4 The Harmonics of Vibrating Strings

4.1 Harmonics and Vibrations

"What I am going to tell you about is what we teach our physics students in the third or fourth year of graduate school . . . It is my task to convince you not to turn away because you don't understand it. You see my physics students don't understand it . . . That is because I don't understand it. Nobody does." Richard Feynman (1918-1988)

Until now we have studied oscillations in several physical systems. These lead to ordinary differential equations describing the time evolution of the systems and required the solution of initial value problems. In this chapter we will extend our study include oscillations in space. The typical example is the vibrating string.

When one plucks a violin, or guitar, string, the string vibrates exhibiting a variety of sounds. These are enhanced by the violin case, but we will only focus on the simpler vibrations of the string. We will consider the one dimensional wave motion in the string. Physically, the speed of these waves depends on the tension in the string and its mass density. The frequencies we hear are then related to the string shape, or the allowed wavelengths across the string. We will be interested the harmonics, or pure sinusoidal waves, of the vibrating string and how a general wave on a string can be represented as a sum over such harmonics. This will take us into the field of spectral, or Fourier, analysis.

Such systems are governed by partial differential equations. The vibrations of a string are governed by the one dimensional wave equation. Another simple partial differential equation is that of the heat, or diffusion, equation. This equation governs heat flow. We will consider the flow of heat through a one dimensional rod. The solution of the heat equation also involves the use of Fourier analysis. However, in this case there are no oscillations in time.

There are many applications that are studied using spectral analysis. At the root of these studies is the belief that continuous waveforms are comprised of a number of harmonics. Such ideas stretch back to the Pythagoreans study of the vibrations of strings, which led to their program of a world of harmony. This idea was carried further by Johannes Kepler (1571-1630) in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis (1616-1703), Brook Taylor (1685-1731) and Jean le Rond d'Alembert (1717-1783).

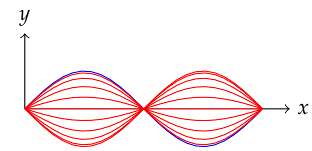

Figure 4.1: Plot of the second harmonic of a vibrating string at different times.

In 1742 d'Alembert solved the wave equation

$$c^2\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the string height and c is the wave speed. However, this solution led himself and others, like Leonhard Euler (1707-1783) and Daniel Bernoulli (1700-1782), to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach to the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x,0) = h(x)$ has a discontinuous derivative! (We will see how this led to important questions in analysis.)

Solutions of the wave equation, such as the one shown, are solved using the Method of Separation of Variables. Such solutions are studies in courses in partial differential equations and mathematical physics.

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x,t) = \sum_k a_k \sin\frac{k\pi x}{L}\cos\frac{k\pi ct}{L},$$

where the string extend over the interval $[0, L]$ with fixed ends at $x = 0$ and $x = L$.

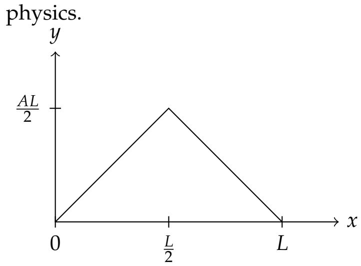

Figure 4.2: Plot of an initial condition for a plucked string.

However, the initial conditions for such superpositions are

$$y(x,0) = \sum_k a_k \sin\frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string given by an initial condition of the form

$$y(x,0) = \begin{cases} Ax, & 0 \le x \le L/2 \\ A(L-x), & L/2 \le x \le L \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's (1768-1830) solution of the heat equation. The use of Fourier expansions has become an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation.

More generally, using a technique called the Method of Separation of Variables, allowed higher dimensional problems to be reduced to one dimensional boundary value problems. However, these studies led to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives be represented by a sum of smooth functions, such as the above sums of trigonometric functions?
3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

The one dimensional version of the heat equation is a partial differential equation for $u(x,t)$ of the form

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Solutions satisfying boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, are of the form

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 t/L^2}.$$

In this case, setting $u(x,0) = f(x)$, one has to satisfy the condition

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

This is similar to where we left off with the wave equation example.

There are many other systems in which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. For example, we can consider ocean waves. Ocean waves are affected by the gravitational pull of the moon and the sun and other numerous forces. These lead to the tides, which in turn have their own periods of motion. In an analysis of wave heights, one can separate out the tidal components by making use of Fourier analysis.

4.2 *Boundary Value Problems*

UNTIL THIS POINT we have solved initial value problems. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, \quad x'(0) = 0. \tag{4.1}$$

In the next chapters we will study boundary value problems and various tools for solving such problems. In this chapter we will motivate our interest in boundary value problems by looking into solving the one-dimensional heat equation, which is a partial differential equation. for the rest of the section, we will use this solution to show that in the background of our solution of boundary value problems is a structure based upon linear algebra and analysis leading to the study of inner product spaces. Though technically, we should be lead to Hilbert spaces, which are complete inner product spaces.

For an initial value problem one has to solve a differential equation subject to conditions on the unknown function or its derivatives at more than one value of the independent variable. As an example, we have a slight modification of the above problem: Find the solution $x = x(t)$ for $0 \leq t \leq 1$ that satisfies the problem

$$x'' + x = 2, \quad x(0) = 1, \quad x(1) = 0. \tag{4.2}$$

Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the endpoints. An example would be a horizontal beam supported at the ends, like a bridge. The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of other types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore the effects of different boundary value conditions in our discussions and exercises.

Let's solve the above boundary value problem. As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so we have that the solution is a sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. The solution of $x'' + x = 0$ is easily found as

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 and c_2 that yield a solution to our problem. The first condition, $x(0) = 0$, gives

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. Using this value for c_1, the second condition, $x(1) = 1$, gives

$$0 = 2 - 2\cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by

$$x(t) = 2\left(1 - \cos t \frac{(\cos 1 - 1)}{\sin 1} \sin t\right).$$

Boundary value problems arise in many physical systems, just as the initial value problems we have seen earlier. We will see in the next sections that boundary value problems for ordinary differential equations often appear in the solutions of partial differential equations. However, there is no guarantee that we will have unique solutions of our boundary value problems as we had found in the example above.

4.3 *Partial Differential Equations*

IN THIS SECTION we will introduce several generic partial differential equations and see how the discussion of such equations leads naturally to the study of boundary value problems for ordinary differential equations. However, we will not derive the particular equations at this time, leaving that for your other courses to cover.

For ordinary differential equations, the unknown functions are functions of a single variable, e.g., $y = y(x)$. Partial differential equations are equations involving an unknown function of several variables, such as $u = u(x,y)$, $u = u(x,y)$, $u = u(x,y,z,t)$, and its (partial) derivatives. Therefore, the derivatives are partial derivatives. We will use the standard notations $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc.

There are a few standard equations that one encounters. These can be studied in one to three dimensions and are all linear differential equations. A list is provided in Table 4.1. Here we have introduced the Laplacian operator, $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems for ordinary differential equations.

Name	2 Vars	3 D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2\nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x,y)$	$\nabla^2 u = F(x,y,z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x,t)u$	$iu_t = \nabla^2 u + F(x,y,z,t)u$

Table 4.1: List of generic partial differential equations.

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L. It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation.

A typical initial-boundary value problem for the heat equation would be that initially one has a temperature distribution $u(x,0) = f(x)$. Placing the bar in an ice bath and assuming the heat flow is only through the ends of the bar, one has the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is plenty of ice to keep that temperature fixed at each end for all time. So, the problem one would need to solve is given as

1D Heat Equation

$$\begin{array}{lcc} \text{PDE} & u_t = k u_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{4.3}$$

Here, k is the heat conduction constant and is determined using properties of the bar.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed. Think of a violin string or a guitar string. Then the string is plucked, giving the string an initial profile. Let $u(x,t)$ be the vertical displacement of the string at position x and time t. The motion of the string is governed by the one dimensional wave equation. The initial-boundary value problem for this problem is given as

1D Wave Equation

$$\begin{array}{lcc} \text{PDE} & u_{tt} = c^2 u_{xx} & 0 < t, \quad 0 \le x \le L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0 \end{array} \tag{4.4}$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string and the tension placed on the string.

4.4 *The 1D Heat Equation*

We would like to see how the solution of such problems involving partial differential equations will lead naturally to studying boundary value problems for ordinary differential equations. We will see this as we attempt the solution of the heat equation problem as shown in (4.3). We will employ a method typically used in studying linear partial differential equations, called *the method of separation of variables*.

Solution of the 1D heat equation using the method of separation of variables.

We assume that u can be written as a product of single variable functions of each independent variable,

$$u(x,t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

Dividing both sides by k and $u = XT$, we then get

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}.$$

We have separated the functions of time on one side and space on the other side. The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ :

$$\underbrace{\frac{1}{k}\frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda}_{\text{constant}}.$$

This leads to two equations:

$$T' = k\lambda T, \tag{4.5}$$

$$X'' = \lambda X. \tag{4.6}$$

These are ordinary differential equations. The general solutions to these equations are readily found as

$$T(t) = Ae^{k\lambda t}, \tag{4.7}$$

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{\sqrt{-\lambda}x}. \tag{4.8}$$

We need to be a little careful at this point. The aim is to force our product solutions to satisfy both the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on u lead to conditions on X.

The first condition is $u(0,t) = 0$. This implies that

$$X(0)T(t) = 0$$

for all t. The only way that this is true is if $X(0) = 0$. Similarly, $u(L,t) = 0$ implies that $X(L) = 0$. So, we have to solve the boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0 = X(L). \tag{4.9}$$

We are seeking nonzero solutions, as $X \equiv 0$ is an obvious and uninteresting solution. We call such solutions *trivial solutions*.

There are three cases to consider, depending on the sign of λ.

Case I. $\underline{\lambda > 0}$

In this case we have the exponential solutions

$$X(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{\sqrt{-\lambda} x}. \tag{4.10}$$

For $X(0) = 0$, we have

$$0 = c_1 + c_2.$$

We will take $c_2 = -c_1$. Then, $X(x) = c_1(e^{\sqrt{\lambda} x} - e^{\sqrt{-\lambda} x}) = 2c_1 \sinh \sqrt{\lambda} x$. Applying the second condition, $X(L) = 0$ yields

$$c_1 \sinh \sqrt{\lambda} L = 0.$$

This will be true only if $c_1 = 0$, since $\lambda > 0$. Thus, the only solution in this case is $X(x) = 0$. This leads to a trivial solution, $u(x,t) = 0$.

