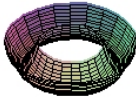
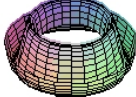
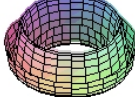
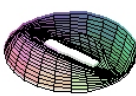

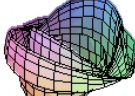





RUSSELL L. HERMAN

AN INTRODUCTION TO  
MATHEMATICAL PHYSICS  
VIA OSCILLATIONS

	<i>n = 1</i>	<i>n = 2</i>	<i>n = 3</i>
<i>m = 0</i>			
<i>m = 1</i>			
<i>m = 2</i>			

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*Dedicated to those students who have endured  
previous editions of AN INTRODUCTION  
TO MATHEMATICAL PHYSICS VIA OSCILLATIONS  
and to those about to embark on the journey.*