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Introduction

“Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say.”
Bertrand Russell (1872-1970)

BEFORE WE BEGIN our study of mathematical physics, perhaps we should review some things from your past classes. You definitely need to know something before taking this class. It is assumed that you have taken Calculus and are comfortable with differentiation and integration. You should also have taken some introductory physics class, preferably the calculus based course. Of course, you are not expected to know every detail from these courses. However, there are some topics and methods that will come up and it would be useful to have a handy reference to what it is you should know, especially when it comes to exams.

Most importantly, you should still have your introductory physics and calculus texts to which you can refer throughout the course. Looking back on that old material, you will find that it appears easier than when you first encountered the material. That is the nature of learning mathematics and physics. Your understanding is continually evolving as you explore topics more in depth. It does not always sink in the first time you see it.

In this chapter we will give a quick review of these topics. We will also mention a few new things that might be interesting. This review is meant to make sure that everyone is at the same level.

1.1 What Do I Need To Know From Calculus?

1.1.1 Introduction

THERE ARE TWO main topics in calculus: derivatives and integrals . You learned that derivatives are useful in providing rates of change in

either time or space. Integrals provide areas under curves, but also are useful in providing other types of sums over continuous bodies, such as lengths, areas, volumes, moments of inertia, or flux integrals. In physics, one can look at graphs of position versus time and the slope (derivative) of such a function gives the velocity. By plotting velocity versus time you can either look at the derivative to obtain acceleration, or you could look at the area under the curve and get the displacement:

$$x = \int_{t_0}^t v dt. \quad (1.1)$$

Of course, you need to know how to differentiate and integrate given functions. Even before getting into differentiation and integration, you need to have a bag of functions useful in physics. Common functions are the polynomial and rational functions. You should be fairly familiar with these. Polynomial functions take the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad (1.2)$$

where $a_n \neq 0$. This is the form of a polynomial of degree n . Rational functions, $f(x) = \frac{g(x)}{h(x)}$, consist of ratios of polynomials. Their graphs can exhibit vertical and horizontal asymptotes.

Next are the exponential and logarithmic functions. The most common are the natural exponential and the natural logarithm. The natural exponential is given by $f(x) = e^x$, where $e \approx 2.718281828\dots$. The natural logarithm is the inverse to the exponential, denoted by $\ln x$. (One needs to be careful, because some mathematics and physics books use log to mean natural exponential, whereas many of us were first trained to use it to mean the common logarithm, which is the 'log base 10'. Here we will use $\ln x$ for the natural logarithm.)

The properties of the exponential function follow from the basic properties for exponents. Namely, we have:

Exponential properties.

$$e^0 = 1, \quad (1.3)$$

$$e^{-a} = \frac{1}{e^a} \quad (1.4)$$

$$e^a e^b = e^{a+b}, \quad (1.5)$$

$$(e^a)^b = e^{ab}. \quad (1.6)$$

The relation between the natural logarithm and natural exponential is given by

$$y = e^x \Leftrightarrow x = \ln y. \quad (1.7)$$

Some common logarithmic properties are

Logarithmic properties.

$$\ln 1 = 0, \quad (1.8)$$

$$\ln \frac{1}{a} = -\ln a, \quad (1.9)$$

$$\ln(ab) = \ln a + \ln b, \quad (1.10)$$

$$\ln \frac{a}{b} = \ln a - \ln b, \quad (1.11)$$

$$\ln \frac{1}{b} = -\ln b. \quad (1.12)$$

We will see further applications of these relations as we progress through the course.

1.1.2 Trigonometric Functions

ANOTHER SET of useful functions are the trigonometric functions. These functions have probably plagued you since high school. They have their origins as far back as the building of the pyramids. Typical applications in your introductory math classes probably have included finding the heights of trees, flag poles, or buildings. It was recognized a long time ago that similar right triangles have fixed ratios of any pair of sides of the two similar triangles. These ratios only change when the non-right angles change.

Thus, the ratio of two sides of a right triangle only depends upon the angle. Since there are six possible ratios (think about it!), then there are six possible functions. These are designated as sine, cosine, tangent and their reciprocals (cosecant, secant and cotangent). In your introductory physics class, you really only needed the first three. You also learned that they are represented as the ratios of the opposite to hypotenuse, adjacent to hypotenuse, etc. Hopefully, you have this down by now.

