Chapter 6

Transform Techniques in Physics

6.1 Introduction

Some of the most powerful tools for solving problems in physics are transform methods. The idea is that one can transform the problem at hand to a new problem in a different space, hoping that the problem in the new space is easier to solve. Such transforms appear in many forms.

6.1.1 The Linearized KdV Equation

As a relatively simple example, we consider the linearized Kortweg-deVries (KdV) equation:

$$u_t + cu_x + \beta u_{xxx} = 0, \quad -\infty < x < \infty. \tag{6.1}$$

This equation governs the propagation of some small amplitude water waves. Its nonlinear counterpart has been at the center of attention in the last 40 years as a generic nonlinear wave equation.

We seek solutions that oscillate in space. So, we assume a solution of the form

$$u(x,t) = A(t)e^{ikx}. \tag{6.2}$$

Such behavior was seen in the last chapter for the wave equation for vibrating strings. In that case, we found plane wave solutions of the form $e^{ik(x-ct)}$, which we could write as $e^{i(kx-\omega t)}$ by defining $\omega = kc$. We further note that one often seeks complex solutions of this form and then takes the real part in order to obtain a real physical solutions.

Inserting the guess into the linearized KdV equation, we find that

$$\frac{dA}{dt} + i(ck - \beta k^3)A = 0. \tag{6.3}$$

Thus, we have converted our problem of seeking a solution of the partial differential equation into seeking a solution to an ordinary differential equation. This new problem is easier to solve. In fact, we have

$$A(t) = A(0)e^{-i(ck-\beta k^3)t}. \tag{6.4}$$

Therefore, the solution of the partial differential equation is

$$u(x,t) = A(0)e^{ik(x-(c-\beta k^2)t)}. \tag{6.5}$$

We note that this takes the form $e^{i(kx-\omega t)}$, where

$$\omega = ck - \beta k^3.$$

In general, the equation $\omega = \omega(k)$ gives the angular frequency as a function of the wave number, k, and is called a *dispersion relation*. For $\beta = 0$, we see that c in nothing but the wave speed. For $\beta \neq 0$, the wave speed is given as

$$v = \frac{\omega}{k} = c - \beta k^2.$$

This suggests that waves with different wave numbers will travel at different speeds. Recalling that wave numbers are related to wavelengths, $k = \frac{2\pi}{\lambda}$, this means that waves with different wavelengths will travel at different speeds. So, a linear combination of such solutions, will not maintain its shape. It is said to disperse, as the waves of differing wavelengths tend to part company.

For a general initial condition, we need to write the solutions to the linearized KdV as a superposition of waves. We can do this as the equation is linear. This should remind you of what we had done when using separation of variables. We first sought product solutions and then took a linear combination of the product solutions to give the general solution.

In this case, we need to sum over all wave numbers. The wave numbers are not restricted to discrete values, so we have a continuous range of values. Thus, summing over k means that we have to integrate over the wave numbers. Thus, we have the general solution

$$u(x,t) = \int_{-\infty}^{\infty} A(k,0) e^{ik(x-(c-\beta k^2)t)}\, dk. \tag{6.6}$$

Note that we have now made A a function of k. This is similar to introducing the A_n's and B_n's in the series solution for waves on a string.

How do we determine the $A(k,0)$'s? We introduce an initial condition. Let $u(x,0) = f(x)$. Then, we have

$$f(x) = u(x,0) = \int_{-\infty}^{\infty} A(k,0) e^{ikx}\, dk. \tag{6.7}$$

Thus, given $f(x)$, we seek $A(k,0)$. This involves what is called the Fourier transform of $f(x)$. This is just one of the so-called integral transforms that we will consider in this section.

6.1.2 The Free Particle Wave Function

A more familiar example in physics comes from quantum mechanics. The Schrödiger equation gives the wave function $\Psi(x,t)$ for a particle under the influence of forces, represented through the corresponding potential function V. The one dimensional time dependent Schrödinger equation is given by

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi. \tag{6.8}$$

We consider the case of a free particle in which there are no forces, $V = 0$. Then we have

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}. \tag{6.9}$$

Taking a hint from the study of the linearized KdV equation, we will assume that solutions of Equation (6.9) take the form

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k,t) e^{ikx}\, dk.$$

[Here we have opted to use the more traditional notation, $\phi(k,t)$ instead of $A(k,t)$ as above.]

Inserting this expression into (6.9), we have

$$i\hbar\int_{-\infty}^{\infty}\frac{d\phi(k,t)}{dt}e^{ikx}\,dk=-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\phi(k,t)(ik)^2e^{ikx}\,dk.$$

Since this is true for all t, we can equate integrands, giving

$$i\hbar\frac{d\phi(k,t)}{dt}=\frac{\hbar^2}{2m}k^2\phi(k,t).$$

This is easily solved. We obtain

$$\phi(k,t)=\phi(k,0)e^{-i\frac{\hbar k^2}{2m}t}.$$

Therefore, we have found the general solution to the time dependent problem for a free particle. It is given as

$$\Psi(x,t)=\int_{-\infty}^{\infty}\phi(k,0)e^{ik(x-\frac{\hbar k}{2m}t)}\,dk.$$

We note that this takes the familiar form

$$\Psi(x,t)=\int_{-\infty}^{\infty}\phi(k,0)e^{i(kx-\omega t)}\,dk,$$

where

$$\omega=\frac{\hbar k^2}{2m}.$$

The wave speed is given as

$$v=\frac{\omega}{k}=\frac{\hbar k}{2m}.$$

As a special note, we see that this is not the particle velocity! Recall that the momentum is given as $p=\hbar k$. So, this wave speed is $v=\frac{p}{2m}$, which is only half the classical particle velocity! A simple manipulation of our result will clarify this "problem". We assume that particles can be represented by a localized wave function. This is the case if the major contributions to the integral are centered about a central wave number, k_0. Thus, we can expand $\omega(k)$ about k_0:

$$\omega(k)=\omega_0+\omega_0'(k-k_0)+\ldots. \tag{6.10}$$

Here $\omega_0 = \omega(k_0)$ and $\omega_0' = \omega'(k_0)$. Inserting this expression into our integral representation for $\Psi(x,t)$, we have

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k,0) e^{i(kx - \omega_0 t - \omega_0'(k-k_0)+\ldots)}\, dk,$$

We make a change of variables, $s = k - k_0$ and rearrange the factors to find

$$\begin{aligned}\Psi(x,t) &\approx \int_{-\infty}^{\infty} \phi(k_0+s,0) e^{i((k_0+s)x - (\omega_0+\omega_0' s)t)}\, ds \\ &= e^{i(-\omega_0 t + k_0\omega_0' t)} \int_{-\infty}^{\infty} \phi(k_0+s,0) e^{i((k_0+s)(x-\omega_0' t))}\, ds \\ &= e^{i(-\omega_0 t + k_0\omega_0' t)} \Psi(x - \omega_0' t, 0). \qquad (6.11)\end{aligned}$$

What we have found is that for a localized wave packet with wave numbers grouped around k_0 the wave function is a translated version of the initial wave function, up to a phase factor. The velocity of the wave packet is seen to be $\omega_0' = \frac{\hbar k}{m}$. This corresponds to the classical velocity of the particle. Thus, one usually defines this to be the *group velocity*,

$$v_g = \frac{d\omega}{dk}$$

and the former velocity as the *phase velocity*,

$$v_p = \frac{\omega}{k}.$$

6.1.3 Transform Schemes

These examples have illustrated one of the features of transform theory. Given a partial differential equation, we can transform the equation from spatial variables to wave number space, or time variables to frequency space. In the new space the time evolution is simpler. In these cases, the evolution is governed by an ordinary differential equation. One solves the problem in the new space and then transforms back to the original space. This is depicted in Figure 6.1 for the Schrödinger equation.

This is similar to the solution of the system of ordinary differential equations in Chapter 3, $\dot{\mathbf{x}} = A\mathbf{x}$. In that case we diagonalized the system using the transformation $\mathbf{x} = S\mathbf{y}$. This lead to a simpler system $\dot{\mathbf{y}} = \Lambda\mathbf{y}$.

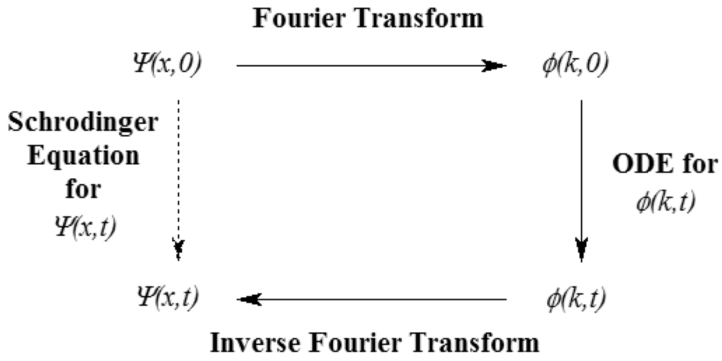

Figure 6.1: The scheme for solving the Schrödiger equation using Fourier transforms. The goal is to solve for $\Psi(x,t)$ given $\Psi(x,0)$. Instead of a direct solution in coordinate space (on the left side), one can first transform the initial condition obtaining $\phi(k,0)$ in wave number space. The governing equation in the new space is found by transforming the PDE to get an ODE. This simpler equation is solved to obtain $\phi(k,t)$. Then an inverse transform yields the solution of the original equation.

Solving for $\mathbf{y}$, we inverted the solution to obtain $\mathbf{x}$. Similarly, one can apply this diagonalization to the solution of linear algebraic systems of equations. The general scheme is shown in Figure 6.2.

Similar transform constructions occur for many other type of problems. We will end this chapter with a study of Laplace transforms, which are useful in the study of initial value problems, particularly for linear ordinary differential equations with constant coefficients. A similar scheme for using Laplace transforms is depicted in Figure 6.20.

In this chapter we will turn to the study of Fourier transforms. These will provide an integral representation of functions defined on the real line. Such functions can also represent analog signals. Analog signals are continuous signals which may be sums over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series were used to represent in an earlier chapter. We will then investigate a related transform, the Laplace transform, which is useful in solving initial value problems such as those encountered in ordinary differential equations.

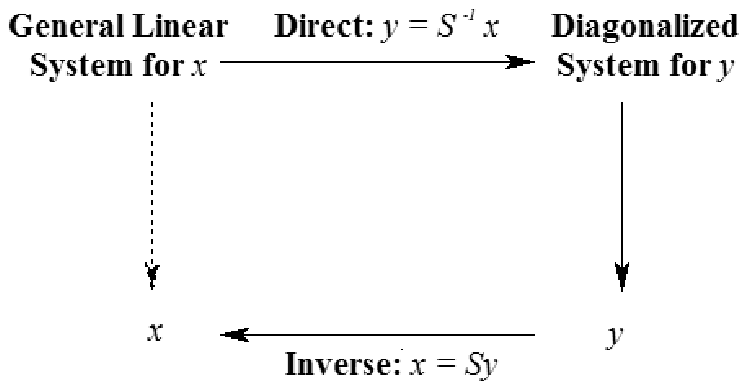

Figure 6.2: The scheme for solving the linear system $A\mathbf{x} = \mathbf{b}$. One finds a transformation between $\mathbf{x}$ and $\mathbf{y}$ of the form $\mathbf{x} = S\mathbf{y}$ which diagonalizes the system. The resulting system is easier to solve for $\mathbf{y}$. Then one uses the inverse transformation to obtain the solution to the original problem. Also, this scheme applies to solving the ODE system $\dot{\mathbf{x}} = A\mathbf{x}$ as we had seen in Chapter 3.

