Chapter 0

Prologue

0.1 Introduction

This is a set of notes originally designed to supplement a standard textbook in Mathematical Physics for undergraduate students who have completed a year long introductory course in physics. The intent of the course is to introduce students to many of the mathematical techniques useful in their undergraduate physics education long before they are exposed to more focused topics in physics.

Most texts on mathematical physics are encyclopedic works which can never be covered in one semester and are often presented as a list of the above topics with some examples from physics inserted to highlight the connection of the particular topic to the real world. The point of these excursions is to introduce the student to a variety of topics and not to delve into the rigor that one would find in some mathematics courses. Most of the topics have equivalent semester long courses which go into the details and proofs of the main conjectures in that topic. Students may decide to later enroll in such courses during their undergraduate, or graduate, study. However, often the relevance to physics must be found in later studies in physics when the particular methods are used for specialized applications.

So, why not teach the methods in the physics courses as they are needed? Part of the reason is that going into the details can take away from the global view of the course. Students often get lost in the mathematical details, as the proverbial tree can be lost in a forest of trees. Many of the mathematical techniques used in one course can be found in other courses. Collecting these techniques in one place, such as a course in mathematical physics, can help to provide a uniform background for students entering later courses in specialized topics in physics. Repeated exposure to standard methods can also help ingrain these methods. Furthermore, in such as course as this, one can see the connections between different fields and one can thus tie together some of the physical ideas between seemingly different courses.

The typical topics covered in a course on mathematical physics are vector analysis, vector spaces, linear algebra, complex variables, power series, ordinary and partial differential equations, Fourier series, Laplace and Fourier transforms, Sturm-Liouville theory, special functions and possibly other more advanced topics, such as tensors, group theory, the calculus of variations, or approximation techniques. We will cover many of these topics, but will do so in the guise of exploring specific physical problems.

0.2 What is Mathematical Physics?

What do you think when you hear the phrase "mathematical physics"? If one does a search on Google, one finds in Wikipedia[1] the following:

> "Mathematical physics is an interdisciplinary field of academic study in between mathematics and physics, aimed at studying and solving problems inspired by physics within a mathematically rigorous framework. Although mathematical physics and theoretical physics are related, these two notions are often distinguished. Mathematical physics emphasizes the mathematical rigor of the same type as found in mathematics while theoretical physics emphasizes the links to actual observations and experimental physics which often requires the theoretical physicists to use heuristic, intuitive, and approximate arguments. Arguably, mathematical physics is closer to mathematics, and theoretical physics is closer to physics.

[1] http://en.wikipedia.org/wiki/Mathematical_ physics

> Because of the required rigor, mathematical physicists often deal with questions that theoretical physicists have considered to be solved for decades. However, the mathematical physicists can sometimes (but neither commonly nor easily) show that the previous solution was incorrect.
>
> Quantum mechanics cannot be understood without a good knowledge of mathematics. It is not surprising, then, that its developed version under the name of quantum field theory is one of the most abstract, mathematically-based areas of physical sciences, being backward-influential to mathematics. Other subjects researched by mathematical physicists include operator algebras, geometric algebra, noncommutative geometry, string theory, group theory, statistical mechanics, random fields etc."

However, we will not adhere to the rigor suggested by this definition of mathematical physics, but will aim more towards the theoretical physics approach and thus this course should really be called "A Course in Mathematical Methods in Physics". With this in mind, the course will be designed as a study of physical topics leading to the use of standard mathematical techniques. However, we should keep in mind Freeman Dyson's words,

> "For a physicist mathematics is not just a tool by means of which phenomena can be calculated, it is the main source of concepts and principles by means of which new theories can be created." from Mathematics in the Physical Sciences

It has not always been the case that we had to think about the differences between mathematics and physics. Until about a century ago people did not view physics and mathematics as separate disciplines. The Greeks did not separate the subjects, but developed an understanding of the natural sciences as part of their philosophical systems. Later, many of the big name physicists and mathematicians actually worked in both areas only to be placed in these categories through historical hindsight. People like Newton and Maxwell made just as many contributions to mathematics as they had to physics while trying to investigate the workings of the physical universe. Mathematicians such as Gauss, Leibniz and Euler had their share of contributions to physics.