You should also know the exact values of these basic trigonometric functions for the special angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$, and their corresponding angles in the second, third and fourth quadrants. This becomes internalized after much use, but we provide these values in Table 1.1 just in case you need a reminder.

θ	$\cos \theta$	$\sin \theta$	$\tan \theta$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{2}$	0	1	undefined

Table 1.1: Table of Trigonometric Values

The problems students often have using trigonometric functions in later courses stem from using, or recalling, identities. We will have many an occasion to do so in this class as well. What is an identity? It is a relation that holds true all of the time. For example, the most common identity for trigonometric functions is the Pythagorean identity

$$\sin^2 \theta + \cos^2 \theta = 1. \quad (1.13)$$

This hold true for every angle θ ! An even simpler identity is

$$\tan \theta = \frac{\sin \theta}{\cos \theta}. \quad (1.14)$$

Other simple identities can be derived from the Pythagorean identity. Dividing the identity by $\cos^2 \theta$, or $\sin^2 \theta$, yields

$$\tan^2 \theta + 1 = \sec^2 \theta, \quad (1.15)$$

$$1 + \cot^2 \theta = \csc^2 \theta. \quad (1.16)$$

Several other useful identities stem from the use of the sine and cosine of the sum and difference of two angles. Namely, we have that

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A, \quad (1.17)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B. \quad (1.18)$$

Note that the upper (lower) signs are taken together.

The double angle formulae are found by setting $A = B$:

$$\sin(2A) = 2 \sin A \cos A, \quad (1.19)$$

$$\cos(2A) = \cos^2 A - \sin^2 A. \quad (1.20)$$

Using Equation (1.13), we can rewrite (1.20) as

$$\cos(2A) = 2 \cos^2 A - 1, \quad (1.21)$$

$$= 1 - 2 \sin^2 A. \quad (1.22)$$

These, in turn, lead to the half angle formulae. Solving for $\cos^2 A$ and $\sin^2 A$, we find that

$$\sin^2 A = \frac{1 - \cos 2A}{2}, \quad (1.23)$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}. \quad (1.24)$$

Finally, another useful set of identities are the product identities.

Sum and difference identities.

Double angle formulae.

Half angle formulae.

Product Identities

For example, if we add the identities for $\sin(A + B)$ and $\sin(A - B)$, the second terms cancel and we have

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$$

Thus, we have that

$$\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B)). \quad (1.25)$$

Similarly, we have

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B)). \quad (1.26)$$

and

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B)). \quad (1.27)$$

These boxed equations are the most common trigonometric identities. They appear often and should just roll off of your tongue.

We will also need to understand the behaviors of trigonometric functions. In particular, we know that the sine and cosine functions are periodic. They are not the only periodic functions, as we shall see. [Just visualize the teeth on a carpenter's saw.] However, they are the most common periodic functions.

A periodic function $f(x)$ satisfies the relation

$$f(x + p) = f(x), \quad \text{for all } x$$

for some constant p . If p is the smallest such number, then p is called the period. Both the sine and cosine functions have period 2π . This means that the graph repeats its form every 2π units. Similarly, $\sin bx$ and $\cos bx$ have the common period $p = \frac{2\pi}{b}$. We will make use of this fact in later chapters.

Related to these are the inverse trigonometric functions. For example, $f(x) = \sin^{-1} x$, or $f(x) = \arcsin x$. Inverse functions give back angles, so you should think

$$\theta = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin \theta. \quad (1.28)$$

Also, you should recall that $y = \sin^{-1} x = \arcsin x$ is only a function if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Similar relations exist for $y = \cos^{-1} x = \arccos x$ and $\tan^{-1} x = \arctan x$.

Once you think about these functions as providing angles, then you can make sense out of more complicated looking expressions, like $\tan(\sin^{-1} x)$. Such expressions often pop up in evaluations of integrals. We can untangle this in order to produce a simpler form by referring

Know the above boxed identities!

