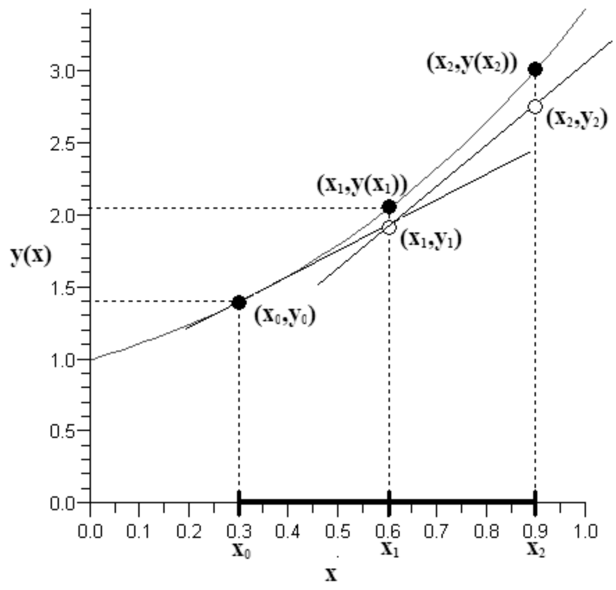

Figure 2.11: The basics of Euler's Method are shown. An interval of the x axis is broken into N subintervals. The approximations to the solutions are found using the slope of the tangent to the solution, given by $f(x, y)$. Knowing previous approximations at (x_{n-1}, y_{n-1}), one can determine the next approximation, y_n.

look now at $x = x_1$. A vertical line intersects both the solution curve and the tangent line. While we do not know the solution, we can determine the tangent line and find the intersection point. As seen in our figure, this intersection point is in theory close to the point on the solution curve. So, we will designate y_1 as the approximation of our solution $y(x_1)$. We just need to determine y_1.

The idea is simple. We approximate the derivative in our differential equation by its difference quotient:

$$\frac{dy}{dx} \approx \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{\Delta x}. \tag{2.80}$$

But, we have by the differential equation that the slope of the tangent to the curve at (x_0, y_0) is

$$y'(x_0) = f(x_0, y_0).$$

Thus,

$$\frac{y_1 - y_0}{\Delta x} \approx f(x_0, y_0). \tag{2.81}$$

So, we can solve this equation for y_1 to obtain

$$y_1 = y_0 + \Delta x f(x_0, y_0). \tag{2.82}$$

This give y_1 in terms of quantities that we know.

We now proceed to approximate $y(x_2)$. Referring to Figure 2.11, we see that this can be done by using the slope of the solution curve at (x_1, y_1). The corresponding tangent line is shown passing though (x_1, y_1) and we can then get the value of y_2. Following the previous argument, we find that

$$y_2 = y_1 + \Delta x f(x_1, y_1). \tag{2.83}$$

Continuing this procedure for all x_n, we arrive at the following numerical scheme for determining a numerical solution to Euler's equation:

$$\begin{aligned} y_0 &= y(x_0), \\ y_n &= y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}), \quad n = 1, \ldots, N. \end{aligned} \tag{2.84}$$

Example We will consider a standard example for which we know the exact solution. This way we can compare our results. The problem is given that

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \tag{2.85}$$

n	x_n	$y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1} = 0.5x_{n-1} + 1.5y_{n-1}$
0	0	1
1	0.5	$0.5(0) + 1.5(1.0) = 1.5$
2	1.0	$0.5(0.5) + 1.5(1.5) = 2.5$

Table 2.1: Application of Euler's Method for $y' = x + y$, $y(0) = 1$ and $\Delta x = 0.5$.

n	x_n	$y_n = 0.2x_{n-1} + 1.2y_{n-1}$
0	0	1
1	0.2	$0.2(0) + 1.2(1.0) = 1.2$
2	0.4	$0.2(0.2) + 1.2(1.2) = 1.48$
3	0.6	$0.2(0.4) + 1.2(1.48) = 1.856$
4	0.8	$0.2(0.6) + 1.2(1.856) = 2.3472$
5	1.0	$0.2(0.8) + 1.2(2.3472) = 2.97664$

Table 2.2: Application of Euler's Method for $y' = x + y$, $y(0) = 1$ and $\Delta x = 0.2$.

find an approximation for $y(1)$.

