

Chapter 7

Electromagnetic Waves

Up to this point we have mainly been confined to problems involving only one or two independent variables. In particular, the heat equation and the wave equation involved one time and one space dimension. However, we live in a world of three spatial dimensions. (Though, some theoretical physicists live in worlds of many more dimensions, or at least they think so.) We will need to extend the study of heat flow and wave theory to three dimensions.

Recall that the one-dimensional wave equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \quad (7.1)$$

We need to generalize the $\frac{\partial^2 u}{\partial x^2}$ term. For the case of electromagnetic waves in a source-free environment, we will derive the three dimensional wave equation. It is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \quad (7.2)$$

This is the generic form of the linear wave equation in Cartesian coordinates. It can be written a more compact form using the Laplacian operator, ∇^2 ,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \quad (7.3)$$

Name	2 Vars	3 D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2 \nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x, y)$	$\nabla^2 u = F(x, y, z)$
Schrödinger's Equation	$i u_t = u_{xx} + F(x, t)u$	$i u_t = \nabla^2 u + F(x, y, z, t)u$

Table 7.1: List of generic partial differential equations.

The introduction of the Laplacian is common when generalizing to higher dimensions. In fact, we already presented some generic equations in Table 4.1, which we reproduce in this chapter in Table 7.1. We have already studied the wave equation and the heat equation. We saw Schrödinger's equation in the last chapter as a natural problem involving Fourier transforms. For steady-state, or equilibrium, heat flow problems, the heat equation no longer involves the time derivative. What is left is called Laplace's equation, which we have also seen in relation to complex functions. Adding an external heat source, Laplace's equation becomes what is known as Poisson's equation.

Using the Laplacian allows us not only to write these equations in a more compact form, but also in a coordinate-free representation. Many problems are more easily cast in other coordinate systems. For example, the propagation of electromagnetic waves in an optical fiber are naturally described in terms of cylindrical coordinates. The heat flow inside a hemispherical igloo can be described using spherical coordinates. The vibrations of a circular drumhead can be described using polar coordinates. In each of these cases the Laplacian has to be written in terms of the needed coordinate systems.

The solution of these partial differential equations can be handled using separation of variables or transform methods. In the next chapter we will look at several examples of applying the separation of variables in higher dimensions. This will lead to the study of ordinary differential equations, which will lead to new sets of functions, other than our typical sines and cosines.

7.1 Maxwell's Equations

There are many applications leading to the equations in Table 7.1. In this chapter our goal is to derive the three dimensional wave equation for electromagnetic waves. This derivation was first carried out by James Clerk Maxwell in 1860. At the time much was known about the relationship between electric and magnetic fields through the work of of such people as Hans Christian Ørsted (1777-1851), Michael Faraday (1791-1867), and André-Marie Ampère. Maxwell provided a mathematical formalism for these relationships consisting of twenty partial differential equations in twenty unknowns. Later these equations were put into more compact notations, namely in terms of quaternions, only later to be cast in vector form. (Quaternions were introduced in 1843 by William Rowan Hamilton (1805-1865) as a four dimensional generalization of complex numbers.)

In vector form, the original Maxwell's equations are given as

$$\begin{aligned}
 \nabla \cdot \mathbf{D} &= \rho \\
 \nabla \times \mathbf{H} &= \mu_0 \mathbf{J}_{\text{tot}} \\
 \mathbf{D} &= \epsilon \mathbf{E} \\
 \mathbf{J} &= \sigma \mathbf{E} \\
 \mathbf{J}_{\text{tot}} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\
 \nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \\
 \mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \\
 \mu \mathbf{H} &= \nabla \times \mathbf{A}.
 \end{aligned} \tag{7.4}$$

Note that Maxwell expressed the electric and magnetic fields in terms of the scalar and vector potentials, ϕ and \mathbf{A} , respectively. Here \mathbf{H} is the magnetic field, \mathbf{D} is the electric displacement field, \mathbf{E} is the electric field, \mathbf{J} is the current density, ρ is the charge density, and σ is the conductivity.