In the 1800's the climate changed. The study of symmetry lead to group theory, problems of convergence of the trigonometric series used by Fourier and others lead to the need for rigor in analysis, the appearance of non-Euclidean geometries challenged the millenia old Euclidean geometry, and the foundations of logic were challenged shortly after the turn of the century. This lead to a whole population of mathematicians interested in abstracting mathematics and putting it on a firmer foundation without much attention to applications in the real world. This split is summarized by Freeman Dyson:

> "I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce." from Missed Opportunities, 1972. (Gibbs Lecture)

In the meantime, many mathematicians are interested in applying and extending their methods to other fields, such as physics, chemistry, biology and economics. These applied mathematicians have helped to mediate the divorce. Likewise, over the past century a number physicists with a strong bent towards mathematics have emerged as mathematical physicists. So, Dyson's report of a divorce might be premature.

Some of the most important fields at the forefront of physics are steeped in mathematics. Einstein's general theory of relativity, a theory of gravitation, involves a good dose of differential geometry. String theory is also highly mathematical. While we will not get into these areas in this course, I would hope that students reading this book at least get a feel for the need to maintain the needed balance between mathematics and physics.

0.3 An Overview of the Course

One of the problems with courses in mathematical physics is that students do not always see the tie with physics. In this class we hope to enable students to see the mathematical techniques needed to enhance their future studies in physics. We will not provide the mathematical topics devoid of physical motivation. We will instead introduce the methods studied in this course while studying one underlying theme from physics. We will tie the class mainly to the idea of oscillation in physics. Even though this theme is not the possible collection of applications seen in physics, it is one of the most pervasive and has proven to be at the center of the revolutions of twentieth century physics.

Generally, some topics in the course might seem difficult the first time through, especially not having had the upper level physics at the time the topics are introduced. However, like all topics in physics, you will eventually grasp certain topics as you see them repeated throughout your studies. The importance of mathematical physics will become clearer as you progress through the curriculum. The successful student will need to develop patience as the story unfolds into graduate school. It will become clear that the more adept one becomes in the mathematical background, the better your understanding of the physics.

You should read through this set of notes and then listen to the lectures. As you read the notes, be prepared to fill in the gaps in derivations and calculations. This is not a spectator sport, but a participatory adventure. Discuss the difficult points with others and your instructor. Work on problems as soon as possible. These are not problems that you can do the night before they are due. This is true of all physics classes. Feel free to go back and reread your old calculus and physics texts.

We conclude this section with an overview of the course in terms of the theme of oscillations even though at this writing there might be other topics introduced as the course is developed. There are many topics that could/might be included in the class depending upon the time that we have set aside. The tentative chapters/topics and their contents are:

1. **Introduction**

   In this chapter we will review some of the key computational tools that you have seen in your first two courses in calculus and recall some of the basic formulae for elementary functions. Then we will provide a short overview of your basic physics background, which will be useful in this course. It is meant to be a reference and additional topics may be added as we get further into the course.

   As the aim of this course is to introduce techniques useful in exploring the basic physics concepts in more detail through computation, we will also provide an overview of how one can use mathematical tables and computer algebra systems to help with the tedious tasks often encountered in solving physics problems.

We will end with an example of how simple estimates in physics can lead to "back of the envelope" computations using simple dimensional analysis. While such computations do not require (at face value) the complex machinery seen in this course, it does use something that can be explained using the more abstract techniques of similarity analysis.

(a) What Do I Need to Know From Calculus?
(b) What I need From my Intro Physics Class?
(c) Using Technology and Tables
(d) Back of the Envelope Computations
(e) What Else is There?

2. **Free Fall and Harmonic Oscillators**

A major theme throughout the course will be to look at problems involving oscillations of various types. We begin with the simplest, simple harmonic motion. We will look at the mass on a spring, LRC circuits and oscillating pendula. We will need to solve constant coefficient differential equations along the way.

Even before introducing differential equations for solving problems involving simple harmonic motion, we will first look at differential equations for simpler examples. We will begin with a discussion of free fall and terminal velocity. As you have been exposed to simple differential equations in your calculus class, we will review some of the basics needed to solve common applications in physics.