First, we will do this by hand. We will break up the interval $[0, 1]$, since we want our solution at $x = 1$ and the initial value is at $x = 0$. Let $\Delta x = 0.50$. Then, $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1.0$. Note that $N = \frac{b-a}{\Delta x} = 2$.

We can carry out Euler's Method systematically. We set up a table for the needed values. Such a table is shown in Table 2.1.

Note how the table is set up. There is a column for each x_n and y_n. The first row is the initial condition. We also made use of the function $f(x, y)$ in computing the y_n's. This sometimes makes the computation easier. As a result, we find that the desired approximation is given as $y_2 = 2.5$.

Is this a good result? Well, we could make the spatial increments smaller. Let's repeat the procedure for $\Delta x = 0.2$, or $N = 5$. The results are in Table 2.2.

Now we see that our approximation is $y_1 = 2.97664$. So, it looks like our value is near 3, but we cannot say much more. Decreasing Δx more shows that we are beginning to converge to a solution. We see this in Table 2.3.

Δx	$y_N \approx y(1)$
0.5	2.5
0.2	2.97664
0.1	3.187484920
0.01	3.409627659
0.001	3.433847864
0.0001	3.436291854

Table 2.3: Results of Euler's Method for $y' = x + y$, $y(0) = 1$ and varying Δx

Of course, these values were not done by hand. The last computation would have taken 1000 lines in our table, or at least 40 pages! One could use a computer to do this. A simple code in Maple would look like the following:

```
> restart:
> f:=(x,y)->y+x;
> a:=0:  b:=1:  N:=100:  h:=(b-a)/N;
> x[0]:=0: y[0]:=1:
  for i from 1 to N do
  y[i]:=y[i-1]+h*f(x[i-1],y[i-1]):
  x[i]:=x[0]+h*(i):
  od:
  evalf(y[N]);
```

In this case we could simply use the exact solution. The exact solution is easily found as

$$y(x) = 2e^x - x - 1.$$

(The reader can verify this.) So, the value we are seeking is

$$y(1) = 2e - 2 = 3.4365636\ldots.$$

Thus, even the last numerical solution was off by about 0.00027.

Adding a few extra lines for plotting, we can visually see how well our approximations compare to the exact solution. The Maple code for doing such a plot is given below.

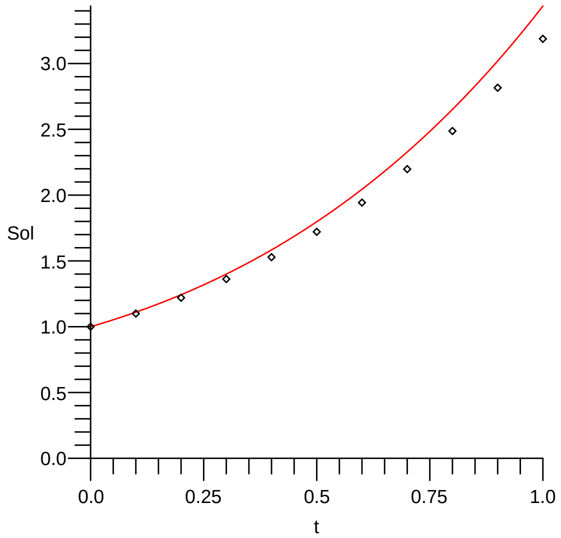

Figure 2.12: A comparison of the results Euler's Method to the exact solution for $y' = x + y$, $y(0) = 1$ and $N = 10$.

```
> with(plots):
> Data:=[seq([x[i],y[i]],i=0..N)]:
> P1:=pointplot(Data,symbol=DIAMOND):
> Sol:=t->-t-1+2*exp(t);
> P2:=plot(Sol(t),t=a..b,Sol=0..Sol(b)):
> display({P1,P2});
```

We show in Figures 2.12-2.13 the results for $N = 10$ and $N = 100$. In Figure 2.12 we can see how quickly our numerical solution diverges from the exact solution. In Figure 2.13 we can see that visually the solutions agree, but we note that from Table 2.3 that for $\Delta x = 0.01$, the solution is still off in the second decimal place with a relative error of about 0.8%.

