8
Vector Analysis and EM Waves

"From a long view of the history of mankind seen from, say, ten thousand years from now, there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics."
The Feynman Lectures on Physics (1964), Richard Feynman (1918-1988)

UP TO THIS POINT we have mainly been confined to problems involving only one or two independent variables. In particular, the heat equation and the wave equation involved one time and one space dimension. However, we live in a world of three spatial dimensions. (Though, some theoretical physicists live in worlds of many more dimensions, or at least they think so.) We will need to extend the study of the heat equation and the wave equation to three spatial dimensions.

Recall that the one-dimensional wave equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{8.1}$$

For higher dimensional problems we will need to generalize the $\frac{\partial^2 u}{\partial x^2}$ term. For the case of electromagnetic waves in a source-free environment, we will derive a three dimensional wave equation for the electric and magnetic fields: It is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{8.2}$$

This is the generic form of the linear wave equation in Cartesian coordinates. It can be written a more compact form using the Laplacian, ∇^2,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{8.3}$$

The introduction of the Laplacian is common when generalizing to higher dimensions. In fact, we have already presented some generic one and three dimensional equations in Table 4.1, which we reproduce in Table 8.1. We have studied the one dimensional wave equation, heat equation, and Schrödinger equation. For steady-state, or equilibrium, heat flow problems, the heat equation no longer involves the time derivative. What is left is called Laplace's equation, which we have also seen in relation to complex functions. Adding an external heat source, Laplace's equation becomes what is known as Poisson's equation.

Name	2 Vars	3 D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2\nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x, y)$	$\nabla^2 u = F(x, y, z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x, t)u$	$iu_t = \nabla^2 u + F(x, y, z, t)u$

Table 8.1: List of generic partial differential equations.

Using the Laplacian allows us not only to write these equations in a more compact form, but also in a coordinate-free representation. Many problems are more easily cast in other coordinate systems. For example, the propagation of electromagnetic waves in an optical fiber are naturally described in terms of cylindrical coordinates. The heat flow inside a hemispherical igloo can be described using spherical coordinates. The vibrations of a circular drumhead can be described using polar coordinates. In each of these cases the Laplacian has to be written in terms of the needed coordinate systems.

The solution of these partial differential equations can be handled using separation of variables or transform methods. In the next chapter we will look at several examples of applying the separation of variables in higher dimensions. This will lead to the study of ordinary differential equations, which in turn leads to new sets of functions, other than the typical sine and cosine solutions.

In this chapter we will review some of the needed vector analysis for the derivation of the three dimensional wave equation from Maxwell's equations. We will review the basic vector operations (the dot and cross products), define the gradient, curl, and divergence and introduce standard vector identities that are often seen in physics courses. Equipped with these vector operations, we will derive the three dimensional waves equation for electromagnetic waves from Maxwell's equations. We will conclude this chapter with a section on curvilinear coordinates and provide the vector differential operators for different coordinate systems.

8.1 Vector Analysis

8.1.1 A Review of Vector Products

AT THIS POINT you might want to reread the first section of Chapter 3. In that chapter we introduced the formal definition of a vector space and some simple properties of vectors. We also discussed one of the common vector products, the dot product, which is defined as

$$\mathbf{u} \cdot \mathbf{v} = uv \cos\theta. \tag{8.4}$$

There is also a component form, which we write as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{k=1}^{3} u_k v_k. \tag{8.5}$$

One of the first physical examples using a cross product is the definition of work. The work done on a body by a constant force $\mathbf{F}$ during a displacement $\mathbf{d}$ is

$$W = \mathbf{F} \cdot \mathbf{d}.$$

In the case of a nonconstant force, we have to add up the incremental contributions to the work, $dW = \mathbf{F} \cdot d\mathbf{r}$ to obtain

$$W = \int_C dW = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{8.6}$$

over the path C. Note how much this looks like a complex path integral. It is a path integral, but the path lies in a real three dimensional space.

Another application of the dot product is the proof of the Law of Cosines. Recall that this law gives the side opposite a given angle in terms of the angle and the other two sides of the triangle:

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{8.7}$$

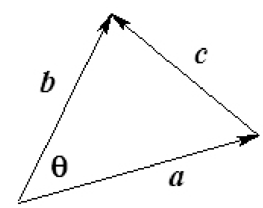

Figure 8.1: $v = r\omega$. The Law of Cosines can be derived using vectors.

Consider the triangle in Figure 8.1. We draw the sides of the triangle as vectors. Note that $\mathbf{b} = \mathbf{c} + \mathbf{a}$. Also, recall that the square of the length any vector can be written as the dot product of the vector with itself. Therefore, we have

$$\begin{aligned} c^2 &= \mathbf{c} \cdot \mathbf{c} \\ &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned} \tag{8.8}$$

We note that this also comes up in writing out inverse square laws in many applications. Namely, the vector $\mathbf{a}$ can locate a mass, or charge, and vector $\mathbf{b}$ points to an observation point. Then the inverse square law would involve vector $\mathbf{c}$, whose length is obtained as $\sqrt{a^2 + b^2 - 2ab\cos\theta}$. Typically, one does not have $\mathbf{a}$'s and $\mathbf{b}$'s, but something like $\mathbf{r}_1$ and $\mathbf{r}_2$, or $\mathbf{r}$ and $\mathbf{R}$. For these problems one is typically interested in approximating the expression of interest in terms of ratios like $\frac{r}{R}$ for $R \gg r$.

Another important vector product is the cross product. The cross product produces a vector, unlike the dot product that results in a scalar. The magnitude of the cross product is given as

$$|\mathbf{a} \times \mathbf{b}| = ab \sin \theta. \tag{8.9}$$

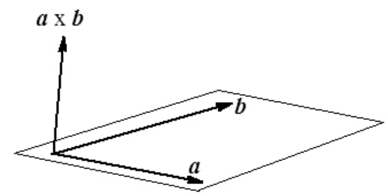

Figure 8.2: The cross product is shown. The direction is obtained using the right hand rule: Curl fingers from **a** through to **b**. The thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

Being a vector, we also have to specify the direction. The cross product produces a vector that is perpendicular to both vectors **a** and **b**. Thus, the vector is normal to the plane in which these vectors live. There are two possible directions. The direction taken is given by the right hand rule. This is shown in Figure 8.2. The direction can also be determined using your right hand. Curl your fingers from **a** through to **b**. The thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

One of the first occurrences of the cross product in physics is in the definition of the torque, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Recall that the torque is the analogue to the force. A net torque will cause an angular acceleration. Consider a rigid body in which a force is applied to to the body at a position **r** from the axis of rotation. (See Figure 8.3.) Then this force produces a torque with respect to the axis. The direction of the torque is given by the right hand rule. Point your fingers in the direction of **r** and rotate them towards **F**. In the figure this would be out of the page. This indicates that the bar would rotate in a counter clockwise direction if this were the only force acting on the bar.

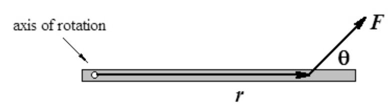

Figure 8.3: A force applied at a point located at **r** from the axis of rotation produces a torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ with respect to the axis.

Another example is that of a body rotating about an axis as shown in Figure 8.4. We locate the body with a position vector pointing from the origin of the coordinate system to the body. The tangential velocity of the body is related to the angular velocity by a cross product $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. The direction of the angular velocity is given be a right hand rule. Curl the fingers of your right hand in the direction of the motion of the rotating mass. Your thumb will point in the direction of $\boldsymbol{\omega}$. Counter clockwise motion produces a positive angular velocity and clockwise will give a negative angular velocity. Note that for the origin at the center of rotation of the mass, we obtain the familiar expression $v = r\omega$.

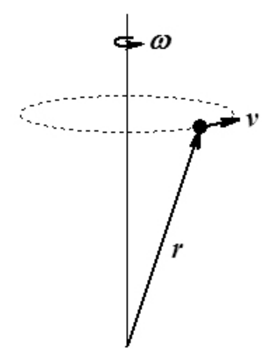

Figure 8.4: A mass rotates at an angular velocity ωabout a fixed axis of rotation. The tangential velocity with respect to a given origin is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

There is also a geometric interpretation of the cross product. Consider the vectors **a** and **b** in Figure 8.5. Now draw a perpendicular from the tip of **b** to vector **a**. This forms a triangle of height h. Slide the triangle over to form a rectangle of base a and height h. The area of this triangle is

$$\begin{aligned} A &= ah \\ &= a(b \sin \theta) \\ &= |\mathbf{a} \times \mathbf{b}|. \end{aligned} \tag{8.10}$$

Therefore, the magnitude of the cross product is the area of the triangle formed by the vectors **a** and **b**.

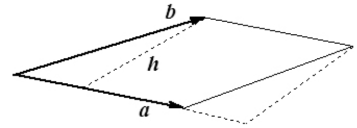

Figure 8.5: The magnitudes of the cross product gives the area of the parallelogram defined by **a** and **b**.

The dot product was shown to have a simple form in terms of the components of the vectors. Similarly, we can write the cross product in component form. Recall that we can expand any vector $\mathbf{v}$ as

$$\mathbf{v} = \sum_{k=1}^{n} v_k \mathbf{e}_k, \tag{8.11}$$

where the $\mathbf{e}_k$'s are the standard basis vectors.

We would like to expand the cross product of two vectors,

$$\mathbf{u} \times \mathbf{v} = \left(\sum_{k=1}^{n} u_k \mathbf{e}_k \right) \times \left(\sum_{k=1}^{n} v_k \mathbf{e}_k \right).$$

In order to do this we need a few properties of the cross product.

Properties of the cross product.

First of all, the cross product is not commutative. In fact, it is anticommutative:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

A simple consequence of this is that $\mathbf{v} \times \mathbf{v} = 0$. Just replace $\mathbf{u}$ with $\mathbf{v}$ in the anticommutativity rule and you have $\mathbf{v} \times \mathbf{v} = -\mathbf{v} \times \mathbf{v}$. Something that is its negative must be zero.

The cross product also satisfies distributive properties:

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}),$$

and

$$\mathbf{u} \times (a\mathbf{v}) = (a\mathbf{u}) \times \mathbf{v} = a\mathbf{u} \times \mathbf{v}.$$

Thus, we can expand the cross product in terms of the components of the given vectors. A simple computation shows that $\mathbf{u} \times \mathbf{v}$ can be expressed in terms of sums over $\mathbf{e}_i \times \mathbf{e}_j$:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \left(\sum_{i=1}^{n} u_i \mathbf{e}_i \right) \times \left(\sum_{j=1}^{n} v_j \mathbf{e}_j \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} u_i v_j \mathbf{e}_i \times \mathbf{e}_j. \end{aligned} \tag{8.12}$$

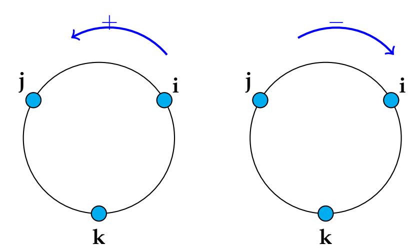

Figure 8.6: The sign for the cross product for basis vectors can be determined from a simple diagram. Arrange the vectors on a circle as above. If the needed computation goes counterclockwise, then the sign is positive. Thus, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$.

