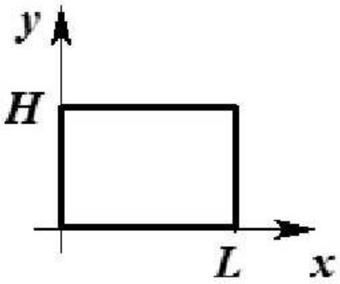

Figure 8.8: Rectangular coordinates in two dimensions.

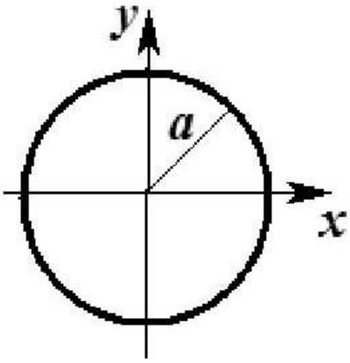

Figure 8.9: A three dimensional view of the vibrating rectangular membrane for the lowest modes.

8.6 Problems in Three Dimensions

need to introduce 3D coordinate systems and theie Laplacians. In this section we will list needed information for various standard coordinate systems.

8.7 Spherical Symmetry

We have solved several types of problems so far. We started with one dimensional problems: heat flow in a long thin rod, vibrations of a one dimensional string. We then moved on to two-dimensional regions and looked at the vibrations of rectangular and circular membranes. The

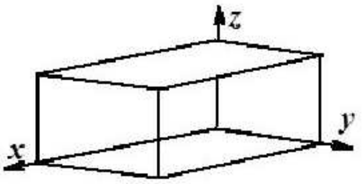

Figure 8.10: Rectangular coordinates in three dimensions.

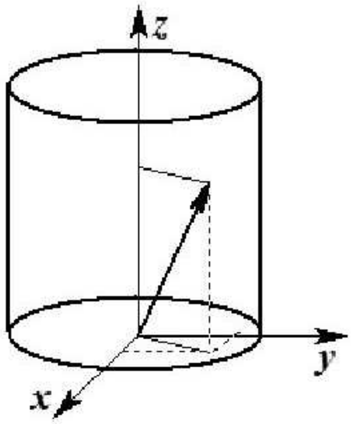

Figure 8.11: Cylindrical coordinates in three dimensions.

circular membrane required the use of polar coordinates due to the circular symmetry in the problem.

In this section we will explore three dimensional symmetries. One type that comes up often in classes like electrodynamics and quantum mechanics is spherical symmetry. We will describe this system in this section.

8.7.1 Laplace's Equation

Laplace's equation, $\nabla^2 u = 0$, arises in electrostatics as an equation for electric potential outside a charge distribution and it occurs as the equation governing equilibrium temperature distributions. Laplace's equation in spherical coordinates is given by

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \quad (8.67)$$

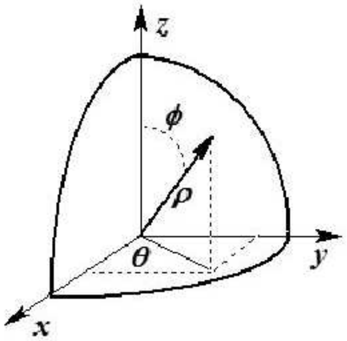

Figure 8.12: Spherical coordinates in three dimensions.

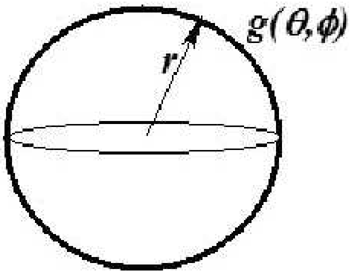

Figure 8.13: A sphere of radius r with the boundary condition $u(r, \theta, \phi) = g(\theta, \phi)$.

We will seek solutions of this equation inside a sphere or radius r subject to the boundary condition $u(\rho, \theta\phi) = g(\theta, \phi)$ as shown in Figure 8.13.

We first do a separation of variables. We seek product solutions of the form $u(\rho, \theta, \phi) = R(\rho)\Theta(\theta)\Phi(\phi)$. Inserting this form into our equation, we obtain

$$\frac{\Theta\Phi}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{R\Phi}{\rho^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{\rho^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0. \tag{8.68}$$

Now, multiply the equation by ρ^2 and divide by $R\Theta\Phi$. This yields

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \tag{8.69}$$

Note that the first term is the only term depending upon ρ. Thus, we can separate out the radial part. However, there is still more work to do on the other two terms, which give the angular part. Thus, we have

$$-\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) = \frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2} = -\lambda, \tag{8.70}$$

where we have introduced our first separation constant.

