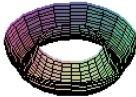
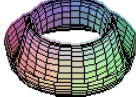
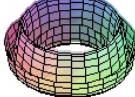
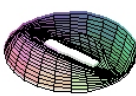

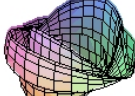





RUSSELL L. HERMAN

AN INTRODUCTION TO
MATHEMATICAL PHYSICS
VIA OSCILLATIONS

	<i>n = 1</i>	<i>n = 2</i>	<i>n = 3</i>
<i>m = 0</i>			
<i>m = 1</i>			
<i>m = 2</i>			

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Fourth printing, January 2012

Contents

0	<i>Prologue</i>	1
0.1	<i>Introduction</i>	1
0.2	<i>What is Mathematical Physics?</i>	2
0.3	<i>An Overview of the Course</i>	4
0.4	<i>Tips for Students</i>	10
0.5	<i>Acknowledgments</i>	10
1	<i>Introduction</i>	11
1.1	<i>What Do I Need To Know From Calculus?</i>	11
1.1.1	<i>Introduction</i>	11
1.1.2	<i>Trigonometric Functions</i>	13
1.1.3	<i>Hyperbolic Functions</i>	16
1.1.4	<i>Derivatives</i>	18
1.1.5	<i>Integrals</i>	19
1.1.6	<i>Geometric Series</i>	27
1.1.7	<i>The Binomial Expansion</i>	30
1.2	<i>What I Need From My Intro Physics Class?</i>	36
1.3	<i>Technology and Tables</i>	37
1.4	<i>Appendix: Dimensional Analysis</i>	38
	<i>Problems</i>	43

2	<i>Free Fall and Harmonic Oscillators</i>	45
2.1	<i>Free Fall and Terminal Velocity</i>	45
2.2	<i>First Order Differential Equations</i>	48
2.2.1	<i>Separable Equations</i>	49
2.2.2	<i>Linear First Order Equations</i>	50
2.2.3	<i>Terminal Velocity</i>	52
2.3	<i>The Simple Harmonic Oscillator</i>	54
2.3.1	<i>Mass-Spring Systems</i>	54
2.3.2	<i>The Simple Pendulum</i>	55
2.4	<i>Second Order Linear Differential Equations</i>	56
2.4.1	<i>Constant Coefficient Equations</i>	57
2.5	<i>LRC Circuits</i>	61
2.5.1	<i>Special Cases</i>	62
2.6	<i>Damped Oscillations</i>	66
2.7	<i>Forced Systems</i>	67
2.7.1	<i>Method of Undetermined Coefficients</i>	68
2.7.2	<i>Forced Oscillations</i>	71
2.7.3	<i>Cauchy-Euler Equations</i>	72
2.7.4	<i>Method of Variation of Parameters</i>	76
2.8	<i>Numerical Solutions of ODEs</i>	79
2.9	<i>Linear Systems</i>	85
2.9.1	<i>Coupled Oscillators</i>	85
2.9.2	<i>Planar Systems</i>	87
2.9.3	<i>Equilibrium Solutions and Nearby Behaviors</i>	89
2.9.4	<i>Polar Representation of Spirals</i>	100
2.10	<i>Appendix: The Nonlinear Pendulum</i>	102
	<i>Problems</i>	106