6.2 Complex Exponential Fourier Series

In this section we will see how to rewrite our trigonometric Fourier series as complex exponential series. Then we will extend our series to problems involving infinite periods.

We first recall the trigonometric Fourier series representation of a function defined on $[-\pi, \pi]$ with period 2π. The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \tag{6.12}$$

where the Fourier coefficients were found as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 0, 1, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots \end{aligned} \tag{6.13}$$

In order to derive the exponential Fourier series, we replace the trigonometric functions with exponential functions and collect like terms.

This gives

$$
\begin{aligned}
f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{inx} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2} \right) e^{-inx}. \quad (6.14)
\end{aligned}
$$

The coefficients can be rewritten by defining

$$c_n = \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots. \quad (6.15)$$

Then, we also have that

$$\bar{c}_n = \frac{1}{2}(a_n - ib_n), \quad n = 1, 2, \ldots. \quad (6.16)$$

This gives our representation as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \bar{c}_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Reindexing the first sum, by letting $k = -n$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=-1}^{-\infty} \bar{c}_{-k} e^{-ikx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Now, we define

$$c_n = \bar{c}_{-n}, \quad n = -1, -2, \ldots.$$

Finally, we note that we can take $c_0 = \frac{a_0}{2}$. So, we can write the complex exponential Fourier series representation as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \quad (6.17)$$

where

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots \\
c_n &= \frac{1}{2}(a_{-n} - ib_{-n}), \quad n = -1, -2, \ldots \\
c_0 &= \frac{a_0}{2}. \quad (6.18)
\end{aligned}
$$

Given such a representation, we would like to write out the integral forms of the coefficients, c_n. So, we replace the a_n's and b_n's with their integral representations and replace the trigonometric functions with a complex exponential function using Euler's formula. Doing this, we have for $n = 1, 2, \ldots$.

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n) \\ &= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx + i\sin nx\right)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx \end{aligned} \tag{6.19}$$

It is a simple matter to determine the c_n's for other values of n. For $n = 0$, we have that

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

For $n = -1, -2, \ldots$, we find that

$$c_n = \bar{c}_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{-inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

Therefore, for all n we have shown that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \tag{6.20}$$

We have converted our trigonometric series for functions defined on $[-\pi, \pi]$ to the complex exponential series in Equation (6.17) with Fourier coefficients given by (6.20). Thus, we have obtained the **Complex Fourier Series** representation

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \tag{6.21}$$

where the complex Fourier coefficients are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \tag{6.22}$$

We can easily extend the above analysis to other intervals. For example, for $x \in [-L, L]$ the Fourier trigonometric series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

with Fourier coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \quad n = 0, 1, \ldots,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots.$$

This can be rewritten in an exponential Fourier series of the form:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}$$

with

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L}\, dx.$$

Finally, we note that these expressions can be put into the form:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-ik_n x}$$

with

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{ik_n x}\, dx,$$

where we have introduced the discrete set of wave numbers

$$k_n = \frac{n\pi}{L}.$$

At times, we will also be interested in functions of time. In this case we will have a function $g(t)$ defined on a time interval $[-T, T]$. The exponential Fourier series will then take the form

$$g(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{-i\omega_n t}$$

with

$$c_n = \frac{1}{2T}\int_{-T}^{T} g(t)e^{i\omega_n t}\,dt.$$

Here we have introduced the discrete set of angular frequencies, which can be related to the corresponding discrete set of frequencies by

$$\omega_n = 2\pi f_n = \frac{n\pi}{T}$$

with

$$f_n = \frac{n}{2T}.$$

6.3 Exponential Fourier Transform

Both the trigonometric and complex exponential Fourier series provide us with representations of a class of functions in term of sums over a discrete set of wave numbers for functions of finite wavelength. On intervals $[-L, L]$ the wavelength is $2L$. Writing the arguments in terms of wavelengths, we have $k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}$, or the sums are over wavelengths $\lambda_n = \frac{2L}{n}$. This is a discrete, or countable, set of wavelengths. A similar argument can be made for time series, or functions of time, which occur more often in signal analysis.

We would now like to extend our interval to $x \in (-\infty, \infty)$ and to extend the discrete set of wave numbers to a continuous set of wave numbers. One can do this rigorously, but it amounts to letting L and n get large and keeping $\frac{n}{L}$ fixed. We define $k = \frac{2\pi}{\lambda}$ and the sum over the continuous set of wave numbers becomes an integral. Formally, we arrive at the *Fourier transform*

$$F[f] = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx. \tag{6.23}$$

This is a generalization of the Fourier coefficients (6.20). Once we know the Fourier transform, then we can *reconstruct* our function using the *inverse Fourier transform*, which is given by

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{-ikx}\,dk. \tag{6.24}$$

We note that it can be proven that the Fourier transform exists when $f(x)$ is *absolutely integrable*, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

Such functions are said to be L_1.

The Fourier transform and inverse Fourier transform are inverse operations. This means that

$$F^{-1}[F[f]] = f(x)$$

and

$$F[F^{-1}[\hat{f}]] = \hat{f}(k).$$

We will now prove the first of these equations. The second follows in a similar way. This is done by inserting the definition of the Fourier transform into the inverse transform definition and then interchanging the orders of integration. Thus, we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F[f]e^{-ikx}\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)e^{ik\xi}\,d\xi\right]e^{-ikx}\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{ik(\xi-x)}\,d\xi dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{ik(\xi-x)}\,dk\right]f(\xi)\,d\xi. \end{aligned} \tag{6.25}$$

In order to complete the proof, we need to evaluate the inside integral, which does not depend upon $f(x)$. This is an improper integral, so we will define

$$D_L(x) = \int_{-L}^{L} e^{ikx}\,dk$$

and compute the inner integral as

$$\int_{-\infty}^{\infty} e^{ik(\xi-x)}\,dk = \lim_{L\to\infty} D_L(\xi - x).$$

We can compute $D_L(x)$. A simple evaluation yields

$$\begin{aligned} D_L(x) &= \int_{-L}^{L} e^{ikx}\,dk \\ &= \frac{e^{ikx}}{ix}\Big|_{-L}^{L} \\ &= \frac{e^{ixL}-e^{-ixL}}{2ix} \\ &= \frac{2\sin xL}{x}. \end{aligned} \tag{6.26}$$

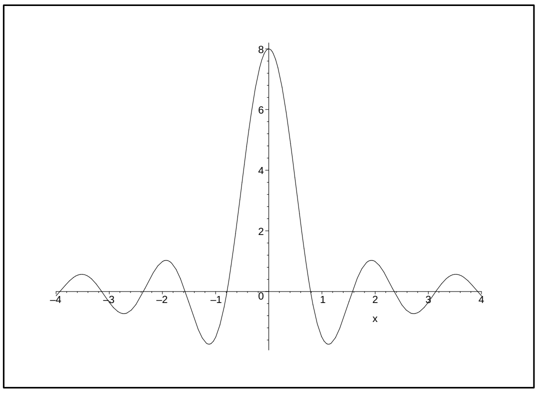

Figure 6.3: A plot of the function $D_L(x)$ for $L = 4$.

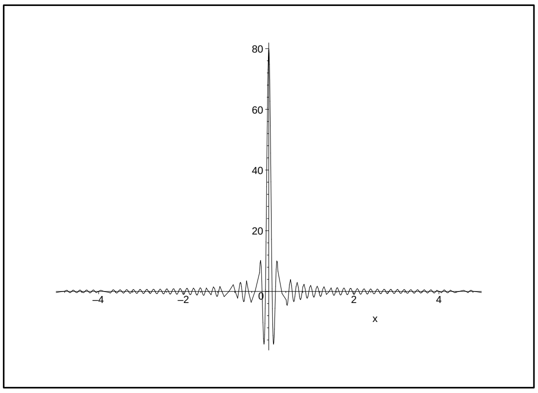

Figure 6.4: A plot of the function $D_L(x)$ for $L = 40$.

We can graph this function. For large x, The function tends to zero. A plot of this function is in Figure 6.3. For large L the peak grows and the values of $D_L(x)$ for $x \neq 0$ tend to zero as show in Figure 6.4. In fact, we can show that as $x \to 0$, $D_L(x) \to 2L$:

$$\begin{aligned}
\lim_{x\to 0} D_L(x) &= \lim_{x\to 0} \frac{2\sin xL}{x} \\
&= \lim_{x\to 0} 2L \frac{\sin xL}{xL} \\
&= 2L\left(\lim_{y\to 0}\frac{\sin y}{y}\right) \\
&= 2L. \qquad (6.27)
\end{aligned}$$

We note that in the limit $L \to \infty$, $D_L(x) = 0$ for $x \neq 0$ and it is infinite at $x = 0$. However, the area is constant for each L. In fact,

$$\int_{-\infty}^{\infty} D_L(x)\, dx = 2\pi.$$

This can be shown using a previous result from complex analysis. In the last chapter we had shown that

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

So,

$$\begin{aligned} \int_{-\infty}^{\infty} D_L(x)\, dx &= \int_{-\infty}^{\infty} \frac{2\sin xL}{x}\, dx \\ &= 2\int_{-\infty}^{\infty} \frac{\sin y}{y}\, dy \\ &= 2\pi, \end{aligned} \tag{6.28}$$

where we had used the substitution $y = Lx$ to carry out the integration.

This behavior can be represented by the limit of other sequences of functions. Define the sequence of functions (not to be confused with frequencies)

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n} \\ \frac{n}{2}, & |x| < \frac{1}{n} \end{cases}$$

This is a sequence of functions as shown in Figure 6.6. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one. Thus, the limiting function is zero at most points but has area one.

The limit is not really a function. It is a *generalized function.* It is called the *Dirac delta function*, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

As a further note, we could have considered the sequence of functions

$$g_n(x) = \begin{cases} 0, & |x| > \frac{1}{n} \\ 2n, & |x| < \frac{1}{n} \end{cases}$$

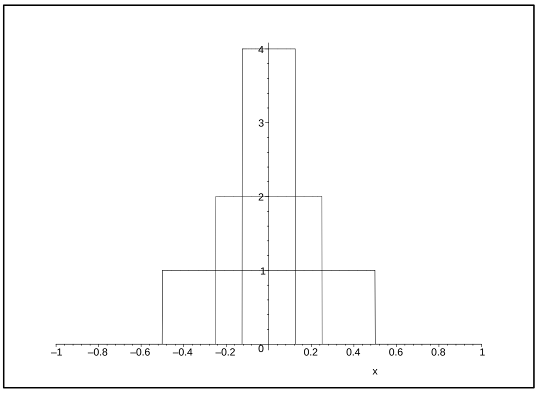

Figure 6.5: A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

This sequence differs from the f_n's by the heights of the functions. As before, the limit as $n \to \infty$ is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is now $2n \times \frac{2}{n} = 4$. So, it is not enough that our sequence of functions consist of nonzero values at just one point. The height might be infinite, but the areas can vary! In this case $\lim_{n\to\infty} g_n(x) = 4\delta(x)$.

Before returning to the proof, we state one more property of the Dirac delta function, which we will prove in the next section. We have that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

This is called the *sifting property* because it sifts out a value of the function $f(x)$.

Returning to the proof, we now have that

$$\int_{-\infty}^{\infty} e^{ik(\xi - x)}\, dk = \lim_{L\to\infty} D_L(\xi - x) = 2\pi\delta(\xi - x).$$

Inserting this into (6.25), we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{ik(\xi-x)}\, dk\right] f(\xi)\, d\xi. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\xi - x) f(\xi)\, d\xi. \\ &= f(x). \end{aligned} \tag{6.29}$$

Thus, we have proven that the inverse transform of the Fourier transform of f is f.