The cross products of basis vectors are simple to compute. First of all, the cross products $\mathbf{e}_i \times \mathbf{e}_j$ vanish when $i = j$ by anticommutativity of the cross product. For $i \neq j$, it is not much more difficult. For the typical basis, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, this is simple. Imagine computing $\mathbf{i} \times \mathbf{j}$. This is a vector of length $|\mathbf{i} \times \mathbf{j}| = |\mathbf{i}||\mathbf{j}| \sin 90^\circ = 1$. The vector $\mathbf{i} \times \mathbf{j}$ is perpendicular to both vectors, $\mathbf{i}$ and $\mathbf{j}$. Thus, the cross product is either $\mathbf{k}$ or $-\mathbf{k}$. Using the right hand rule, we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Similarly, we find the following

$$\begin{aligned} \mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} &= \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \end{aligned} \tag{8.13}$$

Inserting these results into the cross product for vectors in R^3, we have

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}. \quad (8.14)$$

While this form for the cross product is correct and useful, there are other forms that help in verifying identities or making computation simpler with less memorization. However, some of these new expressions can lead to problems for the novice as dealing with indices can be daunting at first sight.

One expression that is useful for computing cross products is the familiar computation using determinants. Namely, we have that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}. \end{aligned} \quad (8.15)$$

A more compact form for the cross product is obtained by introducing the completely antisymmetric symbol, ϵ_{ijk}. This symbol is defined by the relations

The completely antisymmetric symbol, or permutation symbol, ϵ_{ijk}.

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1,$$

and

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1,$$

and all other combinations, like ϵ_{113}, vanish. Note that all indices must differ. Also, if the order is a cyclic permutation of $\{1, 2, 3\}$, then the value is $+1$. For this reason ϵ_{ijk} is also called the permutation symbol or the Levi-Civita symbol. We can also indicate the index permutation more generally using the following identities:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}.$$

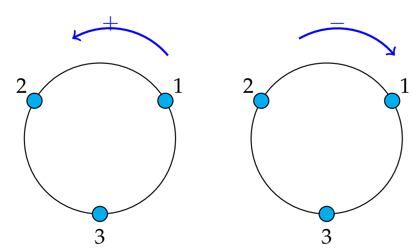

Figure 8.7: The sign for the permutation symbol can be determined from a simple cyclic diagram similar to that for the cross product. Arrange the numbers from 1 to 3 on a circle. If the needed computation goes counterclockwise, then the sign is positive, otherwise it is negative.

Returning to the cross product, we can introduce the standard basis $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, and $\mathbf{e}_3 = \mathbf{k}$. With this notation, we have that

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk} \mathbf{e}_k. \quad (8.16)$$

Example 8.1. *Compute the cross product of the basis vectors $\mathbf{e}_2 \times \mathbf{e}_1$ using the permutation symbol. A straight forward application of the definition of the cross product,*

$$\mathbf{e}_2 \times \mathbf{e}_1 = \sum_{k=1}^{3} \epsilon_{21k} \mathbf{e}_k$$

$$\begin{aligned} &= \epsilon_{211}\mathbf{e}_1 + \epsilon_{212}\mathbf{e}_2 + \epsilon_{213}\mathbf{e}_3 \\ &= -\mathbf{e}_3. \end{aligned} \tag{8.17}$$

It is helpful to write out enough terms in these sums until you get familiar with manipulating the indices. Note that the first two terms vanished because of repeated indices. In the last term we used $\epsilon_{213} = -1$.

We now write out the general cross product as

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j \mathbf{e}_i \times \mathbf{e}_j \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j \left(\sum_{k=1}^{3}\epsilon_{ijk}\mathbf{e}_k\right) \\ &= \sum_{i,j,k=1}^{3}\epsilon_{ijk}u_i v_j \mathbf{e}_k. \end{aligned} \tag{8.18}$$

Note that the last sum is a triple sum over the indices i, j, and k.

Example 8.2. *Let* $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ *and* $\mathbf{v} = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$. *Compute* $\mathbf{u}\times\mathbf{v}$. *We can compute this easily using determinants.*

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 1 & 5 & 4 \end{vmatrix} \\ &= \begin{vmatrix} -3 & 0 \\ 5 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}\mathbf{k} \\ &= -12\mathbf{i} - 8\mathbf{j} + 13\mathbf{k}. \end{aligned} \tag{8.19}$$

Using the permutation symbol to compute this cross product, we have

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \epsilon_{123}u_1v_2\mathbf{k} + \epsilon_{231}u_2v_3\mathbf{i} + \epsilon_{312}u_3v_1\mathbf{j} \\ &\quad +\epsilon_{213}u_2v_1\mathbf{k} + \epsilon_{132}u_1v_3\mathbf{j} + \epsilon_{321}u_3v_2\mathbf{i} \\ &= 2(5)\mathbf{k} + (-3)4\mathbf{i} + (0)1\mathbf{j} - (-3)1\mathbf{k} - (2)4\mathbf{j} - (0)5\mathbf{i} \\ &= -12\mathbf{i} - 8\mathbf{j} + 13\mathbf{k}. \end{aligned} \tag{8.20}$$

Sometimes it is useful to note that the kth component of the cross product is given by

$$(\mathbf{u}\times\mathbf{v})_k = \sum_{i,j=1}^{3}\epsilon_{ijk}u_i v_j.$$

In more advanced texts, or in the case of relativistic computations with tensors, the summation symbol is suppressed. For this case, one writes

$$(\mathbf{u}\times\mathbf{v})_k = \epsilon_{ijk}u_i v_j,$$

where it is understood that summation is performed over repeated indices. This is called the *Einstein summation convention*.

Einstein summation convention is used to suppress summation notation. In general relativity, one also needs to employ raised indices, so that vector components are written in the form u^i. The convention then requires that one only sums over a combination of one lower and one upper index. Thus, we would write $\epsilon_{ijk}u^iv^j$. We will forgo the need for raised indices.

Since the cross product can be written as both a determinant,

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \epsilon_{ij1}u_iv_j\mathbf{i} + \epsilon_{ij2}u_iv_j\mathbf{j} + \epsilon_{ij3}u_iv_j\mathbf{k}. \end{aligned} \tag{8.21}$$

and using the permutation symbol,

$$\mathbf{u}\times\mathbf{v} = \epsilon_{ijk}u_iv_j\mathbf{e}_k,$$

we can define the determinant as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}a_{1i}a_{2j}a_{3k}. \tag{8.22}$$

Here we added the triple sum in order to emphasize the hidden summations.

Example 8.3. *Compute the determinant* $\begin{vmatrix} 1 & 0 & 2 \\ 0 & -3 & 4 \\ 2 & 4 & -1 \end{vmatrix}$.

We insert the components of each row into the expression for the determinant:

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 2 \\ 0 & -3 & 4 \\ 2 & 4 & -1 \end{vmatrix} &= \epsilon_{123}(1)(-3)(-1) + \epsilon_{231}(0)(4)(2) + \epsilon_{312}(2)(0)(4) \\ &\quad +\epsilon_{213}(0)(0)(-1) + \epsilon_{132}(1)(4)(4) + \epsilon_{321}(2)(-3)(2) \\ &= 3 + 0 + 0 - 0 - 14 - (-12) \\ &= 15. \end{aligned} \tag{8.23}$$

Note that if one adds copies of the first two columns, as shown in Figure 8.8, then the products of the first three diagonals, downward to the right (blue), give the positive terms in the determinant computation and the products of the last three diagonals, downward to the left (red), give the negative terms.

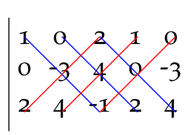

Figure 8.8: Diagram for computing determinants.

One useful identity is

$$\epsilon_{jki}\epsilon_{j\ell m} = \delta_{k\ell}\delta_{im} - \delta_{km}\delta_{i\ell},$$

Product identity satisfied by the permutation symbol, ϵ_{ijk}.

where δ_{ij} is the Kronecker delta. Note that the Einstein summation convention is used in this identity; i.e., summing over j is understood. So, the left side is really a sum of three terms:

$$\epsilon_{jki}\epsilon_{j\ell m} = \epsilon_{1ki}\epsilon_{1\ell m} + \epsilon_{2ki}\epsilon_{2\ell m} + \epsilon_{3ki}\epsilon_{3\ell m}.$$

This identity is simple to understand. For nonzero values of the Levi-Civita symbol, we have to require that all indices differ for each factor on the left side of the equation: $j \neq k \neq i$ and $j \neq \ell \neq m$. Since the first two slots are the same j, and the indices only take values $1, 2$, or 3, then either $k = \ell$ or $k = m$. This will give terms with factors of $\delta_{k\ell}$ or δ_{km}. If the former is true, then there is only one possibility for the third slot, $i = m$. Thus, we have a term $\delta_{k\ell}\delta_{im}$. Similarly, the other case yields the second term on the right side of the identity. We just need to get the signs right. Obviously, changing the order of ℓ and m will introduce a minus sign. A little care will show that the identity gives the correct ordering.

Other identities involving the permutation symbol are

$$\epsilon_{mjk}\epsilon_{njk} = 2\delta_{mn},$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6.$$

We will end this section by recalling triple products. There are only two ways to construct triple products. Starting with the cross product $\mathbf{b} \times \mathbf{c}$, which is a vector, we can multiply the cross product by a $\mathbf{a}$ to either obtain a scalar or a vector.

In the first case we have the triple scalar product, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Actually, we do not need the parentheses. Writing $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ could only mean one thing. If we computed $\mathbf{a} \cdot \mathbf{b}$ first, we would get a scalar. Then the result would be a multiple of $\mathbf{c}$, which is not a scalar. So, leaving off the parentheses would mean that we want the triple scalar product by convention.

Let's consider the component form of this product. We will use the Einstein summation convention and the fact that the permutation symbol is cyclic in ijk. Using $\epsilon_{jki} = \epsilon_{ijk}$,

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{a}_i (\mathbf{b} \times \mathbf{c})_i \\ &= \epsilon_{jki} a_i b_j c_k \\ &= \epsilon_{ijk} a_i b_j c_k \\ &= (\mathbf{a} \times \mathbf{b})_k c_k \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \end{aligned} \tag{8.24}$$

We have proven that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Now, imagine how much writing would be involved if we had expanded everything out in terms of all of the components.

Note that this result suggests that the triple scalar product can be computed by just computing a determinant:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \epsilon_{ijk} a_i b_j c_k \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \end{aligned} \tag{8.25}$$

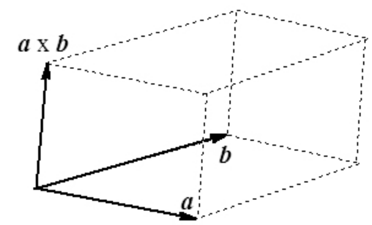

Figure 8.9: Three non-coplanar vectors define a parallelepiped. The volume is given by the triple scalar product, $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$.

There is a geometric interpretation of the scalar triple product. Consider the three vectors drawn as in Figure 8.9. If they do not all lie in a plane, then they form the sides of a parallelepiped. The cross product $\mathbf{a} \times \mathbf{b}$ gives the area of the base as we had seen earlier. The cross product is perpendicular to this base. The dot product of $\mathbf{c}$ with this cross product gives the height of the parallelepiped. So, the volume of the parallelepiped is the height times the base, or the triple scalar product. In general, one gets a signed volume, as the cross product could be pointing below the base.

The second type of triple product is the triple cross product,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \epsilon_{mnj} \epsilon_{ijk} a_i b_m c_n \mathbf{e}_k.$$

In this case we cannot drop the parentheses as this would lead to a real ambiguity. Lets think a little about this product. The vector $\mathbf{b} \times \mathbf{c}$ is a vector that is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$. Computing the triple cross product would then produce a vector perpendicular to $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. But the later vector is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ already. Therefore, the triple cross product must lie in the plane spanned by these vectors. In fact, there is an identity that tells us exactly the right combination of vectors $\mathbf{b}$ and $\mathbf{c}$. It is given by

The BAC-CAB rule.

$$\boxed{\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).} \tag{8.26}$$

This rule is called the BAC-CAB rule because of the order of the right side of this equation.

Example 8.4. *Prove that* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$.