More complicated problems involve coupled systems and this will allow us to explore linear systems of differential equations. Such systems can be posed using matrices and the solutions are then obtained by solving eigenvalue problems. One hot topic in physics is that of nonlinear systems, so we will also see an application near the end of the chapter.

Other techniques for studying such problems involve power series and Laplace transforms. These ideas will be explored later in the course.

(a) Free Fall and Terminal Velocity
(b) The Simple Harmonic Oscillator
(c) LRC Circuits

   (d) Damped and Forced Oscillations
   (e) Coupled Oscillators

3. **Linear Algebra**

   One of the most important mathematical topics in physics is linear algebra. Nowadays, the linear algebra course in most mathematics departments has evolved into a hybrid course covering matrix manipulations and some basics from vector spaces. However, it is seldom the case that applications, especially from physics, are covered. In this chapter we will introduce vector spaces, linear transformations and view matrices as representations of linear transformations. The main theorem of linear algebra is the spectral theorem, which means studying eigenvalue problems.

   The mathematical basis of much of physics relies on an understanding of both finite and infinite dimensional vector spaces. Linear algebra is important in the study of ordinary and partial differential equations, Fourier analysis, quantum mechanics and general relativity. We will return to this idea throughout the text. In the chapter we will first see this in the solution of coupled systems of ordinary differential equations.

   (a) Vector Spaces
   (b) Linear Transformations
   (c) Matrices
   (d) Eigenvalue Problems
   (e) Coupled Systems

4. **The Harmonics of Vibrating Strings**

   The next simplest type of oscillation is provided by finite length vibrating strings. We will derive the one dimensional wave equation and look at techniques for solving the wave equation. The standard technique is to use separation of variables, turning the solution of a partial differential equation into the solution of several ordinary differential equations. The resulting solution will be an infinite series of sinusoidal functions. This leads to the study of Fourier series, which is the basis of many other techniques in which one explore complicated signals to determine their spectral content.

In the meantime, we will also introduce the heat, or diffusion, equation as another example of a generic one dimensional example the methods of this chapter. This will be the beginning of our study of initial-boundary value problems, which pervade upper level physics courses, especially in electromagnetic theory and quantum mechanics.

(a) The Heat Equation in 1D
(b) Boundary Value Problems
(c) Harmonics and Vibrations
(d) The Wave Equation in 1D
(e) Fourier Series and Forced Oscillators
(f) Finite Length Strings

5. **Complex Representations of The Real World**

As a simple example, useful later for studying electromagnetic waves, we will consider the infinite dimensional string. This will lead to the study of Fourier Transforms. However, useful results can only be obtained after first introducing complex variable techniques.

So, we will spend some time exploring complex variable techniques and introducing the calculus of complex functions. We will apply these techniques to solving some special problems. We will first introduce a problem in fluid flow in two dimensions, which involve's solving Laplace's equation. We will explore dispersion relations, relations between frequency and wave number for wave propagation, and the computation of complicated integrals such as those encountered in computing induced current using Faraday's Law. Finally, we will introduce the Gamma function and compute the volume of a hypersphere.

(a) Complex Representations of Waves
(b) Complex Numbers
(c) Complex Functions and Their Derivatives
(d) Harmonic Functions and Laplace's Equation
(e) Complex Series Representations
(f) Fluid Flow in 2D Using Complex Functions
(g) Singularities and Dispersion Relations

(h) Computing Real Integrals Using the Residue Theorem

(i) The Volume of a Hypersphere

6. **Transforms of the Wave and Heat Equations**

   For problems defined on an infinite interval, solutions are no longer given in terms of infinite series. They can be represented in terms of integrals, which as associated with integral transforms. We will explore Fourier and Laplace transform methods for solving both ordinary and partial differential equations. By transforming our equations, we are lead to simpler equations in transform space. We will apply these methods to ordinary differential equations modeling forced oscillations and to the heat and wave equations.

   (a) Transform Theory
   (b) Exponential Fourier Transform
   (c) The Dirac Delta Function
   (d) The Laplace Transform
   (e) Applications to Oscillations

7. **Electromagnetic Waves**

   One of the major theories is that of electromagnetism. We will recall Maxwell's equations and use vector identities and vector theorems to derive the wave equation. This will require us to recall some vector calculus from Calculus III. In particular, we will review vector products, gradients, divergence and curl. This will lead us to some needed vector identities useful for deriving the existence of electromagnetic waves from Maxwell's equations. In the next chapter we will solve the resulting wave equation for some physically interesting systems.