This set of equations differs from what we typically present in physics courses. Several of these equations are defining quantities. While the potentials are part of a course in electrodynamics, they are not cast as the core set of equations now referred to as Maxwell's equations. Also, several equations are given as defining relations between the various variables,

though they have some physical significance of their own, such as the continuity equation, given by $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$. Furthermore, the distinction between the magnetic field strength, \mathbf{H} , and the magnetic flux density, \mathbf{B} , only becomes important in the presence of magnetic materials. Students are typically first introduced to \mathbf{B} in introductory physics classes. In general, $\mathbf{B} = \mu\mathbf{H}$, where μ is the magnetic permeability of a material. In the absence of magnetic materials, $\mu = \mu_0$. In fact, in many applications of the propagation of electromagnetic waves, $\mu \approx \mu_0$.

In the case of a vacuum, these equations can be written in a more familiar form. The equations that we will refer to as Maxwell's equations from now on are

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, & (\text{Gauss' Law}) \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & (\text{Faraday's Law}) \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, & (\text{Maxwell-Ampère Law})\end{aligned}\quad (7.5)$$

We have noted the common names attributed to each law. There are corresponding integral forms of these laws, which are often presented in introductory physics class. The first law is Gauss' law. It allows one to determine the electric field due to specific charge distributions. The second law typically has no name attached to it, but in some cases is called Gauss' law for magnetic fields. It simply states that there are no free magnetic poles. The third law is Faraday's law, indicating that changing magnetic flux induces electric potential differences. Lastly, the fourth law is a modification of ampere's law that states that electric currents produce magnetic fields.

It should be noted that the last term in the fourth equation was introduced by Maxwell. As we will see, the divergence of the curl of any vector is zero,

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0.$$

(for now, we will just rely on the fact that the del operator is a differential operator and will later recall the exact form that you had seen in your third semester calculus class.)

We compute the divergence of the curl of the electric field. We find from

Maxwell's equations that

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{E}) &= -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} \\ &= -\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0.\end{aligned}\tag{7.6}$$

Thus, the relation works here. However, before Maxwell, Ampère's law in differential form would have been written as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Computing the divergence of the curl of the magnetic field, we have

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{B}) &= \mu_0 \nabla \cdot \mathbf{J} \\ &= -\mu_0 \frac{\partial \rho}{\partial t}.\end{aligned}\tag{7.7}$$

Here we made use of the continuity equation. However, the vector identity does not apply here! Maxwell argued that we need to account for a changing distribution. He introduced what he called the displacement current, $\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ into the Ampère Law. Now, we have

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{B}) &= \mu_0 \nabla \cdot \left(\mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= -\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} \\ &= -\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\epsilon_0} \right) = 0.\end{aligned}\tag{7.8}$$

So, Maxwell's introduction of the displacement current was not only physically important, it made the equations mathematically consistent.

In this chapter we will review some of the needed vector analysis for the derivation of the three dimensional wave equation from Maxwell's equations. We will recall some of the basic vector operations (the dot and cross products), define the gradient, curl, and divergence and introduce some of the standard vector identities that are often seen in physics courses.

7.2 Vector Analysis

7.2.1 Vector Products

At this point you might want to reread the first section of Chapter 3. In that chapter we introduced the formal definition of a vector space and some simple properties of vectors. We also discussed one of the common vector products, the dot product, which is defined as

$$\mathbf{u} \cdot \mathbf{v} = uv \cos \theta. \quad (7.9)$$

There is also a component form, which we can write as

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{k=1}^3 u_kv_k. \quad (7.10)$$

One of the first physical examples using a cross product is the definition of work. The work done on a body by a constant force \mathbf{F} during a displacement \mathbf{d} is

$$W = \mathbf{F} \cdot \mathbf{d}.$$

In the case of a nonconstant force, we have to add up the incremental contributions to the work, $dW = \mathbf{F} \cdot d\mathbf{r}$ to obtain

$$W = \int_C dW = \int_C \mathbf{F} \cdot d\mathbf{r} \quad (7.11)$$

over the path C . Note how much this looks like a path integral. It is a path integral, but the path lies in a real three dimensional space.

Another application of the dot product is in proving the Law of Cosines. Recall that this law gives the side opposite a given angle in terms of the angle and the other two sides of the triangle:

$$c^2 = a^2 + b^2 - 2ab \cos \theta. \quad (7.12)$$

Consider the triangle in Figure 7.1. We draw the sides of the triangle as vectors. Note that $\mathbf{b} = \mathbf{c} + \mathbf{a}$. Also, recall that the square of the length any vector can be written as the dot product of the vector with itself.