3	<i>Linear Algebra</i>	111
3.1	<i>Vector Spaces</i>	111
3.2	<i>Linear Transformations</i>	117
3.3	<i>Matrices</i>	119
3.4	<i>Eigenvalue Problems</i>	130
3.4.1	<i>An Introduction to Coupled Systems</i>	130
3.4.2	<i>Example of an Eigenvalue Problem</i>	132
3.4.3	<i>Eigenvalue Problems - A Summary</i>	135
3.5	<i>Matrix Formulation of Planar Systems</i>	136
3.5.1	<i>Solving Constant Coefficient Systems in 2D</i>	137
3.5.2	<i>Examples of the Matrix Method</i>	140
3.5.3	<i>Planar Systems - Summary</i>	144
3.6	<i>Applications</i>	145
3.6.1	<i>Circuits</i>	145
3.6.2	<i>Love Affairs</i>	146
3.6.3	<i>Predator Prey Models</i>	147
3.6.4	<i>Mixture Problems</i>	147
3.6.5	<i>Chemical Kinetics</i>	149
3.6.6	<i>Epidemics</i>	150
3.7	<i>Rotations of Conics</i>	151
3.8	<i>Appendix: Diagonalization and Linear Systems</i>	155
	<i>Problems</i>	160
4	<i>The Harmonics of Vibrating Strings</i>	165
4.1	<i>Harmonics and Vibrations</i>	165
4.2	<i>Boundary Value Problems</i>	167
4.3	<i>Partial Differential Equations</i>	169
4.4	<i>The 1D Heat Equation</i>	170
4.5	<i>The 1D Wave Equation</i>	174

4.6	<i>Introduction to Fourier Series</i>	176
4.7	<i>Fourier Trigonometric Series</i>	180
4.8	<i>Fourier Series Over Other Intervals</i>	187
4.8.1	<i>Fourier Series on $[a, b]$</i>	192
4.9	<i>Sine and Cosine Series</i>	194
4.10	<i>Solution of the Heat Equation</i>	199
4.11	<i>Finite Length Strings</i>	201
4.12	<i>Appendix: The Gibbs Phenomenon</i>	203
	<i>Problems</i>	208
5	<i>Non-sinusoidal Harmonics and Special Functions</i>	211
5.1	<i>Function Spaces</i>	212
5.2	<i>Classical Orthogonal Polynomials</i>	215
5.3	<i>Fourier-Legendre Series</i>	219
5.4	<i>Gamma Function</i>	230
5.5	<i>Fourier-Bessel Series</i>	232
5.6	<i>Sturm-Liouville Eigenvalue Problems</i>	237
5.6.1	<i>Sturm-Liouville Operators</i>	237
5.6.2	<i>Properties of Sturm-Liouville Eigenvalue Problems</i>	241
5.6.3	<i>Adjoint Operators</i>	243
5.6.4	<i>Lagrange's and Green's Identities</i>	245
5.6.5	<i>Orthogonality and Reality</i>	246
5.6.6	<i>The Rayleigh Quotient - optional</i>	247
5.6.7	<i>The Eigenfunction Expansion Method - optional</i>	249
5.7	<i>Appendix: The Least Squares Approximation</i>	251
5.8	<i>Appendix: The Fredholm Alternative Theorem</i>	254
	<i>Problems</i>	257

6	<i>Complex Representations of Functions</i>	265
6.1	<i>Complex Representations of Waves</i>	265
6.2	<i>Complex Numbers</i>	267
6.3	<i>Complex Valued Functions</i>	271
6.4	<i>Complex Differentiation</i>	277
6.5	<i>Complex Integration</i>	280
6.5.1	<i>Complex Path Integrals</i>	280
6.5.2	<i>Cauchy's Theorem</i>	283
6.5.3	<i>Analytic Functions and Cauchy's Integral Formula</i>	287
6.5.4	<i>Laurent Series</i>	291
6.5.5	<i>Singularities and The Residue Theorem</i>	294
6.5.6	<i>Infinite Integrals</i>	302
6.5.7	<i>Integration Over Multivalued Functions</i>	307
6.5.8	<i>Appendix: Jordan's Lemma</i>	311
	<i>Problems</i>	312
A	<i>Review of Sequences and Infinite Series</i>	317
A.1	<i>Sequences of Real Numbers</i>	318
A.2	<i>Convergence of Sequences</i>	318
A.3	<i>Limit Theorems</i>	319
A.4	<i>Infinite Series</i>	321
A.5	<i>Convergence Tests</i>	322
A.6	<i>Sequences of Functions</i>	326
A.7	<i>Infinite Series of Functions</i>	329
A.8	<i>Power Series</i>	331
A.9	<i>The Order of Sequences and Functions</i>	344
	<i>Problems</i>	347
	<i>Bibliography</i>	351
	<i>Index</i>	353