We can prove the BAC-CAB rule the permutation symbol and some identities. We first use the cross products $\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k$ *and* $\mathbf{b} \times \mathbf{c} = \epsilon_{mnj} b_m c_n \mathbf{e}_j$:

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (a_i \mathbf{e}_i) \times ((\mathbf{b} \times \mathbf{c})_j \mathbf{e}_j) \\ &= a_i (\mathbf{b} \times \mathbf{c})_j (\mathbf{e}_i \times \mathbf{e}_j) \\ &= a_i (\mathbf{b} \times \mathbf{c})_j \epsilon_{ijk} \mathbf{e}_k \\ &= \epsilon_{mnj} \epsilon_{ijk} a_i b_m c_n \mathbf{e}_k \end{aligned} \tag{8.27}$$

Now, we use the identity

$$\epsilon_{mnj}\epsilon_{ijk} = \delta_{mk}\delta_{ni} - \delta_{mi}\delta_{nk},$$

the properties of the Kronecker delta functions, and then rearrange the results to finish the proof:

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \epsilon_{mnj}\epsilon_{ijk} a_i b_m c_n \mathbf{e}_k \\
&= a_i b_m c_n \left(\delta_{mk}\delta_{ni} - \delta_{mi}\delta_{nk}\right) \mathbf{e}_k \\
&= a_n b_m c_n \mathbf{e}_m - a_m b_m c_n \mathbf{e}_n \\
&= (b_m \mathbf{e}_m)(c_n a_n) - (c_n \mathbf{e}_n)(a_m b_m) \\
&= \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \qquad (8.28)
\end{aligned}$$

8.1.2 *Differentiation and Integration of Vectors*

YOU HAVE ALREADY BEEN INTRODUCED to the idea that vectors can be differentiated and integrated in your introductory physics course. These ideas are also the major theme encountered in a multivariate calculus class, or Calculus III. We review some of these topics in the next sections. We first recall the differentiation and integration of vector functions.

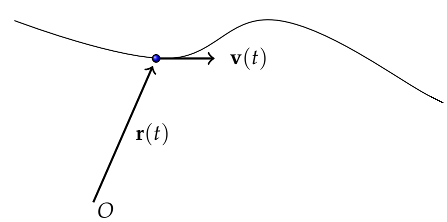

Figure 8.10: Position and velocity vectors of moving particle.

The position vector can change in time, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + x(t)\mathbf{k}$. The rate of change of this vector is the velocity,

$$\begin{aligned}
\mathbf{v}(t) &= \frac{d\mathbf{r}}{dt} \\
&= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\
&= v_x \mathbf{i} + v_y \mathbf{k} + v_z \mathbf{k}. \qquad (8.29)
\end{aligned}$$

The velocity vector is tangent to the path, $\mathbf{r}(t)$, as seen in Figure 8.1.2. The magnitude of this vector gives the speed,

$$|\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

Moreover, differentiating this vector gives the acceleration, $\mathbf{a}(t) = \mathbf{v}'(t)$.

In general, one can differentiate an arbitrary time-dependent vector $\mathbf{v}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ as

$$\frac{d\mathbf{v}}{dt} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}. \qquad (8.30)$$

Example 8.5. *A simple example is given by the motion on a circle. A circle in the xy-plane can be parametrized as* $\mathbf{r}(t) = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}$. *Then the velocity is found as*

$$\mathbf{v}(t) = -r\omega\sin(\omega t)\mathbf{i} + r\omega\cos(\omega t)\mathbf{j}.$$

Its speed is $v = r\omega$, *which is easily recognized as the tangential speed. The acceleration is*

$$\mathbf{a}(t) = -\omega^2 r\cos(\omega t)\mathbf{i} - \omega^2 r\sin(\omega t)\mathbf{j}.$$

The magnitude gives the centripetal acceleration, $a = \omega^2 r$ *and the acceleration vector is pointing towards the center of the circle.*

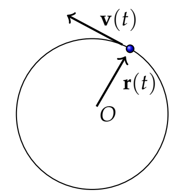

Figure 8.11: Particle on circular path.

Once one can differentiate time-dependent vectors, one can prove some standard properties.

a. $\frac{d}{dt}[\mathbf{u}+\mathbf{v}] = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$.

b. $\frac{d}{dt}[c\mathbf{u}] = c\frac{d\mathbf{u}}{dt}$.

c. $\frac{d}{dt}[f(t)\mathbf{u}] = f'(t)\mathbf{u} + f(t)\frac{d\mathbf{u}}{dt}$.

d. $\frac{d}{dt}[\mathbf{u}\cdot\mathbf{v}] = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}$.

e. $\frac{d}{dt}[\mathbf{u}\times\mathbf{v}] = \frac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\frac{d\mathbf{v}}{dt}$.

f. $\frac{d}{dt}[\mathbf{u}(f(t))] = \frac{d\mathbf{u}}{df}\frac{df}{dt}$.

Example 8.6. *Let* $|\mathbf{r}(t)|$ *=const. Then,* $\mathbf{r}'(t)$ *is perpendicular* $\mathbf{r}(t)$.

Since $|\mathbf{r}|$ *=const,* $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{r}$ *=const. Differentiating this expression, one has* $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0$. *Therefore,* $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0$, *as was to be shown.*

In this discussion, we have referred to t as the time. However, when parametrizing spacecurves, t could represent any parameter. For example, the circle could be parametrized for t the angle swept out along any arc of the circle, $\mathbf{r}(t) = r\cos t\mathbf{i} + r\sin t\mathbf{j}$, for $t_1 \le t \le t_2$. We can still differentiate with respect to this parameter. It not longer has the meaning of velocity. another standard parameter is that of arclength. The arclength of a path is the distance along the path from some starting point. In deriving an expression for arclength, one first considers incremental distances along paths. Moving from point (x, y, z) to point $(x+\Delta x, y+\Delta y, z+\Delta z)$, one has gone a distance of

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.$$

Given a curve parametrized by t, such as the time, one can rewrite this as

$$\Delta s = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\,\Delta t.$$

Letting Δt get small, as well as the other increments, we are led to

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \tag{8.31}$$

We note that the square root is $|\mathbf{r}'(t)|$. So,

$$ds = |\mathbf{r}'(t)| dt,$$

or

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

In order to find the total arclength, we need only integrate over the parameter range,

$$s = \int_{t_1}^{t_2} |\mathbf{r}'(t)|\, dt.$$

If t is time and $\mathbf{r}(t)$ is the position vector of a particle, then $|\mathbf{r}'(t)|$ is the particle speed and we have that the distance traveled is simply an integral of the speed,

$$s = \int_{t_1}^{t_2} v\, dt.$$

If one is interested in knowing the distance traveled from point $\mathbf{r}(t_1)$ to an arbitrary point $\mathbf{r}(t)$, one can define the arclength function

$$s(t) = \int_{t_1}^{t} |\mathbf{r}'(\tau)|\, d\tau.$$

Example 8.7. *Determine the length of the parabolic path described by* $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j}$, $t \in [0, 1]$.

We want to determine the length, $L = \int_0^1 |\mathbf{r}'(t)|\, dt$, *of a path. First, we have* $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$. *Then,* $|\mathbf{r}'(t)| = \sqrt{1 + 4t^2}$. *Using*

$$\int \sqrt{t^2 + a^2}\, dt = \frac{1}{2}\left(t\sqrt{t^2 + a^2} + a^2 \ln(t + \sqrt{t^2 + a^2})\right),$$

$$\begin{aligned} s &= \int_0^1 |\mathbf{r}'(t)|\, dt \\ &= \int_0^1 \sqrt{1 + 4t^2}\, dt \\ &= \left[x\sqrt{x^2 + \frac{1}{4}} + \frac{1}{4}\ln\left(x + \sqrt{x^2 + \frac{1}{4}}\right)\right]_0^1 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2 + \sqrt{5}). \end{aligned} \tag{8.32}$$

Line integrals are defined as integrals of functions along a path, or curve, in space. Let $f(x, y, z)$ be the function, and C a parametrized path. Then we are interested in computing $\int_C f(x,y,z)\,ds$, where s is the arclength parameter. This integral looks similar to the contour integrals that we had studied in Chapter 5. We can compute such integrals in a similar manner by introducing the parametrization:

$$\int_C f(x,y,z)\,ds = \int_C f(x(t),y(t),z(t))|\mathbf{r}'(t)|\,dt.$$

Example 8.8. *Compute* $\int_C (x^2+y^2+z^2)\,ds$ *for the helical path* $\mathbf{r} = (\cos t, \sin t, t)$, $t \in [0,2\pi]$.

In order to do this integral, we have to integrate over the given range of t values. So, we replace ds with $|\mathbf{r}'(t)|dt$. *In this problem* $|\mathbf{r}'(t)| = \sqrt{2}$. *Also, we insert the parametric forms for* $x(t) = \cos t$, $y(t) = \sin t$, *and* $z = t$ *into* $f(x,y,z)$. *Thus,*

$$\int_C (x^2+y^2+z^2)\,ds = \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt = 2\sqrt{2}\pi\left(1+\frac{4\pi^2}{3}\right). \quad (8.33)$$

One can also integrate vector functions. Given the vector function $\mathbf{v}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, we can do a straight forward term by term integration,

$$\int_a^b \mathbf{v}(t)\,dt = \int_a^b f(t)\,dt\mathbf{i} + \int_a^b g(t)\,dt\mathbf{j} + \int_a^b h(t)\,dt\mathbf{k}.$$

If $\mathbf{v}(t)$ is the velocity and t is the time, then

$$\int_a^b \mathbf{v}(t)\,dt = \int_a^b \frac{d\mathbf{r}}{dt}\,dt = \mathbf{r}(b) - \mathbf{r}(a).$$

We can thus interpret this integral as giving the displacement of a particle between times $t = a$ and $t = b$.

At the beginning of this chapter we had recalled the work done on a body by a nonconstant force $\mathbf{F}$ over a path C,

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} \quad (8.34)$$

If the path is parametrized by t, then we can write $d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt$. Thus the prescription for computing line integrals such as this is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt.$$

There are other forms that such line integrals can take. Let $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$. Noting that $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{y} + dz\mathbf{k}$, then we can write

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

Example 8.9. *Compute the work done by the force* $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ *on a particle as it moves around the circle* $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, *for* $0 \leq t \leq \pi$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\,dx - x\,dy.$$

One way to complete this is to note that $dx = -\sin t\,dt$ *and* $dy = \cos t\,dt$. *Then*

$$\int_C y\,dx - x\,dy = \int_0^\pi (-\sin^2 t - \cos^2 t)\,dt = -\pi.$$

8.1.3 Div, Grad, Curl

Throughout physics we see functions which vary in both space and time. A function $f(x, y, z, t)$ is called a scalar function when the output is a scalar, or number. An example of such a function is the temperature. A function $\mathbf{F}(x, y, z, t)$ is called a vector (or vector valued) function if the output of the function is a vector. Let $\mathbf{v}(x, y, z, t)$ represent the velocity of a fluid at position (x, y, z) at time t. This is an example of a vector function. Typically when we assign a number, or a vector, to every point in a domain, we refer to this as a scalar, or vector, field. In this section we discuss how fields change from one point in space to another. Namely, we look at derivatives of multivariate functions with respect to their independent variables and the meanings of these derivatives in a physical context.

In studying functions of one variable in calculus, one is introduced to the derivative, $\frac{df}{dx}$: The derivative has several meanings. The standard mathematical meaning is that the derivative gives the slope of the graph of $f(x)$ at x. The derivative also tells us how rapidly $f(x)$ varies when x is changed by dx. Recall that dx is called a differential. We can think of the differential dx as an infinitesimal increment in x. Then changing x by an amount dx results in a change in $f(x)$ by

$$df = \frac{df}{dx}\,dx.$$

We can extend this idea to functions of several variables. Consider the temperature $T(x, y, z)$ at a point in space. The change in temperature depends on the direction in which one moves in space. Extending the above relation between differentials of the dependent and independent variables, we have

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz. \tag{8.35}$$

Note that if we only changed x, keeping y and z fixed, then we recover the form $dT = \frac{dT}{dx}\,dx$.