In Feynman's *Surely You're Joking Mr. Feynman!*, Richard Feynman (1918-1988) talks about his invention of his own notation for both trigonometric and inverse trigonometric functions as the standard notation did not make sense to him.

to expression (1.28). $\theta = \sin^{-1} x$ is simple an angle whose sine is x . Knowing the sine is the opposite side of a right triangle divided by its hypotenuse, then one just draws a triangle in this proportion. Namely, the side opposite the angle has length x and the hypotenuse has length 1. Using the Pythagorean Theorem, the missing side (adjacent to the angle) is simply $\sqrt{1-x^2}$. Having obtained the lengths for all three sides, we can now produce the tangent of the angle as

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}.$$

1.1.3 Hyperbolic Functions

SO, ARE THERE ANY other functions that are useful in physics? Actually, there are many more. However, you have probably not see many of them to date. We will see by the end of the semester that there are many important functions that arise as solutions of some fairly generic, but important, physics problems. In your calculus classes you have also seen that some relations are represented in parametric form. However, there is at least one other set of elementary functions, which you should already know about. These are the hyperbolic functions. Such functions are useful in representing hanging cables, unbounded orbits, and special traveling waves called solitons. They also play a role in special and general relativity.

Hyperbolic functions are actually related to the trigonometric functions, as we shall see after a little bit of complex function theory. For now, we just want to recall a few definitions and an identity. Just as all of the trigonometric functions can be built from the sine and the cosine, the hyperbolic functions can be defined in terms of the hyperbolic sine and hyperbolic cosine:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad (1.29)$$

$$\cosh x = \frac{e^x + e^{-x}}{2}. \quad (1.30)$$

There are four other hyperbolic functions. These are defined in terms of the above functions similar to the relations between the trigonometric functions. We have

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad (1.31)$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad (1.32)$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad (1.33)$$

Solitons are special solutions to some generic nonlinear wave equations. They typically experience elastic collisions and play special roles in a variety of fields in physics, such as hydrodynamics and optics. A simple soliton solution is of the form $u(x,t) = 2\eta^2 \operatorname{sech}^2 \eta(x - 4\eta^2 t)$.

Hyperbolic functions; We will see later the connection between the hyperbolic and trigonometric functions.

Hyperbolic identities

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \quad (1.34)$$

There are also a whole set of identities, similar to those for the trigonometric functions. For example, the Pythagorean identity for trigonometric functions, $\sin^2 \theta + \cos^2 \theta = 1$, is replaced by the identity

$$\cosh^2 x - \sinh^2 x = 1.$$

This is easily shown by simply using the definitions of these functions. This identity is also useful for providing a parametric set of equations describing hyperbolae. Letting $x = a \cosh t$ and $y = b \sinh t$, one has

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cosh^2 t - \sinh^2 t = 1.$$

A list of commonly needed hyperbolic function identities are given by the following:

$$\cosh^2 x - \sinh^2 x = 1, \quad (1.35)$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1, \quad (1.36)$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B, \quad (1.37)$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \sinh B \cosh A, \quad (1.38)$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x, \quad (1.39)$$

$$\sinh 2x = 2 \sinh x \cosh x, \quad (1.40)$$

$$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x), \quad (1.41)$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1). \quad (1.42)$$

Note the similarity with the trigonometric identities. Other identities can be derived from these.

There also exist inverse hyperbolic functions and these can be written in terms of logarithms. As with the inverse trigonometric functions, we begin with the definition

$$y = \sinh^{-1} x \Leftrightarrow x = \sinh y. \quad (1.43)$$

The aim is to write y in terms of x without using the inverse function. First, we note that

$$x = \frac{1}{2}(e^y - e^{-y}). \quad (1.44)$$

Now, we solve for e^y . This is done by noting that $e^{-y} = \frac{1}{e^y}$ and rewriting the previous equation as

$$0 = (e^y)^2 - 2xe^y - 1. \quad (1.45)$$

Inverse Hyperbolic Functions:

$$\sinh^{-1} x = \ln(x + \sqrt{1+x^2})$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$$

This equation is in quadratic form which we can solve as

$$e^y = x + \sqrt{1 + x^2}.$$

(There is only one root as we expect the exponential to be positive.)
The final step is to solve for y ,

$$y = \ln \left(x + \sqrt{1 + x^2} \right). \quad (1.46)$$

1.1.4 Derivatives

NOW THAT WE KNOW some elementary functions, we seek their derivatives. We will not spend time exploring the appropriate limits in any rigorous way. We are only interested in the results. We provide these in Table 1.2. We expect that you know the meaning of the derivative and all of the usual rules, such as the product and quotient rules.