Case II. $\underline{\lambda = 0}$

For this case it is easier to set λ to zero in the differential equation. So, $X'' = 0$. Integrating twice, one finds

$$X(x) = c_1 x + c_2.$$

Setting $x = 0$, we have $c_2 = 0$, leaving $X(x) = c_1 x$. Setting $x = L$, we find $c_1 L = 0$. So, $c_1 = 0$ and we are once again left with a trivial solution.

Case III. $\underline{\lambda < 0}$

In this case is would be simpler to write $\lambda = -\mu^2$. Then the differential equation is

$$X'' + \mu^2 X = 0.$$

The general solution is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

At $x = 0$ we get $0 = c_1$. This leaves $X(x) = c_2 \sin \mu x$. At $x = L$, we find

$$0 = c_2 \sin \mu L.$$

So, either $c_2 = 0$ or $\sin \mu L = 0$. $c_2 = 0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L = n\pi, \quad n = 1, 2, \ldots.$$

Note that $n = 0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we can find solutions to the boundary value problem (4.9) for particular values of λ. The solutions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Product solutions of the heat equation (4.3) satisfying the boundary conditions are therefore

Product solutions.

$$u_n(x,t) = b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots, \tag{4.11}$$

where b_n is an arbitrary constant. However, these do not necessarily satisfy the initial condition $u(x, 0) = f(x)$. What we do get is

$$u_n(x, 0) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

So, if our initial condition is in one of these forms, we can pick out the right n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, we can write a linear combination of our product solutions and obtain the *general solution* satisfying the given boundary conditions as

General solution.

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}. \tag{4.12}$$

The only thing to impose is the initial condition:

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, if we are given $f(x)$, can we find the constants b_n? If we can, then we will have the solution to the full initial-boundary value problem. This will be the subject of the next chapter. However, first we will look at the general form of our boundary value problem and relate what we have done to the theory of infinite dimensional vector spaces.

Before moving on to the wave equation, we should note that (4.9) is an eigenvalue problem. We can recast the differential equation as

$$LX = \lambda X,$$

where

$$L = D^2 = \frac{d^2}{dx^2}$$

is a linear differential operator. The solutions, $X_n(x)$, are called eigenfunctions and the λ_n's are the eigenvalues. We will elaborate more on this characterization later in the book.

4.5 The 1D Wave Equation

IN THIS SECTION we will apply the method of separation of variables to the one dimensional wave equation, given by

Solution of the 1D wave equation using the method of separation of variables.

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x} \tag{4.13}$$

and subject to the conditions

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \\ u(0,t) &= 0, \\ u(L,t) &= 0. \end{aligned} \tag{4.14}$$

This problem applies to the propagation of waves on a string of length L with both ends fixed so that they do not move. $u(x,t)$ represents the vertical displacement of the string over time. The derivation of the wave equation assumes that the vertical displacement is small and the string is uniform. The constant c is the wave speed, given by

$$c = \sqrt{\frac{T}{\mu}},$$

where T is the tension in the string and μ is the mass per unit length. We can understand this in terms of string instruments. The tension can be adjusted to produce different tones and the makeup of the string (nylon or steel, thick or thin) also has an effect. In some cases the mass density is changed simply by using thicker strings. Thus, the thicker strings in a piano produce lower frequency notes.

The u_{tt} term gives the acceleration of a piece of the string. The u_{xx} is the concavity of the string. Thus, for a positive concavity the string is curved upward near the point of interest. Thus, neighboring points tend to pull upward towards the equilibrium position. If the concavity is negative, it would cause a negative acceleration.

The solution of this problem is easily found using separation of variables. We let $u(x,t) = X(x)T(t)$. Then we find

$$XT'' = c^2 X'' T,$$

which can be rewritten as

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X}.$$

Again, we have separated the functions of time on one side and space on the other side. Therefore, we set each function equal to a constant.

λ :

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda}_{\text{constant}}.$$

This leads to two equations:

$$T'' = c^2 \lambda T, \tag{4.15}$$

$$X'' = \lambda X. \tag{4.16}$$

As before, we have the boundary conditions on $X(x)$:

$$X(0) = 0, \quad \text{and} \quad X(L) = 0.$$

Again, this gives us that

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

The main difference from the solution of the heat equation is the form of the time function. Namely, from Equation (4.15) we have to solve

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0. \tag{4.17}$$

This equation takes a familiar form. We let

$$\omega_n = \frac{n\pi c}{L},$$

then we have

$$T'' + \omega_n^2 T = 0.$$

The solutions are easily found as

$$T(t) = A_n \cos \omega_n t + B_n \sin \omega_n t. \tag{4.18}$$

Therefore, we have found that the product solutions of the wave equation take the forms $\sin \frac{n\pi x}{L} \cos \omega_n t$ and $\sin \frac{n\pi x}{L} \sin \omega_n t$. The general solution, a superposition of all product solutions, is given by

General solution.

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \tag{4.19}$$

This solution satisfies the wave equation and the boundary conditions. We still need to satisfy the initial conditions. Note that there are two initial conditions, since the wave equation is second order in time.

First, we have $u(x,0) = f(x)$. Thus,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{4.20}$$

In order to obtain the condition on the initial velocity, $u_t(x,0) = g(x)$, we need to differentiate the general solution with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left[-A_n \sin \frac{n\pi ct}{L} + B_n \cos \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \quad (4.21)$$

Then, we have

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L}. \quad (4.22)$$

In both cases we have that the given functions, $f(x)$ and $g(x)$, are represented as Fourier sine series. In order to complete the problem we need to determine the constants A_n and B_n for $n = 1, 2, 3, \ldots$. Once we have these, we have the complete solution to the wave equation.

We had seen similar results for the heat equation. In the next section we will find out how to determine the Fourier coefficients for such series of sinusoidal functions.

4.6 *Introduction to Fourier Series*

IN THIS CHAPTER we will look at trigonometric series. In your calculus courses you have probably seen that many functions could have series representations as expansions in powers of x, or $x - a$. This led to MacLaurin or Taylor series. When dealing with Taylor series, you often had to determine the expansion coefficients. For example, given an expansion of $f(x)$ about $x = a$, you learned that the Taylor series was given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where the expansion coefficients are determined as

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Then you found that the Taylor series converged for a certain range of x values. (We review Taylor series in the book appendix and later when we study series representations of complex valued functions.)

In a similar way, we will investigate the Fourier trigonometric series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}.$$

We will find expressions useful for determining the Fourier coefficients $\{a_n, b_n\}$ given a function $f(x)$ defined on $[-L, L]$. We will also see if the resulting infinite series reproduces $f(x)$. However, we first begin with some basic ideas involving simple sums of sinusoidal functions.

The natural appearance of such sums over sinusoidal functions is in music, or sound. A pure note can be represented as

$$y(t) = A\sin(2\pi f t),$$

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume of the sound. The larger the amplitude, the louder the sound. In Figure 4.3 we show plots of two such tones with $f = 2$ Hz in the top plot and $f = 5$ Hz in the bottom one.

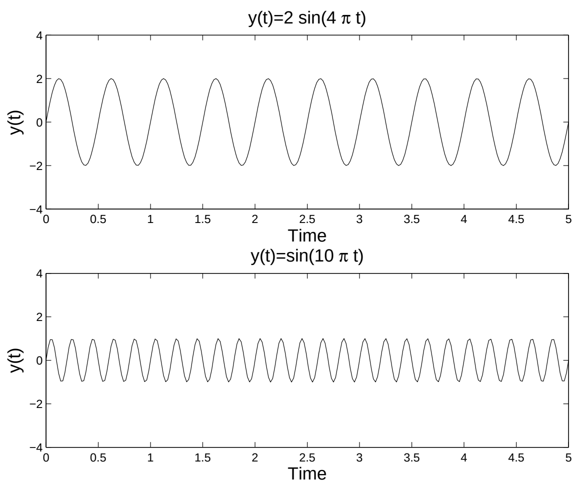

Figure 4.3: Plots of $y(t) = A\sin(2\pi f t)$ on $[0, 5]$ for $f = 2$ Hz and $f = 5$ Hz.

In these plots you should notice the difference due to the amplitudes and the frequencies. You can easily reproduce these plots and others in your favorite plotting utility.

As an aside, you should be cautious when plotting functions, or sampling data. The plots you get might not be what you expect, even for a simple sine function. In Figure 4.4 we show four plots of the function $y(t) = 2\sin(4\pi t)$. In the top left you see a proper rendering of this function. However, if you use a different number of points to plot this function, the results may be surprising. In this example we show what happens if you use $N = 200, 100, 101$ points instead of the 201 points used in the first plot. Such disparities are not only possible when plotting functions, but are also present when collecting data. Typically, when you sample a set of data, you only gather a finite amount of information at a fixed rate. This could happen when getting data on ocean wave heights, digitizing music and other audio to put on your computer, or any other process when you attempt to analyze a continuous signal.

Next, we consider what happens when we add several pure tones.

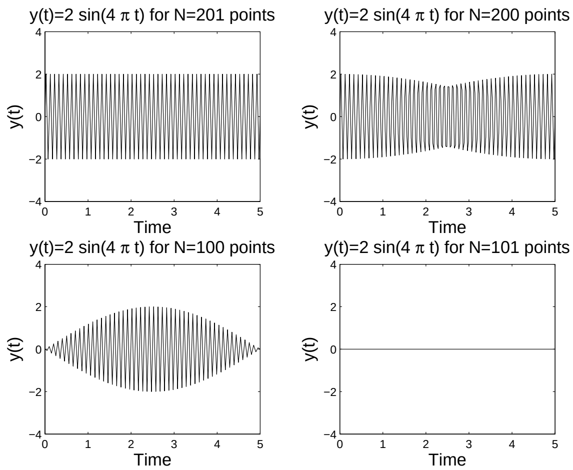

Figure 4.4: Problems can occur while plotting. Here we plot the function $y(t) = 2\sin 4\pi t$ using $N = 201, 200, 100, 101$ points.

After all, most of the sounds that we hear are in fact a combination of pure tones with different amplitudes and frequencies. In Figure 4.5 we see what happens when we add several sinusoids. Note that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. However, we still have a function of time. In this chapter we will ask, "Given a function $f(t)$, can we find a set of sinusoidal functions whose sum converges to $f(t)$?"

Looking at the superpositions in Figure 4.5, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the *period*. We can define this more precisely.