6.4 The Dirac Delta Function

In the last section we introduced the Dirac delta function, $\delta(x)$. This is one example of what is known as a *generalized function* or a distribution. Dirac had introduced this function in the 1930's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral.

Two properties were used in the last section. First one has that the area under the delta function is one,

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

More generally, the integration over a more general interval gives

$$\int_{a}^{b} \delta(x)\, dx = 1, \quad 0 \in [a, b]$$

and

$$\int_{a}^{b} \delta(x)\, dx = 0, \quad 0 \text{ not in } [a, b]$$

.

The other property that was used was that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = \int_{-\infty}^{\infty} \delta(x-a) f(a)\, dx = f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a).$$

Other occurrences of the delta function are integrals of the form $\int_{-\infty}^{\infty} \delta(f(x))\, dx$. Such integrals can be converted into a useful form depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1) = 0$, then one has that

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \frac{1}{|f'(x)|} \delta(x - x_1)\, dx.$$

This can be proven using the substitution $y = f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x)) = \frac{1}{|f'(x_1)|} \delta(x - x_1).$$

More generally, one can show that when $f(x_j) = 0$ for x_j, $j = 1, 2, \ldots, n$, then

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{1}{|f'(x_j)|} \delta(x - x_j).$$

Finally, one can show that there is a relationship between the Heaviside, or step, function and the Dirac delta function. We define the Heaviside function as

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Then, it is easy to see that $H'(x) = \delta(x)$. For $x \neq 0$, $H'(x) = 0$. It has an infinite slope at $x = 0$. We need only check that the area is one. Thus,

$$\int_{-\infty}^{\infty} H'(x)\, dx = \lim_{L \to \infty} [H(L) - H(-L)] = 1.$$

In some texts the notation $\theta(x)$ is used for the step function, $H(x)$.

Example Evaluate $\int_{-\infty}^{\infty} \delta(3x - 2) x^2\, dx$.

This is not a simple $\delta(x - a)$. So, we need to find the zeros of $f(x) = 3x - 2$. There is only one, $x = \frac{2}{3}$. Also, $|f'(x)| = 3$. Therefore, we have

$$\int_{-\infty}^{\infty} \delta(3x - 2) x^2\, dx = \int_{-\infty}^{\infty} \frac{1}{3} \delta(x - \frac{2}{3}) x^2\, dx = \frac{1}{3} \left(\frac{2}{3}\right)^2 = \frac{4}{27}.$$

Example Evaluate $\int_{-\infty}^{\infty} \delta(2x)(x^2 + 4)\, dx$.

This problem is deceiving. One cannot just plug in $x = 0$ into the function $x^2 + 4$. One has to use the fact that the derivative of $2x$ is 2. So, $\delta(2x) = \frac{1}{2}\delta(x)$. So,

$$\int_{-\infty}^{\infty} \delta(2x)(x^2+4)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)(x^2+4)\,dx = 2.$$

6.5 Properties of the Fourier Transform

We now return to the Fourier transform. Before actually computing the Fourier transform of some functions, we prove a few of the properties of the Fourier transform.

First we recall that there are several forms that one may encounter for the Fourier transform. In applications our functions can either be functions of time, $f(t)$, or space, $f(x)$. The corresponding Fourier transforms are then written as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt, \tag{6.30}$$

or

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx. \tag{6.31}$$

ω is called the angular frequency and is related to the frequency ν by $\omega = 2\pi\nu$. The units of frequency are typically given in Hertz (Hz). Sometimes the frequency is denoted by f when there is no confusion. Recall that k is called the wavenumber. It has units of inverse length and is related to the wavelength, λ, by $k = \frac{2\pi}{\lambda}$.

1. **Linearity**

   For any functions $f(x)$ and $g(x)$ for which the Fourier transform exists and constant a, we have

   $$F[f+g] = F[f] + F[g]$$

   and

   $$F[af] = aF[f].$$

   These simply follow from the properties of integration and establish the linearity of the Fourier transform.

2. $F\left[\frac{df}{dx}\right] = -ik\hat{f}(k)$

   This property can be shown using integration by parts.

$$\begin{aligned} F\left[\frac{df}{dx}\right] &= \int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx}\, dx \\ &= \lim_{L\to\infty} \left(f(x)e^{ikx}\right)|_{-L}^{L} - ik \int_{-\infty}^{\infty} f(x)e^{ikx}\, dx. \end{aligned} \quad (6.32)$$

   The limit will vanish if we assume that $\lim_{x\to\pm\infty} f(x) = 0$. The integral is recognized as the Fourier transform of f, proving the given property.

3. $F\left[\frac{d^n f}{dx^n}\right] = (-ik)^n \hat{f}(k)$

   The proof of this property follows from the last result, or doing several integration by parts. We will consider the case when $n = 2$. Noting that the second derivative is the derivative of $f'(x)$ and applying the last result, we have

$$\begin{aligned} F\left[\frac{d^2 f}{dx^2}\right] &= F\left[\frac{d}{dx} f'\right] \\ &= -ikF\left[\frac{df}{dx}\right] = (-ik)^2 \hat{f}(k). \end{aligned} \quad (6.33)$$

   This result will be true if both $\lim_{x\to\pm\infty} f(x) = 0$ and $\lim_{x\to\pm\infty} f'(x) = 0$. Generalizations to the transform of the nth derivative easily follows.

4. $F[xf(x)] = -i\frac{d}{dk}\hat{f}(k)$

   This property can be shown by using the fact that $\frac{d}{dk} e^{ikx} = ixe^{ikx}$ and being able to differentiate an integral with respect to a parameter.

$$\begin{aligned} F[xf(x)] &= \int_{-\infty}^{\infty} xf(x)e^{ikx}\, dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{d}{dk}\left(\frac{1}{i}e^{ikx}\right) dx \\ &= -i\frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\, dx \\ &= -i\frac{d}{dk}\hat{f}(k). \end{aligned} \quad (6.34)$$

5. **Shifting Properties** For constant a, we have the following shifting properties:

$$f(x-a) \leftrightarrow e^{ika}\hat{f}(k), \tag{6.35}$$
$$f(x)e^{-iax} \leftrightarrow \hat{f}(k-a). \tag{6.36}$$

   Here we have denoted the Fourier transform pairs as $f(x) \leftrightarrow \hat{f}(k)$. These are easily proven by inserting the desired forms into the definition of the Fourier transform, or inverse Fourier transform. The first shift property is shown by the following argument. We evaluate the Fourier transform.

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{ikx}\, dx.$$

   Now perform the substitution $y = x - a$. This will make f a function on asingle variable. Then,

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(y)e^{ik(y+a)}\, dy = e^{ika}\int_{-\infty}^{\infty} f(y)e^{iky}\, dy = e^{ika}\hat{f}(k).$$

   The second shift property follows in a similar way.

6. **Convolution** We define the convolution of two functions $f(x)$ and $g(x)$ as

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\, dx. \tag{6.37}$$

   Then

$$F[f*g] = \hat{f}(k)\hat{g}(k). \tag{6.38}$$

   We will return to the proof and examples of this property in a later section.

6.5.1 Fourier Transform Examples

In this section we will compute some Fourier transforms of several functions.

Example 1 $f(x) = e^{-ax^2/2}$.

This function is called the Gaussian function. It has many applications in areas such as quantum mechanics, molecular theory, probability and heat diffusion. We will compute the Fourier transform of this function and show that the Fourier transform of a Gaussian is a Gaussian. In the derivation we will introduce classic techniques for computing such integrals.

We begin by applying the definition of the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx. \tag{6.39}$$

The first step in computing this integral is to complete the square in the argument of the exponential. Our goal is to rewrite this integral so that a simple substitution will lead to a classic integral of the form $\int_{-\infty}^{\infty} e^{\beta y^2}\,dy$, which we can integrate. The completion of the square follows as usual:

$$\begin{aligned}
-\frac{a}{2}x^2 + ikx &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x\right] \\
&= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x + \left(-\frac{ik}{a}\right)^2 - \left(-\frac{ik}{a}\right)^2\right] \\
&= -\frac{a}{2}\left(x - \frac{ik}{a}\right)^2 - \frac{k^2}{2a}
\end{aligned} \tag{6.40}$$

Using this result in the integral and making the substitution $y = x - \frac{ik}{a}$, we have

$$\begin{aligned}
\hat{f}(k) &= e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x-\frac{ik}{a}\right)^2}\,dx \\
&= e^{-\frac{k^2}{2a}} \int_{-\infty-\frac{ik}{a}}^{\infty-\frac{ik}{a}} e^{-\beta y^2}\,dy, \quad \beta = \frac{a}{2}.
\end{aligned} \tag{6.41}$$

One would be tempted to absorb the $-\frac{ik}{a}$ terms in the limits of integration. In fact, this is what is usually done in texts. However, we need to be careful. We know from our previous study that the integration takes place over a contour in the complex plane. We can deform this horizontal contour to a contour along the real axis since we will not cross any singularities of the integrand. So, we can safely write

$$\hat{f}(k) = e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\beta y^2}\,dy.$$

The resulting integral is a classic integral and can be performed using a standard trick. Let I be given by

$$I = \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy.$$

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-\beta y^2}\, dy \int_{-\infty}^{\infty} e^{-\beta x^2}\, dx.$$

Note that we needed to introduce a second integration variable. We can now write this product as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta(x^2+y^2)}\, dxdy.$$

This is an integral over the entire xy-plane. Since this is a function of x^2+y^2, it is natural to transform to polar coordinates. We have that $r^2 = x^2 + y^2$ and the area element is given by $dxdy = r\, drd\theta$. Therefore, we have that

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\beta r^2}\, rdrd\theta.$$

This integral is doable. Letting $z = r^2$,

$$\int_0^{\infty} e^{-\beta r^2}\, rdr = \frac{1}{2}\int_0^{\infty} e^{-\beta z}\, dz.$$

This gives $I^2 = \frac{\pi}{\beta}$. So, the final result is found by taking the square root of both sides:

$$I = \sqrt{\frac{\pi}{\beta}}.$$

We can now insert this result into Equation (6.41) to give the Fourier transform of the Gaussian function:

$$\hat{f}(k) = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}. \tag{6.42}$$

Example 2 $f(x) = \begin{cases} b, & |x| \le a \\ 0, & |x| > a \end{cases}.$

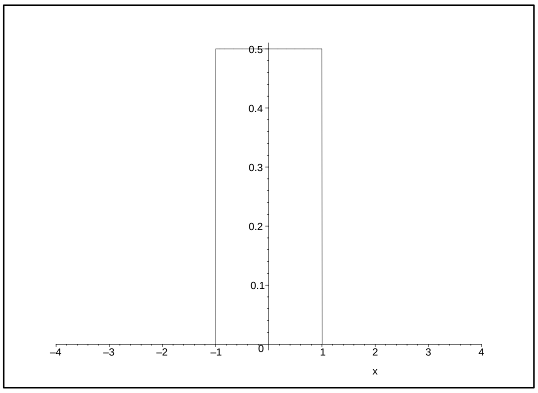

Figure 6.6: A plot of the box function in Example 2.

This function is called the box function, or gate function. It is shown in Figure 6.6. The Fourier transform of the box function is relatively easy to compute. It is given by

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= \int_{-a}^{a} be^{ikx}\,dx \\ &= \frac{b}{ik}e^{ikx}|_{-a}^{a} \\ &= \frac{2b}{k}\sin ka. \end{aligned} \tag{6.43}$$

We can rewrite this using the sinc function, $\operatorname{sinc} x \equiv \frac{\sin x}{x}$, as

$$\hat{f}(k) = 2ab\frac{\sin ka}{ka} = 2ab \operatorname{sinc} ka.$$

The sinc function appears often in signal analysis. A plot of this function is shown in Figure 6.7. We will consider special limiting values for the box function and its transform.