List of Figures

1.1 A photograph of the first atomic bomb test	41
2.1 Free fall on the Earth.	48
2.2 Family of exponential solutions	50
2.3 Family of circles.	50
2.4 Spring-Mass system.	55
2.5 A simple pendulum.	55
2.6 The pendulum forces.	55
2.7 Series LRC Circuit.	61
2.8 Parallel LRC Circuit.	62
2.9 Charge vs time for charging capacitor.	63
2.10 Charge vs time for charging capacitor.	64
2.11 Charge vs time for discharging capacitor.	65
2.12 A plot of underdamped oscillation.	67
2.13 The basics of Euler's Method.	80
2.14 Comparison of Euler's Method with the solution, $N = 10$.	83
2.15 Comparison of Euler's Method to the solution, $N = 100$.	84
2.16 Spring-Mass system.	85
2.17 Spring-Mass system.	86
2.18 Solutions of Example 2.22 for several initial conditions.	90
2.19 Orbits for Example 2.22.	91
2.20 Sketch of tangent vectors for Example 2.22.	92
2.21 Direction field for Example 2.22.	93
2.22 Phase portrait for Example 2.22.	93
2.23 Solutions of Example 2.24 for several initial conditions.	94
2.24 Phase portrait for Example 2.24, a saddle.	94
2.25 Solutions of Example 2.25 for several initial conditions.	94
2.26 Phase portrait for Example 2.25, an unstable node or source.	95
2.27 Solutions of Example 2.26 for several initial conditions.	95
2.28 Phase portrait for Example 2.26, a center.	96
2.29 Solutions of Example 2.27 for several initial conditions.	96
2.30 Solutions of Example 2.27 for several initial conditions.	96
2.31 Phase portrait for Example 2.27 with $\alpha = 0.1$.	98

2.32	Phase portrait for Example 2.27 with $\alpha = -0.2$.	98
2.33	Solutions of Example 2.28 for several initial conditions.	99
2.34	Plots of direction field of Example 2.29.	99
2.35	The relative error in approximating $\sin \theta$ by θ .	103
2.36	Relative error for the nonlinear pendulum period.	106
3.1	Vector \mathbf{v} in a standard coordinate system.	117
3.2	Vector \mathbf{v} in a rotated coordinate system.	117
3.3	Comparison of the coordinate systems.	118
3.4	Rotation of vector \mathbf{v}	118
3.5	A circuit with two loops.	145
3.6	Parallel circuit with the directions indicated.	145
3.7	A typical mixing problem.	148
3.8	The two tank problem.	148
3.9	Plot of the ellipse given by $13x^2 - 10xy + 13y^2 - 72 = 0$.	152
3.10	Comparison of rotated ellipses.	153
3.11	Principal eigenvectors of an ellipse.	154
3.12	Plot of the hyperbola given by $xy = 6$.	155
3.13	Plot of the rotated hyperbola given by $x^2 - y^2 = 12$.	155
4.1	Plot of the second harmonic of a vibrating string.	166
4.2	Plot of an initial condition for a plucked string.	166
4.3	Plots of $y(t) = A \sin(2\pi ft)$.	177
4.4	Problems can occur while plotting.	178
4.5	Superposition of several sinusoids.	178
4.6	Functions $y(t) = 2 \sin(4\pi t)$ and $y(t) = 2 \sin(4\pi t + 7\pi/8)$.	179
4.7	Plot of a functions and its periodic extension.	180
4.8	Plot of discontinuous function in Example 4.3.	185
4.9	Transformation between intervals.	187
4.10	Areas under even and odd functions.	190
4.11	Partial sums of the Fourier series for $f(x) = x $.	191
4.12	Plot of the first 10 terms of the Fourier series for $f(x) = x $	191
4.13	Plot of the first 200 terms of the Fourier series for $f(x) = x$	191
4.14	A sketch of a function and its various extensions.	195
4.15	The periodic extension of $f(x) = x^2$ on $[0, 1]$.	198
4.16	The even periodic extension of $f(x) = x^2$ on $[0, 1]$.	198
4.17	The odd periodic extension of $f(x) = x^2$ on $[0, 1]$.	199
4.18	The evolution of the initial condition $f(x) = \sin x$.	200
4.19	The evolution of the initial condition $f(x) = x(1 - x)$.	201
4.20	Initial profile for a string plucked at $x = 0.25$.	203
4.21	Odd extension about the right end of a plucked string.	203
4.22	Summing the odd periodic extensions.	203
4.23	A plucked string at six successive times.	204
4.24	The Fourier representation of a step function, $N = 10$.	204