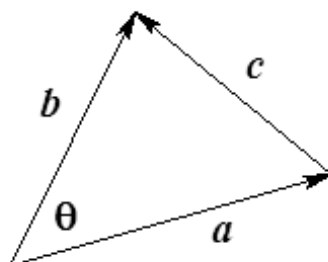


Figure 7.1: The Law of Cosines can be derived using vectors.

Therefore, we have

$$\begin{aligned}
 c^2 &= \mathbf{c} \cdot \mathbf{c} \\
 &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\
 &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} \\
 &= a^2 + b^2 - 2ab \cos \theta.
 \end{aligned} \tag{7.13}$$

We note that this also comes up in writing out inverse square laws in many applications. Namely, the vector \mathbf{a} can locate a mass, or charge, and vector \mathbf{b} points to an observation point. Then the inverse law would involve vector \mathbf{c} , whose length is obtained as $\sqrt{a^2 + b^2 - 2ab \cos \theta}$. Typically, one does not have \mathbf{a} 's and \mathbf{b} 's, but something like \mathbf{r}_1 and \mathbf{r}_2 , or \mathbf{r} and \mathbf{R} . Then one is interested in approximating the expression of interest in terms of ratios like $\frac{r}{R}$ when $R \gg r$.

Another important vector product is the cross product. The cross product produces a vector, unlike the dot product that results in a scalar. The magnitude of the cross product is given as

$$|\mathbf{a} \times \mathbf{b}| = ab \sin \theta. \tag{7.14}$$

Being a vector, we also have to specify the direction. The cross product produces a vector that is perpendicular to both vectors \mathbf{a} and \mathbf{b} . Thus, the vector is normal to the plane in which these vectors live. There are two possible directions. The direction taken is given by the right hand rule. This is shown in Figure 7.2. The direction can also be determined using your right hand. Curl your fingers from \mathbf{a} through to \mathbf{b} . The thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

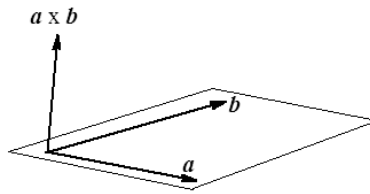


Figure 7.2: The cross product is shown. The direction is obtained using the right hand rule: Curl fingers from \mathbf{a} through to \mathbf{b} . The thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

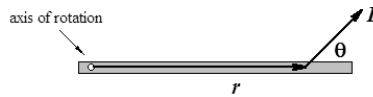


Figure 7.3: A force applied at a point located at \mathbf{r} from the axis of rotation produces a torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ with respect to the axis.

One of the first occurrences of the cross product in physics is in the definition of the torque, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Recall that the torque is the analogue to the force. A net torque will cause an angular acceleration. Consider a rigid body in which a force is applied to it at a position \mathbf{r} from the axis of rotation. (See Figure 7.3.) Then this force produces a torque with respect to the axis. The direction of the torque is given by the right hand rule. Point your fingers in the direction of \mathbf{r} and rotate them towards \mathbf{F} . In the figure this would be out of the page. This indicates that the bar would rotate in a counter clockwise direction if this were the only force acting on the bar.

Another example is that of a body rotating about an axis as shown in Figure 7.4. We locate the body with a position vector pointing from the origin of the coordinate system to the body. The tangential velocity of the body is related to the angular velocity by a cross product $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. The direction of the angular velocity is given by a right hand rule. Curl the fingers of your right hand in the direction of the motion of the rotating mass. Your thumb will point in the direction of $\boldsymbol{\omega}$. Counter clockwise motion produces a positive angular velocity and clockwise will give a negative angular velocity. Note that for the origin at the center of rotation of the mass, we obtain the familiar expression $v = r\omega$.

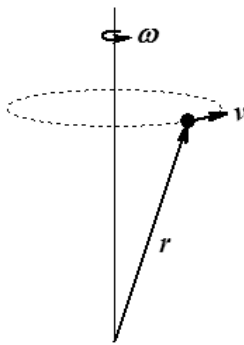


Figure 7.4: A mass rotates at an angular velocity ω about a fixed axis of rotation. The tangential velocity with respect to a given origin is given by $\mathbf{v} = \omega \times \mathbf{r}$.

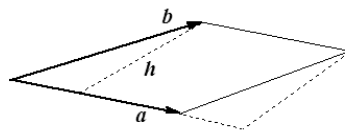


Figure 7.5: The magnitudes of the cross product gives the area of the parallelogram defined by \mathbf{a} and \mathbf{b} .

