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Prologue

"All science is either physics or stamp collecting." Ernest Rutherford (1871-1937)

0.1 Introduction

This is a set of notes on mathematical physics for undergraduate students who have completed a year long introductory course in physics. The intent of the course is to introduce students to many of the mathematical techniques useful in their undergraduate physics education long before they are exposed to more focused topics in physics.

This is an introduction to topics in mathematical physics, introduced using the physics of oscillations and waves. It is based upon a one semester junior level course in mathematics physics taught at the University of North Carolina Wilmington and originally set to book form in 2005. The notes were later modified and used in 2006, 2011, and 2012.

Most texts on mathematical physics are encyclopedic works which can never be covered in one semester and are often presented as a list of the seemingly unrelated topics with some examples from physics inserted to highlight the connection of the particular topic to the real world. The point of these excursions is to introduce the student to a variety of topics and not to delve into the rigor that one would find in some mathematics courses. Most of these topics have equivalent semester long courses which go into the details and proofs of the main conjectures in that topic. Students may decide to later enroll in such courses during their undergraduate, or graduate, study. Often the relevance to physics must be found in more advanced courses in physics when the particular methods are used for specialized applications.

So, why not teach the methods in the physics courses as they are needed? Part of the reason is that going into the details can take away from the global view of the course. Students often get lost in the mathematical details, as the proverbial tree can be lost in a forest of trees. Many of the mathematical techniques used in one course can be found in other courses. Collecting these techniques in one place, such as a course in mathematical physics, can provide a uniform background for students entering later courses in specialized topics in physics. Repeated exposure to standard methods also helps to ingrain these methods. Furthermore, in such a course as this, the student first sees both the physical and mathematical connections between different fields. Instructors can use this course as an opportunity to show students how the physics curriculum ties together what otherwise might appear to be a group of seemingly different courses.

The typical topics covered in a course on mathematical physics are vector analysis, vector spaces, linear algebra, complex variables, power series, ordinary and partial differential equations, Fourier series, Laplace and Fourier transforms, Sturm-Liouville theory, special functions and possibly other more advanced topics, such as tensors, group theory, the calculus of variations, or approximation techniques. We will cover many of these topics, but will do so in the guise of exploring specific physical problems. In particular, we will introduce these topics in the context of the physics of oscillations and waves.

0.2 *What is Mathematical Physics?*

WHAT DO YOU THINK when you hear the phrase "mathematical physics"? If one does a search on Google, one finds in Wikipedia the following:

> "Mathematical physics is an interdisciplinary field of academic study in between mathematics and physics, aimed at studying and solving problems inspired by physics within a mathematically rigorous framework. Although mathematical physics and theoretical physics are related, these two notions are often distinguished. Mathematical physics emphasizes the mathematical rigor of the same type as found in mathematics while theoretical physics emphasizes the links to actual observations and experimental physics which often requires the theoretical physicists to use heuristic, intuitive, and approximate arguments. Arguably, mathematical physics is closer to mathematics, and theoretical physics is closer to physics.
>
> Because of the required rigor, mathematical physicists often deal with questions that theoretical physicists have considered to be solved for decades. However, the mathematical physicists can sometimes (but neither commonly nor easily) show that the previous solution was incorrect.
>
> Quantum mechanics cannot be understood without a good knowledge of mathematics. It is not surprising, then, that its developed version under the name of quantum field theory is one of the most abstract, mathematically-based areas of physical sciences, being backward-influential to mathematics. Other subjects researched by mathematical physicists include operator algebras, geometric algebra, noncommutative geometry, string theory, group theory, statistical mechanics, random fields etc."

However, we will not adhere to the rigor suggested by this definition of mathematical physics, but will aim more towards the theoretical physics approach. Thus, this course could be called "A Course in Mathematical Methods in Physics." With this approach in mind, the course will be designed as a study of physical topics leading to the use of standard mathematical techniques. However, we should keep in mind Freeman Dyson's (b. 1923) words,

> "For a physicist mathematics is not just a tool by means of which phenomena can be calculated, it is the main source of concepts and principles by means of which new theories can be created." from *Mathematics in the Physical Sciences Mathematics in the Physical Sciences,* **Scientific American**, 211(3), September 1964, pp. 129-146.

It has not always been the case that we had to think about the differences between mathematics and physics. Until about a century ago people did not view physics and mathematics as separate disciplines. The Greeks did not separate the subjects, but developed an understanding of the natural sciences as part of their philosophical systems. Later, many of the big name physicists and mathematicians actually worked in both areas only to be placed in these categories through historical hindsight. People like Newton and Maxwell made just as many contributions to mathematics as they had to physics while trying to investigate the workings of the physical universe. Mathematicians such as Gauss, Leibniz and Euler had their share of contributions to physics.