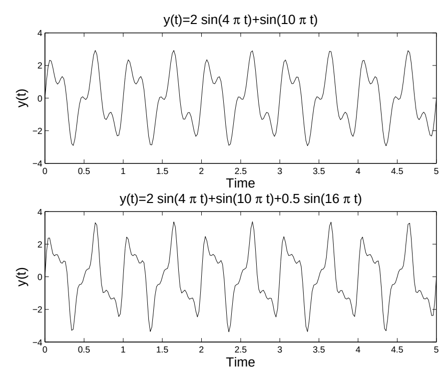

Figure 4.5: Superposition of several sinusoids. Top: Sum of signals with $f = 2$ Hz and $f = 5$ Hz. Bottom: Sum of signals with $f = 2$ Hz, $f = 5$ Hz, and and $f = 8$ Hz.

Definition 4.1. A function is said to be *periodic with period* T if $f(t + T) = f(t)$ for all t and the smallest such positive number T is called the *period*.

For example, we consider the functions used in Figure 4.5. We began with $y(t) = 2\sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 4.3 we can verify this result. (You can count the full number of cycles in the graph and divide this into the total time to get a more accurate value of the period.)

In general, if $y(t) = A\sin(2\pi f t)$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1}, or cycles per second.

Returning to Figure 4.5, the functions $y(t) = 2\sin(4\pi t)$, $y(t) = \sin(10\pi t)$, and $y(t) = 0.5\sin(16\pi t)$ have periods of 0.5s, 0.2s, and 0.125s, respectively. Each superposition in Figure 4.5 retains a period that is the least common multiple of the periods of the signals added. For both plots, this is $1.0\text{s} = 2(0.5)\text{s} = 5(.2)\text{s} = 8(.125)\text{s}$.

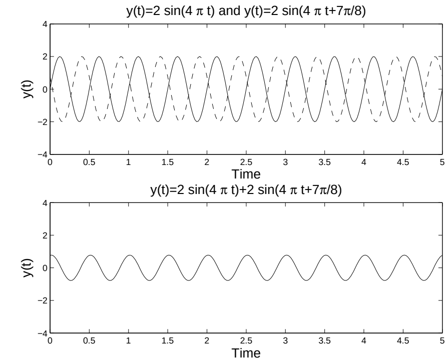

Figure 4.6: Plot of the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum.

Our goal will be to start with a function and then determine the amplitudes of the simple sinusoids needed to sum to that function. We will see that this might involve an infinite number of such terms. T hus, we will be studying an infinite series of sinusoidal functions.

Secondly, we will find that using just sine functions will not be enough either. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We will consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like "Row, Row, Row your Boat" starting at slightly different times.

We can easily add shifted sine functions. In Figure 4.6 we show the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be written as $y(t) = 2\sin(4\pi(t + 7/32))$. Thus, this corresponds to a time shift of $-7/32$.

We should note that the form in the lower plot of Figure 4.6 looks like a simple sinusoidal function for a reason. Let

$$y_1(t) = 2\sin(4\pi t),$$

$$y_2(t) = 2\sin(4\pi t + 7\pi/8).$$

Then,

$$\begin{aligned} y_1 + y_2 &= 2\sin(4\pi t + 7\pi/8) + 2\sin(4\pi t) \\ &= 2[\sin(4\pi t + 7\pi/8) + \sin(4\pi t)] \\ &= 4\cos\frac{7\pi}{16}\sin\left(4\pi t + \frac{7\pi}{16}\right). \end{aligned}$$

This can be confirmed using the identity

$$2\sin x\cos y = \sin(x+y) + \sin(x-y).$$

So, we should account for shifted sine functions in our general sum. Of course, we would then need to determine the unknown time shift as well as the amplitudes of the sinusoidal functions that make up our signal, $f(t)$. While this is one approach that some researchers use to analyze signals, there is a more common approach. This results from another reworking of the shifted function.

Consider the general shifted function

$$y(t) = A\sin(2\pi f t + \phi). \tag{4.23}$$

Note that $2\pi f t + \phi$ is called the *phase* of the sine function and ϕ is called the *phase shift*. We can use the trigonometric identity for the sine of the sum of two angles[1] to obtain

$$y(t) = A\sin(2\pi f t + \phi) = A\sin(\phi)\cos(2\pi f t) + A\cos(\phi)\sin(2\pi f t).$$

Defining $a = A\sin(\phi)$ and $b = A\cos(\phi)$, we can rewrite this as

$$y(t) = a\cos(2\pi f t) + b\sin(2\pi f t).$$

Thus, we see that the signal in Equation (4.23) is a sum of sine and cosine functions with the same frequency and different amplitudes. If we can find a and b, then we can easily determine A and ϕ:

$$A = \sqrt{a^2 + b^2}, \quad \tan\phi = \frac{b}{a}.$$

We are now in a position to state our goal in this chapter.

[1] Recall the identities (4.30)-(4.31)

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \sin y\cos x, \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y. \end{aligned}$$

Goal

Given a signal $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function. This is called *Fourier Analysis.*

4.7 *Fourier Trigonometric Series*

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]. \qquad (4.24)$$

Notice that we have opted to drop the references to the time-frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi f_n t$ when applying these ideas to applications.

The series representation in Equation (4.24) is called a *Fourier trigonometric series.* We will simply refer to this as a *Fourier series* for now. The set of constants a_0, a_n, b_n, $n = 1, 2, \ldots$ are called the *Fourier coefficients.* The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0. Our goal is to find the Fourier series representation given $f(x)$. Having found the Fourier series representation, we will be interested in determining when the Fourier series converges and to what function it converges.

From our discussion in the last section, we see that The Fourier series is periodic. The periods of $\cos nx$ and $\sin nx$ are $\frac{2\pi}{n}$. Thus, the largest period, $T = 2\pi.$, comes from the $n = 1$ terms and the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π.

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 4.7 we show a function defined on $[0, 2\pi]$. In the same figure, we show its periodic extension. These are just copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to $[0, 2\pi]$ will give a representation of the original function. Therefore, we will first consider Fourier series representations of functions defined on this interval. Note that we could just as easily considered functions defined on $[-\pi, \pi]$ or any interval of length 2π.

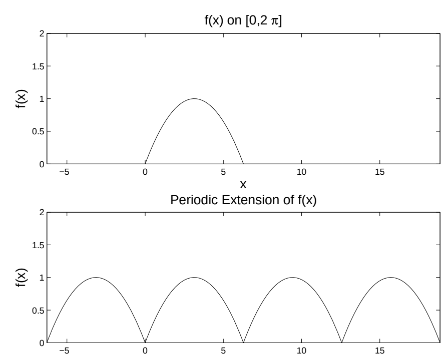

Figure 4.7: Plot of the functions $f(t)$ defined on $[0, 2\pi]$ and its periodic extension.

Fourier Coefficients

Theorem 4.1. *The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$, when it exists, is given by (4.24) with Fourier coefficients*

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{4.25}$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will look at the derivations of the a_n's. First we obtain a_0.

We begin by integrating the Fourier series term by term in Equation (4.24).

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \int_0^{2\pi}\sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right]\,dx. \tag{4.26}$$

We assume that we can integrate the infinite sum term by term. Then we need to compute

$$\begin{aligned} \int_0^{2\pi}\frac{a_0}{2}\,dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi}\cos nx\,dx &= \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0, \\ \int_0^{2\pi}\sin nx\,dx &= \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0. \end{aligned} \tag{4.27}$$

From these results we see that only one term in the integrated sum does not vanish leaving

$$\int_0^{2\pi} f(x)\,dx = \pi a_0.$$

This confirms the value for a_0.[2]

[2] Note that $\frac{a_0}{2}$ is the average of $f(x)$ over the interval $[0, 2\pi]$. Recall from the first semester of calculus, that the average of a function defined on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For $f(x)$ defined on $[0, 2\pi]$, we have

$$f_{\text{ave}} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}.$$

Next, we need to find a_n. We will multiply the Fourier series (4.24) by $\cos mx$ for some positive integer m. This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions for different integers m all at the same time. We will see that this will allow us to solve for the a_n's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\begin{aligned} \int_0^{2\pi} f(x)\cos mx\,dx &= \int_0^{2\pi}\frac{a_0}{2}\cos mx\,dx \\ &+ \int_0^{2\pi}\sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right]\cos mx\,dx. \end{aligned} \tag{4.28}$$

Integrating term by term, the right side becomes

$$\int_0^{2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos mx\,dx$$
$$+\sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos mx\,dx + b_n\int_0^{2\pi}\sin nx\cos mx\,dx\right]. \tag{4.29}$$

We have already established that $\int_0^{2\pi}\cos mx\,dx = 0$, which implies that the first term vanishes.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some trigonometric identities. While you have seen such integrals before in your calculus class, we will review how to carry out such integrals. For future reference, we list several useful identities, some of which we will prove along the way.

Useful Trigonometric Identities

$$\sin(x \pm y) = \sin x\cos y \pm \sin y\cos x \tag{4.30}$$
$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y \tag{4.31}$$
$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \tag{4.32}$$
$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \tag{4.33}$$
$$\sin x\sin y = \frac{1}{2}(\cos(x-y)-\cos(x+y)) \tag{4.34}$$
$$\cos x\cos y = \frac{1}{2}(\cos(x+y)+\cos(x-y)) \tag{4.35}$$
$$\sin x\cos y = \frac{1}{2}(\sin(x+y)+\sin(x-y)) \tag{4.36}$$

We first want to evaluate $\int_0^{2\pi}\cos nx\cos mx\,dx$. We do this by using the product identity (4.35). In case you forgot how to derive this identity, we will first review the proof. Recall the addition formulae for cosines:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B,$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Adding these equations gives

$$2\cos A\cos B = \cos(A+B)+\cos(A-B).$$

We can use this identity with $A = mx$ and $B = nx$ to complete the integration. We have

$$\int_0^{2\pi}\cos nx\cos mx\,dx = \frac{1}{2}\int_0^{2\pi}[\cos(m+n)x+\cos(m-n)x]\,dx$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} \\
&= 0. \qquad (4.37)
\end{aligned}
$$

There is one caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For our problem $m+n \neq 0$, since both m and n are positive integers. However, it is possible for $m=n$. This means that the vanishing of the integral can only happen when $m \neq n$. So, what can we do about the $m=n$ case? One way is to start from scratch with our integration. (Another way is to compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

For $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx\,dx$. This can also be handled using a trigonometric identity. Recall identity (4.33):

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta).$$

Letting $\theta = mx$ and inserting the identity into the integral, we find

$$
\begin{aligned}
\int_0^{2\pi} \cos^2 mx\,dx &= \frac{1}{2}\int_0^{2\pi}(1+\cos 2mx)\,dx \\
&= \frac{1}{2}\left[x+\frac{1}{2m}\sin 2mx\right]_0^{2\pi} \\
&= \frac{1}{2}(2\pi) = \pi. \qquad (4.38)
\end{aligned}
$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases} \qquad (4.39)$$

This holds true for $m, n = 0, 1, \ldots$. [Why did we include $m, n = 0$?] When we have such a set of functions, they are said to be an orthogonal set over the integration interval.