This leads to two equations:

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \lambda R = 0 \tag{8.71}$$

and

$$\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2} = -\lambda. \tag{8.72}$$

The final separation can be performed by multiplying the last equation by $\sin^2\theta$, rearranging the terms, and introducing a second separation constant:

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \mu. \tag{8.73}$$

This gives equations for $\Theta(\theta)$ and $\Phi(\phi)$:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + (\lambda\sin^2\theta - \mu)\Theta = 0, \tag{8.74}$$

and

$$\frac{d^2\Phi}{d\phi^2} + \mu\Phi = 0. \tag{8.75}$$

We now have three ordinary differential equations to solve. These are the radial equation (8.71) and the two angular equations (8.74)-(8.75). We note that all three are in the form of Sturm-Liouville problems. We need to solve each one and impose the boundary conditions.

The simplest equation is the one for $\Phi(\phi)$). We have seen equations of this form many times. The boundary condition in ϕ is not specified in the problem. It is one of those unstated conditions. We know that $\phi \in [0, 2\pi]$. Similar to the vibrating circular membrane problem, we expect that the solution and its derivative should agree at the endpoints of this interval, as these are physically the same point. Thus, we impose periodic boundary conditions:

$$u(\rho, \theta, 0) = u(\rho, \theta, 2\pi), \quad u_\phi(\rho, \theta, 0) = u\phi(\rho, \theta, 2\pi).$$

As this should hold for all ρ and θ, we have

$$\Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi).$$

As we have seen before, the eigenfunctions and eigenvalue are found as

$$\Phi(\phi) = \{\cos m\phi, \sin m\phi\}, \quad \mu = m^2. \tag{8.76}$$

The next simplest looking equation is the radial equation. However, as we will note later, λ will need to take the form $\lambda = \ell(\ell+1)$ for $\ell = 0, 1, 2, \ldots$. So, we will use this form in the derivation of the radial function.

The radial equation (8.71) can be written as

$$\rho^2 R'' + 2\rho R' - \ell(\ell+1)R = 0. \tag{8.77}$$

This equation is a Cauchy-Euler type of equation. So, we can guess the form of the solution to be $R(\rho) = \rho^s$, where s is a yet to be determined constant. Inserting our guess, we obtain the characteristic equation

$$s(s+1) = \ell(\ell+1).$$

Solving for s, we have

$$s = \ell, -(\ell+1).$$

Thus, the general solution of the radial equation is

$$R(\rho) = a\rho^\ell + b\rho^{-(\ell+1)}. \tag{8.78}$$

We would normally apply boundary conditions at this point. Recall that we gave that for $\rho = r$, $u(r, \theta, \phi) = g(\theta, \phi)$. This is not a homogeneous boundary condition, so we will need to hold off using it until we have the general solution to the three dimensional problem. However, we do have a hidden condition. Since we are interested in solutions inside the sphere, we need to consider what happens at $\rho = 0$. Note that $\rho^{-(\ell+1)}$ is not defined at the origin. Since the solution is expected to be bounded at the origin, we can set $b = 0$. In some applications we are interested in solutions outside the sphere. In this case we set $a = 0$ so that our solution is bounded at infinity. So, in the current problem we have established that

$$R(\rho) = a\rho^{\ell}.$$

Now we come to our last equation, (8.75). We will need to transform this equation in order to identify the solutions. Let $x = \cos\theta$. Then the derivatives transform as

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = \sin\theta\frac{d}{dx}.$$

Letting $y(x) = Theta(\theta)$ and noting that $\sin^2\theta = 1x^2$, Equation (8.75) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)y = 0. \tag{8.79}$$

We note that $x \in [-1, 1]$, as can be easily confirmed by the reader.

This is another Sturm-Liouville eigenvalue problem. The solutions consist of a set of orthogonal eigenfunctions. For the special case that $m = 0$ this equation becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \ell(\ell+1)y = 0. \tag{8.80}$$

The solutions of this equation are described in the next chapter. They are the Legendre polynomials, denoted by $P_\ell(x)$. The more general case, $m \neq 0$ has solutions called the associated Legendre functions. The two linearly independent solutions are denoted by $P_\ell^m(x)$ and $Q_\ell^m(x)$. The latter functions are not well behaved at $x = \pm 1$, corresponding to the north and south poles of the original problem. So, we can throw out these solutions and are left with

$$\Theta(\theta) = P_\ell^m(\cos\theta)$$

as the needed solutions.