Introducing the vectors,

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}, \tag{8.36}$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}, \tag{8.37}$$

The gradient of a function,

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}, .$$

we can write Equation (8.35) as

$$dT = \nabla T \cdot d\mathbf{r} \tag{8.38}$$

Equation (8.37) defines the gradient of a scalar function, T. Equation (8.38) gives the change in T as one moves in the direction $d\mathbf{r}$.

Using the definition of the dot product, we also have

$$dT = |\nabla T||d\mathbf{r}|\cos\theta.$$

Note that by fixing $|d\mathbf{r}|$ and varying θ, the maximum value of dT is obtained when $\cos\theta = 1$. Thus, the maximum value of dT is in the direction of the gradient. Similarly, since $\cos\pi = -1$, the minimum value of dT is in a direction 180° from the gradient.

The greatest change is a function is in the direction of its gradient.

Example 8.10. *Let $f(x, y, z) = x^2y + ze^{xy}$. Compute ∇f.*

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \\ &= (2xy + yze^{xy})\mathbf{i} + (x^2 + xze^{xy})\mathbf{j} + e^{xy}\mathbf{k}. \end{aligned} \tag{8.39}$$

From this analysis, we see that the rate of change of a function, such as $T(x, y, z,)$, depends on the direction one heads away from a given point. So, if one moves an infinitesimal distance ds in some direction $d\mathbf{r}$, then how does T change with respect to s? Another way to ask this is to ask what is the directional derivative of T in direction $\mathbf{n}$? We define this directional derivative as

The directional derivative of a function, $D_{\mathbf{n}}T = \frac{dT}{ds} = \nabla T \cdot \mathbf{n}$.

$$D_{\mathbf{n}}T = \frac{dT}{ds}. \tag{8.40}$$

We can develop an operational definition of the directional derivative. From Equation (8.38) we have

$$\frac{dT}{ds} = \nabla T \cdot \frac{d\mathbf{r}}{ds}. \tag{8.41}$$

We note that

$$\frac{d\mathbf{r}}{ds} = \left(\frac{dx}{ds}\right)\mathbf{i} + \left(\frac{dy}{ds}\right)\mathbf{j} + \left(\frac{dz}{ds}\right)\mathbf{k}$$

and

$$\left|\frac{d\mathbf{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = 1.$$

Thus, $\mathbf{n} = \frac{d\mathbf{r}}{ds}$ is a unit vector pointing in the direction of interest and the directional derivative of $T(x, y, z)$ in direction $\mathbf{n}$ can be written as

$$D_{\mathbf{n}}T = \nabla T \cdot \mathbf{n}. \tag{8.42}$$

Example 8.11. *Let the temperature in a rectangular plate be given by* $T(x,y) = 5.0 \sin \frac{3\pi x}{2} \sin \frac{\pi y}{2}$. *Determine the directional derivative at* $(x,y) = (1,1)$ *in the following directions: (a)* $\mathbf{i}$, *(b)* $3\mathbf{i} + 4\mathbf{j}$.

In part (a) we have

$$D_{\mathbf{i}} T = \nabla T \cdot \mathbf{i} = \frac{\partial T}{\partial x}.$$

So,

$$D_{\mathbf{i}} T \bigg|_{(1,1)} = \frac{15}{2} \cos \frac{3\pi}{2} \sin \frac{\pi}{2} = 0.$$

In part (b) the direction given is not a unit vector, $|3\mathbf{i} + 4\mathbf{j}| = 5$. *Dividing by the length of the vector, we obtain a unit normal vector,* $\mathbf{n} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$. *The directional derivative can now be computed:*

$$\begin{aligned} D_{\mathbf{n}} T &= \nabla T \cdot \mathbf{n} \\ &= \frac{3}{5}\frac{\partial T}{\partial x} + \frac{4}{5}\frac{\partial T}{\partial y} \\ &= \frac{9\pi}{2} \cos \frac{3\pi x}{2} \sin \frac{\pi y}{2} + 2\pi \sin \frac{3\pi x}{2} \cos \frac{\pi y}{2}. \end{aligned} \tag{8.43}$$

Evaluating this result at $(x,y) = (1,1)$, *we have*

$$D_{\mathbf{n}} T \bigg|_{(1,1)} = \frac{9\pi}{2} \cos \frac{3\pi}{2} \sin \frac{\pi}{2} + 2\pi \sin \frac{3\pi}{2} \cos \frac{\pi}{2} = 0.$$

We can write the gradient in the form

$$\nabla T = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) T. \tag{8.44}$$

Thus, we see that the gradient can be viewed as an operator acting on T. The operator,

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k},$$

is called the *del*, or *gradient*, operator. It is a differential vector operator. It can act on scalar functions to produce a vector field. Recall, if the gravitational potential is given by $\Phi(\mathbf{r})$, then the gravitational force is found as $\mathbf{F} = -\nabla\Phi$.

We can also allow the del operator to act on vector fields. Recall that a *vector field* is simply a vector valued function. For example, a force field is a function defined at points in space indicating the force that would act on a mass placed at that location. We could denote it as $\mathbf{F}(x,y,z)$. Again, think about the gravitational force above. The force acting on a mass in the Earth's gravitational field is a given by a vector field. At each point in space one would see that the force vector takes on different magnitudes and directions depending upon the location of the mass in space.

How can we combine the (vector) del operator and a vector field? Well, we could "multiply" them. We could either compute the dot product, $\nabla \cdot \mathbf{F}$, or we could compute the cross product $\nabla \times \mathbf{F}$. The first expression is called the *divergence* of the vector field and the second is called the *curl* of the vector field. These are typically encountered in a third semester calculus course. In some texts they are denoted by div $\mathbf{F}$ and curl $\mathbf{F}$.

The divergence is computed the same as any other dot product. Writing the vector field in component form,

The divergence, div $\mathbf{F} = \nabla \cdot \mathbf{F}$.

$$\mathbf{F} = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k},$$

we find the divergence is simply given as

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \end{aligned} \tag{8.45}$$

Similarly, we can compute the curl of $\mathbf{F}$. Using the determinant form, we have

The curl $\mathbf{F} = \nabla \times \mathbf{F}$.

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \end{aligned} \tag{8.46}$$

Example 8.12. *Compute the divergence and curl of the vector field:* $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

$$\nabla \cdot \mathbf{F} = \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} = 0.$$

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ y & -x & 0 \end{vmatrix} \\ &= \left(-\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right)\mathbf{k} = -2. \end{aligned} \tag{8.47}$$

These operations also have interpretations. The divergence measures how much the vector field $\mathbf{F}$ spreads from a point. When the divergence of a vector field is nonzero around a point, that is an indication that there is a source (div $\mathbf{F} > 0$) or a sink (div $\mathbf{F} < 0$). For example, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ indicates that there are sources contributing to the electric ﬁled. For a single charge, the field lines are radially pointing towards (sink) or away from (source) the charge. A field in which the divergence is zero is called divergenceless or solenoidal.

The curl is an indication of a rotational field. It is a measure of how much a field curls around a point. Consider the flow of a stream. The velocity of each element of fluid can be represented by a velocity field. If the curl of the field is nonzero, then when we drop a leaf into the stream we will see it begin to rotate about some point. A field that has zero curl is called irrotational.

The last common differential operator is the Laplace operator. This is the common second derivative operator, the divergence of the gradient,

$$\nabla^2 f = \nabla \cdot \nabla f.$$

The Laplace operator, $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

It is easily computed as

$$\begin{aligned} \nabla^2 f &= \nabla \cdot \nabla f \\ &= \nabla \cdot \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned} \tag{8.48}$$

8.1.4 The Integral Theorems

MAXWELL'S EQUATIONS ARE GIVEN LATER IN THIS CHAPTER in differential form and only describe electric and magnetic fields locally. At times we would like to also provide global information, or information over an finite region. In this case one can derive various integral theorems. These are the finale in a three semester calculus sequence. We will not delve into these theorems here, as this will take us away from our goal of deriving a three dimensional wave equation. However, these integral theorems are important and useful in deriving local conservation laws.

These theorems are all different versions of a generalized Fundamental Theorem of Calculus:

(a) $\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$, The Fundamental Theorem of Calculus in 1D.

(b) $\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{r} = T(\mathbf{b}) - T(\mathbf{a})$, The Fundamental Theorem of Calculus for Vector Fields.

(c) $\oint_C (P\,dx + Q\,dy) = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$, Green's Theorem in the Plane.

(d) $\int_V \nabla \cdot \mathbf{F}\,dV = \oint_S \mathbf{F} \cdot d\mathbf{a}$, Gauss' Divergence Theorem.

(e) $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint_C \mathbf{F} \cdot d\mathbf{r}$, Stoke's Theorem.

The connections between these integral theorems are probably more easily seen by thinking in terms of fluids. Consider a fluid with mass density $\rho(x,y,z)$ and fluid velocity $\mathbf{v}(x,y,z,t)$. We define $(Q) = \rho\mathbf{v}$ as the mass flow rate. [Note the units are kg/m^2/s indicating the mass per area per time.]

Now consider the fluid flowing through an imaginary rectangular box. Let the fluid flow into the left face and out the right face. The rate at which the fluid mass flows through a face can be represented by $\mathbf{Q}\cdot d\boldsymbol{\sigma}$, where $d\boldsymbol{\sigma} = \mathbf{n}d\sigma$ represents the differential area element normal to the face. The rate of flow across the left face is

$$\mathbf{Q}\cdot d\boldsymbol{\sigma} = -Q_y\,dxdz\Big|_y$$

and that flowing across the right face is

$$\mathbf{Q}\cdot d\boldsymbol{\sigma} = Q_y\,dxdz\Big|_{y+dy}.$$

The net flow rate is the sum of these

$$Q_y dxdz\Big|_{y+dy} - Q_y dxdz\Big|_y = \frac{\partial Q_y}{\partial y}\,dxdydz.$$

A similar computation can be done for the other faces, leading to the result that the total rate of flow is $\nabla\cdot\mathbf{Q}\,d\tau$, where $d\tau = dxdydz$ is the volume element. So, the rate of flow per volume from the volume element gives

$$\nabla\cdot\mathbf{Q} = -\frac{\partial\rho}{\partial t}.$$

Note that if more fluid is flowing out the right face than is flowing into the left face, then the amount of fluid inside the region will decrease. That is why the right hand side of this equation has the negative sign.

Conservation of mass equation,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{Q} = 0.$$

If the fluid is incompressible, i.e., ρ =const., then $\nabla\cdot\mathbf{Q} = 0$, which implies $\nabla\cdot\mathbf{v} = 0$ assuming there are no sinks or sources. If there were a sink in the rectangular box, then there would be a loss of fluid not accounted for. Likewise, is a hose were inserted and fluid were supplied, then one would have a source.