Mathematics and physics are intimately related.

In the 1800's the climate changed. The study of symmetry lead to group theory, problems of convergence of the trigonometric series used by Fourier and others lead to the need for rigor in analysis, the appearance of non-Euclidean geometries challenged the millennia old Euclidean geometry, and the foundations of logic were challenged shortly after the turn of the century. This lead to a whole population of mathematicians interested in abstracting mathematics and putting it on a firmer foundation without much attention to applications in the real world. This split is summarized by Freeman Dyson:

> "I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce." from Missed Opportunities, 1972. (Gibbs Lecture)

In the meantime, many mathematicians have been interested in applying and extending their methods to other fields, such as physics, chemistry, biology and economics. These applied mathematicians have helped to mediate the divorce. Likewise, over the past century a number physicists with a strong bent towards mathematics have emerged as mathematical physicists. So, Dyson's report of a divorce might be premature.

Some of the most important fields at the forefront of physics are steeped in mathematics. Einstein's general theory of relativity, a theory of gravitation, involves a good dose of differential geometry. String theory is also highly mathematical, delving well beyond the topics in this book. While we will not get into these areas in this course, I would hope that students reading this book at least get a feel for the need to maintain the needed balance between mathematics and physics.

It is difficult to list all of the topics needed to study a subject like string theory. However, it is safe to say that a good grasp of the topics in this and more advance books on mathematical physics would help. A solid background in complex analysis, differential geometry, Lie groups and algebras, and variational principles should provide a good start.

0.3 An Overview of the Course

One of the problems with courses in mathematical physics and some of the courses taught in mathematics departments is that students do not always see the tie with physics. In this class we hope to enable students to see the mathematical techniques needed to enhance their future studies in physics. We will try not provide the mathematical topics devoid of physical motivation. We will instead introduce the methods studied in this course while studying one underlying theme from physics. We will tie the class mainly to the idea of oscillation in physics. Even though this theme is not the only possible collection of applications seen in physics, it is one of the most pervasive and has proven to be at the center of the revolutions of twentieth century physics.

In this section we provide an overview of the course in terms of the theme of oscillations even though at this writing there might be other topics introduced as the course is developed. There are many topics that could/might be included in the class depending upon the time that we have set aside. The current chapters/topics and their contents are:

1. **Introduction**

   In this chapter we review some of the key computational tools that you have seen in your first two courses in calculus and recall some of the basic formulae for elementary functions. Then we provide a short overview of your basic physics background, which will be useful in this course. It is meant to be a reference and additional topics may be added as we get further into the course.

   As the aim of this course is to introduce techniques useful in exploring the basic physics concepts in more detail through computation, we will also provide an overview of how one can use mathematical tables and computer algebra systems to help with the tedious tasks often encountered in solving physics problems.

   We will end with an example of how simple estimates in physics can lead to "back of the envelope" computations using dimensional analysis. While such computations do not require (at face value) the complex machinery seen in this course, it does use something that can be explained using the more abstract techniques of similarity analysis.

(a) What Do I Need to Know From Calculus?

(b) What I Need From My Intro Physics Class?

(c) Using Technology and Tables

(d) Dimensional Analysis

2. **Free Fall and Harmonic Oscillators**

A major theme throughout this book is that of oscillations, starting with simple vibrations and ending with the vibrations of membranes, electromagnetic fields, and the electron wave function. We will begin the first chapter by studying the simplest type of oscillation, simple harmonic motion. We will look at examples of a mass on a spring, LRC circuits, and oscillating pendula. These examples lead to constant coefficient differential equations, whose solutions we study along the way. We further consider the effects of damping and forcing in such systems, which are key ingredients to understanding the qualitative behavior of oscillating systems. Another important topic in physics is that of a nonlinear system. We will touch upon such systems in a few places in the text.

Even before introducing differential equations for solving problems involving simple harmonic motion, we will first look at differential equations for simpler examples, beginning with a discussion of free fall and terminal velocity. As you have been exposed to simple differential equations in your calculus class, we need only review some of the basic methods needed to solve standard applications in physics.

More complicated physical problems involve coupled systems. In fact, the problems in this chapter can be formulated as linear systems of differential equations. Such systems can be posed using matrices and the solutions are then obtained by solving eigenvalue problems, which is treated in the next chapter.

Other techniques for studying such problems described by differential equations involve power series methods and Laplace and other integral transforms. These ideas will be explored later in the book when we move on to exploring partial differential equations in higher dimensions. We will also touch on numerical solutions of differential equations, as not all problems can be solved analytically.