(a) $a \to \infty$ and b fixed.

In this case, as a gets large the box function approaches the constant function $f(x) = b$. At the same time, we see that the Fourier transform approaches a Dirac delta function. We had seen this function earlier when we first defined the Dirac delta function. Compare Figure 6.7 with Figure 6.3. In fact,

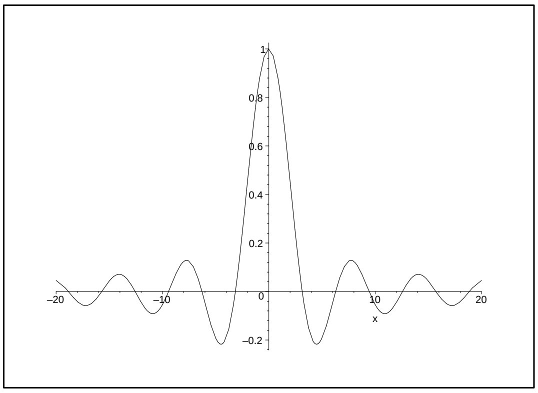

Figure 6.7: A plot of the Fourier transform of the box function in Example 2.

$\hat{f}(k) = bD_a(k)$. So, in the limit we obtain $\hat{f}(k) = 2\pi b\delta(k)$. This limit gives us the fact that the Fourier transform of $f(x) = 1$ is $\hat{f}(k) = 2\pi\delta(k)$. As the width of the box becomes wider, the Fourier transform becomes more localized. Namely, we have arrived at the result that

$$\int_{-\infty}^{\infty} e^{ikx}\, dx = 2\pi\delta(k). \tag{6.44}$$

(b) $b \to \infty$, $a \to 0$, and $2ab = 1$.

In this case our box narrows and becomes steeper while maintaining a constant area of one. This is the way we had found a representation of the Dirac delta function previously. The Fourier transform approaches a constant in this limit. As a approaches zero, the sinc function approaches one, leaving $\hat{f}(k) \to 2ab = 1$. Thus, the Fourier transform of the Dirac delta function is one. Namely, we have

$$\int_{-\infty}^{\infty} \delta(x)e^{ikx}\, dx = 1. \tag{6.45}$$

In this case we have that the more localized the function $f(x)$ is, the more spread out the Fourier transform is.

(c) **The Uncertainty Principle**

The widths of the box function and its Fourier transform are related as we have seen in the last two limiting cases. It is natural to define the width, Δx, of the box function as

$$\Delta x = 2a.$$

The width of the Fourier transform is a little trickier. This function actually extends along the entire k-axis. However, as $\hat{f}(k)$ becomes more localized, the central peak becomes narrower. So, we define the width of this function, Δk as the distance between the first zeros on either side of the main lobe. Since $\hat{f}(k) = \frac{2b}{k} \sin ka$, the zeros are at the zeros of the sine function, $\sin ka = 0$. The first zeros are at $k = \pm\frac{\pi}{a}$. Thus,

$$\Delta k = \frac{2\pi}{a}.$$

Combining the expressions for the two widths, we find that

$$\Delta x \Delta k = 4\pi.$$

Thus, the more localized a signal (smaller δx), the less localized its transform (larger δk). This notion is referred to as the Uncertainty Principle. For more general signals, one needs to define the effective widths more carefully, but the main idea holds:

$$\Delta x \Delta k \geq 1.$$

While this is a result of Fourier transforms, the uncertainty principle arises in other forms elsewhere. In particular, it appears in quantum mechanics, where is it most known. In quantum mechanics (or modern physics), one finds that the momentum is given in terms of the wave number, $p = \hbar k$, where $\hbar$ is Planck's constant divided by 2π. Inserting this into the above condition, one obtains

$$\Delta x \Delta p \geq \hbar.$$

This gives the famous uncertainty relation between the uncertainties in position and momentum.

Example 3 $f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \ a > 0.$

The Fourier transform of this function is

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= \int_{o}^{\infty} e^{ikx-ax}\,dx \\ &= \frac{1}{a-ik}. \end{aligned} \tag{6.46}$$

Next, we will compute the inverse Fourier transform of this result and recover the original function.

Example 4 $\hat{f}(k) = \frac{1}{a-ik}$.
The inverse Fourier transform of this function is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{-ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\,dk.$$

This integral can be evaluated using contour integral methods. We recall Jordan's Lemma from the last chapter:

> If $f(z)$ converges uniformly to zero as $z \to \infty$, then
>
> $$\lim_{R\to\infty}\int_{C_R} f(z)e^{ikz}\,dz = 0$$
>
> where $k > 0$ and C_R is the upper half of the circle $|z| = R$. A similar result applies for $k < 0$, but one closes the contour in the lower half plane.

In this example, we have to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-ixz}}{a-iz}\,dz.$$

According to Jordan's Lemma, we need to enclose the contour with a semicircle in the upper half plane for $x < 0$ and in the lower half plane for $x > 0$. The integrations along the semicircles will vanish and we will have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\,dk \\ &= \pm\frac{1}{2\pi}\oint_{C} \frac{e^{-ixz}}{a-iz}\,dz \end{aligned}$$

$$= \begin{cases} 0, & x < 0 \\ -\frac{1}{2\pi} 2\pi i \text{ Res } [z = -ia], & x > 0 \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ e^{-ax}, & x > 0 \end{cases} . \tag{6.47}$$

Example 5 $\hat{f}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

We would like to find the inverse Fourier transform of this function. Instead of carrying out any integration, we will make use of the properties of Fourier transforms. Since the transforms of sums are the sums of transforms, we can look at each term individually. Consider $\delta(\omega - \omega_0)$. This is a shifted function. From the Shift Theorems we have

$$e^{i\omega_0 t} f(t) \leftrightarrow \hat{f}(\omega - \omega_0).$$

Recalling from a previous example that

$$\int_{-\infty}^{\infty} 1 e^{i\omega t}\, dt = 2\pi\delta(\omega),$$

we have

$$F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} e^{-i\omega_0 t}.$$

The other term can be transformed similarly. Therefore, we have

$$F^{-1}[\pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0] = \frac{1}{2} e^{i\omega_0 t} + \frac{1}{2} e^{-i\omega_0 t} = \cos\omega_0 t.$$

Example 6 The Finite Wave Train $f(x) = \begin{cases} \cos\omega_0 t, & |t| \le a \\ 0, & |t| > a \end{cases}$.

For our last example, we consider the finite wave train, which often appears in signal analysis. A straight forward computation gives

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt \\ &= \int_{-a}^{a} \cos\omega_0 t e^{i\omega t}\, dt \\ &= \int_{-a}^{a} \cos\omega_0 t \cos\omega t\, dt \\ &= \frac{1}{2}\int_{-a}^{a} [\cos(\omega_0 + \omega)t + \cos(\omega_0 - \omega)t]\; dt \\ &= \frac{\sin(\omega_0 + \omega)a}{\omega + \omega_0} + \frac{\sin(\omega_0 - \omega)a}{\omega - \omega_0}. \end{aligned} \tag{6.48}$$

6.6 The Convolution Theorem - *Optional*

In our list of properties, we defined the convolution of two functions, $f(x)$ and $g(x)$ to be the integral

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi. \tag{6.49}$$

In some sense one is looking at a sum of the overlaps of one of the functions and all of the shifted versions of the other function. The German word for convolution is *faltung*, which means 'folding'.

First, we note that the convolution is commutative: $f * g = g * f$. This is easily shown by replacing $x - \xi$ with a new variable, y.

$$\begin{aligned}
(g * f)(x) &= \int_{-\infty}^{\infty} g(\xi)f(x-\xi)\,d\xi \\
&= -\int_{\infty}^{-\infty} g(x-y)f(y)\,dy \\
&= \int_{-\infty}^{\infty} f(y)g(x-y)\,dy \\
&= (f * g)(x).
\end{aligned} \tag{6.50}$$

Example Graphical Convolution.

In order to understand the convolution operation, we need to apply it to several functions. We will do this graphically for the box function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and the triangular function

$$g(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

as shown in Figures 6.8 and 6.9.

In order to determine the contributions to the integrand, we look at the shifted and reflected function $g(\xi - x)$ for various values of ξ. For $\xi = 0$, we

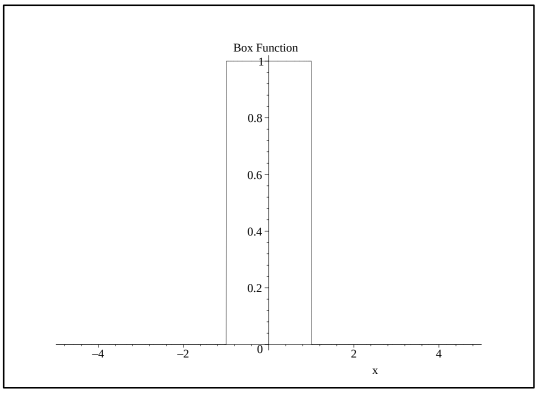

Figure 6.8: A plot of the box function $f(x)$.

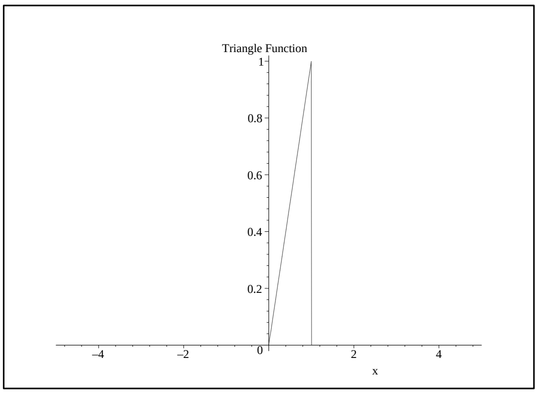

Figure 6.9: A plot of the triangle function.

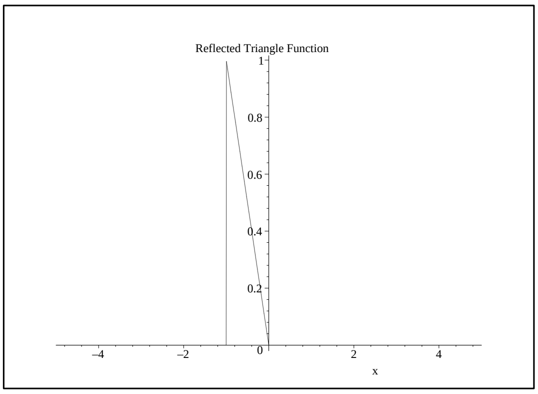

Figure 6.10: A plot of the reflected triangle function.

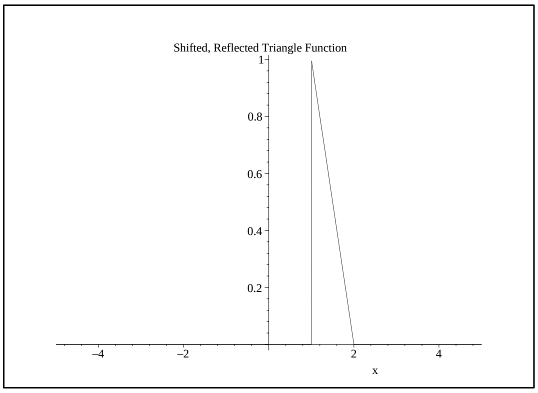

Figure 6.11: A plot of the reflected triangle function shifted by 2 units.