Definition 4.2. [3] A set of (real) functions $\{\phi_n(x)\}$ is said to be *orthogonal* on $[a,b]$ if $\int_a^b \phi_n(x)\phi_m(x)\,dx = 0$ when $n \neq m$. Furthermore, if we also have that $\int_a^b \phi_n^2(x)\,dx = 1$, these functions are called *orthonormal.*

[3] Definition of an orthogonal set of functions and orthonormal functions.

The set of functions $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by $\sqrt{\pi}$ as indicated by Equation (4.38). This is sometimes referred to normalization of the set of functions.

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)f(x)\,dx$ is the generalization of the dot product, and is called the scalar product of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. But that is another topic for later.

Returning to the evaluation of the integrals in equation (4.29), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx\, dx$. This can also be evaluated using trigonometric identities. In this case, we need an identity involving products of sines and cosines, (4.36). Such products occur in the addition formulae for sine functions, using (4.30):

$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$

$$\sin(A-B) = \sin A \cos B - \sin B \cos A.$$

Adding these equations, we find that

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$

Setting $A = nx$ and $B = mx$, we find that

$$\begin{aligned} \int_0^{2\pi} \sin nx \cos mx\, dx &= \frac{1}{2} \int_0^{2\pi} [\sin(n+m)x + \sin(n-m)x]\, dx \\ &= \frac{1}{2} \left[\frac{-\cos(n+m)x}{n+m} + \frac{-\cos(n-m)x}{n-m} \right]_0^{2\pi} \\ &= (-1+1) + (-1+1) = 0. \end{aligned} \tag{4.40}$$

So,

$$\boxed{\int_0^{2\pi} \sin nx \cos mx\, dx = 0.} \tag{4.41}$$

For these integrals we also should be careful about setting $n = m$. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx\, dx = \frac{1}{2} \int_0^{2\pi} \sin 2mx\, dx = \frac{1}{2} \left[\frac{-\cos 2mx}{2m} \right]_0^{2\pi} = 0.$$

Finally, we can finish our evaluation of (4.29). We have determined that all but one integral vanishes. In that case, $n = m$. This leaves us with

$$\int_0^{2\pi} f(x) \cos mx\, dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx\, dx.$$

Since this is true for all $m = 1, 2, \ldots$, we have proven this part of the theorem. The only part left is finding the b_n's This will be left as an exercise for the reader.

We now consider examples of finding Fourier coefficients for given functions. In all of these cases we define $f(x)$ on $[0, 2\pi]$.

Example 4.1. $f(x) = 3\cos 2x$, $x \in [0, 2\pi]$.

We first compute the integrals for the Fourier coefficients.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x\, dx = 0. \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \cos nx\, dx = 0, \quad n \neq 2. \\ a_2 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos^2 2x\, dx = 3, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} 3\cos 2x \sin nx\, dx = 0, \forall n. \end{aligned} \tag{4.42}$$

The integrals for a_0, $a_n, n \neq 2$, and b_n are the result of orthogonality. For a_2, the integral evaluation can be done as follows:

$$\begin{aligned} a_2 &= \frac{1}{\pi}\int_0^{2\pi} 3\cos^2 2x\, dx \\ &= \frac{3}{2\pi}\int_0^{2\pi} \left[1 + \cos 4x\right]\, dx \\ &= \frac{3}{2\pi}\left[x + \underbrace{\frac{1}{4}\sin 4x}_{\text{This term vanishes!}}\right]_0^{2\pi} = 3. \end{aligned} \tag{4.43}$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3\cos 2x$. Well, we should have known this before doing all of these integrals. So, if we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work.

Example 4.2. $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

We could determine the Fourier coefficients by integrating as in the last example. However, it is easier to use trigonometric identities. We know that

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \ldots$. There is a constant term, implying $a_0/2 = 1/2$. So, $a_0 = 1$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$. So, $a_0 = 1$, $a_2 = -\frac{1}{2}$, and all other Fourier coefficients vanish.

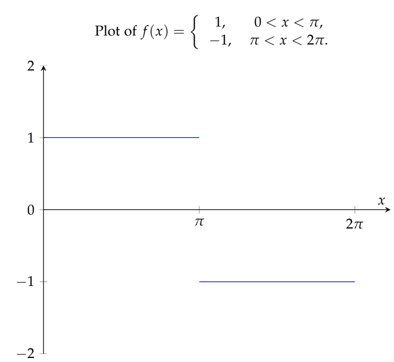

Figure 4.8: Plot of discontinuous function in Example 4.3.

Example 4.3. $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$.

This example will take a little more work. We cannot bypass evaluating any integrals at this time. This function is discontinuous, so we will have to compute each integral by breaking up the integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)\,dx \\ &= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi+\pi) = 0. \end{aligned} \tag{4.44}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx - \int_{\pi}^{2\pi}\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\ &= 0. \end{aligned} \tag{4.45}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx - \int_{\pi}^{2\pi}\sin nx\,dx\right] \\ &= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\ &= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\ &= \frac{2}{n\pi}(1-\cos n\pi). \end{aligned} \tag{4.46}$$

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (4.24), we note that $\cos n\pi = (-1)^n$. *Therefore,*

Often we see expressions involving $\cos n\pi = (-1)^n$ and $1 \pm \cos n\pi = 1 \pm (-1)^n$. This is an example showing how to re-index series containing such a factor.

$$1 - \cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd}. \end{cases} \tag{4.47}$$

So, half of the b_n*'s are zero. While we could write the Fourier series representation as*

$$f(x) \sim \frac{4}{\pi}\sum_{n=1,\ odd}^{\infty} \frac{1}{n}\sin nx,$$

we could let $n = 2k-1$ *and write*

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1},$$

But does this series converge? Does it converge to $f(x)$*? We will discuss this question later in the chapter.*

4.8 Fourier Series Over Other Intervals

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$, or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such intervals arise often in applications. For example, one can study vibrations of a one dimensional string of length L and set up the axes with the left end at $x = 0$ and the right end at $x = L$. Another problem would be to study the temperature distribution along a one dimensional rod of length L. Such problems lead to the original studies of Fourier series. As we will see later, symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling our interval. Then we could apply this transformation to the Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

We define $x \in [0, 2\pi]$ and $t \in [0, L]$. A linear transformation relating these intervals is simply $x = \frac{2\pi t}{L}$ as shown in Figure 5.12. So, $t = 0$ maps to $x = 0$ and $t = L$ maps to $x = 2\pi$. Furthermore, this transformation maps $f(x)$ to a new function $g(t) = f(x(t))$, which is defined on $[0, L]$. We will determine the Fourier series representation of this function using the representation for $f(x)$.

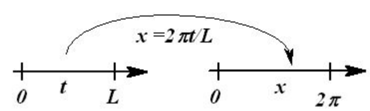

Figure 4.9: A sketch of the transformation between intervals $x \in [0, 2\pi]$ and $t \in [0, L]$.

Recall the form of the Fourier representation for $f(x)$ in Equation (4.24):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]. \tag{4.48}$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L}\right]. \tag{4.49}$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to determine the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

We need to make a substitution in the integral of $x = \frac{2\pi t}{L}$. We also will need to transform the differential, $dx = \frac{2\pi}{L} dt$. Thus, the resulting form for the Fourier coefficients is

$$a_n = \frac{2}{L}\int_0^L g(t)\cos\frac{2n\pi t}{L}\,dt. \tag{4.50}$$

Similarly, we find that

$$b_n = \frac{2}{L}\int_0^L g(t)\sin\frac{2n\pi t}{L}\,dt. \tag{4.51}$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos\frac{2n\pi t}{L}$ is L/n, which means that the representation for $g(t)$ has a period of L.

At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader. In Table 4.2 we summarize some commonly used Fourier series representations.

We will end our discussion for now with some special cases and an example for a function defined on $[-\pi, \pi]$.

At this point we need to remind the reader about the integration of even and odd functions.

1. **Even Functions:** In this evaluation we made use of the fact that the integrand is an even function. Recall that $f(x)$ is an *even function* if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis as shown in Figure 4.10(A). If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx. \tag{4.58}$$

   One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \\ &= -\int_a^0 f(-y)\,dy + \int_0^a f(x)\,dx \\ &= \int_0^a f(y)\,dy + \int_0^a f(x)\,dx \\ &= 2\int_0^a f(x)\,dx. \end{aligned} \tag{4.59}$$

   This can be visually verified by looking at Figure 4.10(A).

2. **Odd Functions:** A similar computation could be done for odd functions. $f(x)$ is an *odd function* if $f(-x) = -f(x)$ for all x. The graphs of such functions are symmetric with respect

Table 4.2: Special Fourier Series Representations on Different Intervals

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right]. \tag{4.52}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{4.53}$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right]. \tag{4.54}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{4.55}$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]. \tag{4.56}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{4.57}$$

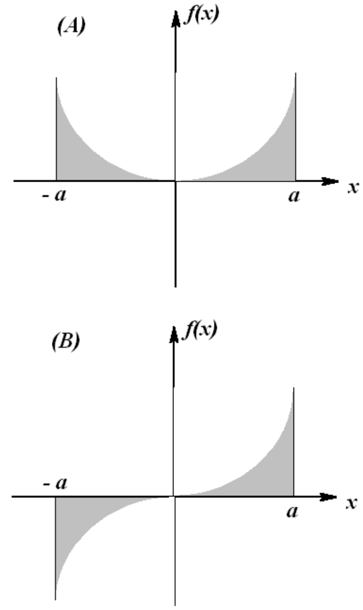

Figure 4.10: Examples of the areas under (A) even and (B) odd functions on symmetric intervals, $[-a, a]$.

to the origin as shown in Figure 4.10(B). If one integrates an odd function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 0. \tag{4.60}$$

Example 4.4. *Let $f(x) = |x|$ on $[-\pi, \pi]$ We compute the coefficients, beginning as usual with a_0. We have, using the fact that $|x|$ is an even function,*

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \pi \end{aligned} \tag{4.61}$$

We continue with the computation of the general Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx. \tag{4.62}$$

Here we have made use of the fact that $|x|\cos nx$ is an even function. In order to compute the resulting integral, we need to use integration by parts ,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du,$$

by letting $u = x$ and $dv = \cos nx\,dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n}\sin nx$. Continuing with the computation, we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx. \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right] \\ &= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} \\ &= -\frac{2}{\pi n^2}(1-(-1)^n). \end{aligned} \tag{4.63}$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n. This leads to a factor $(1-(-1)^n)$. This factor can be simplified as

$$1-(-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \tag{4.64}$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n's is simpler. We note that we have to integrate $|x|\sin nx$ from $x = -\pi$ to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n.