(a) Free Fall and Terminal Velocity; First Order ODEs

(b) The Simple Harmonic Oscillator; Second Order ODEs

(c) LRC Circuits

(d) Damped and Forced Oscillations; Nonhomogeneous ODEs

(e) Coupled Oscillators; Planar Systems

(f) The Nonlinear Pendulum

3. **Linear Algebra**

One of the most important mathematical topics in physics is linear algebra. Nowadays, the linear algebra course in most mathematics departments has evolved into a hybrid course covering matrix manipulations and some basics from vector spaces. However, it is seldom the case that applications, especially from physics, are covered. In this chapter we will introduce vector spaces, linear transformations and view matrices as representations of linear transformations. The main theorem of linear algebra is the spectral theorem, which means studying eigenvalue problems. Essentially, when can a given operator (or, matrix representation) be diagonalized? As operators act between vector spaces, it is useful to understand the concepts of both finite and infinite dimensional vector spaces and linear transformations on them.

The mathematical basis of much of physics relies on an understanding of both finite and infinite dimensional vector spaces. Linear algebra is important in the study of ordinary and partial differential equations, Fourier analysis, quantum mechanics and general relativity. We will return to this idea throughout the text. In this chapter we will introduce the concepts of vector spaces, linear transformations, and eigenvalue problems. We will also show how techniques from linear algebra are useful in solving coupled linear systems of differential equations. Later we shall see how much of what we do in physics involves linear transformations on vector spaces and eigenvalue problems.

(a) Finite Dimensional Vector Spaces

(b) Linear Transformations

(c) Matrices

(d) Eigenvalue Problems

(e) More Coupled Systems

(f) Diagonalization or The Spectral Theorem

4. **The Harmonics of Vibrating Strings**

The next type of oscillations which we will study are solutions of the one dimensional wave equation. A key example is provided by the finite length vibrating string. We will study traveling wave solutions and look at techniques for solving the wave equation. The standard technique is to use separation of variables, turning the solution of a partial differential equation into the solution of several ordinary differential equations. The resulting general solution will be written as an infinite series of sinusoidal functions, leading us to the study of Fourier series, which in turn provides the basis for studying the spectral content of complex signals.

In the meantime, we will also introduce the heat, or diffusion, equation as another example of a generic one dimensional partial differential equation exploiting the methods of this chapter. These problems begin our study of initial-boundary value problems, which pervade upper level physics courses, especially in electromagnetic theory and quantum mechanics.

(a) The Wave Equation in 1D

(b) Harmonics and Vibrations

(c) Fourier Trigonometric Series

(d) The Heat Equation in 1D

(e) Finite Length Strings

5. **Special Functions and the Space in Which They Live**

In our studies of systems in higher dimensions we encounter a variety of new solutions of boundary value problems. These collectively are referred to as Special Functions and have been known for a long time. They appear later in the undergraduate curriculum and we will cover several important examples. At the same time, we will see that these special functions may provide bases for infinite dimensional function spaces. Understanding these functions spaces goes a long way to understanding generalized Fourier theory, differential equations, and applications in electrodynamics and quantum theory.

In order to fully appreciate the special functions typically encountered in solving problem in higher dimensions, we will develop the Sturm-Liouville theory with some further excursion into the theory of infinite dimensional vector spaces.

(a) Infinite Dimensional Function Spaces

(b) Classical Orthogonal Polynomials

(c) Legendre Polynomials

(d) Gamma Function

(e) Bessel Functions

(f) Sturm-Liouville Eigenvalue Problems

6. **Complex Representations of The Real World**

   Another simple example, useful later for studying electromagnetic waves, is the infinite one-dimensional string. We begin with the solution of the finite length string, which consists of an infinite sum over a discrete set of frequencies, or a Fourier series. Allowing for the string length to get large will turn the infinite sum into a sum over a continuous set of frequencies. Such a sum is now an integration and the resulting integrals are defined as Fourier Transforms. Fourier transforms are useful in representing analog signals and localized waves in electromagnetism and quantum mechanics. Such integral transforms will be explored in the next chapter. However, useful results can only be obtained after first introducing complex variable techniques.

   So, we will spend some time exploring complex variable techniques and introducing the calculus of complex functions. In particular, we will become comfortable manipulating complex expressions and learn how to use contour methods to aid in the computation of integrals. We can apply these techniques to solving some special problems. We will first introduce a problem in fluid flow in two dimensions, which involve's solving Laplace's equation. We will explore dispersion relations, relations between frequency and wave number for wave propagation, and the computation of complicated integrals such as those encountered in computing induced current using Faraday's Law.