Why would we use a numerical method when we have the exact solution? Exact solutions can serve as test cases for our methods. We can make sure our code works before applying them to problems whose solution is not known.

There are many other methods for solving first order equations. One commonly used method is the fourth order Runge-Kutta method. This

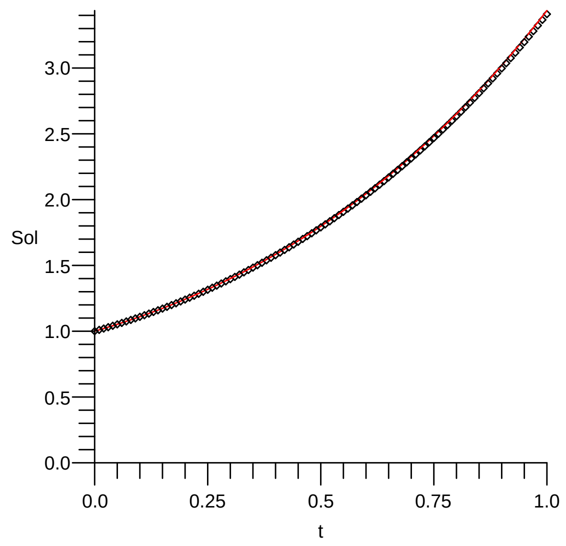

Figure 2.13: A comparison of the results Euler's Method to the exact solution for $y' = x + y$, $y(0) = 1$ and $N = 100$.

method has smaller errors at each step as compared to Euler's Method. It is well suited for programming and comes built-in in many packages like Maple and Matlab. Typically, it is set up to handle systems of first order equations.

In fact, it is well known that nth order equations can be written as a system of n first order equations. Consider the simple second order equation

$$y'' = f(x, y).$$

This is a larger class of equations than our second order constant coefficient equation. We can turn this into a system of two first order differential equations by letting $u = y$ and $v = y' = u'$. Then, $v' = y'' = f(x, u)$. So, we have the first order system

$$\begin{aligned} u' &= v, \\ v' &= f(x, u). \end{aligned} \tag{2.86}$$

We will not go further into the Runge-Kutta Method here. You can find more about it in a numerical analysis text. However, we will see that systems of differential equations do arise naturally in physics. Such systems are often coupled equations and lead to interesting behaviors.

2.8 Coupled Oscillators

In the last section we saw that the numerical solution of second order equations, or higher, can be cast into systems of first order equations. Such systems are typically coupled in the sense that the solution of at least one of the equations in the system depends on knowing one of the other solutions in the system. In many physical systems this coupling takes place naturally. We will introduce a simple model in this section to illustrate the coupling of simple oscillators. However, we will reserve solving the coupled system until the next chapter after exploring the needed mathematics.

There are many problems in physics that result in systems of equations. This is because the most basic law of physics is given by Newton's Second Law, which states that if a body experiences a net force, it will accelerate. Thus,

$$\sum \mathbf{F} = m\mathbf{a}.$$

Since $\mathbf{a} = \ddot{\mathbf{x}}$ we have a system of second order differential equations in general for three dimensional problems, or one second order differential equation for one dimensional problems.

We have already seen the simple problem of a mass on a spring as shown in Figure 2.2. Recall that the net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant and x is the elongation of the spring. When it is positive, the spring force is negative and when it is negative the spring force is positive. The equation for simple harmonic motion for the mass-spring system was found to be given by

$$m\ddot{x} + kx = 0.$$

This second order equation can be written as a system of two first order equations in terms of the unknown position and velocity. We first set

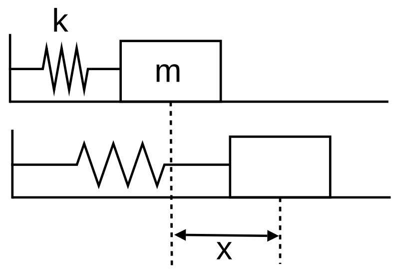

Figure 2.14: Spring-Mass system.

$y = \dot{x}$ and then rewrite the second order equation in terms of x and y. Thus, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{2.87}$$

The coefficient matrix for this system is $\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$, where $\omega^2 = \frac{k}{m}$.