Putting this all together, the Fourier series representation of $f(x) = |x|$ on $[-\pi, \pi]$ is given as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,\ odd}^{\infty}\frac{\cos nx}{n^2}. \tag{4.65}$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let $n = 2k - 1$ for $k = 1, 2, 3, \ldots$. Then we only get the odd integers. The series can then be written as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}. \tag{4.66}$$

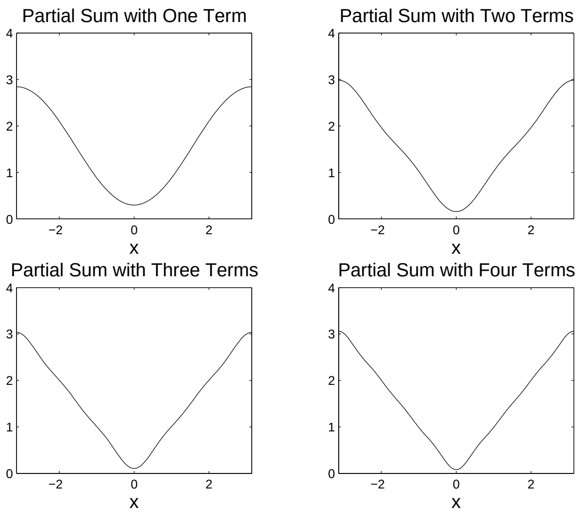

Figure 4.11: Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. Here is an example of an infinite series of functions. What does this series sum to? We show in Figure 4.11 the first few partial sums. They appear to be converging to $f(x) = |x|$ fairly quickly.

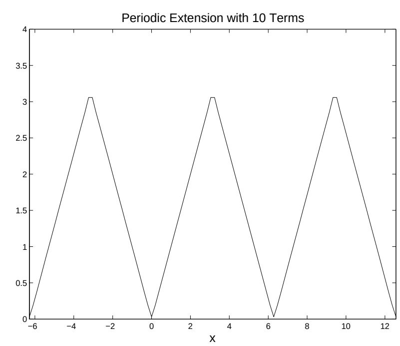

Figure 4.12: Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

Even though $f(x)$ was defined on $[-\pi, \pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 4.12, we see that the representation agrees with $f(x)$ on the interval $[-\pi, \pi]$. Outside this interval we have a periodic extension of $f(x)$ with period 2π.

Another example is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$ as left for Problem 7. This is determined to be

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{4.67}$$

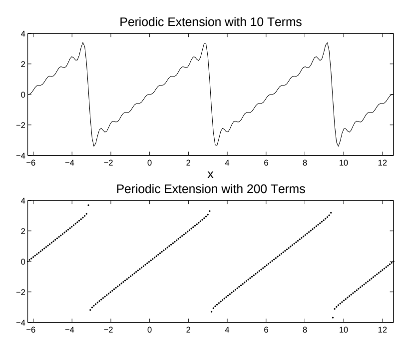

Figure 4.13: Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

As seen in Figure 4.13 we again obtain the periodic extension of our function. In this case we needed many more terms. Also, the vertical parts of the first plot are nonexistent. In the second plot we only plot the points and not the typical connected points that most software packages plot as the default style.

Example 4.5. *It is interesting to note that one can use Fourier series to obtain sums of some infinite series. For example, in the last example we found that*

$$x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Now, what if we chose $x = \frac{\pi}{2}$? Then, we have

$$\frac{\pi}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} = 2 \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \right].$$

This gives a well known expression for π:

$$\pi = 4 \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \right].$$

4.8.1 Fourier Series on $[a, b]$

This section can be skipped on first reading. It is here for completeness and the end result, Theorem 4.2 provides the result of the section.

A Fourier series representation is also possible for a general interval, $t \in [a, b]$. As before, we just need to transform this interval to $[0, 2\pi]$. Let

$$x = 2\pi \frac{t-a}{b-a}.$$

Inserting this into the Fourier series (4.24) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right]. \tag{4.68}$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, mathematics students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned}
g(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\cos \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} + \sin \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right. \\
&\quad \left. + b_n \left(\sin \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} - \cos \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\cos \frac{2n\pi t}{b-a} \left(a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \right) \right. \\
&\quad \left. + \sin \frac{2n\pi t}{b-a} \left(a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a} \right) \right].
\end{aligned} \tag{4.69}$$

Defining $A_0 = a_0$ and

$$\begin{aligned} A_n &\equiv a_n \cos\frac{2n\pi a}{b-a} - b_n \sin\frac{2n\pi a}{b-a} \\ B_n &\equiv a_n \sin\frac{2n\pi a}{b-a} + b_n \cos\frac{2n\pi a}{b-a}, \end{aligned} \tag{4.70}$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a, b]$.

$$g(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\frac{2n\pi t}{b-a} + B_n \sin\frac{2n\pi t}{b-a} \right]. \tag{4.71}$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi\frac{t-a}{b-a}$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{2}{b-a}\int_a^b g(t)\cos\frac{2n\pi(t-a)}{b-a}\,dt, \end{aligned} \tag{4.72}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{2}{b-a}\int_a^b g(t)\sin\frac{2n\pi(t-a)}{b-a}\,dt. \end{aligned} \tag{4.73}$$

Then, inserting these integrals in A_n, combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$\begin{aligned} A_n &\equiv a_n \cos\frac{2n\pi a}{b-a} - b_n \sin\frac{2n\pi a}{b-a} \\ &= \frac{2}{b-a}\int_a^b g(t)\left[\cos\frac{2n\pi(t-a)}{b-a}\cos\frac{2n\pi a}{b-a} - \sin\frac{2n\pi(t-a)}{b-a}\sin\frac{2n\pi a}{b-a}\right] dt \\ &= \frac{2}{b-a}\int_a^b g(t)\cos\frac{2n\pi t}{b-a}\,dt. \end{aligned} \tag{4.74}$$

A similar computation gives

$$B_n = \frac{2}{b-a}\int_a^b g(t)\sin\frac{2n\pi t}{b-a}\,dt. \tag{4.75}$$

Summarizing, we have shown that:

Theorem 4.2. *The Fourier series representation of $f(x)$ defined on $[a,b]$ when it exists, is given by*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right]. \tag{4.76}$$

with Fourier coefficients

$$\begin{aligned} a_n &= \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx. \quad n=0,1,2,\ldots, \\ b_n &= \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx. \quad n=1,2,\ldots. \end{aligned} \tag{4.77}$$

4.9 *Sine and Cosine Series*

In the last two examples ($f(x)=|x|$ and $f(x)=x$ on $[-\pi,\pi]$) we have seen Fourier series representations that contain only sine or cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called sine series and those involving just cosines (and the constant term) are called cosine series.

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0,\pi]$. Note from Figures 4.11-4.13 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.
2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0,L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

In Figure 4.14 we show a function defined on $[0,1]$. To the right is its periodic extension to the whole real axis. This representation has a period of $L=1$. The bottom left plot is obtained by first reflecting

f about the y-axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

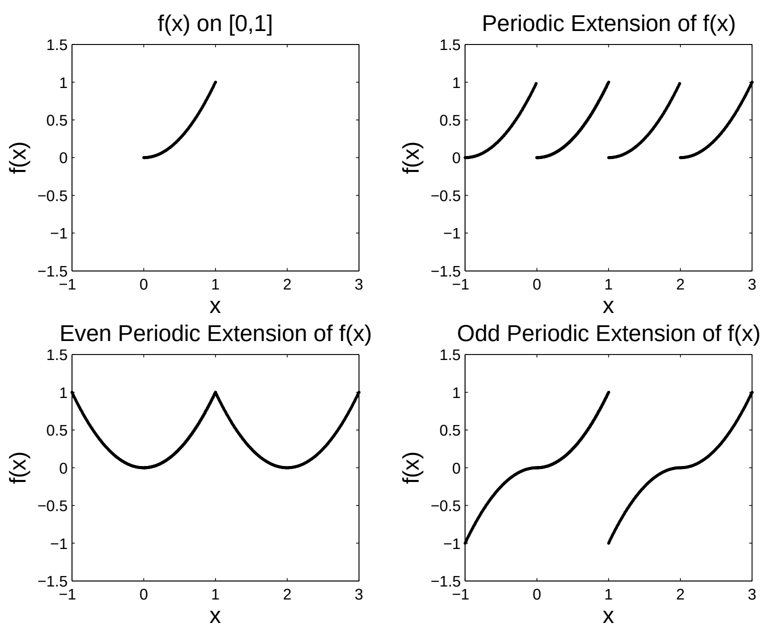

Figure 4.14: This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0,1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the periodic, even and odd extensions. Now, starting with $f(x)$ defined on $[0, L]$, we would like to determine the Fourier series representations leading to these extensions. [For easy reference, the results are summarized in Table 4.3]

We have already seen from Table (4.2) that the periodic extension of $f(x)$, defined on $[0, L]$, is obtained through the Fourier series representation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right], \tag{4.84}$$

where

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{4.85}$$

Given $f(x)$ defined on $[0, L]$, the *even periodic extension* is obtained by simply computing the Fourier series representation for the even

Table 4.3: Fourier Cosine and Sine Series Representations on $[0, L]$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{4.78}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{4.79}$$

Fourier Cosine Series on $[0, L]$

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{4.80}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{4.81}$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{4.82}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{4.83}$$

function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ f(-x) & -L < x < 0. \end{cases} \tag{4.86}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L, L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (4.76) with $a = -L$ and $b = L$.]:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{4.87}$$

with Fourier coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f_e(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{4.88}$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the *Cosine Series Representation* of $f(x)$ for $x \in [0, L]$ is given as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{4.89}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{4.90}$$

Similarly, given $f(x)$ defined on $[0, L]$, the *odd periodic extension* is obtained by simply computing the Fourier series representation for the odd function

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \tag{4.91}$$

The resulting series expansion leads to defining the *Sine Series Representation* of $f(x)$ for $x \in [0, L]$ as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{4.92}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{4.93}$$

Example 4.6. *In Figure 4.14 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.*

For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is shown in Table 4.4.