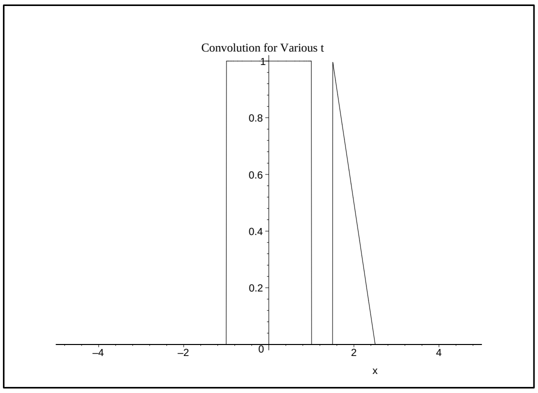

Figure 6.12: A plot of the box and triangle functions with the convolution indicated by the shaded area.

have $g(-x)$. This is a reflection of the triangle function as shown in Figure 6.10.

We then translate this function through horizontal shifts by ξ. In Figure 6.11 we show such a shifted and reflected $g(x)$ for $\xi = 2$. The following figures show other shifts superimposed on $f(x)$. The integrand is the product of $f(x)$ and $g(\xi - x)$ and the convolution evaluated at ξ is given by the shaded areas. In Figures 6.12 and 6.16 the area is zero, as there is no overlap of the functions. Intermediate shift values are displayed in Figures 6.13-6.15 and the convolution is shown by the area under the product of the two functions.

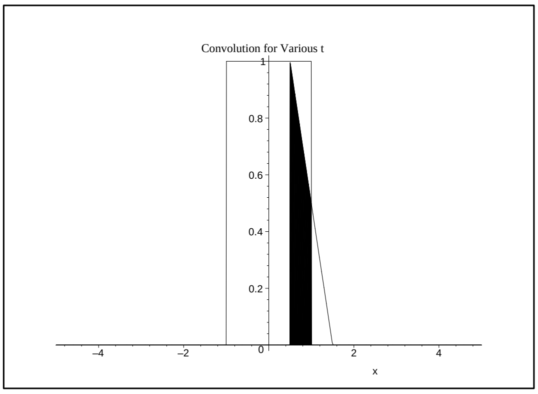

Figure 6.13: A plot of the box and triangle functions with the convolution indicated by the shaded area.

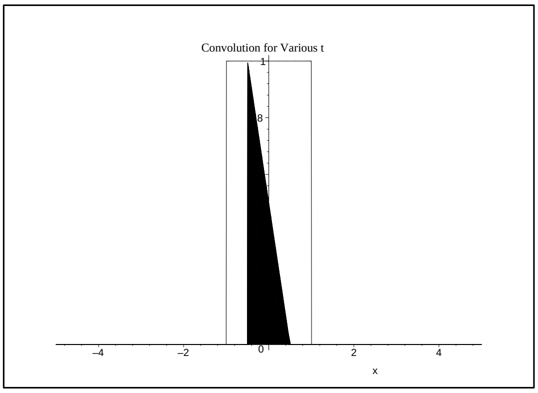

Figure 6.14: A plot of the box and triangle functions with the convolution indicated by the shaded area.

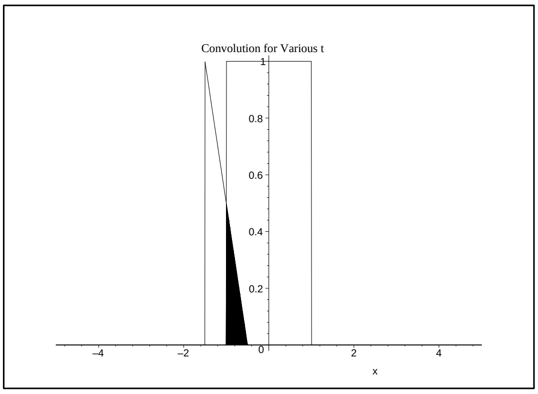

Figure 6.15: A plot of the box and triangle functions with the convolution indicated by the shaded area.

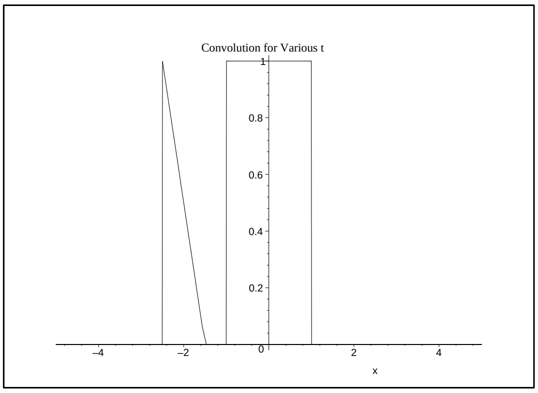

Figure 6.16: A plot of the box and triangle functions with the convolution indicated by the shaded area.

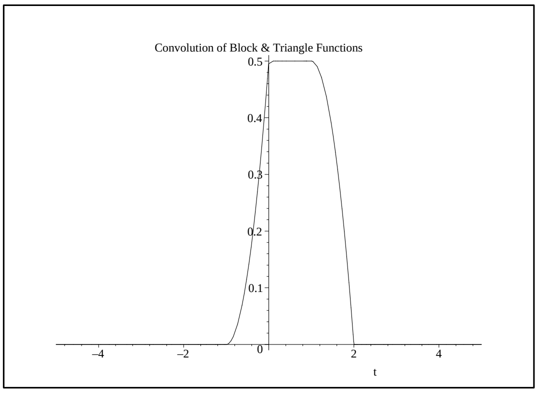

Figure 6.17: A plot of the convolution of the box and triangle functions.

We see that the value of the convolution integral builds up and then quickly drops to zero. The plot of the convolution of the box and triangle functions is given in Figure 6.17.

Next we would like to compute the Fourier transform of the convolution integral. First, we use the definitions of Fourier transform and convolution to write the transform as

$$\begin{aligned} F[f*g] &= \int_{-\infty}^{\infty} (f*g)(x) e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) g(x-\xi)\,d\xi \right) e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-\xi) e^{ikx}\,dx \right) f(\xi)\,d\xi. \end{aligned} \tag{6.51}$$

Next, we substitute $y = x - \xi$ on the inside integral and separate the integrals:

$$\begin{aligned} F[f*g] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-\xi) e^{ikx}\,dx \right) f(\xi)\,d\xi \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{ik(y+\xi)}\,dy \right) f(\xi)\,d\xi \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{iky}\,dy \right) f(\xi) e^{ik\xi}\,d\xi \qquad (6.52) \\ &= \left(\int_{-\infty}^{\infty} f(\xi) e^{ik\xi}\,d\xi \right) \left(\int_{-\infty}^{\infty} g(y) e^{iky}\,dy \right). \qquad (6.53) \end{aligned}$$

We see the the two integral factors are just the Fourier transforms of f and

g. Therefore, the Fourier transform of a convolution is the product of the Fourier transforms of the functions involved:

$$F[f * g] = \hat{f}(k)\hat{g}(k). \tag{6.54}$$

Example Convolution of two Gaussian functions.

We will compute the convolution of two Gaussian functions with different widths. Let $f(x) = e^{-ax^2}$ and $g(x) = e^{-bx^2}$. A direct evaluation of the integral would be to compute

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\, d\xi = \int_{-\infty}^{\infty} e^{-a\xi^2 - b(x-\xi)^2}\, d\xi.$$

This integral can be rewritten as

$$(f * g)(x) = e^{-bx^2} \int_{-\infty}^{\infty} e^{-(a+b)\xi^2 + 2bx\xi}\, d\xi.$$

One could proceed to complete the square and finish carrying out the integration. However, we will use the Convolution Theorem to evaluate the convolution. Recalling the Fourier transform of a Gaussian, we have

$$\hat{f}(k) = F[e^{-ax^2}] = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a} \tag{6.55}$$

and

$$\hat{g}(k) = F[e^{-bx^2}] = \sqrt{\frac{2\pi}{b}} e^{-k^2/2b}.$$

Denoting the convolution function by $h(x) = (f * g)(x)$, the Convolution Theorem gives

$$\hat{h}(k) = \hat{f}(k)\hat{g}(k) = \frac{2\pi}{\sqrt{ab}} e^{-k^2/2a} e^{-k^2/2b}.$$

This is another Gaussian function, as seen by rewriting the Fourier transform of $h(x)$ as

$$\hat{h}(k) = \frac{2\pi}{\sqrt{ab}} e^{-\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)k^2} = \frac{2\pi}{\sqrt{ab}} e^{-\frac{a+b}{2ab}k^2}. \tag{6.56}$$

To complete the evaluation of the convolution of these two Gaussian functions, we need to find the inverse transform of the Gaussian in

Equation (6.56). We can do this by looking at Equation (6.55). We have first that

$$F^{-1}\left[\sqrt{\frac{2\pi}{a}}e^{-k^2/2a}\right] = e^{-ax^2}.$$

Moving the constants, we then obtain

$$F^{-1}[e^{-k^2/2a}] = \sqrt{\frac{a}{2\pi}}e^{-ax^2}.$$

We now make the substitution $\alpha = \frac{1}{2a}$,

$$F^{-1}[e^{-\alpha k^2}] = \sqrt{\frac{1}{4\pi\alpha}}e^{-x^2/2\alpha}.$$

This is in the form needed to invert (6.56). Thus, for $\alpha = \frac{a+b}{2ab}$ we find

$$(f * g)(x) = h(x) = \sqrt{\frac{2\pi}{a+b}}e^{-\frac{ab}{a+b}x^2}.$$

6.7 Applications of the Convolution Theorem - *Optional*

There are many applications of the convolution operation. In this section we will describe a few of the applications.

The first application is filtering signals. For a given signal there might be some noise in the signal, some undesirable high frequencies, or the device used for recording an analog signal might naturally not be able to record high frequencies. Let $f(t)$ denote the amplitude of a given analog signal and $\hat{f}(\omega)$ be the Fourier transform of this signal. An example is provided in Figure 6.18. Recall that the Fourier transform gives the frequency content of the signal and that $\omega = 2\pi\nu$, where ν is the frequency in Hertz, or cycles per second (cps).

There are many ways to filter out unwanted frequencies. The simplest would be to just drop all of the high frequencies, $|\omega| > \omega_0$ for some cutoff frequency ω_0. The Fourier transform of the filtered signal would then be zero for $|\omega| > \omega_0$. This could be accomplished by multiplying the Fourier

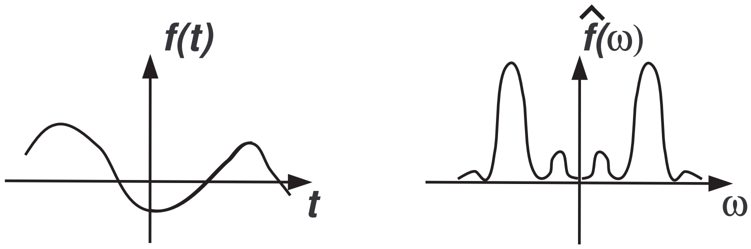

Figure 6.18: Plot of a signal $f(t)$ and its Fourier transform $\hat{f}(\omega)$.

transform of the signal by a function that vanishes for $|\omega| > \omega_0$. For example, we could consider the gate function

$$p_{\omega_0}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases} . \tag{6.57}$$

Figure 6.19 shows how the gate function is used to filter the signal.