One can look at more complicated spring-mass systems. Consider two blocks attached with two springs as in Figure 2.15. In this case we apply Newton's second law for each block. We will designate the elongations of each spring from equilibrium as x_1 and x_2. These are shown in Figure 2.15.

For mass m_1, the forces acting on it are due to each spring. The first spring with spring constant k_1 provides a force on m_1 of $-k_1x_1$. The second spring is stretched, or compressed, based upon the relative locations of the two masses. So, it will exert a force on m_1 of $k_2(x_2 - x_1)$.

Similarly, the only force acting directly on mass m_2 is provided by the restoring force from spring 2. So, that force is given by $-k_2(x_2 - x_1)$. The reader should think about the signs in each case.

Putting this all together, we apply Newton's Second Law to both masses. We obtain the two equations

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_1 - x_2) \\ m_2\ddot{x}_2 &= -k_2(x_1 - x_2). \end{aligned} \tag{2.88}$$

Thus, we see that we have a coupled system of two second order differential equations.

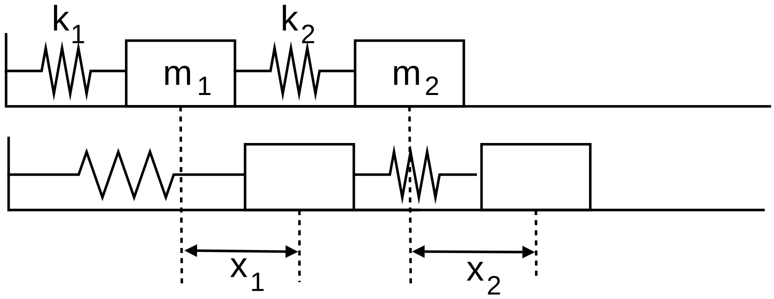

Figure 2.15: Spring-Mass system.

One can rewrite this system of two second order equations as a system of four first order equations by letting $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$. This leads to the system

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_1 - x_2) \\
\dot{x}_4 &= -\frac{k_2}{m_2}(x_1 - x_2).
\end{aligned} \tag{2.89}$$

As we will see, this system can be written more compactly in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1-k_2}{m_1} & -\frac{k_2}{m_1} & 0 & 0 \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.90}$$

However, before we can solve this system of first order equations, we need to recall a few things from linear algebra. This will be done in the next chapter.

2.9 The Nonlinear Pendulum *Optional*

We can also make the system more realistic by adding damping. This could be due to energy loss in the way the string is attached to the support or due to the drag on the mass, etc. Assuming that the damping is proportional to the angular velocity, we have equations for the damped nonlinear and damped linear pendula:

$$L\ddot{\theta} + b\dot{\theta} + g\sin\theta = 0. \tag{2.91}$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \tag{2.92}$$

Finally, we can add forcing. Imagine that the support is attached to a device to make the system oscillate horizontally at some frequency. Then we could have equations such as

$$L\ddot{\theta} + b\dot{\theta} + g\sin\theta = F\cos\omega t. \tag{2.93}$$

We will look at these and other oscillation problems later in our discussion.

Before returning to studying the equilibrium solutions of the nonlinear pendulum, we will look at how far we can get at obtaining analytical solutions. First, we investigate the simple linear pendulum.

The linear pendulum equation (2.29) is a constant coefficient second order linear differential equation. The roots of the characteristic equations are $r = \pm\sqrt{\frac{g}{L}}i$. Thus, the general solution takes the form

$$\theta(t) = c_1\cos(\sqrt{\frac{g}{L}}t) + c_2\sin(\sqrt{\frac{g}{L}}t). \tag{2.94}$$

We note that this is usually simplified by introducing the angular frequency

$$\omega \equiv \sqrt{\frac{g}{L}}. \tag{2.95}$$

One consequence of this solution, which is used often in introductory physics, is an expression for the period of oscillation of a simple pendulum. The period is found to be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{g}{L}}. \tag{2.96}$$

As we have seen, this value for the period of a simple pendulum was derived assuming a small angle approximation. How good is this