   (a) Maxwell's Equations
   (b) Vector Analysis
   (c) Electromagnetic Waves

8. **Problems in Higher Dimensions**

   Having studied one dimensional oscillations, we will be prepared to move on to higher dimensional applications. These will involve solving problems in different geometries and thus we pause to look at generalized coordinate systems and then extend our use of separation of variables to solve these problems. In order to fully appreciate the general techniques, we will develop the Sturm-Liouville theory with some excursion into the theory of infinite dimensional vector spaces.

We will apply these methods to the solution of the wave and heat equations in higher dimensions. Another major equation of interest that you will encounter in Modern physics is the Schrödinger equation. We will introduce this equation and explore solution techniques obtaining the relevant special functions involved in describing the wavefunction for a hydrogenic electron.

(a) Vibrations of a Rectangular Membrane
(b) Vibrations of a Kettle Drum
(c) Sturm-Liouville Problems
(d) Curvilinear Coordinates
(e) Waveguides
(f) The Hydrogen Atom
(g) Geopotential Theory
(h) Temperature Distribution in Igloos
(i) Optical Fibers

9. **Special Functions**

In our studies of systems in higher dimensions we encounter a variety of new solutions of boundary value problems. These collectively are referred to as Special Functions and have been known for a long time. They appear later in the undergraduate curriculum and we will cover a couple of the important examples.

(a) Classical Orthogonal Polynomials
(b) Legendre Polynomials
(c) Spherical Harmonics
(d) Gamma Function
(e) Bessel Functions

10. **Describing Vibrations in Higher Dimensions**

There are many other topics that can be explored beyond the standard topics above. If there is time, we will look at some topics that the student can take up on their own, or in honors research. One possible topic is the introduction of tensors. Tensors appear in both classic and modern physics. In particular, tensors are important in the study of dynamics, electrical properties of materials, and general relativity.

(a) The Inertia Tensor

(b) Metrics in Relativity and Beyond

Not all problems can be solved exactly. We try to introduce some approximation techniques in parts of the course.

0.4 Famous Quotes

There are many quotes that can be found about physics, or physics and mathematics. Here are a few for you to read and think about.

"Physics is much too hard for physicists." David Hilbert (1862-1943) from C. Reid Hilbert, London: Allen and Unwin, 1970.

"All science is either physics or stamp collecting." Ernest Rutherford (1871-1937) 1st Baron Rutherford of Nelson British physicist, prof. at Manchester and Cambridge

"Physics isn't a religion. If it were, we'd have a much easier time raising money." Leon Lederman

"Mathematics began to seem too much like puzzle solving. Physics is puzzle solving, too, but of puzzles created by nature, not by the mind of man." Maria Goeppert Mayer (1906-1972) from J. Dash, Maria Goeppert-Mayer, A Life of One's Own.

"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say." Bertrand Russell (1872-1970) British philosopher, mathematician and social reformer from The Scientific Outlook, 1931.

"What I am going to tell you about is what we teach our physics students in the third or fourth year of graduate school . . . It is my task to convince you not to turn away because you don't understand it. You see my physics students don't understand it . . . That is because I don't understand it. Nobody does." Richard Feynman (1918-1988) US physicist, quantum electrodynamics, Nobel Prize, Physics, 1965 from QED, The Strange Theory of Light and Matter, Penguin Books, London, 1990, p 9.

"There are many examples of old, incorrect theories that stubbornly persisted, sustained only by the prestige of foolish but well-connected scientists. . . . Many of these theories have been killed off only when some decisive experiment exposed their incorrectness. .. Thus the yeoman work in any science, and especially physics, is done by the experimentalist, who must keep the theoreticians honest." Michio Kaku from Michio Kaku Hyperspace, Oxford University Press, 1995, p 263.