In general, we multiply the Fourier transform of the signal by some filtering function $\hat{h}(t)$ to get the Fourier transform of the filtered signal,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{h}(\omega).$$

The new signal, $g(t)$ is then the inverse Fourier transform of this product, giving the new signal as a convolution:

$$g(t) = F^{-1}[\hat{f}(\omega)\hat{h}(\omega)] = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)\,d\tau. \tag{6.58}$$

Such processes occur often in systems theory as well. One thinks of $f(t)$ as the input signal into some filtering device which in turn produces the output, $g(t)$. The function $h(t)$ is called the *impulse response.* This is because it is a response to the impulse function, $\delta(t)$. In this case, one has

$$\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)\,d\tau = h(t).$$

Another application of the convolution is in windowing. This represents what happens when one measures a real signal. Real signals cannot be recorded for all values of time. Instead data is collected over a finite time

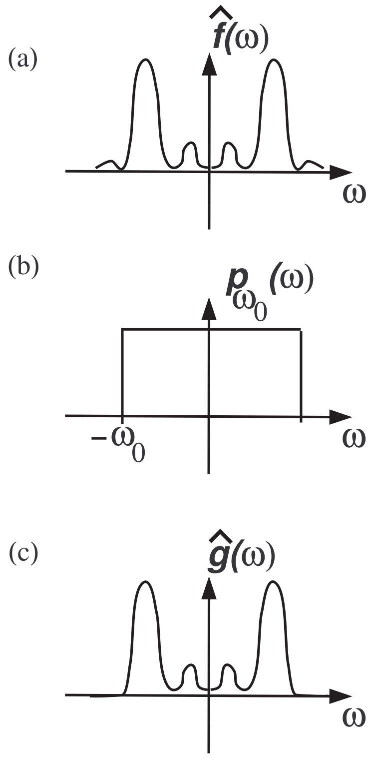

Figure 6.19: (a) Plot of the Fourier transform $\hat{f}(\omega)$ of a signal. (b) The gate function $p_{\omega_0}(\omega)$ used to filter out high frequencies. (c) The product of the functions, $\hat{g}(\omega) = \hat{f}(\omega)p_{\omega_0}(\omega)$, in (a) and (b).

interval. If the length of time the data is collected is T, then the resulting signal is zero outside this time interval. This can be modeled in the same way as with filtering, except the new signal will be the product of the old signal with the windowing function. The resulting Fourier transform of the new signal will be a convolution of the Fourier transforms of the original signal and the windowing function.

We will later see that the effect of windowing would be to change the spectral content of the signal we are trying to analyze. We will study these natural windowing and filtering effects from recording data in the last chapter.

We can also use the convolution theorem to derive **Parseval's Equality**:

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega. \tag{6.59}$$

This equality has a physical meaning for signals. The integral on the left side is a measure of the energy content of the signal in the time domain. The right side provides a measure of the energy content of the transform of the signal. Parseval's equality, sometimes referred as Plancherel's formula, is simply a statement that the energy is invariant under the transform.

Let's rewrite the Convolution Theorem in the form

$$F^{-1}[\hat{f}(k)\hat{g}(k)] = (f * g)(t). \tag{6.60}$$

Then, by the definition of the inverse Fourier transform, we have

$$\int_{-\infty}^{\infty} f(t-u)g(u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{-i\omega t}\, d\omega.$$

Setting $t = 0$,

$$\int_{-\infty}^{\infty} f(-u)g(u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)\, d\omega. \tag{6.61}$$

Now, let $g(t) = \overline{f(-t)}$, or $f(-t) = \overline{g(t)}$. Then, the Fourier transform of $g(t)$ is related to the Fourier transform of $f(t)$:

$$\begin{aligned}\hat{g}(\omega) &= \int_{-\infty}^{\infty} \overline{f(-t)}e^{i\omega t}\, dt \\ &= -\int_{\infty}^{-\infty} \overline{f(\tau)}e^{-i\omega\tau}\, d\tau \\ &= \overline{\int_{-\infty}^{\infty} f(\tau)e^{i\omega\tau}\, d\tau} = \overline{\hat{f}(\omega)}.\end{aligned} \tag{6.62}$$

So, inserting this result into Equation (6.61), we find that

$$\int_{-\infty}^{\infty} f(-u)\overline{f(-u)}\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega$$

which implies Parseval's Equality.

6.8 The Laplace Transform

Up until this point we have only explored Fourier exponential transforms as one type of integral transform. The Fourier transform is useful on infinite domains. However, students are often introduced to another integral transform, called the Laplace transform, in their introductory differential equations class. These transforms are defined over semi-infinite domains and are useful for solving ordinary differential equations. They also have proven useful in engineering for solving circuit problems and doing systems analysis.

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^{\infty} f(t)e^{-st}\, dt, \quad s > 0. \tag{6.63}$$

This is an improper integral and one needs

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

to guarantee convergence.

It is typical that one makes use of Laplace transforms by referring to a Table of transform pairs. A sample of such pairs is given in Table 6.8. Combining some of these simple Laplace transforms with the properties of the Laplace transform, as shown in Table 6.8, we can deal with many applications of the Laplace transform. We will first prove a few of the given Laplace transforms and show how they can be used to obtain new transform pairs. In the next section we will show how these can be used to solve ordinary differential equations.

We begin with some simple transforms. These are found by simply using the definition of the Laplace transform.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
c	$\frac{c}{s}$	e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > 0$	$\delta(t-a)$	$e^{-as}, \quad a \geq 0, s > 0$

Table 6.1: Table of selected Laplace transform pairs.

Example 1: $\mathcal{L}[1]$

For this example, we insert $f(t) = 1$ into our integral:

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\, dt.$$

This is an improper integral and the computation is understood by introducing an upper limit of a and then letting $a \to \infty$. We will not always write this limit, but it will be understood that this is how one computes such improper integrals. Thus, we have

$$\begin{aligned} \mathcal{L}[1] &= \int_0^\infty e^{-st}\, dt \\ &= \lim_{a\to\infty} \int_0^a e^{-st}\, dt \\ &= \lim_{a\to\infty} \left(-\frac{1}{s} e^{-st}\right)_0^a \\ &= \lim_{a\to\infty} \left(-\frac{1}{s} e^{-sa} + \frac{1}{s}\right) = \frac{1}{s}. \end{aligned} \tag{6.64}$$

Thus, we have that the Laplace transform of 1 is $\frac{1}{s}$. This can be extended to any constant c, using the property of linearity of the transform. The Laplace transform is simply an integral. So, $\mathcal{L}[c] = c\mathcal{L}[1]$, since we can pull a constant factor out from under the integral. Therefore, we have

$$\mathcal{L}[c] = \frac{c}{s}.$$

Example 1: $\mathcal{L}[e^{at}]$,

For this example, we can easily compute the transform. It again is simply the integral of an exponential function.

$$\begin{aligned}
\mathcal{L}[e^{at}] &= \int_0^\infty e^{at}e^{-st}\,dt \\
&= \int_0^\infty e^{(a-s)t}\,dt \\
&= \left(\frac{1}{a-s}e^{(a-s)t}\right)_0^\infty \\
&= \lim_{t\to\infty}\frac{1}{a-s}e^{(a-s)t} - \frac{1}{a-s} = \frac{1}{s-a}. \qquad (6.65)
\end{aligned}$$

Note that the last limit was computed as $\lim_{t\to\infty} e^{(a-s)t} = 0$. This is only true if $a - s < 0$, or $s > a$. [Actually, a could be complex. In this case we would only need that s is greater than the real part of a.]

Example 2: $\mathcal{L}[\cos at]$ and $\mathcal{L}[\sin at]$

In these cases, we could again insert the functions directly into the transform. For example,

$$\mathcal{L}[\cos at] = \int_0^\infty e^{-st}\cos at\,dt.$$

Recall how one does such integrals involving both the trigonometric function and the exponential function. One integrates by parts two times and then obtains an integral of the form with which one started. Rearranging the result the answer can be obtained. However, there is a much simpler way to compute these transforms.

Recall that $e^{iat} = \cos at + i\sin at$. Making use of the linearity of the Laplace transform, we have

$$\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at] + i\mathcal{L}[\sin at].$$

Thus, transforming this complex exponential and looking at the real and imaginary parts of the results will give both transforms at the same time! The transform is simply computed as

$$\mathcal{L}[e^{iat}] = \int_0^\infty e^{iat}e^{-st}\,dt = \int_0^\infty e^{-(s-ia)t}\,dt = \frac{1}{s-ia}.$$

Note that we could easily have used the result for the transform of an exponential, which was already proven. In this case $s > Re(ia) = 0$.

We now extract the real and imaginary parts of the result using the complex conjugate of the denominator:

$$\frac{1}{s-ia} = \frac{1}{s-ia}\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}.$$

Reading off the real and imaginary parts gives

$$\begin{aligned}\mathcal{L}[\cos at] &= \frac{s}{s^2+a^2} \\ \mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}.\end{aligned} \tag{6.66}$$

Example 3: $\mathcal{L}[t]$

For this example we need to evaluate

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt.$$

This integration can be done using integration by parts. (Pick $u = t$ and $dv = e^{-st}\,dt$. Then, $du = dt$ and $v = -\frac{1}{s}e^{-st}$.)

$$\begin{aligned}\int_0^\infty te^{-st}\,dt &= -t\frac{1}{s}e^{-st}|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2}.\end{aligned} \tag{6.67}$$

Example 4: $\mathcal{L}[t^n]$

We can generalize the last example to powers greater than $n = 1$. In this case we have to do the integral

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt.$$

Following the previous example, we integrate by parts:

$$\begin{aligned}\int_0^\infty t^n e^{-st}\,dt &= -t^n\frac{1}{s}e^{-st}|_0^\infty + n\frac{1}{s}\int_0^\infty t^{-n}e^{-st}\,dt \\ &= n\frac{1}{s}\int_0^\infty t^{-n}e^{-st}\,dt.\end{aligned} \tag{6.68}$$

We could continue to integrate by parts until the final integral can be computed. However, look at the integral that resulted after one integration by parts. It is just the Laplace transform of t^{n-1}. So, we can write the result as

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}].$$

This is an example of an recursive definition of a sequence, in this case a sequence of integrals. Denoting

$$I_n = \mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$$

and noting that $I[0] = \mathcal{L}[1] = \frac{1}{s}$, we have the following:

$$I_n = \frac{n}{s}I_{n-1}, \quad I_0 = \frac{1}{s}. \tag{6.69}$$

This is also what is called a difference equation. It is a *first order difference equation* with an "initial condition", I_0. There is a whole theory of difference equations, which we will not get into here.

Our goal is to solve the above difference equation. It is easy to do by simple iteration. Note that replacing n with $n-1$, we have

$$I_{n-1} = \frac{n-1}{s}I_{n-2}.$$

So, repeating the process we find

$$\begin{aligned} I_n &= \frac{n}{s}I_{n-1} \\ &= \frac{n}{s}\left(\frac{n-1}{s}I_{n-2}\right) \\ &= \frac{n(n-1)}{s^2}I_{n-2}. \end{aligned} \tag{6.70}$$

We can repeat this process until we get to I_0, which we know. In some cases you need to be careful so that you can count the number of iterations of the process. So, we first ask what the result is after k steps. This can be seen by watching for patterns. Continuing the iteration process, we have

$$I_n = \frac{n}{s}I_{n-1}$$

$$\begin{aligned} &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \\ &= \ldots \\ &= \frac{n(n-1)(n-2)\ldots(n-k+1)}{s^k} I_{n-k}. \end{aligned} \tag{6.71}$$

Since we know I_0, we choose to stop at $k = n$ obtaining

$$I_n = \frac{n(n-1)(n-2)\ldots(2)(1)}{s^n} I_0 = \frac{n!}{s^{n+1}}.$$

Therefore, we have shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$. [Such iterative techniques are useful in obtaining a variety of of integrals, such as $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$.]