Example 4.7. Periodic Extension - Trigonometric Fourier Series *Using the code in Table 4.4, we have that* $a_0 = \frac{2}{3}$ $a_n = \frac{1}{n^2\pi^2}$ *and* $b_n = -\frac{1}{n\pi}$. *Thus, the resulting series is given as*

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}\cos 2n\pi x - \frac{1}{n\pi}\sin 2n\pi x\right].$$

In Figure 4.15 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f. *There appear to be some problems with the convergence around integer values of* x. *We will later see that this is because of the discontinuities in the periodic extension and the resulting overshoot is referred to as the Gibbs phenomenon which is discussed in the appendix to this chapter.*

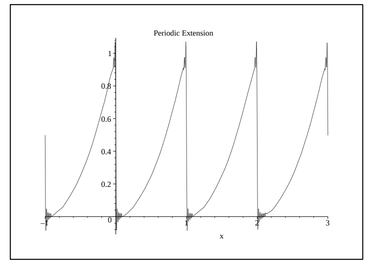

Figure 4.15: The periodic extension of $f(x) = x^2$ on $[0, 1]$.

Example 4.8. Even Periodic Extension - Cosine Series

In this case we compute $a_0 = \frac{2}{3}$ *and* $a_n = \frac{4(-1)^n}{n^2\pi^2}$. *Therefore, we have*

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

In Figure 4.16 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

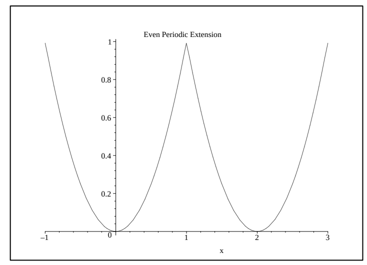

Figure 4.16: The even periodic extension of $f(x) = x^2$ on $[0, 1]$.

Table 4.4: Maple code for computing Fourier coefficients and plotting partial sums of the Fourier series.

```
> restart:
> L:=1:
> f:=x^2:
> assume(n,integer):
> a0:=2/L*int(f,x=0..L);
                            a0 := 2/3
> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);
                                   1
                          an := -------
                                  2   2
                                n~  Pi
> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);
                                   1
                          bn := - -----
                                  n~ Pi
> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
     -1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title=`Periodic Extension`,
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);
```

Example 4.9. Odd Periodic Extension - Sine Series

Finally, we look at the sine series for this function. We find that $b_n = -\frac{2}{n^3\pi^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)$. *Therefore,*

$$f(x) \sim -\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)\sin n\pi x.$$

Once again we see discontinuities in the extension as seen in Figure 4.17. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 4.14.

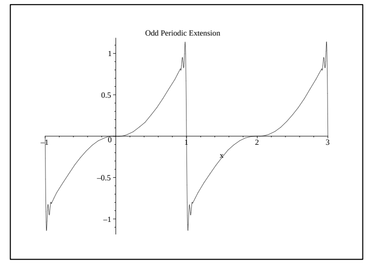

Figure 4.17: The odd periodic extension of $f(x) = x^2$ on $[0,1]$.

4.10 Solution of the Heat Equation

We started out the chapter seeking the solution of an initial-boundary value problem involving the heat equation and the wave equation. In particular, we found the general solution for the problem of heat flow in a one dimensional rod of length L with fixed zero temperature ends. The problem was given by

$$\begin{array}{ccc} \text{PDE} & u_t = ku_{xx} & 0 < t, \quad 0 \leq x \leq L \\ \text{IC} & u(x,0) = f(x) & 0 < x < L \\ \text{BC} & u(0,t) = 0 & t > 0 \\ & u(L,t) = 0 & t > 0. \end{array} \tag{4.94}$$

We found the solution using separation of variables. This resulted in a sum over various product solutions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t}\sin\frac{n\pi x}{L},$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

This equation satisfies the boundary conditions. However, we had only gotten to state initial condition using this solution. Namely,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

We were left with having to determine the constants b_n. Once we know them, we have the solution.

Now we can get the Fourier coefficients when we are given the initial condition, $f(x)$. They are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

We consider a couple of examples with different initial conditions.

Example 1 $f(x) = \sin x$ for $L = \pi$.

In this case the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin nx,$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

However, the initial condition takes the form of the first term in the expansion; i.e., the $n = 1$ term. So, we need not carry out the integral because we can immediately write $b_1 = 1$ and $b_n = 0$, $n = 2, 3, \ldots$. Therefore, the solution consists of just one term,

$$u(x,t) = e^{-kt} \sin x.$$

In Figure 4.18 we see that how this solution behaves for $k = 1$ and $t \in [0, 1]$.

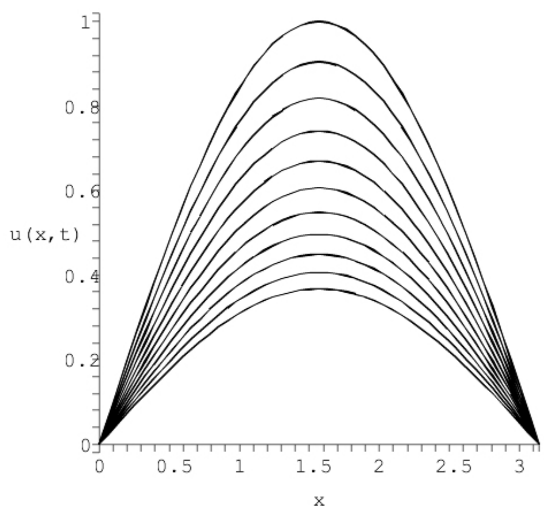

Figure 4.18: The evolution of the initial condition $f(x) = \sin x$ for $L = \pi$ and $k = 1$.

Example 2 $f(x) = x(1-x)$ for $L = 1$.

This example requires a bit more work. The solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt} \sin n\pi x,$$

where

$$b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx.$$

This integral is easily computed using integration by parts

$$\begin{aligned} b_n &= 2 \int_0^1 x(1-x) \sin n\pi x \, dx \\ &= \left[2x(1-x) \left(-\frac{1}{n\pi} \cos n\pi x \right) \right]_0^1 + \frac{2}{n\pi} \int_0^1 (1-2x) \cos n\pi x \, dx \end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{n^2\pi^2}\left\{[(1-2x)\sin n\pi x]_0^1 + 2\int_0^1 \sin n\pi x\, dx\right\} \\
&= \frac{4}{n^3\pi^3}\left[\cos n\pi x\right]_0^1 \\
&= \frac{4}{n^3\pi^3}\left(\cos n\pi - 1\right) \\
&= \begin{cases} 0, & n \text{ even} \\ -\frac{8}{n^3\pi^3}, & n \text{ odd} \end{cases}.
\end{aligned} \tag{4.95}
$$

So, we have that the solution can be written as

$$u(x,t) = \frac{8}{\pi^3}\sum_{\ell=1}^{\infty}\frac{1}{(2\ell-1)^3}e^{-(2\ell-1)^2\pi^2 kt}\sin(2\ell-1)\pi x.$$

In Figure 4.18 we see that how this solution behaves for $k = 1$ and $t \in [0,1]$. Twenty terms were used. We see that this solution diffuses much faster that the last example. Most of the terms damp out quickly as the solution asymptotically approaches the first term.

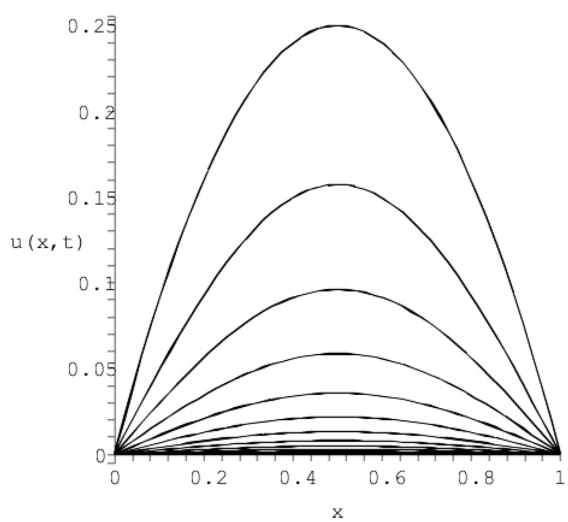

Figure 4.19: The evolution of the initial condition $f(x) = x(1-x)$ for $L = 1$ and $k = 1$.

4.11 *Finite Length Strings*

We now return to the physical example of wave propagation in a string. We have found that the general solution can be represented as a sum over product solutions. We will restrict our discussion to the special case that the initial velocity is zero and the original profile is given by $u(x,0) = f(x)$. The solution is then

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \tag{4.96}$$

satisfying

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{4.97}$$

We have learned that the Fourier sine series coefficients are given by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{4.98}$$

Note that we are using A_n's only because of the development of the solution.

We can rewrite this solution in a more compact form. First, we define the wave numbers,

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots,$$

and the angular frequencies,

$$\omega_n = ck_n = \frac{n\pi c}{L}.$$

Then the product solutions take the form

$$\sin k_n x \cos \omega_n t.$$

Using trigonometric identities, these products can be written as

$$\sin k_n x \cos \omega_n t = \frac{1}{2} \left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t) \right].$$

Inserting this expression in our solution, we have

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} A_n \left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t) \right]. \tag{4.99}$$

Since $\omega_n = ck_n$, we can put this into a more suggestive form:

$$u(x,t) = \frac{1}{2} \left[\sum_{n=1}^{\infty} A_n \sin k_n(x + ct) + \sum_{n=1}^{\infty} A_n \sin k_n(x - ct) \right]. \tag{4.100}$$

We see that each sum is simply the sine series for $f(x)$ but evaluated at either $x + ct$ or $x - ct$. Thus, the solution takes the form

$$u(x,t) = \frac{1}{2} \left[f(x + ct) + f(x - ct) \right]. \tag{4.101}$$

If $t = 0$, then we have $u(x,0) = \frac{1}{2} \left[f(x) + f(x) \right] = f(x)$. So, the solution satisfies the initial condition. At $t = 1$, the sum has a term $f(x - c)$. Recall from your mathematics classes that this is simply a shifted version of $f(x)$. Namely, it is shifted to the right. For general times, the function is shifted by ct to the right. For larger values of t, this shift is further to the right. The function (wave) shifts to the right with velocity c. Similarly, $f(x+ct)$ is a wave traveling to the left with velocity $-c$.