There is also a geometric interpretation of the cross product. Consider the vectors \mathbf{a} and \mathbf{b} in Figure 7.5. Now draw a perpendicular from the tip of \mathbf{b} to vector \mathbf{a} . This forms a triangle of height h . Slide the triangle over to form a rectangle of base a and height h . The area of this triangle is

$$\begin{aligned} A &= ah \\ &= a(b \sin \theta) \\ &= |\mathbf{a} \times \mathbf{b}|. \end{aligned} \tag{7.15}$$

Therefore, the magnitude of the cross product is the area of the triangle formed by the vectors \mathbf{a} and \mathbf{b} .

The dot product was shown to have a simple form in terms of the components of the vectors. Similarly, we can write the cross product in component form. Recall that we can expand any vector \mathbf{v} as

$$\mathbf{v} = \sum_{k=1}^n v_k \mathbf{e}_k, \tag{7.16}$$

where the \mathbf{e}_k 's are the standard basis vectors.

We would like to expand the cross product of two vectors,

$$\mathbf{u} \times \mathbf{v} = \left(\sum_{k=1}^n u_k \mathbf{e}_k \right) \times \left(\sum_{k=1}^n v_k \mathbf{e}_k \right).$$

In order to do this we need a few properties of the cross product.

First of all, the cross product is not commutative. In fact, it is anticommutative:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

A simple consequence of this is that $\mathbf{v} \times \mathbf{v} = 0$. Just replace \mathbf{u} with \mathbf{v} in the anticommutativity rule and you have $\mathbf{v} \times \mathbf{v} = -\mathbf{v} \times \mathbf{v}$. Something that is its negative must be zero.

The cross product also satisfies distributive properties:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w},$$

and

$$\mathbf{u} \times (a\mathbf{v}) = (a\mathbf{u}) \times \mathbf{v} = a\mathbf{u} \times \mathbf{v}.$$

Thus, we can expand the cross product in terms of the components of the given vectors. However, we will end up with cross products of the basis vectors:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \left(\sum_{i=1}^n u_i \mathbf{e}_i \right) \times \left(\sum_{j=1}^n v_j \mathbf{e}_j \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n u_i v_j \mathbf{e}_i \times \mathbf{e}_j.\end{aligned}\quad (7.17)$$

These cross products are simple to compute. First of all, the cross products $\mathbf{e}_i \times \mathbf{e}_j$ vanish when $i = j$ by anticommutativity of the cross product. For $i \neq j$, it is slightly more difficult in this general index formalism. For the typical basis, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, it is simple. Imagine computing $\mathbf{i} \times \mathbf{j}$. This is a vector of length $|\mathbf{i} \times \mathbf{j}| = |\mathbf{i}||\mathbf{j}| \sin 90^\circ = 1$. It is perpendicular to both vectors. Thus, it is either \mathbf{k} or $-\mathbf{k}$. Using the right hand rule, we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Similarly, we find the following

$$\begin{aligned}\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.\end{aligned}\quad (7.18)$$

Inserting this into our cross product for vectors in R^3 , we have

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}.\quad (7.19)$$

While this form is correct and useful, there are other forms that help in verifying identities or making computation simpler with less memorization. However, some of these can lead to problems with the novice as dealing with indices are at first daunting.

First of all, there is the familiar computation using determinants. The same result as above can be obtained by noting that

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}.\end{aligned}\quad (7.20)$$

A more compact form can be obtained by introducing the completely antisymmetric symbol, ϵ_{ijk} . This symbol is defined by the relations

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1,$$

and

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1,$$

and all other combinations, like ϵ_{113} , vanish. Note that all indices must differ. Also, if the order is a cyclic permutation of $\{1, 2, 3\}$, then the value is $+1$.

With this notation, we have that

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^3 \epsilon_{ijk} \mathbf{e}_k. \quad (7.21)$$

For example, consider the product $\mathbf{e}_2 \times \mathbf{e}_1$:

$$\begin{aligned} \mathbf{e}_2 \times \mathbf{e}_1 &= \sum_{k=1}^3 \epsilon_{21k} \mathbf{e}_k \\ &= \epsilon_{211} \mathbf{e}_1 + \epsilon_{212} \mathbf{e}_2 + \epsilon_{213} \mathbf{e}_3 \\ &= -\mathbf{e}_3. \end{aligned} \quad (7.22)$$

It is helpful to write out enough terms in these sums until you get familiar with manipulating the indices. Note that the first two terms vanished because of repeated indices. In the last term we used $\epsilon_{213} = -1$.