4.25	The Fourier representation of a step function, $N = 10$.	205
4.26	The Fourier representation of a step function, $N = 20$.	205
4.27	The Fourier representation of a step function, $N = 100$.	205
4.28	The Fourier representation of a step function, $N = 500$.	205
5.1	Basis vectors \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 .	216
5.2	Vectors \mathbf{e}_1 , \mathbf{a}_2 , and \mathbf{e}_2 .	217
5.3	A plot of vectors for determining \mathbf{e}_3 .	217
5.4	Plots of the Legendre polynomials	221
5.5	Earth-moon system.	223
5.6	Fourier-Legendre series expansion of Heaviside function.	230
5.7	Plot of the Gamma function.	231
5.8	Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.	233
5.9	Plots of the Neumann functions, $N_0(x), \dots, N_3(x)$.	234
5.10	Fourier-Bessel series expansion for $f(x) = 1$ on $0 < x < 1$.	236
5.11	Plots of the eigenfunctions, $y(x) = \sqrt{2} \sin(n\pi \ln x)$.	251
5.12	Plot of the solution in Example 5.16.	251
6.1	The Argand diagram for plotting complex numbers	267
6.2	Locating $1 + i$ in the complex z -plane.	268
6.3	Locating the cube roots of unity in the complex z -plane.	271
6.4	Defining a complex valued function on \mathbb{C}	271
6.5	$f(z) = z^2$ maps a grid in the z -plane into the w -plane.	272
6.6	$f(z) = e^z$ maps the z -plane into the w -plane.	273
6.7	Domain coloring of the complex z -plane	273
6.8	Domain coloring for $f(z) = z^2$.	273
6.9	Domain coloring for $f(z) = 1/z(1 - z)$.	274
6.10	Domain coloring for $f(z) = z$.	274
6.11	Domain coloring for $f(z) = z^2$.	274
6.12	Domain coloring for several functions.	275
6.13	Domain coloring for $f(z) = z^2 - 0.75 - 0.2i$.	275
6.14	Domain coloring for six iterations of $f(z) = (1 - i/2) \sin x$.	276
6.15	There are many paths that approach z as $\Delta z \rightarrow 0$.	277
6.16	A path that approaches z with $y = \text{constant}$.	277
6.17	Integration of a complex function $f(z)$ over the path Γ .	280
6.18	Examples of (a) a connected set and (b) a disconnected set.	280
6.19	Open disks inside and on the boundary of a region.	280
6.20	Contour for Example 6.10.	281
6.21	Contour for Example 6.11 with $\Gamma = \gamma_1 \cup \gamma_2$.	282
6.22	Contour for Example 6.12.	282
6.23	$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz$ for all paths from z_1 to z_2 .	282
6.24	The integral $\oint_{\mathbb{C}} f(z) dz$ is zero if path independent.	283
6.25	The contours needed to prove $\oint_{\mathbb{C}} f(z) dz = \oint_{\mathbb{C}'} f(z) dz$.	285
6.26	The contours used to compute $\oint_{\mathbb{R}} \frac{dz}{z}$.	285