As a final note, one can extend this result to cases when n is not an integer. To do this, one introduces what is called the Gamma function. This function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{6.72}$$

Note the similarity to the Laplace transform of t^{x-1} :

$$\mathcal{L}[t^{x-1}] = \int_0^\infty t^{x-1} e^{-st}\, dt.$$

For $x - 1$ an integer and $s = 1$, we have that

$$\Gamma(x) = (x-1)!.$$

Thus, the Gamma function seems to be a generalization of the factorial. In fact, we show this result later and state here that

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

for values of $p > -1$.

Example 5: $\mathcal{L}[\frac{df}{dt}]$

We have to compute

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st}\, dt.$$

Laplace Transform Properties
$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$
$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$
$\mathcal{L}[\frac{dy}{dt}] = sY(s)-y(0)$
$\mathcal{L}[\frac{d^2y}{dt^2}] = s^2Y(s)-sy(0)-y'(0)$
$\mathcal{L}[e^{at}f(t)] = F(s-a)$
$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$
$\mathcal{L}[(f*g)(t)] = \mathcal{L}[\int_0^t f(t-u)g(u)\,du] = F(s)G(s)$

Table 6.2: Table of selected Laplace transform properties.

We can move the derivative off of f by integrating by parts. This is similar to what we had done when finding the Fourier transform of the derivative of a function. Thus letting $u = e^{-st}$ and $v = f(t)$, we have

$$\begin{aligned} \mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt}e^{-st}\,dt \\ &= f(t)e^{-st}|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\ &= -f(0)+sF(s). \end{aligned} \tag{6.73}$$

Here we have assumed that $f(t)e^{-st}$ vanishes for large t.

The final result is that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0).$$

Example 6: $\mathcal{L}[\frac{d^2f}{dt^2}]$

We can compute this using two integrations by parts, or we could make use of the last result. Letting $g(t) = \frac{df(t)}{dt}$, we have

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \mathcal{L}\left[\frac{dg}{dt}\right] = sG(s)-g(0) = sG(s)-f'(0).$$

But,

$$G(s) = \mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0).$$

So,

$$\begin{aligned} \mathcal{L}\left[\frac{d^2f}{dt^2}\right] &= sG(s) - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf(0) - f'(0). \end{aligned} \tag{6.74}$$

6.8.1 Solution of ODEs Using Laplace Transforms

One of the typical applications of Laplace transforms is the solution of nonhomogeneous linear constant coefficient differential equations. In the following examples we will show how this works.

The general idea is that one transforms the equation for an unknown function $y(t)$ into an algebraic equation for its transform, $Y(t)$. Typically, the algebraic equation is easy to solve for $Y(s)$ as a function of s. Then one transforms back into t-space using Laplace transform tables and the properties of Laplace transforms. The scheme is shown in Figure 6.20.

Later we will see that there is an integral form for the inverse transform. This is typically not covered in introductory differential equations classes as one needs carry out integrations in the complex plane.

Example 1: Solve the initial value problem $y' + 3y = e^{2t}$, $y(0) = 1$.

The first step is to perform a Laplace transform of the initial value problem. The transform of the left side of the equation is

$$\mathcal{L}[y' + 3y] = sY - y(0) + 3Y = (s+3)Y - 1.$$

Transforming the right hand side, we have

$$\mathcal{L}[e^{2t}] = \frac{1}{s-2}.$$

Combining these, we obtain

$$(s+3)Y - 1 = \frac{1}{s-2}.$$

The next step is to solve for $Y(s)$:

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)}.$$

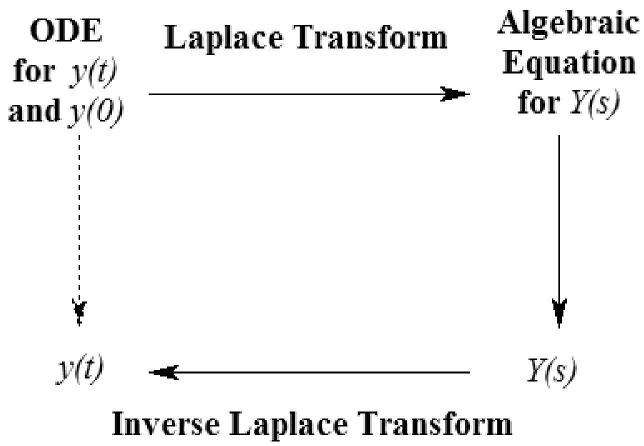

Figure 6.20: The scheme for solving an ordinary differential equation using Laplace transforms. One transforms the initial value problem for $y(t)$ and obtains an algebraic equation for $Y(s)$. Solve for $Y(s)$ and the inverse transform give the solution to the initial value problem.

Now, we need to find the inverse Laplace transform. Namely, we need to figure out what function has a Laplace transform of the above form. It is easy to do if we only had the first term. The inverse transform of the first term is e^{-3t}.

We have not seen anything that looks like the second form in the table of transforms that we have compiled so far. However, we are not stuck. We know that we can rewrite the second term by using a *partial fraction decomposition.* Let's recall how to do this. The goal is to find constants, A and B, such that

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}.$$

We picked this form because we know that recombining the two terms into one term will have the same denominator. We just need to make sure the numerators agree afterwards. So, adding the two terms, we have

$$\frac{1}{(s-2)(s+3)} = \frac{A(s+3)+B(s-2)}{(s-2)(s+3)}.$$

Equating numerators,

$$1 = A(s+3) + B(s-2).$$

This has to be true for all s. Rewriting the equation by gathering terms with common powers of s, we have

$$(A+B)s + 3A - 2B = 1.$$

The only way that this can be true for all s is that the coefficients of the different powers of s agree on both sides. This leads to two equations for A and B:

$$\begin{aligned} A + B &= 0 \\ 3A - 2B &= 1. \end{aligned} \tag{6.75}$$

The first equation gives $A = -B$, so the second equation becomes $-5B = 1$. The solution is then $A = -B = \frac{1}{5}$.

Returning to the problem, we have found that

$$Y(s) = \frac{1}{s+3}\frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right).$$

[Of course, we could have tried to guess the form of the partial fraction decomposition as we had done earlier when talking about Laurent series.] In order to finish the problem at hand, we find a function whose Laplace transform is of this form. We easily see that

$$y(t) = e^{-3t} + \frac{1}{5}\left(e^{2t} - e^{-3t}\right)$$

works. Simplifying, we have the solution of the initial value problem

$$y(t) = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}.$$

Example 2: Solve the initial value problem $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

We can probably solve this without Laplace transforms, but it is a simple exercise. Transforming the equation, we have

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) + 4Y \\ &= (s^2+4)Y - s - 3. \end{aligned} \tag{6.76}$$

Solving for Y, we have

$$Y(s) = \frac{s+3}{s^2+4}.$$

We now ask if we recognize the transform pair needed. The denominator looks like the type needed for the transform of a sine or cosine. We just need to play with the numerator. Splitting the expression into two terms, we have

$$Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4}.$$

The first term is now recognizable as the transform of $\cos 2t$. The second term is not the transform of $\sin 2t$. It would be if the numerator were a 2. This can be corrected by multiplying and dividing by 2:

$$\frac{2}{s^2+4} = \frac{3}{2}\frac{2}{s^2+4}.$$

So, out solution is then found as

$$y(t) = \mathcal{L}[\frac{s}{s^2+4} + \frac{3}{2}\frac{2}{s^2+4}] = \cos 2t + \frac{3}{2}\sin 2t.$$

6.8.2 Step and Impulse Functions

The initial value problems that we have solved so far can be solved using the Method of Undetermined Coefficients or the Method of Variation of Parameters. However, using variation of parameters can be messy and involves some skill with integration. Many circuit designs can be modelled with systems of differential equations using Kirchoff's Rules. Such systems can get fairly complicated. However, Laplace transforms can be used to solve such systems and electrical engineers have long used such methods in circuit analysis.

In this section we add a couple of more transform pairs and transform properties that are useful in accounting for things like turning on a driving force, using periodic functions like a square wave, or introducing an impulse force.

We first recall the Heaviside step function, given by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{6.77}$$

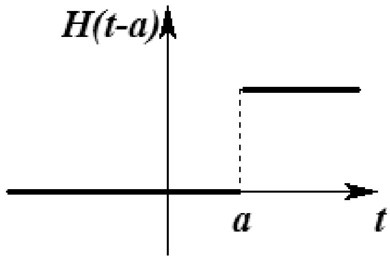

Figure 6.21: A shifted Heaviside function, $H(t-a)$.

A more general version of the step function is the horizontally shifted step function, $H(t-a)$. This function is shown in Figure 6.21. The Laplace transform of this function is found for $a>0$ as

$$\begin{aligned} \mathcal{L}[H(t-a)] &= \int_0^\infty H(t-a)e^{-st}\,dt \\ &= \int_a^\infty H(t-a)e^{-st}\,dt \\ &= \frac{e^{-st}}{s}\Big|_a^\infty = \frac{e^{-as}}{s}. \end{aligned} \tag{6.78}$$

Just like the Fourier transform, the Laplace transform has two shift theorems involving multiplication of $f(t)$ or $F(s)$ by exponentials. These are given by

$$\mathcal{L}[e^{at}f(t)] = F(s-a) \tag{6.79}$$

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s). \tag{6.80}$$

We prove the first shift theorem and leave the other proof as an exercise for the reader. Namely,

$$\begin{aligned} \mathcal{L}[e^{at}f(t)] &= \int_0^\infty e^{at}f(t)e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a). \end{aligned} \tag{6.81}$$

Example: Compute the Laplace transform of $e^{-at}\sin\omega t$.

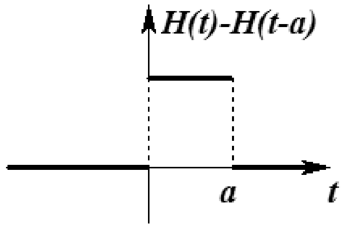

Figure 6.22: The box function, $H(t) - H(t-a)$.

This function arises as the solution of the underdamped harmonic oscillator. We first note that the exponential multiplies a sine function. The shift theorem tells us that we need the transform of this function. So,

$$F(s) = \frac{\omega}{s^2 + \omega^2}.$$

Knowing this, we can write the solution as

$$\mathcal{L}[e^{-at} \sin \omega t] = F(s+a) = \frac{\omega}{(s+a)^2 + \omega^2}.$$

More interesting examples can be found in piecewise functions. First we consider the function $H(t) - H(t-a)$. For $t < 0$ both terms are zero. In the interval $[0, a]$ the function $H(t) = 1$ and $H(t-a) = 0$. Therefore, $H(t) - H(t-a) = 1$ for $t \in [0, a]$. Finally, for $t > a$, both functions are one and therefore the difference is zero. This function is shown in Figure 6.22.

We now consider the piecewise defined function

$$g(t) = \begin{cases} f(t), & 0 \leq t \leq a, \\ 0, & t < 0, t > a. \end{cases}$$

This function can be rewritten in terms of step functions. We only need to multiply $f(t)$ by the above box function, $g(t) = f(t)[H(t) - H(t-a)]$. We depict this in Figure 6.23.

Even more complicated functions can be written out in terms of step functions. We only need to look at sums of functions of the form $f(t)[H(t-a) - H(t-b)]$ for $b > a$. This is just a box between a and b of height $f(t)$. An example of a square wave function is shown in Figure 6.24. It can be represented as a sum of an infinite number of boxes, $f(t) = \sum_{n=-\infty}^{\infty} [H(t - 2na) - H(t - (2n+1)a)]$.