The associated Legendre functions are related to the Legendre polynomials by

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x). \tag{8.81}$$

Also, we have that $\ell = 0, 1, 2, , \ldots$ and $m = 0, 1, \ldots, \ell$. They satisfy the orthogonality condition

$$\int_{-1}^{1} P_\ell^m(x) P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}. \tag{8.82}$$

We have carried out the full separation of Laplace's equation in spherical coordinates. The product solutions consist of the forms

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta) \cos m\phi$$

and

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta) \sin m\phi$$

for $\ell = 0, 1, 2, \ldots$ and $m = 0, \pm 1, , \ldots, \pm \ell$. These are often combined into a complex representation as

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta) e^{im\phi}.$$

The solutions of the angular parts of the problem are often combined, as the main differences in problems with spherical symmetry arise in the radial equation.

The general solution is given as a linear combination of these solutions. As there are two indices, we have a double sum:

$$u(\rho, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \rho^\ell P_\ell^m(\cos\theta) e^{im\phi}. \tag{8.83}$$

Due to the frequency of occurrence of the angular contributions, one often encounters what are called the spherical harmonics functions,

$$Y_{\ell m}(\theta, \phi),$$

which are defined as

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{2} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m e^{im\phi}. \tag{8.84}$$

These satisfy the simple orthogonality relation

$$int_{-1}^{1}\int_0^{2\pi} Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)\sin\theta\, d\phi d\theta = \delta_{\ell\ell'}\delta_{mm'}.$$

We will not go into these further at this time.

8.7.2 Example

As a simple example, we consider the solution of Laplace's equation in which there is azimuthal symmetry. Let $u(r,\theta,\phi) = g(\theta) = 1-\cos 2\theta$. This function is zero at the poles and has a maximum at the equator. In problems in which there is no ϕ-dependence, only the $m=0$ term of the general solution survives. Thus, we have that

$$u(\rho,\theta,\phi) = \sum_{\ell=0}^{\infty} a_\ell \rho^\ell P_\ell(\cos\theta). \tag{8.85}$$

Here we have used the fact that $P_\ell^0(x) = P_\ell(x)$. We just need to determine the unknown expansion coefficients, a_ℓ. Imposing the boundary condition at $\rho = r$, we are lead to

$$g(\theta) = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta). \tag{8.86}$$

This is a Fourier-Legendre series representation of $g(\theta)$. Since the Legendre polynomials are an orthogonal set of eigenfunctions, we can extract the coefficients just as we had done for Fourier sine series. In the next chapter we prove that

$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta = \int_{-1}^{1} P_n(x)P_m(x)\, dx = \frac{2}{2n+1}\delta_{nm}.$$

So, multiplying our expression for $g(\theta)$ by $P_m(\cos\theta)\sin\theta$ and integrating, we obtain the desired solution:

$$a_\ell = \frac{2\ell+1}{2}\int_0^{\Pi} g(\theta)P_\ell(\cos\theta)\sin\theta\, d\theta. \tag{8.87}$$

Sometimes it is easier to rewrite $g(\theta)$ as a polynomial in $\cos\theta$ and avoid the integration. For our example this is possible. Note that

$$\begin{aligned} g(\theta) &= 1-\cos 2\theta \\ &= 2\sin^2\theta \\ &= 2-2\cos^2\theta. \end{aligned} \tag{8.88}$$

Thus, setting $x = \cos\theta$, we have $g(\theta) = 2 - 2x^2$. In the next chapter we show that $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. We seek the form

$$g(\theta) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x).$$

Since $2 - 2x^2$ does not have any x terms, we know that $c_1 = 0$. So,

$$2 - 2x^2 = c_0(1) + c_2\frac{1}{2}(3x^2 - 1) = c0 - \frac{1}{2}c_2 + \frac{3}{2}c_2x^2.$$

By observation we have $c2 = -\frac{4}{3}$ and thus, $c0 = 2 + \frac{1}{2}c_2 = \frac{4}{3}$. This gives our sought expansion for g:

$$g() = \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}P_2(\cos\theta). \tag{8.89}$$

Therefore, $a_0 = \frac{4}{3}$, $a_2 = \frac{4}{3}\frac{1}{r^2}$, and the rest of the coefficients are zero. Inserting into the general solution, we have

$$u(\rho, \theta\phi) = \frac{4}{3}P_0(cos) - \frac{4}{3}\left(\frac{\rho}{r}\right)^2 P_2(\cos\theta) = \frac{4}{3}P_0(cos) - \frac{2}{3}\left(\frac{\rho}{r}\right)^2 (3\cos^2\theta - 1).$$

8.8 Other Applications

8.8.1 Temperature Distribution in Igloos

8.8.2 Waveguides

8.8.3 Optical Fibers

8.8.4 The Hydrogen Atom