We can now write out the general cross product as

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \sum_{i=1}^n \sum_{j=1}^n u_i v_j \mathbf{e}_i \times \mathbf{e}_j \\ &= \sum_{i=1}^n \sum_{j=1}^n u_i v_j \sum_{k=1}^3 \epsilon_{ijk} \mathbf{e}_k \\ &= \sum_{i,j,k=1}^3 \epsilon_{ijk} u_i v_j \mathbf{e}_k. \end{aligned} \quad (7.23)$$

Sometimes it is useful to note that the k th component of the cross product is given by

$$(\mathbf{u} \times \mathbf{v})_k = \sum_{i,j=1}^3 \epsilon_{ijk} u_i v_j.$$

Also, in some more advanced texts, or in the case of relativistic computations with tensors, the summation symbol is suppressed. For this case, one has

$$(\mathbf{u} \times \mathbf{v})_k = \epsilon_{ijk} u_i v_j,$$

where it is understood that summation is performed over repeated indices. This is called the *Einstein summation convention*.

We should also note that since the cross product is given as a determinant, then we could write

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i,j,k=1}^3 \epsilon_{ijk} a_{1i} a_{2j} a_{3k}.$$

One useful identity is

$$\epsilon_{jki} \epsilon_{jlm} = \delta_{kl} \delta_{im} - \delta_{km} \delta_{il},$$

where δ_{ij} is the Kronecker delta. Note that the Einstein summation convention is used in this identity; i.e., summing over j is understood. So, the left side is really a sum of three terms:

$$\epsilon_{jki} \epsilon_{jlm} = \epsilon_{1ki} \epsilon_{1lm} + \epsilon_{2ki} \epsilon_{2lm} + \epsilon_{3ki} \epsilon_{3lm}.$$

This identity is simple to understand. For nonzero values of the Levi-Civita symbol, we have to require that all indices differ for each factor on the left side of the equation: $j \neq k \neq i$ and $j \neq \ell \neq m$. Since the first two slots are the same j , and the indices only take values 1, 2, or 3, then either $k = \ell$ or $k = m$. This will give terms with $\delta_{k\ell}$ or δ_{km} . If the former is true, then there is only one possibility for the third slot, $i = m$. Thus, we have a term $\delta_{kl} \delta_{im}$. Similarly, the other case yield the second term on the right side of the identity. We just need to get the signs right. Obviously, changing the order of ℓ and m will introduce a minus sign. A little care will show that the above is the correct ordering.

We will end this section by looking at the triple products. There are only two ways to construct triple products. One starts with the cross product $\mathbf{b} \times \mathbf{c}$. This is a vector. The only thing we can do is to multiply this times vector \mathbf{a} to either yield a scalar or a vector.

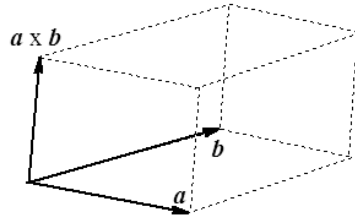


Figure 7.6: Three non-coplanar vectors define a parallelepiped. The volume is given by the triple scalar product, $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$.

In the first case we obtain the triple scalar product, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Actually, we do not need the parentheses. Writing $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ could only mean one thing. If we computed $\mathbf{a} \cdot \mathbf{b}$ first, then we would get a scalar. Then the result would be a multiple of \mathbf{c} , which is not a scalar. So, leaving off the parentheses would mean that we want the triple scalar product by convention.

Let's consider the component form of this product. We will use the Einstein summation convention and the fact that the permutation symbol is cyclic in ijk .

$$\begin{aligned}
 \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{a}_i (\mathbf{b} \times \mathbf{c})_i \\
 &= \epsilon_{jki} a_i b_j c_k \\
 &= \epsilon_{ijk} a_i b_j c_k \\
 &= (\mathbf{b} \times \mathbf{c})_k c_k \\
 &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c},
 \end{aligned} \tag{7.24}$$

Thus, we have proven that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Now, imagine how much writing would be involved if we had expanded everything out in terms of all of the components.