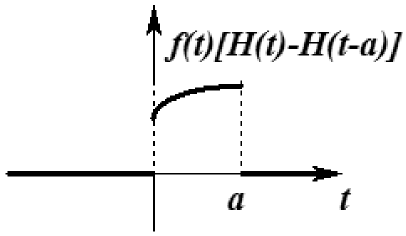

Figure 6.23: Formation of a piecewise function, $f(t)[H(t) - H(t-a)]$.

Example: Laplace Transform of a square wave turned on at $t = 0$,

$$f(t) = \sum_{n=0}^{\infty} [H(t-2na) - H(t-(2n+1)a)].$$

Using the properties of the Heaviside function, we have

$$\begin{aligned}
\mathcal{L}[\delta(t-a)] &= \sum_{n=0}^{\infty} [\mathcal{L}[H(t-2na)] - \mathcal{L}[H(t-(2n+1)a)]] \\
&= \sum_{n=0}^{\infty} \left[\frac{e^{-2nas}}{s} - \frac{e^{-2(n+1)as}}{s} \right] \\
&= \frac{1-e^{-as}}{s} \sum_{n=0}^{\infty} \left(e^{-2as} \right)^n \\
&= \frac{1-e^{-as}}{s} \left(\frac{1}{1-e^{-2as}} \right) \\
&= \frac{1-e^{-as}}{s(1-e^{-2as})}. \qquad (6.82)
\end{aligned}$$

Note that the third line in the derivation is a geometric series. We summed this series to get our answer in a compact form.

Another interesting example is the delta function. The delta function represents a point impulse, or point driving force. For example, while a mass on a spring is undergoing simple harmonic motion, one could hit it for an instant at time $t = a$. In such a case, we could represent the force as a multiple of $\delta(t-a)$. One would then need the Laplace transform of the delta function to solve the associated differential equation.

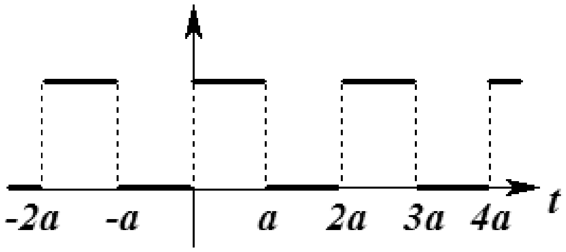

Figure 6.24: A square wave, $f(t) = \sum_{n=-\infty}^{\infty}[H(t-2na) - H(t-(2n+1)a)]$.

We find that for $a > 0$

$$\begin{aligned} \mathcal{L}[\delta(t-a)] &= \int_0^\infty \delta(t-a)e^{-st}\,dt \\ &= \int_{-\infty}^\infty \delta(t-a)e^{-st}\,dt \\ &= e^{-as}. \end{aligned} \tag{6.83}$$

Example: Solve the initial value problem $y'' + 4\pi^2 y = \delta(t-2)$, $y(0) = y'(0) = 0$.

In this case we see that we have a nonhomogeneous spring problem. Without the forcing term, given by the delta function, this spring is initially at rest and not stretched. The delta function models a unit impulse at $t = 2$. Of course, we anticipate that at this time the spring will begin to oscillate. We will solve this problem using Laplace transforms.

First, transform the differential equation:

$$s^2Y - sy(0) - y'(0) + 4\pi^2 Y = e^{-2s}.$$

Inserting the initial conditions, we have

$$(s^2 + 4\pi^2)Y = e^{-2s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{e^{-2s}}{s^2 + 4\pi^2}.$$

We now seek the function for which this is the Laplace transform. The form of this function is an exponential times some $F(s)$. Thus, we need the second shift theorem. First we need to find the $f(t)$ corresponding to

$$F(s) = \frac{1}{s^2 + 4\pi^2}.$$

The denominator suggests a sine or cosine. Since the numerator is constant, we pick sine. From the tables of transforms, we have

$$\mathcal{L}[\sin 2\pi t] = \frac{2\pi}{s^2 + 4\pi^2}.$$

So, we write

$$F(s) = \frac{1}{2\pi}\frac{2\pi}{s^2 + 4\pi^2}.$$

This gives $f(t) = (2\pi)^{-1}\sin 2\pi t$.

We now apply the second shift theorem, $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$.

$$\begin{aligned} y(t) &= H(t-2)f(t-2) \\ &= \frac{1}{2\pi}H(t-2)\sin 2\pi(t-2). \end{aligned} \tag{6.84}$$

This solution tells us that the mass is at rest until $t = 2$ and then begins to oscillate at its natural frequency.

Finally, we consider the convolution of two functions. Often we are faced with having the product of two Laplace transforms that we know and we seek the inverse transform of the product. For example, let's say you end up with $Y(s) = \frac{1}{(s-1)(s-2)}$ while trying to solve a differential equation. We know how to do this if we only have one of the denominators present. Of course, we could do a partial fraction decomposition. But, there is another way to find the inverse transform, especially if we cannot perform a partial fraction decomposition.

We define the convolution of two functions defined on $[0, \infty)$ much the same way as we had done for the Fourier transform. We define

$$(f * g)(t) = \int_0^t f(u)g(t-u)\,du.$$

The convolution operation has two important properties:

1. The convolution is commutative: $f * g = g * f$

Proof: The key is to make a substitution $y = t - u$ int the integral to make f a simple function of the integration variable.

$$\begin{aligned} (g * f)(t) &= \int_0^t g(u) f(t-u)\, du \\ &= -\int_t^0 g(t-y) f(y)\, dy \\ &= \int_0^t f(y) g(t-y)\, dy \\ &= (f * g)(t). \end{aligned} \tag{6.85}$$

2. **The Convolution Theorem:** The Laplace transform of a convolution is the product of the Laplace transforms of the individual functions:

$$\mathcal{L}[f * g] = F(s)G(s)$$

Proving this theorem takes a bit more work. We will make some assumptions that will work in many cases. First, we assume that our functions are causal, $f(t) = 0$ and $g(t) = 0$ for $t < 0$. Secondly, we will assume that we can interchange integrals, which needs more rigorous attention than will be provided here. The first assumption will allow us to write the finite integral as an infinite integral. Then a change of variables will allow us to split the integral into the product of two integrals that are recognized as a product of two Laplace transforms.

$$\begin{aligned} \mathcal{L}[f * g] &= \int_0^\infty \left(\int_0^t f(u) g(t-u)\, du \right) e^{-st}\, dt \\ &= \int_0^\infty \left(\int_0^\infty f(u) g(t-u)\, du \right) e^{-st}\, dt \\ &= \int_0^\infty f(u) \left(\int_0^\infty g(t-u) e^{-st}\, dt \right) du \\ &= \int_0^\infty f(u) \left(\int_0^\infty g(\tau) e^{-s(\tau+u)}\, d\tau \right) du \\ &= \int_0^\infty f(u) e^{-su} \left(\int_0^\infty g(\tau) e^{-s\tau}\, d\tau \right) du \\ &= \left(\int_0^\infty f(u) e^{-su}\, du \right) \left(\int_0^\infty g(\tau) e^{-s\tau}\, d\tau \right) \\ &= F(s)G(s). \end{aligned} \tag{6.86}$$

We make use of the Convolution theorem to do the following example.

Example $y(t) = \mathcal{L}^{-1}[\frac{1}{(s-1)(s-2)}]$.

We note that this is a product of two functions

$$Y(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-1}\frac{1}{s-2} = F(s)G(s).$$

We know the inverse transforms of the factors: $f(t) = e^t$ and $g(t) = e^{2t}$.

Using the Convolution Theorem, we find that $y(t) = (f * g)(t)$. We compute the convolution:

$$\begin{aligned} y(t) &= \int_0^t f(u)g(t-u)\,du \\ &= \int_0^t e^u e^{2(t-u)}\,du \\ &= e^{2t}\int_0^t e^{-u}\,du \\ &= e^{2t}[-e^t+1] = e^{2t} - e^t. \end{aligned} \tag{6.87}$$

You can confirm this by carrying out the partial fraction decomposition.

6.8.3 The Inverse Laplace Transform

Up until this point we have seen that the inverse Laplace transform can be found by making use of Laplace transform tables and properties of Laplace transforms. This is typically the way Laplace transforms are taught and used. One can do the same for Fourier transforms. However, in that case we introduced an inverse transform in the form of an integral. does such an inverse exist for the Laplace transform? Yes, it does. In this section we will introduce the inverse Laplace transform integral and show how it is used.

We begin by considering a function $f(t)$ which vanishes for $t < 0$. We define the function $g(t) = f(t)e^{-ct}$. For $g(t)$ absolutely integrable,

$$\int_{-\infty}^{\infty} |g(t)|\,dt = \int_0^{\infty} |f(t)|e^{-ct}\,dt < \infty,$$

we can write the Fourier transform,

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \int_0^{\infty} f(t)e^{i\omega t - ct}dt$$

and the inverse Fourier transform,

$$g(t) = f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)e^{-i\omega t}\,d\omega.$$

Multiplying by e^{ct} and inserting $\hat{g}(\omega)$ into the integral for $g(t)$, we find

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(\tau)e^{(i\omega-c)\tau}d\tau e^{-(i\omega-c)t}\,d\omega.$$

Letting $s = c - i\omega$ (so $d\omega = ids$), we have

$$f(t) = \frac{i}{2\pi}\int_{c+i\infty}^{c-i\infty}\int_{0}^{\infty} f(\tau)e^{-s\tau}d\tau e^{st}\,ds.$$

Note that the inside integral is simply $F(s)$. So, we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds.$$

This is the inverse Laplace transform, called the Bromwich integral. This integral is evaluated along a path in the complex plane. The typical way to compute this integral is to chose c so that all poles are to the left of the contour and to close the contour with a semicircle enclosing the poles. One then relies on Jordan's lemma extended into the second and third quadrants.

Example: Find the inverse Laplace transform of $F(s) = \frac{1}{s(s+1)}$.

The integral we have to compute is

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(s+1)}\,ds.$$

This integral has poles at $s = 0$ and $s = -1$. The contour we will use is shown in Figure 6.25. We enclose the contour with a semicircle to the left of the path in the complex s-plane. One has to verify that the integral over the semicircle vanishes as the radius goes to infinity. Assuming that we have done this, then the result is simply obtained as $2\pi i$ times the sum of the residues. The residues in this case are:

$$Res[\frac{e^{zt}}{z(z+1)}; z = 0] = \lim_{z\to 0}\frac{e^{zt}}{(z+1)} = 1$$

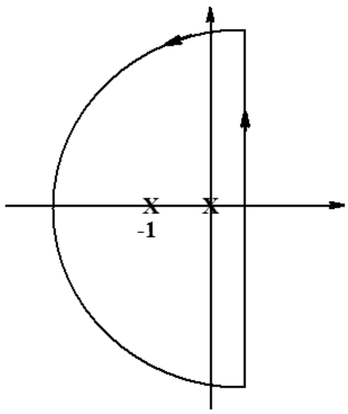

Figure 6.25: The contour used for applying the Bromwich integral to $F(s) = \frac{1}{s(s+1)}$.

and

$$Res[\frac{e^{zt}}{z(z+1)}; z=-1] = \lim_{z \to -1} \frac{e^{zt}}{z} = -e^{-t}.$$

Therefore, we have

$$f(t) = 2\pi i \left(\frac{1}{2\pi i}(1) + \frac{1}{2\pi i}(-e^{-t}) \right) = 1 - e^{-t}.$$

We can verify this result using the Convolution Theorem or using partial fraction decomposition. The decomposition is simplest:

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

The first term leads to an inverse transform of 1 and the second term gives an e^{-t}. Thus, we have verified the result from doing a contour integration.