"To the pure geometer the radius of curvature is an incidental characteristic like the grin of the Cheshire cat. To the physicist it is an indispensable characteristic. It would be going too far to say that to the physicist the cat is merely incidental to the grin. Physics is concerned with interrelatedness such as the interrelatedness of cats and grins. In this case the "cat without a grin" and the "grin without a cat" are equally set aside as purely mathematical phantasies." Sir Arthur Stanley Eddington (1882-1944) British astronomer and physicist, director of Cambridge observatory from The Expanding Universe

"One needn't be a crank to miss the scientific boat. The very paragon of genius, Albert Einstein, couldn't be persuaded to give quantum physics his unreserved endorsement. Here is Einstein's most frequently paraphrased statement of dissatisfaction with the theory: Quantum mechanics is very impressive. But an inner voice tells me that it is not yet the real thing. The theory yields a lot, but it hardly brings us any closer to the secret of the Old One. In any case I am convinced that He doesn't play dice." Albert Einstein (1879-1955) German-Swiss-American mathematical physicist, famous for his theories of relativity. from Letter to Max Born, December 4, 1926

"When a distinguished but elderly scientist states that something is possible, he is almost certainly right. When he states that something is impossible, he is very probably wrong. Perhaps the adjective 'elderly' requires definition. In physics, mathematics, and astronautics it means over thirty; in the other disciplines, senile decay is sometimes postponed to the forties. There are, of course, glorious exceptions; but as every researcher just out of college knows, scientists of over fifty are good for nothing but board meetings, and should at all costs be kept out of the laboratory!" Arthur C. Clarke (1917-) English author of science fiction from 'Profiles of the Future' 1962 (Clarke's First Law)

"He is not a true man of science who does not bring some sympathy to his studies, and expect to learn something by behavior as well as by application. It is childish to rest in the discovery of mere coincidences, or of partial and extraneous laws. The study of geometry is a petty and idle exercise of the mind, if it is applied to no larger system than the starry one. Mathematics should be mixed not only with physics but with ethics; that is mixed mathematics. The fact which interests us most is the life of the naturalist. The purest science is still biographical." Henry David Thoreau (1817-1862) American philosopher and naturalist, writer of Walden

"The doctrine, as I understand it, consists in maintaining that the language of daily life, with words used in their ordinary meanings, suffices for philosophy, which has no need of technical terms or of changes in the significance of common terms. I find myself totally unable to accept this view. I object to it: 1.Because it is insincere; 2.Because it is capable of excusing ignorance of mathematics, physics and neurology in those who have had only a classical education; 3.Because it is advanced by some in a tone of unctuous rectitude, as if opposition to it were a sin against democracy; 4.Because it makes philosophy trivial; 5.Because it makes almost inevitable the perpetuation amongst philosophers of the muddle-headedness they have taken over from common sense." Bertrand Russell (1872-1970) British philosopher, mathematician and social reformer from Portraits from Memory, Russell

"Einstein, twenty-six years old, only three years away from crude privation, still a patent examiner, published in the Annalen der Physik in 1905 five papers on entirely different subjects. Three of them were among the greatest in the history of physics. One, very simple, gave the quantum explanation of the photoelectric effectit was this work for which, sixteen years later he was awarded the Nobel prize. Another dealt with the phenomenon of Brownian motion, the apparently erratic movement of tiny particles suspended in a liquid: Einstein showed that these movements satisfied a clear statistical law. This was like a conjuring trick, easy when explained: before it, decent scientists could still doubt the concrete existence of atoms and molecules: this paper was as near direct proof of their concreteness as a theoretician could give. The third paper was the special theory of relativity, which quietly amalgamated space, time and matter into one fundamental unity. This last paper contains no references and quotes no authority. All of them are written in a style unlike any other theoretical physicist's. They contain very little mathematics. There is a good deal of verbal commentary. The conclusions, the bizarre conclusions, emerge as though with the greatest of ease: the reasoning is unbreakable. It looks as though he had reached the conclusions by pure thought, unaided, without listening to the opinions of others. To a surprisingly large extent, that is precisely what he had done. It is pretty safe to say that, so long as physics lasts, no one will again hack out three major breakthroughs in one year." Charles Percy Snow (1905-1980) Baron Snow of Leicester English author and physicist from C.P. Snow, Variety of Men, Penguin Books, Harmondsworth, U.K. 1969, pp 85-86.