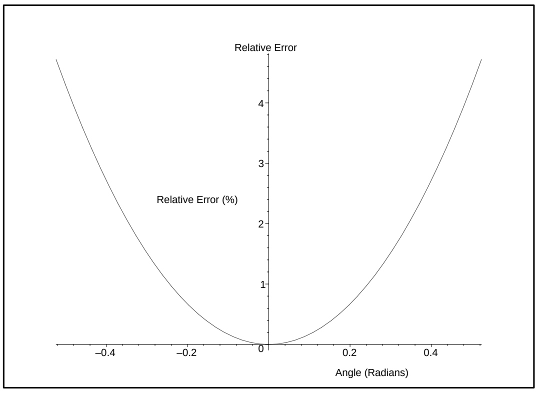

Figure 2.16: The relative error in percent when approximating $\sin\theta$ by θ..

approximation? What is meant by a *small* angle? We could recall from calculus that the Taylor series approximation of $\sin\theta$ about $\theta = 0$:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots. \tag{2.97}$$

One can obtain a bound on the error when truncating this series to one term after taking a numerical analysis course. But we can just simply plot the relative error, which is defined as

$$\text{Relative Error } = \frac{\sin\theta - \theta}{\sin\theta}.$$

A plot of the relative error is given in Figure 2.16. Thus for $\theta \approx 0.4$ radians (or, degrees) we have that the relative error is about 4%.

We would like to do better than this. So, we now turn to the nonlinear pendulum. We first rewrite Equation (2.93) is the simpler form

$$\ddot{\theta} + \omega^2\theta = 0. \tag{2.98}$$

We next employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 + \int^{\theta(t)} F(\phi)\, d\phi\right] = (\ddot{\theta} + F(\theta))\dot{\theta}.$$

For our problem, we multiply Equation (2.98) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2\theta\dot{\theta} = 0$$

and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \qquad (2.99)$$

Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \sqrt{2(c + \omega^2\cos\theta)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2\cos\theta)}}.$$

Of course, one needs to be able to do the integral. When one gets a solution in this implicit form, one says that the problem has been solved by quadratures. Namely, the solution is given in terms of some integral.

In fact, the above integral can be transformed into what is know as an elliptic integral of the first kind. We will rewrite our result and then use it to obtain an approximation to the period of oscillation of our nonlinear pendulum, leading to corrections to the linear result found earlier.

We will first rewrite the constant found in (2.99). This requires a little physics. The swinging of a mass on a string, assuming no energy loss at the pivot point, is a conservative process. Namely, the total mechanical energy is conserved. Thus, the total of the kinetic and gravitational potential energies is a constant. The kinetic energy of the masses on the string is given as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

The potential energy is the gravitational potential energy. If we set the potential energy to zero at the bottom of the swing, then the potential energy is $U = mgh$, where h is the height that the mass is from the bottom of the swing. A little trigonometry gives that $h = L(1 - \cos\theta)$. So,

$$U = mgL(1 - \cos\theta).$$

So, the total mechanical energy is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos\theta). \tag{2.100}$$

We note that a little rearranging shows that we can relate this to Equation (2.99):

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \frac{1}{mL^2}E - \omega^2 = c.$$

We can use Equation (2.100) to get a value for the total energy. At the top of the swing the mass is not moving, if only for a moment. Thus, the kinetic energy is zero and the total energy is pure potential energy. Letting θ_0 denote the angle at the highest position, we have that

$$E = mgL(1 - \cos\theta_0) = mL^2\omega^2(1 - \cos\theta_0).$$

Therefore, we have found that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = \omega^2(1 - \cos\theta_0). \tag{2.101}$$

Using the half angle formula,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta),$$

we can rewrite Equation (2.101) as

$$\frac{1}{2}\dot{\theta}^2 = 2\omega^2\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right]. \tag{2.102}$$

Solving for θ', we have

$$\frac{d\theta}{dt} = 2\omega\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right]^{1/2}. \tag{2.103}$$

One can now apply separation of variables and obtain an integral similar to the solution we had obtained previously. Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, we have that

$$T = \frac{2}{\omega} \int_0^{\theta_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}. \tag{2.104}$$

This result is not much different than our previous result, but we can now easily transform the integral into an elliptic integral. We define

$$z = \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta_0}{2}}$$

and

$$k = \sin \frac{\theta_0}{2}.$$

Then Equation (2.104) becomes

$$T = \frac{4}{\omega} \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}. \tag{2.105}$$

This is done by noting that $dz = \frac{1}{2k} \cos \frac{\theta}{2}\, d\theta = \frac{1}{2k}(1 - k^2 z^2)^{1/2}\, d\theta$ and that $\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} = k^2(1 - z^2)$. The integral in this result is an elliptic integral of the first kind. In particular, the elliptic integral of the first kind is defined as

$$F(\phi, k) \equiv= \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{\sin \phi} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$

In some contexts, this is known as the incomplete elliptic integral of the first kind and $K(k) = F(\frac{\pi}{2}, k)$ is called the complete integral of the first kind.