   (a) Complex Representations of Waves
   (b) Complex Numbers
   (c) Complex Functions and Their Derivatives
   (d) Harmonic Functions and Laplace's Equation
   (e) Complex Series Representations
   (f) Complex Integration
   (g) Applications to 2D Fluid Flow and AC Circuits

7. **Transforms of the Wave and Heat Equations**

   For problems defined on an infinite interval, solutions are no longer given in terms of infinite series. They can be represented in terms of integrals, which are associated with integral transforms. We will explore Fourier and Laplace transform methods for solving both ordinary and partial differential equations. By transforming our equations, we are lead to simpler equations in transform space. We will apply these methods to ordinary differential equations modeling forced oscillations and to the heat and wave equations.

(a) Transform Theory
(b) Exponential Fourier Transform
(c) The Dirac Delta Function
(d) The Laplace Transform and Its Applications
(e) Solution of Initial Value Problems; Circuits Problems
(f) The Inverse Laplace Transform
(g) Green's Functions and the Heat Equation

8. **Electromagnetic Waves**

One of the major theories is that of electromagnetism. In this chapter we will recall Maxwell's equations and use vector identities and vector theorems to derive the wave equation for electromagnetic waves. This will require us to recall some vector calculus from Calculus III. In particular, we will review vector products, gradients, divergence, curl, and standard vector identities useful in physics. In the next chapter we will solve the resulting wave equation for some physically interesting systems.

In preparation for solving problems in higher dimensions, we will pause to look at generalized coordinate systems and the transformation of gradients and other differential operators in these new systems. This will be useful in the next chapter for solving problems in other geometries.

(a) Maxwell's Equations
(b) Vector Analysis
(c) Electromagnetic Waves
(d) Curvilinear Coordinates

9. **Problems in Higher Dimensions**

Having studied one dimensional oscillations, we will now be prepared to move on to higher dimensional applications. These will involve heat flow and vibrations in different geometries primarily using the method of separation of variables. We will apply these methods to the solution of the wave and heat equations in higher dimensions.

Another major equation of interest that you will encounter in upper level physics is the Schrödinger equation. We will introduce this equation and explore solution techniques obtaining the relevant special functions involved in describing the wavefunction for a hydrogenic electron.

(a) Vibrations of a Rectangular Membrane

(b) Vibrations of a Kettle Drum

(c) Laplace's Equation in 3D

(d) Heat Equation in 3D

(e) Spherical Harmonics

(f) The Hydrogen Atom

0.4 Tips for Students

GENERALLY, SOME TOPICS in the course might seem difficult the first time through, especially not having had the upper level physics at the time the topics are introduced. However, like all topics in physics, you will understand many of the topics at deeper levels as you progress through your studies. It will become clear that the more adept one becomes in the mathematical background, the better your understanding of the physics.

You should read through this set of notes and then listen to the lectures. As you read the notes, be prepared to fill in the gaps in derivations and calculations. This is not a spectator sport, but a participatory adventure. Discuss the difficult points with others and your instructor. Work on problems as soon as possible. These are not problems that you can do the night before they are due. This is true of all physics classes. Feel free to go back and reread your old calculus and physics texts.

0.5 Acknowledgments

Most, if not all, of the ideas and examples are not my own. These notes are a compendium of topics and examples that I have used in teaching not only mathematical physics, but also in teaching numerous courses in physics and applied mathematics. Some of the notions even extend back to when I first learned them in courses I had taken. Some references to specific topics are included within the book, while other useful references [1], [2], [3], [4], [5] are provided in the bibliography for further study.

I would also like to express my gratitude to the many students who have found typos, or suggested sections needing more clarity. This not only applies to this set of notes, but also to my other two sets of notes, *An Introduction to Fourier and Complex Analysis with Application to the Spectral Analysis of Signals* and *A Second Course in Ordinary Differential Equations: Dynamical Systems and Boundary Value Problems* with which this text has some significant overlap.

[1] Mary L. Boas. *Mathematical Methods in the Physical Sciences*. John Wiley & Sons, Inc, third edition, 2006

[2] George Arfken. *Mathematical Methods for Physicists*. Academic Press, second edition, 1970

[3] Sadri Hassani. *Foundations of Mathematical Physics*. Allyn and Bacon, 1991

[4] Abdul J. Jerri. *Integral and Discrete Transforms with Applications and Error Analysis*. Marcal Dekker, Inc, 1992

[5] Susan M. Lea. *Mathematics for Physicists*. Brooks/Cole, 2004