If there are sinks, or sources, then the net mass due to these would contribute to an overall flow through the surrounding surface. This is captured by the equation

Gauss' Divergence Theorem

$$\underbrace{\int_V \nabla\cdot\mathbf{Q}\,d\tau}_{\text{Net mass due to sink/sources}} = \underbrace{\oint_S \mathbf{Q}\cdot\mathbf{n}\,d\sigma}_{\text{Net flow outward from } S}. \quad (8.49)$$

Dividing out the constant mass density, since $\mathbf{Q} = \rho\mathbf{v}$, this becomes

$$\boxed{\int_V \nabla\cdot\mathbf{v}\,d\tau = \oint_S \mathbf{v}\cdot\mathbf{n}\,d\sigma.} \quad (8.50)$$

The surface integral on the right had side is called the flux of the vector field through surface S. This is nothing other than Gauss' Divergence Theorem.[1]

The unit normal can be written in terms of the direction cosines,

$$\mathbf{n} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k},$$

where the angles are the directions between $\mathbf{n}$ and the coordinate axes. For example, $\mathbf{n}\cdot\mathbf{i} = \cos\alpha$. For vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, we have

$$\begin{aligned}\int_S \mathbf{v}\cdot\mathbf{n}\,d\sigma &= \int_S (v_1\cos\alpha + v_2\cos\beta + v_3\cos\gamma)\,d\sigma \\ &= \int_S (u_1 dydz + u_2 dzdx + u_3 dxdy). \qquad (8.51)\end{aligned}$$

[1] We should note that the Divergence Theorem holds provided $\mathbf{v}$ is a continuous vector field and has continuous partial derivatives in a domain containing V. Also, $\mathbf{n}$ is the outward normal to the surface S.

Example 8.13. *Use the Divergence Theorem to compute*

$$\int_S (x^2 dydz + y^2 dzdx + z^2 dxdy)$$

for S the surface of the unit cube, $[0,1]\times[0,1]\times[0,1]$.

We first compute the divergence of the vector $\mathbf{v} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, *which we obtained from the coefficients in the given integral. Then*

$$\nabla\cdot\mathbf{v} = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 2(x+y+z).$$

Then,

$$\begin{aligned}\int_S (x^2 dydz + y^2 dzdx + z^2 dxdy) &= \int_V 2(x+y+z)\,d\tau \\ &= 2\int_0^1\int_0^1\int_0^1 (x+y+z)\,dxdydz \\ &= 2\int_0^1\int_0^1 (\frac{1}{2}+y+z)\,dydz \\ &= 2\int_0^1 (\frac{1}{2}+\frac{1}{2}+z)\,dz \\ &= 2(1+\frac{1}{2}) = 3. \qquad (8.52)\end{aligned}$$

The other integral theorem's are just a variation of the divergence theorem. For example, a two dimensional version of this obtained by considering a simply connected region, D, bounded by a simple closed curve, C. One could think of the laminar flow of a thin sheet of fluid. Then the total mass in contained in D and the net mass would be related to the next flow through the boundary, C. The integral theorem for this situation is given as

$$\int_D \nabla\cdot\mathbf{v}\,dA = \oint_C \mathbf{v}\cdot\mathbf{n}\,ds. \qquad (8.53)$$

The tangent vector to the curve at point $\mathbf{r}$ on the curve C, is

$$\frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}.$$

Therefore, the outward normal at that point is given by

$$\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

Letting $\mathbf{v} = Q(x,y)\mathbf{i} - P(x,y)\mathbf{j}$, the two dimensional version of the Divergence Theorem becomes

Green's Theorem in the Plane, which is a special case of Stoke's Theorem.

$$\oint_C (P\,dx + Q\,dy) = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \tag{8.54}$$

This is just Green's Theorem in the Plane.

Example 8.14. *Evaluate $\int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy$ for C given by $x^2 + 4y^2 = 4$.*

Green's Theorem in the Plane gives

$$\begin{aligned}\int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy &= \int_S \left(\frac{\partial}{\partial x}(e^y + 6x) - \frac{\partial}{\partial y}(e^x - 3y)\right) dxdy \\ &= \int_S (6+3)\ dxdy \\ &= 9\int_S dxdy. \end{aligned} \tag{8.55}$$

The integral that we need to compute is simply the area of the ellipse $x^2 + 4y^2 = 4$. Recall that the area of an ellipse with semimajor axis a and semiminor axis b is πab. For this ellipse $a = 2$ and $b = 1$. So,

$$\int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy = 18\pi.$$

We can obtain Stoke's Theorem by applying the Divergence Theorem to the vector $\mathbf{v} \times \mathbf{n}$.

$$\int_V \nabla \cdot (\mathbf{v} \times \mathbf{n})\,d\tau = \oint_S \mathbf{n}_s \cdot (\mathbf{v} \times \mathbf{n})\,d\sigma. \tag{8.56}$$

Here $\mathbf{n}_s = \mathbf{u} \times \mathbf{n}$ where $\mathbf{u}$ is tangent to the curve C, and $\mathbf{n}$ is normal to the domain D. Noting that $(\mathbf{u} \times \mathbf{n}) \times (\mathbf{v} \times \mathbf{n}) = \mathbf{v} \cdot \mathbf{u}$ and $\nabla \cdot (\mathbf{v} \times \mathbf{n}) = \mathbf{n} \cdot \nabla \times \mathbf{v}$, then

$$\int_0^h \left(\int_D \mathbf{n} \cdot \nabla \times \mathbf{v}\,d\sigma\right) dh = \int_0^h \left(\oint_C \mathbf{v} \cdot \mathbf{u}\,ds\right) dh. \tag{8.57}$$

Since h is arbitrary, then we obtain Stoke's Theorem:

Stoke's Theorem.

$$\int_D \mathbf{n} \cdot \nabla \times \mathbf{v}\,d\sigma = \oint_C \mathbf{v} \cdot \mathbf{u}\,ds. \tag{8.58}$$

Example 8.15. *Evaluate $\int_C (z\,dx + x\,dy + y\,dz)$ for C the boundary of the triangle with vertices (1,0,0), (0,1,0), (0,0,1) using Stoke's Theorem.*

We first identify the vector $\mathbf{v} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$. *Then, we compute the curl,*

$$\begin{aligned}\nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ z & x & y \end{vmatrix} \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k}. \end{aligned} \tag{8.59}$$

Stoke's Theorem then gives

$$\int_C (z\,dx + x\,dy + y\,dz) = \int_D \mathbf{n} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\, d\sigma,$$

where $\mathbf{n}$ *is the outward normal to the surface of the triangle. For a surface defined by* $\phi(x,y,z) =$*const, the normal is in the direction of* $\nabla\phi$. *In this case the triangle lives in the plane* $x + y + z = 1$. *Thus,* $\phi(x,y,z) = x + y + z$ *and* $\nabla\phi = \mathbf{i} + \mathbf{j} + \mathbf{k}$. *Thus,*

$$\int_C (z\,dx + x\,dy + y\,dz) = 3\int_D d\sigma.$$

The remaining integral is just the area of the triangle. We can determine this area as follows. Imagine the vectors $\mathbf{a}$ *and* $\mathbf{b}$ *pointing from (1,0,0) to (0,1,0) and from (1,0,0) to (0,0,1), respectively. So,* $\mathbf{a} = \mathbf{j} - \mathbf{i}$ *and* $\mathbf{b} = \mathbf{k} - \mathbf{i}$.

These vectors are the sides of a parallelogram whose area is twice that of the triangle. The area of the parallelogram is given by $|\mathbf{a} \times \mathbf{b}|$. *The area of the triangle is thus*

$$\begin{aligned}\int_D d\sigma &= \frac{1}{2}|\mathbf{a} \times \mathbf{b}| \\ &= \frac{1}{2}|(\mathbf{j} - \mathbf{i}) \times (\mathbf{k} - \mathbf{i})| \\ &= \frac{1}{2}|\mathbf{i} + \mathbf{j} + \mathbf{k}| = \frac{3}{2}. \end{aligned} \tag{8.60}$$

Finally, we have

$$\int_C (z\,dx + x\,dy + y\,dz) = \frac{9}{2}.$$

8.1.5 Vector Identities

IN THIS SECTION we will list some common vector identities and show how to prove a few of them. We will introduce two triple products and list first derivative and second derivative identities. These are useful in reducing some equations into simpler forms.

Proving these identities can be straight forward, though sometimes tedious in the more complicated cases. You should try to prove these yourself. Sometimes it is useful to write out the components on each side of the identity and see how one can fill in the needed arguments which would provide the proofs. We will provide a couple of examples of this process.

1. Triple Products
   (a) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$
   (b) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$
2. First Derivatives
   (a) $\nabla(fg) = f\nabla g + g\nabla f$
   (b) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
   (c) $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$
   (d) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
   (e) $\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} - \mathbf{A} \times \nabla f$
   (f) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$
3. Second Derivatives
   (a) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ — div curl $= 0$.
   (b) $\nabla \times (\nabla f) = 0$ — curl grad$= \mathbf{0}$.
   (c) $\nabla \cdot (\nabla f \times \nabla g) = 0$
   (d) $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$
   (e) $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
   (f) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

Example 8.16. *Prove* $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$.

In such problems one can write out the components on both sides of the identity. Using the determinant form of the triple scalar, the left hand side becomes

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \\ &= A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1). \end{aligned} \tag{8.61}$$

Similarly, the right hand side is given as

$$\begin{aligned} \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) &= \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= B_1(C_2A_3 - C_3A_2) - B_2(C_1A_3 - C_3A_1) + B_3(C_1A_2 - C_2A_1). \end{aligned} \tag{8.62}$$

We can rearrange this result by separating out the components of $\mathbf{A}$.

$$\begin{aligned} B_1(C_2A_3 &- C_3A_2) - B_2(C_1A_3 - C_3A_1) + B_3(C_1A_2 - C_2A_1) \\ &= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1). \end{aligned} \tag{8.63}$$

Upon inspection, we see that we obtain the same result as we had for the left hand side.

This problem can also be solved using the completely antisymmetric symbol, ϵ_{ijk}. Recall that the scalar triple product is given by

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \epsilon_{ijk} A_i B_j C_k.$$

(We have employed the Einstein summation convention.) Since $\epsilon_{ijk} = \epsilon_{jki}$, we have

$$\epsilon_{ijk} A_i B_j C_k = \epsilon_{jki} A_i B_j C_k = \epsilon_{jki} B_j C_k A_i.$$

But,

$$\epsilon_{jki} B_j C_k A_i = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}).$$

So, we have once again proven the identity. However, it took a little less work and an understanding of the antisymmetric symbol. Furthermore, you should note that this identity was proven earlier in the chapter.

Example 8.17. *Prove $\nabla(fg) = f\nabla g + g\nabla f$. In this problem we compte the gradient of fg. Then we note that each derivative is the derivative of a product and apply the Product Rule. Carefully writing out the terms, we obtain the desired result.*

$$\begin{aligned} \nabla(fg) &= \frac{\partial fg}{\partial x}\mathbf{i} + \frac{\partial fg}{\partial y}\mathbf{j} + \frac{\partial fg}{\partial z}\mathbf{k} \\ &= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) g + f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) \\ &= f\nabla g + g\nabla f. \end{aligned} \tag{8.64}$$

8.2 Electromagnetic Waves

8.2.1 Maxwell's Equations

There are many applications leading to the equations in Table 8.1. One goal of this chapter is to derive the three dimensional wave equation for electromagnetic waves. This derivation was first carried out by James Clerk Maxwell in 1860. At the time much was known about the relationship between electric and magnetic fields through the work of of such people as Hans Christian Ørstead (1777-1851), Michael Faraday (1791-1867), and André-Marie Ampère. Maxwell provided a mathematical formalism for these relationships consisting of twenty partial differential equations in twenty unknowns. Later these equations were put into more compact notations, namely in terms of quaternions, only later to be cast in vector form.

Quaternions were introduced in 1843 by William Rowan Hamilton (1805-1865) as a four dimensional generalization of complex numbers.