Thus, the waves on the string consist of waves traveling to the right and to the left. However, the story does not stop here. We have a problem when needing to shift $f(x)$ across the boundaries. The original problem only defines $f(x)$ on $[0, L]$. If we are not careful, we would think that the function leaves the interval leaving nothing left inside. However, we have to recall that our sine series representation for $f(x)$ has a period of $2L$. So, before we apply this shifting, we need to account for its periodicity. In fact, being a sine series, we really have the odd periodic of $f(x)$ being shifted. The details of such analysis would take us too far from our current goal. However, we can illustrate this with a few figures.

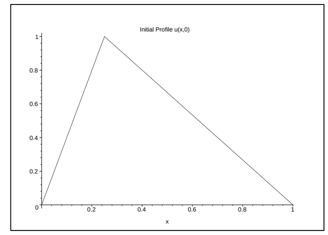

Figure 4.20: The initial profile for a string of length one plucked at $x = 0.25$.

We begin by plucking a string of length L. This can be represented by the function

$$f(x) = \begin{cases} \frac{x}{a} & 0 \leq x \leq a \\ \frac{L-x}{L-a} & a \leq x \leq L \end{cases} \tag{4.102}$$

where the string is pulled up one unit at $x = a$. This is shown in Figure 4.20.

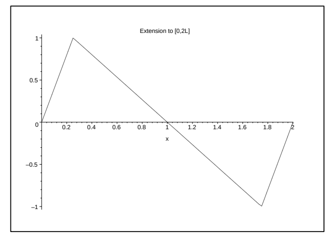

Figure 4.21: Odd extension about the right end of a plucked string.

Next, we create an odd function by extending the function to a period of $2L$. This is shown in Figure 4.21.

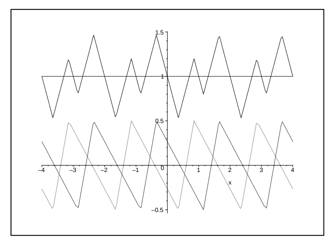

Figure 4.22: Summing the odd periodic extensions. The lower plot shows copies of the periodic extension, one moving to the right and the other moving to the left. The upper plot is the sum.

Finally, we construct the periodic extension of this to the entire line. In Figure 4.22 we show in the lower part of the figure copies of the periodic extension, one moving to the right and the other moving to the left. (Actually, the copies are $\frac{1}{2}f(x \pm ct)$.) The top plot is the sum of these solutions. The physical string lies in the interval [0,1].

The time evolution for this plucked string is shown for several times in Figure 4.23. This results in a wave that appears to reflect from the ends as time increases.

The relation between the angular frequency and the wave number, $\omega = ck$, is called a dispersion relation. In this case ω depends on k linearly. If one knows the dispersion relation, then one can find the wave speed as $c = \frac{\omega}{k}$. In this case, all of the harmonics travel at the same speed. In cases where they do not, we have nonlinear dispersion, which we will discuss later.

4.12 *Appendix: The Gibbs Phenomenon*

We have seen the Gibbs phenomenon when there is a jump discontinuity in the periodic extension of a function, whether the function originally had a discontinuity or developed one due to a mismatch in the values of the endpoints. This can be seen in Figures 4.13, 4.15 and 4.17. The Fourier series has a difficult time converging at the

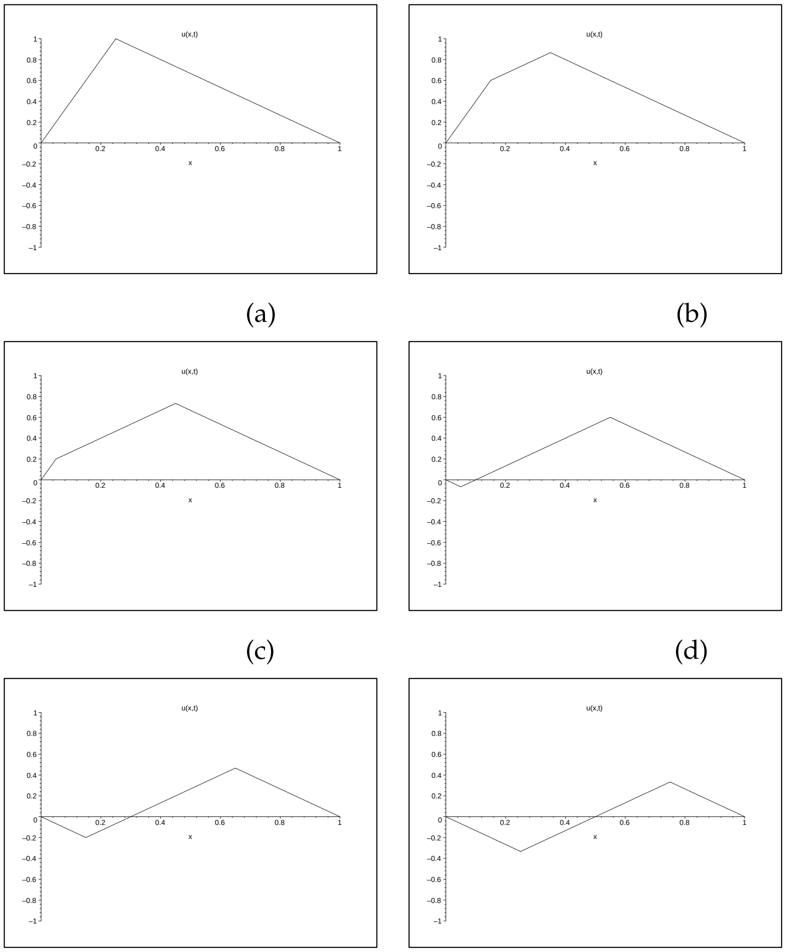

Figure 4.23: This Figure shows the plucked string at six successive times from (a) to (f).

point of discontinuity and these graphs of the Fourier series show a distinct overshoot which does not go away. This is called the Gibbs phenomenon[4] and the amount of overshoot can be computed.

[4] The Gibbs phenomenon was named after Josiah Willard Gibbs (1839-1903) even though it was discovered earlier by the Englishman Henry Wilbraham (1825-1883). Wilbraham published a soon forgotten paper about the effect in 1848. In 1889 Albert Abraham Michelson (1852-1931), an American physicist,observed an overshoot in his mechanical graphing machine. Shortly afterwards J. Willard Gibbs published papers describing this phenomenon, which was later to be called the Gibbs phenomena. Gibbs was a mathematical physicist and chemist and is considered the father of physical chemistry.

In one of our first examples, Example 4.3, we found the Fourier series representation of the piecewise defined function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

to be

$$f(x) \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

In Figure 4.24 we display the sum of the first ten terms. Note the wiggles, overshoots and under shoots. These are seen more when we plot the representation for $x \in [-3\pi, 3\pi]$, as shown in Figure 4.25.

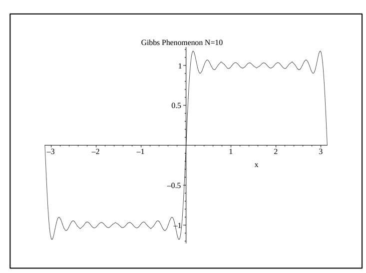

Figure 4.24: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$.

We note that the overshoots and undershoots occur at discontinuities in the periodic extension of $f(x)$. These occur whenever $f(x)$ has a discontinuity or if the values of $f(x)$ at the endpoints of the domain do not agree.

One might expect that we only need to add more terms. In Figure 4.26 we show the sum for twenty terms. Note the sum appears to converge better for points far from the discontinuities. But, the overshoots and undershoots are still present. In Figures 4.27 and 4.28 show magnified plots of the overshoot at $x = 0$ for $N = 100$ and $N = 500$, respectively. We see that the overshoot persists. The peak is at about the same height, but its location seems to be getting closer to the origin. We will show how one can estimate the size of the overshoot.

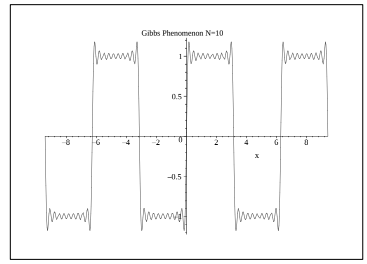

Figure 4.25: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$ plotted on $[-3\pi, 3\pi]$ displaying the periodicity.

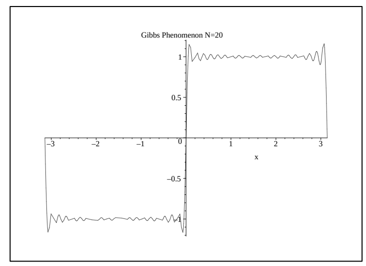

Figure 4.26: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 20$.

We can study the Gibbs phenomenon by looking at the partial sums of general Fourier trigonometric series for functions $f(x)$ defined on the interval $[-L, L]$. Some of this is done discussed in the options section **??** where we talk more about the convergence.

Writing out the partial sums, inserting the Fourier coefficients and rearranging, we have

$$\begin{aligned}
S_N(x) &= a_0 + \sum_{n=1}^{N}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(y)\,dy + \sum_{n=1}^{N}\left[\left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\frac{n\pi y}{L}\,dy\right)\cos\frac{n\pi x}{L}\right. \\
&\quad \left. + \left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\frac{n\pi y}{L}\,dy.\right)\sin\frac{n\pi x}{L}\right] \\
&= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\left(\cos\frac{n\pi y}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi y}{L}\sin\frac{n\pi x}{L}\right)\right\} f(y)\,dy \\
&= \frac{1}{L}\int_{-L}^{L}\left\{\frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi(y-x)}{L}\right\} f(y)\,dy \\
&\equiv \frac{1}{L}\int_{-L}^{L} D_N(y-x) f(y)\,dy
\end{aligned}$$

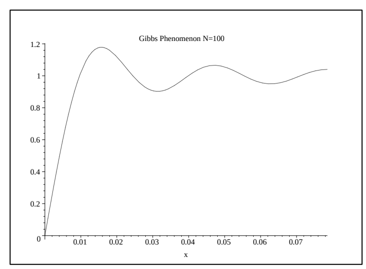

Figure 4.27: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 100$.