There is a geometric interpretation of the scalar triple product. Consider the three vectors drawn as in Figure 7.6. If they do not all lie in a plane, then they form the sides of a parallelepiped. The cross product $\mathbf{a} \times \mathbf{b}$ gives the area of the base as we had seen earlier. The cross product is perpendicular to this base. The dot product of \mathbf{c} with this cross product gives the height of the parallelepiped. So, the volume of the parallelepiped

is the height times the base, or the triple scalar product. In general, one gets a signed volume, as the cross product could be pointing below the base.

The second type of triple product is the triple cross product, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. In this case we cannot drop the parentheses as this would lead to a real ambiguity. Lets think a little about this product. The vector $\mathbf{b} \times \mathbf{c}$ is a vector that is perpendicular to both \mathbf{b} and \mathbf{c} . Computing the triple cross product would then produce a vector perpendicular to \mathbf{a} and $\mathbf{b} \times \mathbf{c}$. But the later vector is perpendicular to both \mathbf{b} and \mathbf{c} already. Therefore, the triple cross product must lie in the plane spanned by these vectors. In fact, there is an identity that tells us exactly the right combination of vectors \mathbf{b} and \mathbf{c} . It is given by

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \quad (7.25)$$

This rule is called the BAC-CAB rule because of the order of the right side of this equation. It will be left to the reader to prove this result.

7.2.2 Div, Grad, Curl

In studying functions of one variable in calculus, one is introduced to the derivative, $\frac{df}{dx}$: The derivative has several meanings. The standard meaning is that it gives the slope of the graph of $f(x)$ at x . The derivative also tells us how rapidly $f(x)$ varies when x is changed by dx . Recall that dx is called a differential. We can think of it as an infinitesimal increment in x . Then changing x by dx results in a change in f by

$$df = \frac{df}{dx} dx.$$

We can extend this idea to functions of several variables. Consider the temperature $T(x, y, z)$ at a point in space. The change in temperature depends on the direction in which one moves in space. Extending the above relation between differentials, we have

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz. \quad (7.26)$$

If we introduce the following vectors,

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}, \quad (7.27)$$

$$\nabla T = \frac{\partial T}{\partial x} \mathbf{i} + \frac{\partial T}{\partial y} \mathbf{j} + \frac{\partial T}{\partial z} \mathbf{k}, \quad (7.28)$$

then we can write

$$dT = \nabla T \cdot d\mathbf{r} \quad (7.29)$$

Equation (7.28) defines the gradient of a scalar function, T . Equation (7.29) gives the change in T as one moves in the direction $d\mathbf{r}$. Using the definition of the dot product, we also have

$$dT = |\nabla T| |d\mathbf{r}| \cos \theta.$$

Note that by fixing $|d\mathbf{r}|$ and varying θ , the maximum value of dT is obtained in the direction of the gradient.

We can write the gradient in the form

$$\nabla T = \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) T. \quad (7.30)$$

Thus, we see that the gradient can be viewed as an operator acting on T . The operator,

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k},$$

is called the *del*, or *gradient*, operator. It is a differential vector operator. It can act on scalar functions to produce a vector field.

However, we can also allow the del operator to act on vector fields. Recall that a *vector field* is simply a vector valued function. For example, a force field is a function defined at points in space indicating the force that would act on a mass placed at that location. We could denote it a $\mathbf{F}(x, y, z)$.

How can we combine the vector operator and a vector field? Well, we could “multiply” them. We could either compute the dot product, $\nabla \cdot \mathbf{F}$, or we could compute the cross product $\nabla \times \mathbf{F}$. The first expression is called the *divergence* of the vector field and the second is called the *curl* of the vector field. These are typically encountered in a third semester calculus course. In some texts they are denoted by $\text{div } \mathbf{F}$ and $\text{curl } \mathbf{F}$.

The divergence is computed the same as any other dot product. Writing the vector field in component form,

$$\mathbf{F} = F_1(x, y, z) \mathbf{i} + F_2(x, y, z) \mathbf{j} + F_3(x, y, z) \mathbf{k},$$

we find the divergence is simply given as

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \cdot (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\end{aligned}\quad (7.31)$$

Similarly, we can compute the curl of \mathbf{F} . Using the determinant form, we have

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \times (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}.\end{aligned}\quad (7.32)$$

These operations also have interpretations. The divergence measures how much the vector field \mathbf{F} spreads from a point. When the divergence of a vector field is nonzero around a point, that is an indication that there is a source ($\text{div } \mathbf{F} > 0$) or a sink ($\text{div } \mathbf{F} < 0$). For example, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ indicates that there are sources contributing to the electric field. For a single charge, the field lines are radially pointing towards (sink) or away from (source) the charge. A field in which the divergence is zero is called divergenceless.