In vector form, the original Maxwell's equations are given as

$$\nabla \cdot \mathbf{D} = \rho$$

$$\begin{aligned}
\nabla \times \mathbf{H} &= \mu_0 \mathbf{J}_{\text{tot}} \\
\mathbf{D} &= \epsilon \mathbf{E} \\
\mathbf{J} &= \sigma \mathbf{E} \\
\mathbf{J}_{\text{tot}} &= \mathbf{J} \frac{\partial \mathbf{D}}{\partial t} \\
\nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \\
\mathbf{E} &= -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \\
\mu \mathbf{H} &= \nabla \times \mathbf{A}.
\end{aligned} \tag{8.65}$$

Note that Maxwell expressed the electric and magnetic fields in terms of the scalar and vector potentials, ϕ and $\mathbf{A}$, respectively, as defined in the last two equation. Here $\mathbf{H}$ is the magnetic field, $\mathbf{D}$ is the electric displacement field, $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the current density, ρ is the charge density, and σ is the conductivity.

This set of equations differs from what we typically present in physics courses. Several of these equations are defining quantities. While the potentials are part of a course in electrodynamics, they are not cast as the core set of equations now referred to as Maxwell's equations. Also, several equations are given as defining relations between the various variables, though they have some physical significance of their own, such as the continuity equation, given by $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$.

Furthermore, the distinction between the magnetic field strength, $\mathbf{H}$, and the magnetic flux density, $\mathbf{B}$, only becomes important in the presence of magnetic materials. Students are typically first introduced to $\mathbf{B}$ in introductory physics classes. In general, $\mathbf{B} = \mu \mathbf{H}$, where μ is the magnetic permeability of a material. In the absence of magnetic materials, $\mu = \mu_0$. In fact, in many applications of the propagation of electromagnetic waves, $\mu \approx \mu_0$.

These equations can be written in a more familiar form. The equations that we will refer to as Maxwell's equations from now on are

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \quad \text{(Gauss' Law)} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{(Faraday's Law)} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad \text{(Maxwell-Ampère Law)}
\end{aligned} \tag{8.66}$$

We have noted the common names attributed to each law. There are corresponding integral forms of these laws, which are often presented in introductory physics class. The first law is Gauss' law. It allows one to determine the electric field due to specific charge distributions. The second law typically has no name attached to it, but in some cases is called Gauss' law for magnetic fields. It simply states that there are no free magnetic poles. The third law is Faraday's law, indicating that changing magnetic flux induces electric potential differences. Lastly, the fourth law is a modification of Ampere's law that states that electric currents produce magnetic fields.

It should be noted that the last term in the fourth equation was introduced by Maxwell. As we have seen, the divergence of the curl of any vector is zero,

The divergence of the curl of any vector is zero.

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0.$$

Computing the divergence of the curl of the electric field, we find from Maxwell's equations that

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{E}) &= -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} \\ &= -\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0. \end{aligned} \tag{8.67}$$

Thus, the relation works here.

However, before Maxwell, Ampère's law in differential form would have been written as

Ampère's law in differential form.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Computing the divergence of the curl of the magnetic field, we have

The introduction of the displacement current makes Maxwell's equations mathematically consistent.

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{B}) &= \mu_0 \nabla \cdot \mathbf{J} \\ &= -\mu_0 \frac{\partial \rho}{\partial t}. \end{aligned} \tag{8.68}$$

Here we made use of the continuity equation,

$$\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \nabla \cdot \mathbf{J} = 0.$$

As you can see, the vector identity, div curl $= 0$, does not work here! Maxwell argued that we need to account for a changing charge distribution. He introduced what he called the displacement current, $\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ into the Ampère Law. Now, we have

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{B}) &= \mu_0 \nabla \cdot \left(\mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= -\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} \\ &= -\mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\epsilon_0} \right) = 0. \end{aligned} \tag{8.69}$$

So, Maxwell's introduction of the displacement current was not only physically important, it made the equations mathematically consistent.

8.2.2 *Electromagnetic Wave Equation*

WE ARE NOW READY to derive the wave equation for electromagnetic waves. We will consider the case of free space in which there are no free charges or currents and the waves propagate in a vacuum. We then have Maxwell's equations in the form

Maxwell's equations in a vacuum.

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{8.70}$$

We will derive the wave equation for the electric field. You should confirm that a similar result can be obtained for the magnetic field. Consider the expression $\nabla \times (\nabla \times \mathbf{E})$. We note that the identities give

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

However, the divergence of $\mathbf{E}$ is zero, so we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}. \tag{8.71}$$

We can also use Faraday's Law on the right side of this equation to obtain

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t} \right).$$

Interchanging the time and space derivatives, and using the Ampere-Maxwell Law, we find

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\ &= -\frac{\partial}{\partial t} \left(\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \end{aligned} \tag{8.72}$$

The three dimensional wave equations for electric and magnetic fields in a vacuum.

Combining the two expressions for $\nabla \times (\nabla \times \mathbf{E})$, we have the sought result:

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}.$$

This is the three dimensional equation for an oscillating electric field. A similar equation can be found for the magnetic field,

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla^2 \mathbf{B}.$$

Recalling that $\epsilon_0 = 8.85 \times 10^{-12}$ $\mathrm{C^2/Nm^2}$ and $\mu_0 = 4\pi \times 10^{-7}$ $\mathrm{N/A^2}$, one finds that $c = 3 \times 10^8$ m/s.

One can derive more general equations. For example, we could look for waves in what are called linear media. In this case one has $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. Here ϵ is called the electric permittivity and μ is the magnetic permeability of the material. Then, the wave speed in a vacuum, c, is replaced by the wave speed in the medium, v. It is given by

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}.$$

Here, n is the index of refraction, $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$. In many materials $\mu \approx \mu_0$. Introducing the dielectric constant, $\kappa = \frac{\epsilon}{\epsilon_0}$, one finds that $n \approx \sqrt{\kappa}$.

The wave equations lead to many of the properties of the electric and magnetic fields. We can also study systems in which these waves are confined, such as waveguides. In such cases we can impose boundary conditions and determine what modes are allowed to propagate within certain structures, such as optical fibers. However, these equation involve unknown vector fields. We have to solve for several inter-related component functions. In the next chapter we will look at simpler models in order to get some ideas as to how one can solve scalar wave equations in higher dimensions. However, we will first explore how the differential operators introduced in this chapter appear in different coordinate systems.

8.2.3 *Potential Functions and Helmholtz's Theorem*

Another application of the use of vector analysis for studying electromagnetism is that of potential theory. In this section we describe the use of a scalar potential, $\phi(\mathbf{r}, t)$ and a vector potential, $\mathbf{A}(\mathbf{r}, t)$ to solve problems in electromagnetic theory. Helmholtz's theorem says that a vector field is uniquely determined by knowing its divergence and its curl. Combining this result with the definitions of the electric and magnetic potentials, we will show that Maxwell's equations will the electric and magnetic fields can be found by simply solving a set of Poisson equations, $\nabla^2 u = f$, for the potential functions.

Hermann Ludwig Ferdinand von Helmholtz (1821-1894) made many contributions to physics. There are several theorems named after him.

A vector field is uniquely determined by knowing its divergence and its curl.

Electric and magnetic potentials.

In the case of static fields, we have from Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = 0.$$

We saw earlier in this chapter that the curl of a gradient is zero and the divergence of a curl is zero. This suggests that $\mathbf{E}$ is the gradient of a scalar function and $\mathbf{B}$ is the curl of a vector function:

$$\mathbf{E} = -\nabla\phi,$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

ϕ is called the electric potential and $\mathbf{A}$ is called the magnetic potential.

The remaining Maxwell equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Inserting the potential functions, we have

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}, \quad \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}.$$

Thus, ϕ satisfies a Poisson equation, which is a simple partial differential equation which can be solved using separation of variables, or other techniques.

The equation satisfied by the magnetic potential looks a little more complicated. However, we can use the identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

If $\nabla \cdot \mathbf{A} = 0$, then we find that

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

Thus, the components of the magnetic potential also satisfy Poisson equations!

It turns out that requiring $\nabla \cdot \mathbf{A} = 0$ is not as restrictive as one might first think. Potential functions are not unique. For example, adding a constant to a potential function will still give the same fields. For example

$$\nabla(\phi + c) = \nabla \phi = -\mathbf{E}.$$

This is not too alarming because it is the field that is physical and not the potential. In the case of the magnetic potential, adding the gradient of some field gives the same magnetic field, $\nabla \times (\mathbf{A} + \nabla \psi) = \nabla \times \mathbf{A} = \mathbf{B}$. So, we can choose ψ such that the new magnetic potential is divergenceless, $\nabla \cdot \mathbf{A} = 0$. A particular choice of the scalar and vector potentials is a called a gauge and the process is called fixing, or choosing, a gauge. The choice of $\nabla \cdot \mathbf{A} = 0$ is called the Coulomb gauge.

Coulomb gauge: $\nabla \cdot \mathbf{A} = 0$.

If the fields are dynamic, i.e., functions of time, then the magnetic potential also contributes to the electric field. In this case, we have

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t},$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Thus, two of Maxwell's equations are automatically satisfied,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

The other two equations become

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 \phi + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0},$$

and

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \Rightarrow$$
$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \frac{1}{c^2} \frac{\partial}{\partial t} \left(\nabla \phi + \frac{\partial \mathbf{A}}{\partial t} \right).$$

Rearranging, we have

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = -\mu_0 \mathbf{J}.$$

If we choose the Lorentz gauge, by requiring

Lorentz gauge: $\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$.

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t},$$

then

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = -\frac{\rho}{\epsilon_0},$$
$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = -\mu_0 \mathbf{J}.$$

In relativity, one defines the d'Alembertian by $\Box \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$. Then, the equations for the potentials become

$$\Box \phi = \frac{\rho}{\epsilon_0},$$

and

$$\Box \mathbf{A} = \mu_0 \mathbf{J}.$$

Thus, the potential satisfy nonhomogeneous wave equations, which can be solved with standard methods as one will see in a course in electrodynamics.

The above introduction of potentials to describe the electric and magnetic fields is a special case of Helmholtz's Theorem for vectors. This theorem states that "any sufficiently smooth, rapidly decaying vector field in three dimensions can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field."[2] This is known as the Helmholtz decomposition. Namely, given any nice vector field $\mathbf{v}$, we can write it as

[2] Wikipedia entry for the Helmholtz decomposition.

$$\mathbf{v} = \underbrace{-\nabla \phi}_{\text{irrotational}} + \underbrace{\nabla \times \mathbf{A}}_{\text{solenoidal}}.$$

Given

$$\nabla \cdot \mathbf{v} = \rho, \quad \nabla \times \mathbf{v} = \mathbf{F},$$

then one has

$$\nabla^2 \phi = \rho$$

and

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mathbf{F}.$$

Forcing $\nabla \cdot \mathbf{A} = 0$,

$$\nabla^2 \mathbf{A} = -\mathbf{F}.$$

Thus, one obtains Poisson equations for ϕ and $\mathbf{A}$. This is just repeating the above procedure which we had seen in the special case of static electromagnetic fields.

8.3 Curvilinear Coordinates

In order to study solutions of the wave equation, the heat equation, or even Schrödinger's equation in different geometries, we need to see how differential operators, such as the Laplacian, appear in these geometries. The most common coordinate systems arising in physics are polar coordinates, cylindrical coordinates, and spherical coordinates. These reflect the common geometrical symmetries often encountered in physics.

In such systems it is easier to describe boundary conditions and to make use of these symmetries. For example, specifying that the electric potential is 10.0 V on a spherical surface of radius one, we would say $\phi(x,y,z) = 10$ for $x^2 + y^2 + z^2 = 1$. However, if we use spherical coordinates, (r,θ,ϕ), then we would say $\phi(r,\theta,\phi) = 10$ for $r = 1$, or $\phi(1,\theta,\phi) = 10$. This is a much simpler representation of the boundary condition.

However, this simplicity in boundary conditions leads to a more complicated looking partial differential equation in spherical coordinates. In this section we will consider general coordinate systems and how the differential operators are written in the new coordinate systems. In the next chapter we will solve some of these new problems.