There are table of values for elliptic integrals and now one can use a computer algebra system to compute values of such integrals. For small angles, we have that k is small. So, we can develop a series expansion for the period, T, for small k. This is simply done by first expanding

$$(1 - k^2 z^2)^{-1/2} = 1 + \frac{1}{2} k^2 z^2 + \frac{3}{8} k^2 z^4 + O((kz)^6)$$

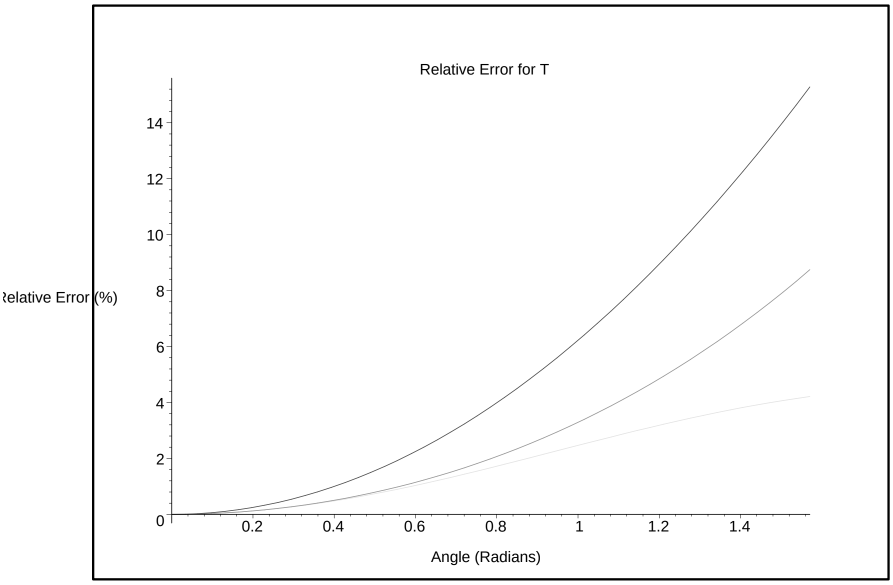

Figure 2.17: The relative error in percent when approximating the exact period of a nonlinear pendulum with one, two, or three terms in Equation (2.106).

using the binomial expansion which we review later in the text. Inserting the expansion in the integrand and integrating term by term, one finds that

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right]. \tag{2.106}$$

This expression gives further corrections to the linear result, which only provides the first term. In Figure 2.17 we show the relative errors incurred when keeping the k^2 and k^4 terms versus not keeping them.

2.10 Cauchy-Euler Equations - *Optional*

Another class of solvable second order differential equations that are of interest are the Cauchy-Euler equations. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \tag{2.107}$$

Note that in such equations the power of x in the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. This leads to the characteristic equation

$$ar(r-1) + br + c = 0. \tag{2.108}$$

Again, one has a quadratic equation and the nature of the roots leads to three classes of solutions.

1. **Real, distinct roots:** r_1, r_2. In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.
2. **Real, equal roots:** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln|x|$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln|x|)x^r$.
3. **Complex conjugate roots:** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^\alpha \cos(\beta \ln|x|)$ and $x^\alpha \sin(\beta \ln|x|)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^\alpha(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|))$.

Example 1. $x^2y'' + 5xy' + 12y = 0$

As with the constant coefficient equations, we begin by writing down the characteristic equation. A simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2 \end{aligned} \tag{2.109}$$

and one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is
$y(x) = \left[c_1 \cos(2\sqrt(2)\ln|x|) + c_2 \sin(2\sqrt(2)\ln|x|)\right] x^{-2}$