The curl is an indication of a rotational field. It is a measure of how much a field curls around a point. Consider the flow of a stream. The velocity of each element of fluid can be represented by a velocity field. If the curl of the field is nonzero, then when we drop a leaf into the stream we will see it begin to rotate about some point. A field that has zero curl is called irrotational.

Maxwell's equations as given in this chapter are in differential form and only describe the physics locally. At times we would like to also provide global information, or information over an finite region. In this case one can derive various integral theorems. These are the finale in a three semester calculus sequence. We will not delve into these theorems here, as this will take us away from our goal of deriving a three dimensional wave

equation. However, these integral theorems are important and useful in deriving local conservation laws.

These theorems are all different versions of a generalized Fundamental Theorem of Calculus:

1. $\int_a^b \frac{\partial f}{\partial x} dx = f(b) - f(a)$, The Fundamental Theorem of Calculus
2. $\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{r} = T(\mathbf{b}) - T(\mathbf{a})$,
3. $\int_V \nabla \cdot \mathbf{F} dV = \oint_S \mathbf{F} \cdot d\mathbf{a}$, Gauss' Divergence Theorem
4. $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint_C \mathbf{F} \cdot d\mathbf{r}$, Stoke's Theorem

7.2.3 Vector Identities

In this section we list the common vector identities.

1. Triple Products

- (a) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$
- (b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

2. First Derivatives

- (a) $\nabla(fg) = f\nabla g + g\nabla f$
- (b) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
- (c) $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$
- (d) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
- (e) $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} - \mathbf{A} \times \nabla f$
- (f) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

3. Second Derivatives

- (a) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
- (b) $\nabla \times (\nabla f) = 0$
- (c) $\nabla \cdot (\nabla f \times \nabla g) = 0$
- (d) $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$
- (e) $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
- (f) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

7.3 Electromagnetic Waves

We are now ready to derive the wave equation for electromagnetic waves. We will consider the case of free space in which there are no free charges or currents and the waves propagate in a vacuum. We then have Maxwell's equations in the form

$$\begin{aligned}
 \nabla \cdot \mathbf{E} &= 0, \\
 \nabla \cdot \mathbf{B} &= 0, \\
 \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
 \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.
 \end{aligned} \tag{7.33}$$

We will derive the wave equation for the electric field. You should confirm that a similar result can be obtained for the magnetic field. Consider the expression $\nabla \times (\nabla \times \mathbf{E})$. We note that our identities give

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

However, the divergence of \mathbf{E} is zero, so we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}. \tag{7.34}$$

We can also use Faraday's Law on the right side of this equation to obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t} \right).$$

Interchanging the time and space derivatives, and using the Ampere-Maxwell Law, we find

$$\begin{aligned}
 \nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\
 &= -\frac{\partial}{\partial t} \left(\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\
 &= -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.
 \end{aligned} \tag{7.35}$$

Combining the two expressions for $\nabla \times (\nabla \times \mathbf{E})$, we have the sought result:

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}.$$

This is the three dimensional equation for an oscillating electric field. A similar equation can be found for the magnetic field,

$$\epsilon_0\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2} = \nabla^2\mathbf{B}.$$

Recalling that $\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2$ and $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$, one finds that $c = 3 \times 10^8 \text{ m/s}$.

One can derive more general equations. For example, we could look for waves in what are called linear media. In this case one has $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. Here ϵ is called the electric permittivity and μ is the magnetic permeability of the material. Then, the wave speed in a vacuum, c , is replaced by the wave speed in the medium, v . It is given by

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}.$$

Here, n is the index of refraction, $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$.

In many materials $\mu \approx \mu_0$. Introducing the dielectric constant, $\kappa = \frac{\epsilon}{\epsilon_0}$, one finds that $n \approx \sqrt{\kappa}$.

The wave equations lead to many of the properties of the electric and magnetic fields. We can also study systems in which these waves are confined, such as waveguides. In such cases we can impose boundary conditions and determine what modes are allowed to propagate within certain structures, such as optical fibers. However, these equations involve unknown vector fields. We have to solve for several inter-related component functions. In the next chapter we will look at simpler models in order to get some ideas as to how one can solve scalar wave equations in higher dimensions.