We begin by introducing the general coordinate transformations between Cartesian coordinates and the more general curvilinear coordinates. Let the Cartesian coordinates be designated by (x_1, x_2, x_3) and the new coordinates by (u_1, u_2, u_3). We will assume that these are related through the transformations

$$\begin{aligned} x_1 &= x_1(u_1, u_2, u_3), \\ x_2 &= x_2(u_1, u_2, u_3), \\ x_3 &= x_3(u_1, u_2, u_3). \end{aligned} \tag{8.73}$$

Thus, given the curvilinear coordinates (u_1, u_2, u_3) for a specific point in space, we can determine the Cartesian coordinates, (x_1, x_2, x_3), of that point. We will assume that we can invert this transformation: Given the Cartesian coordinates, one can determine the corresponding curvilinear coordinates.

Need to insert figures depicting this.

In the Cartesian system we can assign an orthogonal basis, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. As a particle traces out a path in space, one locates its position by the coordinates (x_1, x_2, x_3). Picking x_2 and x_3 constant, the particle lies on the curve $x_1 =$ value of the x_1 coordinate. This line lies in the direction of the basis vector $\mathbf{i}$. We can do the same with the other coordinates and essentially map out a grid in three dimensional space. All of the x_i-curves intersect at each point orthogonally and the basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ lie along the grid lines and are mutually orthogonal.

We would like to mimic this construction for general curvilinear coordinates. Requiring the orthogonality of the resulting basis vectors leads to orthogonal curvilinear coordinates.

As for the Cartesian case, we consider u_2 and u_3 constant. This leads to a curve parametrized by $u_1 : \mathbf{r} = x_1(u_1)\mathbf{i} + x_2(u_1)\mathbf{j} + x_3(u_1)\mathbf{k}$. We call this the u_1-curve. Similarly, when u_1 and u_3 are constant we obtain a u_2-curve and for u_1 and u_2 constant we obtain a u_3-curve. We will assume that these curves intersect such that each pair of curves intersect orthogonally. Furthermore, we will assume that the unit tangent vectors to these curves form a right handed system similar to the $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ systems for Cartesian coordinates. We will denote these as $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$.

We can quantify all of this. Consider the position vector as a function of the new coordinates,

$$\mathbf{r}(u_1, u_2, u_3) = x_1(u_1, u_2, u_3)\mathbf{i} + x_2(u_1, u_2, u_3)\mathbf{j} + x_3(u_1, u_2, u_3)\mathbf{k}.$$

Then the infinitesimal change in position is given by

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u_i} du_i.$$

We note that the vectors $\frac{\partial \mathbf{r}}{\partial u_i}$ are tangent to the u_i-curves. Thus, we define the unit tangent vectors

$$\hat{\mathbf{u}}_i = \frac{\frac{\partial \mathbf{r}}{\partial u_i}}{\left|\frac{\partial \mathbf{r}}{\partial u_i}\right|}.$$

Solving for the tangent vector, we have

$$\frac{\partial \mathbf{r}}{\partial u_i} = h_i \hat{\mathbf{u}}_i,$$

where

$$h_i \equiv \left|\frac{\partial \mathbf{r}}{\partial u_i}\right|$$

are called the *scale factors* for the transformation.

The scale factors, $h_i \equiv \left|\frac{\partial \mathbf{r}}{\partial u_i}\right|$.

Example 8.18. *Determine the scale factors for the polar coordinate transformation.*

Show an annotated polar plot here.

The transformation for polar coordinates is

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Here we note that $x_1 = x$, $y_1 = y$, $u_1 = r$, and $u_2 = \theta$. The u_1-curves are curves with $\theta =$ const. Thus, these curves are radial lines. Similarly, the u_2-curves have $r =$ const. These curves are concentric circles about the origin.

The unit vectors are easily found. We will denote them by $\hat{\mathbf{u}}_r$ *and* $\hat{\mathbf{u}}_\theta$*. We can determine these unit be first computing* $\frac{\partial \mathbf{r}}{\partial u_i}$*. Let*

$$\mathbf{r} = x(r,\theta)\mathbf{i} + y(r,\theta)\mathbf{j} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.$$

Then,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial \theta} &= -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}. \end{aligned} \tag{8.74}$$

The first vector already is a unit vector. So,

$$\hat{\mathbf{u}}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}.$$

The second vector has length r since $|-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}| = r$*. Dividing* $\frac{\partial \mathbf{r}}{\partial \theta}$ *by* r*, we have*

$$\hat{\mathbf{u}}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}.$$

We can see these vectors are orthogonal and form a right hand system. That they form a right hand system can be seen by either drawing the vectors, or computing the cross product,

$$(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \times (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \mathbf{k}.$$

Since

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \hat{\mathbf{u}}_r, \\ \frac{\partial \mathbf{r}}{\partial \theta} &= r\hat{\mathbf{u}}_\theta, \end{aligned}$$

The scale factors are $h_r = 1$ *and* $h_\theta = r$*.*

We have determined that once we know the scale factors, we have that

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i.$$

The infinitesimal arclength is then given by

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = \sum_{i=1}^{3} h_i^2 du_i^2$$

when the system is orthogonal. Also, along the u_i-curves,

$$d\mathbf{r} = h_i du_i \hat{\mathbf{u}}_i, \quad \text{(no summation)}.$$

So, we consider at a given point (u_1, u_2, u_3) an infinitesimal parallelepiped of sides $h_i du_i$, $i = 1, 2, 3$. This infinitesimal parallelepiped has a volume of size

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| du_1 du_2 du_3.$$

The triple scalar product can be computed using determinants and the resulting determinant is call the Jacobian, and is given by

$$\begin{aligned} J &= \left|\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}\right| \\ &= \left|\frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3}\right| \\ &= \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}. \end{aligned} \tag{8.75}$$

Therefore, the volume element can be written as

$$dV = J\,du_1du_2du_3 = \left|\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}\right| du_1du_2du_3.$$

Example 8.19. *Determine the volume element for cylindrical coordinates (r,θ,z), given by*

$$x = r\cos\theta, \tag{8.76}$$

$$y = r\sin\theta, \tag{8.77}$$

$$z = z. \tag{8.78}$$

Here, we have $(u_1,u_2,u_3) = (r,\theta,z)$. Then, the Jacobian is given by

$$\begin{aligned} J &= \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| \\ &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= r \end{aligned} \tag{8.79}$$

Thus, the volume element is given as

$$dV = r dr d\theta dz.$$

This result should be familiar from multivariate calculus.

Next we will derive the forms of the gradient, divergence, and curl in curvilinear coordinates. The results are given here for quick reference.

Gradient, divergence and curl in orthogonal curvilinear coordinates.

$$\nabla\phi = \sum_{i=1}^{3} \frac{\hat{\mathbf{u}}_i}{h_i}\frac{\partial\phi}{\partial u_i} = \frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial\phi}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial\phi}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial\phi}{\partial u_3}. \quad (8.80)$$

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_1h_3F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right). \quad (8.81)$$

$$\nabla\times\mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{u}}_1 & h_2\hat{\mathbf{u}}_2 & h_3\hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix}. \quad (8.82)$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right) \quad (8.83)$$

$$(8.84)$$

We begin the derivations of these formulae by looking at the gradient, $\nabla\phi$, of the scalar function $\phi(u_1, u_2, u_3)$. We recall that the gradient operator appears in the differential change of a scalar function,

Derivation of the gradient form.

$$d\phi = \nabla\phi\cdot d\mathbf{r} = \sum_{i=1}^{3}\frac{\partial\phi}{\partial u_i}du_i.$$

Since

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i,$$

we also have that

$$d\phi = \nabla\phi\cdot d\mathbf{r} = \sum_{i=1}^{3}(\nabla\phi)_i h_i du_i.$$

Comparing these two expressions for $d\phi$, we determine that the components of the del operator can be written as

$$(\nabla\phi)_i = \frac{1}{h_i}\frac{\partial\phi}{\partial u_i}$$

and thus the gradient is given by

$$\nabla\phi = \frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial\phi}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial\phi}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial\phi}{\partial u_3}.$$

Next we compute the divergence,

Derivation of the divergence form.

$$\nabla\cdot\mathbf{F} = \sum_{i=1}^{3}\nabla\cdot(F_i\hat{\mathbf{u}}_i).$$

We can do this by computing the individual terms in the sum. We will compute $\nabla\cdot(F_1\hat{\mathbf{u}}_1)$.

We first note that the gradients of the coordinate functions are found as $\nabla u_i = \frac{\hat{\mathbf{u}}_i}{h_i}$. (This results from a direct application of the gradient operator form just derived.) Then

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3}{h_2 h_3} = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}.$$

This gives

$$\begin{aligned} \nabla \cdot (F_1 \hat{\mathbf{u}}_1) &= \nabla \cdot (F_1 h_2 h_3 \nabla u_2 \times \nabla u_3) \\ &= \nabla (F_1 h_2 h_3) \cdot \nabla u_2 \times \nabla u_3 + F_1 h_2 h_2 \nabla \cdot (\nabla u_2 \times \nabla u_3). \end{aligned} \tag{8.85}$$

Here we used the vector identity

$$\nabla \cdot (f\mathbf{A}) = f \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

The second term can be handled using the identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

where **A** and **B** are gradients. However, each term the curl of a gradient, which are identically zero! Or, you could just use the third identity in the previous list of second derivative identities,

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

Using the expression $\nabla u_2 \times \nabla u_3 = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}$ and the expression for the gradient operator in curvilinear coordinates, we have

$$\nabla \cdot (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_2 h_3) \cdot \frac{\hat{\mathbf{u}}_1}{h_2 h_3} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (F_1 h_2 h_3).$$

Similar computations can be done for the remaining components, leading to the sought expression for the divergence in curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_1 h_3 F_2) + \frac{\partial}{\partial u_2} (h_1 h_2 F_3) \right).$$

We now turn to the curl operator. In this case, we need to simplify

Derivation of the curl form.

$$\nabla \times \mathbf{F} = \sum_{i=1}^{3} \nabla \times (F_i \hat{\mathbf{u}}_i).$$

Using the identity

$$\nabla \times (f\mathbf{A}) = f \nabla \times \mathbf{A} - \mathbf{A} \times \nabla f,$$

we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla \times (F_1 h_1 \nabla u_1) \\ &= \nabla (F_1 h_1) \times \nabla u_1 + F_1 h_1 \nabla \times \nabla u_1. \end{aligned} \tag{8.86}$$

Again, the curl of the gradient vanishes, leaving

$$\nabla \times (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_1) \times \nabla u_1.$$

Since $\nabla u_1 = \frac{\hat{\mathbf{u}}_1}{h_1}$, we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla (F_1 h_1) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \left(\sum_{i=1}^{3} \frac{\hat{\mathbf{u}}_i}{h_i} \frac{\partial (F_1 h_1)}{\partial u_i} \right) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \frac{\hat{\mathbf{u}}_2}{h_3 h_1} \frac{\partial (F_1 h_1)}{\partial u_3} - \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \frac{\partial (F_1 h_1)}{\partial u_2}. \end{aligned} \tag{8.87}$$

The other terms can be handled in a similar manner. The overall result is that

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{\hat{\mathbf{u}}_1}{h_2 h_3} \left(\frac{\partial (h_3 F_3)}{\partial u_2} - \frac{\partial (h_2 F_2)}{\partial u_3} \right) + \frac{\hat{\mathbf{u}}_2}{h_1 h_3} \left(\frac{\partial (h_1 F_1)}{\partial u_3} - \frac{\partial (h_3 F_3)}{\partial u_1} \right) \\ &\quad + \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \left(\frac{\partial (h_2 F_2)}{\partial u_1} - \frac{\partial (h_1 F_1)}{\partial u_2} \right) \end{aligned} \tag{8.88}$$

This can be written more compactly as

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix} \tag{8.89}$$

Finally, we turn to the Laplacian. In the next chapter we will solve higher dimensional problems in various geometric settings such as the wave equation, the heat equation, and Laplace's equation. These all involve knowing how to write the Laplacian in different coordinate systems. Since $\nabla^2 \phi = \nabla \cdot \nabla \phi$, we need only combine the above results for the gradient and the divergence in curvilinear coordinates. This is straight forward and gives

$$\begin{aligned} \nabla^2 \phi &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial u_2} \right) \right. \\ &\quad \left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right). \end{aligned} \tag{8.90}$$

The results of rewriting the standard differential operators in cylindrical and spherical coordinates are shown in Problems 28 and 29. In particular, the Laplacians are given as

Cylindrical Coordinates:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{8.91}$$

Spherical Coordinates:

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{8.92}$$

These forms will be used in the next chapter for the solution of Laplace's equation, the heat equation, and the wave equation in these coordinate systems.