We have defined

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N}\cos\frac{n\pi x}{L},$$

which is called the ***N*-th Dirichlet Kernel** . We now prove

Lemma 4.1.

$$D_N(x) = \begin{cases} \frac{\sin((N+\frac{1}{2})\frac{\pi x}{L})}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0, \\ N + \frac{1}{2}, & \sin\frac{\pi x}{2L} = 0. \end{cases}$$

Proof. Let $\theta = \frac{\pi x}{L}$ and multiply $D_N(x)$by $2\sin\frac{\theta}{2}$to obtain:

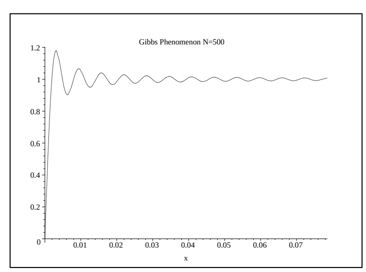

Figure 4.28: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 500$.

$$\begin{aligned}
2\sin\frac{\theta}{2}D_N(x) &= 2\sin\frac{\theta}{2}\left[\frac{1}{2}+\cos\theta+\cdots+\cos N\theta\right] \\
&= \sin\frac{\theta}{2}+2\cos\theta\sin\frac{\theta}{2}+2\cos 2\theta\sin\frac{\theta}{2}+\cdots+2\cos N\theta\sin\frac{\theta}{2} \\
&= \sin\frac{\theta}{2}+\left(\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}\right)+\left(\sin\frac{5\theta}{2}-\sin\frac{3\theta}{2}\right)+\cdots \\
&\quad +\left[\sin\left(N+\frac{1}{2}\right)\theta-\sin\left(N-\frac{1}{2}\right)\theta\right] \\
&= \sin\left(N+\frac{1}{2}\right)\theta. \qquad (4.103)
\end{aligned}$$

Thus,

$$2\sin\frac{\theta}{2}D_N(x)=\sin\left(N+\frac{1}{2}\right)\theta,$$

or if $\sin\frac{\theta}{2}\neq 0$,

$$D_N(x)=\frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \qquad \theta=\frac{\pi x}{L}.$$

If $\sin\frac{\theta}{2}=0$, then one needs to apply L'Hospital's Rule as $\theta\to 2m\pi$:

$$\begin{aligned}
\lim_{\theta\to 2m\pi}\frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} &= \lim_{\theta\to 2m\pi}\frac{(N+\frac{1}{2})\cos\left(N+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} \\
&= \frac{(N+\frac{1}{2})\cos\left(2m\pi N+m\pi\right)}{\cos m\pi} \\
&= \frac{(N+\frac{1}{2})(\cos 2m\pi N\cos m\pi-\sin 2m\pi N\sin m\pi)}{\cos m\pi} \\
&= N+\frac{1}{2}. \qquad (4.104)
\end{aligned}$$

□

We further note that $D_N(x)$ is periodic with period $2L$ and is an even function.

So far, we have found that

$$S_N(x)=\frac{1}{L}\int_{-L}^{L}D_N(y-x)f(y)\,dy. \qquad (4.105)$$

Now, make the substitution $\xi=y-x$. Then,

$$\begin{aligned}
S_N(x) &= \frac{1}{L}\int_{-L-x}^{L-x}D_N(\xi)f(\xi+x)\,d\xi \\
&= \frac{1}{L}\int_{-L}^{L}D_N(\xi)f(\xi+x)\,d\xi. \qquad (4.106)
\end{aligned}$$

In the second integral we have made use of the fact that $f(x)$ and $D_N(x)$ are periodic with period $2L$ and shifted the interval back to $[-L, L]$.

Now split the integration and use the fact that $D_N(x)$ is an even function. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L}^{0} D_N(\xi)f(\xi+x)\,d\xi + \frac{1}{L}\int_{0}^{L} D_N(\xi)f(\xi+x)\,d\xi \\ &= \frac{1}{L}\int_{0}^{L} \left[f(x-\xi)+f(\xi+x)\right] D_N(\xi)\,d\xi. \end{aligned} \tag{4.107}$$

We can use this result to study the Gibbs phenomenon whenever it occurs. In particular, we will only concentrate on our earlier example. For this case, we have

$$S_N(x) = \frac{1}{\pi}\int_{0}^{\pi} \left[f(x-\xi)+f(\xi+x)\right] D_N(\xi)\,d\xi \tag{4.108}$$

for

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos nx.$$

Also, one can show that

$$f(x-\xi)+f(\xi+x) = \begin{cases} 2, & 0 \le \xi < x, \\ 0, & x \le \xi < \pi - x, \\ -2, & \pi - x \le \xi < \pi. \end{cases}$$

Thus, we have

$$\begin{aligned} S_N(x) &= \frac{2}{\pi}\int_{0}^{x} D_N(\xi)\,d\xi - \frac{2}{\pi}\int_{\pi-x}^{\pi} D_N(\xi)\,d\xi \\ &= \frac{2}{\pi}\int_{0}^{x} D_N(z)\,dz + \frac{2}{\pi}\int_{0}^{x} D_N(\pi - z)\,dz. \end{aligned} \tag{4.109}$$

Here we made the substitution $z = \pi - \xi$ in the second integral. The Dirichlet kernel for $L = \pi$ is given by

$$D_N(x) = \frac{\sin(N+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$

For N large, we have $N + \frac{1}{2} \approx N$, and for small x, we have $\sin\frac{x}{2} \approx \frac{x}{2}$. So, under these assumptions,

$$D_N(x) \approx \frac{\sin Nx}{x}.$$

Therefore,

$$S_N(x) \to \frac{2}{\pi}\int_{0}^{x} \frac{\sin N\xi}{\xi}\,d\xi \qquad \text{for large } N, \text{ and small } x.$$

If we want to determine the locations of the minima and maxima, where the undershoot and overshoot occur, then we apply the first derivative test for extrema to $S_N(x)$. Thus,

$$\frac{d}{dx}S_N(x) = \frac{2}{\pi}\frac{\sin Nx}{x} = 0.$$

The extrema occur for $Nx = m\pi$, $m = \pm 1, \pm 2, \ldots$. One can show that there is a maximum at $x = \pi/N$ and a minimum for $x = 2\pi/N$. The value for the overshoot can be computed as

$$\begin{aligned} S_N(\pi/N) &= \frac{2}{\pi}\int_0^{\pi/N} \frac{\sin N\xi}{\xi}\, d\xi \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\, dt \\ &= \frac{2}{\pi}\mathrm{Si}(\pi) \\ &= 1.178979744\ldots. \end{aligned} \tag{4.110}$$

Note that this value is independent of N and is given in terms of the sine integral,

$$\mathrm{Si}(x) \equiv \int_0^x \frac{\sin t}{t}\, dt.$$

Problems

1. Solve the following boundary value problem:

$$x'' + x = 2, \quad x(0) = 0, \quad x'(1) = 0.$$

2. Find product solutions, $u(x,t) = b(t)\phi(x)$, to the heat equation satisfying the boundary conditions $u_x(0,t) = 0$ and $u(L,t) = 0$. Use these solutions to find a general solution of the heat equation satisfying these boundary conditions.

3. Consider the following boundary value problems. Determine the eigenvalues, λ, and eigenfunctions, $y(x)$ for each problem.[5]

a. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0.$

b. $y'' - \lambda y = 0, \quad y(-\pi) = 0, \quad y'(\pi) = 0.$

c. $x^2 y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0.$

d. $(x^2 y')' + \lambda y = 0, \quad y(1) = 0, \quad y'(e) = 0.$

[5] In problem d you will not get exact eigenvalues. Show that you obtain a transcendental equation for the eigenvalues in the form $\tan z = 2z$. Find the first three eigenvalues numerically.

4. Consider the boundary value problem for the deflection of a horizontal beam fixed at one end,

$$\frac{d^4 y}{dx^4} = C, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0.$$

Solve this problem assuming that C is a constant.

5. Write $y(t) = 3\cos 2t - 4\sin 2t$ in the form $y(t) = A\cos(2\pi f t + \phi)$.

6. Derive the coefficients b_n in Equation(4.25).

7. For the following sets of functions: i) show that each is orthogonal on the given interval, and ii) determine the corresponding orthonormal set. [See page 183]

a. $\{\sin 2nx\}, \quad n = 1,2,3,\ldots, \quad 0 \le x \le \pi.$

b. $\{\cos n\pi x\}, \quad n = 0,1,2,\ldots, \quad 0 \le x \le 2.$

c. $\{\sin \frac{n\pi x}{L}\}, \quad n = 1,2,3,\ldots, \quad x \in [-L, L].$

8. Consider $f(x) = 4\sin^3 2x$.

a. Derive the trigonometric identity giving $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$ using DeMoivre's Formula.

b. Find the Fourier series of $f(x) = 4\sin^3 2x$ on $[0, 2\pi]$ without computing any integrals.

9. Find the Fourier series of the following:

a. $f(x) = x$, $x \in [0, 2\pi]$.

b. $f(x) = \frac{x^2}{4}$, $|x| < \pi$.

c. $f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi, \\ -\frac{\pi}{2}, & \pi < x < 2\pi. \end{cases}$

10. Find the Fourier Series of each function $f(x)$ of period 2π. For each series, plot the Nth partial sum,

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos nx + b_n \sin nx\right],$$

for $N = 5, 10, 50$ and describe the convergence (is it fast? what is it converging to, etc.) [Some simple Maple code for computing partial sums is shown in the notes.]

a. $f(x) = x$, $|x| < \pi$.

b. $f(x) = |x|$, $|x| < \pi$.

c. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

11. Find the Fourier series of $f(x) = x$ on the given interval. Plot the Nth partial sums and describe what you see.

a. $0 < x < 2$.

b. $-2 < x < 2$.

c. $1 < x < 2$.

12. The result in problem 9b above gives a Fourier series representation of $\frac{x^2}{4}$. By picking the right value for x and a little arrangement of the series, show that [See Example 4.5.]

a.
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

b.
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Hint: Consider how the series in part a. can be used to do this.

13. Sketch (by hand) the graphs of each of the following functions over four periods. Then sketch the extensions each of the functions as both an even and odd periodic function. Determine the corresponding Fourier sine and cosine series and verify the convergence to the desired function using Maple.

a. $f(x) = x^2, 0 < x < 1$.

b. $f(x) = x(2-x), 0 < x < 2$.

c. $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$

d. $f(x) = \begin{cases} \pi, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$