6.27 The contours used to compute $\oint_R \frac{dz}{z}$.	286
6.28 Regions of convergence for expansions of $f(z) = \frac{1}{1+z}$.	289
6.29 Circular contour used for the Cauchy Integral Formula.	290
6.30 Circular contour used in computing $\oint_{ z =4} \frac{\cos z}{z^2-6z+5} dz$.	291
6.31 Sketch of the annulus, $R_1 < z - z_0 < R_2$.	293
6.32 Contour for computing $\oint_{ z =1} \frac{dz}{\sin z}$.	298
6.33 A depiction of how one cuts out poles.	299
6.34 Contour for computing $\oint_{ z =2} \frac{dz}{z^2-1}$.	299
6.35 Contour for computing $\oint_{ z =3} \frac{z^2+1}{(z-1)^2(z+2)} dz$.	299
6.36 Contour for computing $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.	301
6.37 Contours for computing $P \int_{-\infty}^{\infty} f(x) dx$.	303
6.38 Contour for computing $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.	304
6.39 Contour for computing $P \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$.	304
6.40 Contour for computing $P \int_{-\infty}^{\infty} \frac{e^{-ix}}{x} dx$.	305
6.41 Example with poles on contour.	306
6.42 Example using a rectangular contour.	306
6.43 Mapping points in UHP under $f(z) = z^{1/2}$.	308
6.44 Mapping points in LHP under $f(z) = z^{1/2}$.	308
6.45 Mapping points on unit circle under $f(z) = z^{1/2}$.	308
6.46 Riemann surface for $f(z) = z^{1/2}$.	309
6.47 An example of a contour which accounts for a branch cut.	309
6.48 Contour needed to compute $\oint_C f(z) \ln(a-z) dz$.	310
6.49 Example using a rectangular contour.	316
A.1 Plot of $a_n = n - 1$ for $n = 1 \dots 10$.	318
A.2 Plot of $a_n = \frac{1}{2^n}$ for $n = 1 \dots 10$.	318
A.3 Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \dots 10$.	319
A.4 Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \dots 10$ showing the tail.	319
A.5 Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \dots 10$ showing the tail.	319
A.6 Plot of $s_n = \sum_{k=1}^n \frac{1}{2^{k-1}}$ for $n = 1 \dots 10$.	321
A.7 Plot of the partial sums, $s_k = \sum_{n=1}^k \frac{1}{n}$.	323
A.8 Plot of $f(x) = x$ and boxes of height $\frac{1}{n}$ and width 1.	324
A.9 For $f_n(x) = x^n$ we see how N depends on x and ϵ .	328
A.10 Plot defining uniform convergence.	328
A.11 $f_n(x) = x^n$ does not converge uniformly.	329
A.12 $f_n(x) = \cos(nx)/n^2$ converges uniformly.	329
A.13 Comparison of $\frac{1}{1-x}$ to $1+x$.	332
A.14 Comparison of $\frac{1}{1-x}$ to $1+x+x^2$.	332
A.15 Comparison of $\frac{1}{1-x}$ to $1+x+x^2$ and $1+x+x^2+x^3$.	332
A.16 Comparison of $\frac{1}{1-x}$ to $\sum_{n=0}^{20} x^n$.	333

List of Tables

1.1	Table of Trigonometric Values	13
1.2	Table of Derivatives (a is a constant.)	18
1.3	Table of Integrals	20
1.4	Tabular Method	23
1.5	Tabular Method - Non-terminating Example.	24
2.1	Euler's Method numerical tabulation.	82
2.2	Euler's Method numerical tabulation.	82
2.3	Euler's Method numerical tabulation.	82
3.1	List of typical behaviors in planar systems.	144
3.2	Three examples of systems with a repeated root of $\lambda = 2$.	144
4.1	List of generic partial differential equations.	169
4.2	Fourier Series Representations on Different Intervals	189
4.3	Fourier Cosine and Sine Series Representations on $[0, L]$	196
4.4	Maple code for computing and plotting Fourier series.	198
5.1	Table of common classical orthogonal polynomials.	219
5.2	Tabular computation of the Legendre polynomials	221
5.3	The zeros of Bessel Functions	235

*Dedicated to those students who have endured
previous editions of AN INTRODUCTION
TO MATHEMATICAL PHYSICS VIA OSCILLATIONS
and to those about to embark on the journey.*