Problems

1. Compute $\mathbf{u} \times \mathbf{v}$ using the permutation symbol. Verify your answer by computing these products using traditional methods.

a. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$.

b. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$.

c. $\mathbf{u} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

2. Compute the following determinants using the permutation symbol. Verify your answer.

a. $\begin{vmatrix} 3 & 2 & 0 \\ 1 & 4 & -2 \\ -1 & 4 & 3 \end{vmatrix}$

b. $\begin{vmatrix} 1 & 2 & 2 \\ 4 & -6 & 3 \\ 2 & 3 & 1 \end{vmatrix}$

3. For the given expressions, write out all values for $i, j = 1, 2, 3$.

a. ϵ_{i2j}.

b. ϵ_{i13}.

c. $\epsilon_{ij1}\epsilon_{i32}$.

4. Show that

a. $\delta_{ii} = 3$.

b. $\delta_{ij}\epsilon_{ijk} = 0$

c. $\epsilon_{imn}\epsilon_{jmn} = 2\delta_{ij}$.

d. $\epsilon_{ijk}\epsilon_{ijk} = 6$.

5. Show that the vector $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ lies on the line of intersection of the two planes: (1) the plane containing **a** and **b** and (2) the plane containing **c** and **d**.

6. Prove the following vector identities:

a. $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$.

b. $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{b}\times\mathbf{d})\mathbf{c} - (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})\mathbf{d}$.

7. Use problem 6a to prove that $|\mathbf{a}\times\mathbf{b}| = ab\sin\theta$.

8. A particle moves on a straight line, $\mathbf{r} = t\mathbf{u}$, from the center of a disk. If the disk is rotating with angular velocity ω, then **u** rotates. Let $\mathbf{u} = (\cos\omega t)\mathbf{i} + (\sin\omega t)\mathbf{j}$.

a. Determine the velocity, **v**.

b. Determine the acceleration, **a**.

c. Describe the resulting acceleration terms identifying the centripetal acceleration and Coriolis acceleration.

9. Compute the gradient of the following:

a. $f(x,y) = x^2 - y^2$.

b. $f(x,y,z) = yz + xy + xz$.

c. $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.

d. $f(x,y,z) =$

10. Find the directional derivative of the given function at the indicated point in the given direction.

a. $f(x,y) = x^2 - y^2$, $(3,2)$, $\mathbf{u} = \mathbf{i} + \mathbf{j}$.

b. $f(x,y) = \frac{y}{x}$, $(2,1)$, $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

c. $f(x,y,z) = x^2 + y^2 + z^2$, $(1,0,2)$, $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$.

11. Zaphod Beeblebrox was in trouble after the infinite improbability drive caused the Heart of Gold, the spaceship Zaphod had stolen when he was President of the Galaxy, to appear between a small insignificant planet and its hot sun. The temperature of the ship's hull is given by $T(x,y,z) = e^{-k(x^2+y^2+z^2)}$ Nivleks. He is currently at $(1,1,1)$, in units of globs, and $k = 2$ globs^{-2}. (Check the *Hitchhikers Guide* for the current conversion of globs to kilometers and Nivleks to Kelvins.)

a. In what direction should he proceed so as to decrease the temperature the quickest?

b. If the Heart of Gold travels at e^6 globs per second, then how fast will the temperature decrease in the direction of fastest decline?

12. For the given vector field, find the divergence and curl of the field.

a. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.

b. $\mathbf{F} = \frac{y}{r}\mathbf{i} - \frac{x}{r}\mathbf{j}$, for $r = \sqrt{x^2 + y^2}$.

c. $\mathbf{F} = x^2y\mathbf{i} + z\mathbf{j} + xyz\mathbf{k}$.

13. Write the following using ϵ_{ijk} notation and simplify if possible.

a. $\mathbf{C} \times (\mathbf{A} \times (\mathbf{A} \times \mathbf{C}))$.

b. $\nabla \cdot (\nabla \times \mathbf{A})$.

c. $\nabla \times \nabla\phi$.

14. Prove the identities:

a. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

b. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

c. $\nabla r^n = nr^{n-2}\mathbf{r}, \quad n \geq 2$.

15. For $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$, simplify the following.

a. $\nabla \times (\mathbf{k} \times \mathbf{r})$.

b. $\nabla \cdot \left(\frac{\mathbf{r}}{r}\right)$.

c. $\nabla \times \left(\frac{\mathbf{r}}{r}\right)$.

d. $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right)$.

16. Newton's Law of Gravitation gives the gravitational force between two masses as

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r}.$$

a. Prove that $\mathbf{F}$ is irrotational.

b. Find a scalar potential for $\mathbf{F}$.

17. Consider an electric dipole moment $\mathbf{p}$ at the origin. It produces an electric potential of $\phi = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3}$ outside the dipole. Noting that $\mathbf{E} = -\nabla\phi$, find the electric field at $\mathbf{r}$.

18. In fluid dynamics the Euler equations govern inviscid fluid flow and provide quantitative statements on the conservation of mass, momentum and energy. The continuity equation is given by

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0,$$

where $\rho(x, y, z, t)$ is the mass density and $\mathbf{v}(x, y, z, t)$ is the fluid velocity. The momentum equations are given by

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla(\rho\mathbf{v}) = \mathbf{f} - \nabla p.$$

Here $p(x, y, z, t)$ is the pressure and $\mathbf{F}$ is the external force per volume.

a. Show that the continuity equation can be rewritten as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot (\mathbf{v}) + \mathbf{v} \cdot \nabla \rho = 0.$$

b. Prove the identity $\frac{1}{2}\nabla v^2 = \mathbf{v} \cdot \nabla \mathbf{v}$ for irrotational $\mathbf{v}$.

c. Assume that

- the external forces are conservative ($\mathbf{F} = -\nabla \phi$),
- the velocity field is irrotational ($\nabla \times \mathbf{v} = \mathbf{0}$).
- the fluid is incompressible ($\rho =$const), and
- the flow is steady, $\frac{\partial \mathbf{v}}{\partial t} = 0$.

Under these assumptions, prove Bernoulli's Principle:

$$\frac{1}{2}v^2 + \phi + \frac{p}{\rho} = \text{ const.}$$

19. Find the lengths of the following curves:

a. $y(x) = x$ for $x \in [0, 2]$.

b. $(x, y, z) = (t, \ln t, 2\sqrt{2}t)$ for $1 \le t \le 2$.

c. $y(x) = 2 \cosh 3x$, $x \in [-2, 2]$. (Recall the hanging chain example from classical dynamics.)

20. Consider the integral $\int_C y^2\, dx - 2x^2\, dy$. Evaluate this integral for the following curves:

a. C is a straight line from (0,2) to (1,1).

b. C is the parabolic curve $y = x^2$ from (0,0) to (2,4).

c. C is the circular path from (1,0) to (0,1) in a clockwise direction.

21. Evaluate $\int_C (x^2 - 2xy + y^2)\, ds$ for the curve $x(t) = 2 \cos t$, $y(t) = 2 \sin t$, $0 \le t \le \pi$.

22. Prove that the magnetic flux density, $\mathbf{B}$, satisfies the wave equation.

23. Prove the identity

$$\int_C \phi \nabla \phi \cdot \mathbf{n}\, ds = \int_D (\phi \nabla^2 \phi + \nabla \cdot \nabla \phi)\, dA.$$

24. Compute the work done by the force $\mathbf{F} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$ in moving a particle counterclockwise around the boundary of the rectangle $R = [0, 3] \times [0, 5]$.

25. Compute the following integrals:

a. $\int_C (x^2 + y)\, dx + (3x + y^3)\, dy$ for C the ellipse $x^2 + 4y^2 = 4$.

b. $\int_S (x-y)\,dydz + (y^2+z^2)\,dzdx + (y-x^2)\,dxdy$ for S the positively oriented unit sphere.

c. $\int_C (y-z)\,dx + (3x+z)\,dy + (x+2y)\,dz$, where C is the curve of intersection between $z = 4 - x^2 - y^2$ and the plane $x + y + z = 0$.

d. $\int_C x^2 y\,dx - xy^2\,dy$ for C a circle of radius 2 centered about the origin.

e. $\int_S x^2 y\,dydz + 3y^2\,dzdx - 2xz^2\,dxdy$, where S is the surface of the cube $[-1,1] \times [-1,1] \times [-1,1]$.

26. Use Stoke's theorem to evaluate the integral

$$\int_C -y^3\,dx + x^3\,dy - z^3\,dz$$

for C the (positively oriented) curve of intersection between the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$.

27. Use Stoke's theorem to derive the integral form of Faraday's law,

$$\int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int \int_S \mathbf{H} \cdot d\mathbf{S}$$

from the differential form of Maxwell's equations.

28. For cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned} \tag{8.93}$$

find the scale factors and derive the following expressions:

Note that it is customary to write the basis as $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ instead of $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$.

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z. \tag{8.94}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}. \tag{8.95}$$

$$\nabla \times \mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\mathbf{e}_z \tag{8.96}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{8.97}$$

29. For spherical coordinates,

$$\begin{aligned} x &= \rho\sin\theta\cos\phi, \\ y &= \rho\sin\theta\sin\phi, \\ z &= \rho\cos\theta. \end{aligned} \tag{8.98}$$

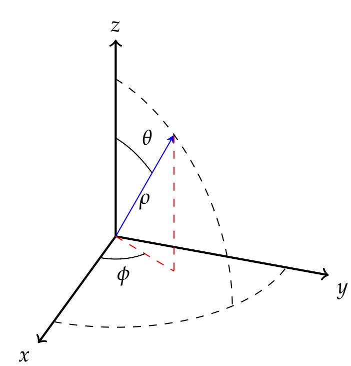

Figure 8.12: Definition of spherical coordinates for Problem 29.

find the scale factors and derive the following expressions:

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi. \tag{8.99}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2 F_\rho)}{\partial \rho} + \frac{1}{\rho \sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial \theta} + \frac{1}{\rho \sin\theta}\frac{\partial F_\phi}{\partial \phi}. \tag{8.100}$$

$$\begin{aligned} \nabla \times \mathbf{F} \quad = \quad & \frac{1}{\rho \sin\theta}\left(\frac{\partial(\sin\theta F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi}\right)\mathbf{e}_\rho + \frac{1}{\rho}\left(\frac{1}{\sin\theta}\frac{\partial F_\rho}{\partial \phi} - \frac{\partial(\rho F_\phi)}{\partial \rho}\right)\mathbf{e}_\theta \\ & + \frac{1}{\rho}\left(\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}\right)\mathbf{e}_\phi \end{aligned} \tag{8.101}$$

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{8.102}$$