Function	Derivative
a	0
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a \sec^2 ax$
$\csc ax$	$-a \csc ax \cot ax$
$\sec ax$	$a \sec ax \tan ax$
$\cot ax$	$-a \csc^2 ax$
$\sinh ax$	$a \cosh ax$
$\cosh ax$	$a \sinh ax$
$\tanh ax$	$a \operatorname{sech}^2 ax$
$\operatorname{csch} ax$	$-a \operatorname{csch} ax \operatorname{coth} ax$
$\operatorname{sech} ax$	$-a \operatorname{sech} ax \tanh ax$
$\operatorname{coth} ax$	$-a \operatorname{csch}^2 ax$

Table 1.2: Table of Derivatives (a is a constant.)

Also, you should be familiar with the Chain Rule. Recall that this rule tells us that if we have a composition of functions, such as the elementary functions above, then we can compute the derivative of the composite function. Namely, if $h(x) = f(g(x))$, then

$$\frac{dh}{dx} = \frac{d}{dx} (f(g(x))) = \left. \frac{df}{dg} \right|_{g(x)} \frac{dg}{dx} = f'(g(x))g'(x). \quad (1.47)$$

For example, let $H(x) = 5 \cos(\pi \tanh 2x^2)$. This is a composition of three functions, $H(x) = f(g(h(x)))$, where $f(x) = 5 \cos x$, $g(x) =$

$\pi \tanh x$, and $h(x) = 2x^2$. Then the derivative becomes

$$\begin{aligned} H'(x) &= 5 \left(-\sin \left(\pi \tanh 2x^2 \right) \right) \frac{d}{dx} \left(\left(\pi \tanh 2x^2 \right) \right) \\ &= -5\pi \sin \left(\pi \tanh 2x^2 \right) \operatorname{sech}^2 2x^2 \frac{d}{dx} \left(2x^2 \right) \\ &= -20\pi x \sin \left(\pi \tanh 2x^2 \right) \operatorname{sech}^2 2x^2. \end{aligned} \quad (1.48)$$

1.1.5 Integrals

INTEGRATION IS TYPICALLY a bit harder. Imagine being given the last result result in (1.48) and having to figure out what I differentiated in order to get that function. As you may recall from the Fundamental Theorem of Calculus, the integral is the inverse operation to differentiation:

$$\int \frac{df}{dx} dx = f(x) + C. \quad (1.49)$$

It is not always easy to evaluate a given integral. In fact some integrals are not even doable! However, you learned in calculus that there are some methods that could yield an answer. While you might be happier using a computer algebra system, such as Maple or WolframAlpha.com, or a fancy calculator, you should know a few basic integrals and know how to use tables for some of the more complicated ones. In fact, it can be exhilarating when you can do a given integral without reference to a computer or a Table of Integrals. However, you should be prepared to do some integrals using what you have been taught in calculus. We will review a few of these methods and some of the standard integrals in this section.

First of all, there are some integrals you are expected to know without doing any work. These integrals appear often and are just an application of the Fundamental Theorem of Calculus to the previous Table 1.2. The basic integrals that students should know off the top of their heads are given in Table 1.3.

These are not the only integrals you should be able to do. However, we can expand the list by recalling a few of the techniques that you learned in calculus. There are just a few: The Method of Substitution, Integration by Parts, Integration Using Partial Fraction Decomposition, and Trigonometric Integrals.

Example 1.1. *When confronted with an integral, you should first ask if a simple substitution would reduce the integral to one you know how to do. So, as an example, consider the following integral*

$$\int \frac{x}{\sqrt{x^2 + 1}} dx.$$

Function	Indefinite Integral
a	ax
x^n	$\frac{x^{n+1}}{n+1}$
e^{ax}	$\frac{1}{a}e^{ax}$
$\frac{1}{x}$	$\ln x$
$\sin ax$	$-\frac{1}{a} \cos ax$
$\cos ax$	$\frac{1}{a} \sin ax$
$\sec^2 ax$	$\frac{1}{a} \tan ax$
$\sinh ax$	$\frac{1}{a} \cosh ax$
$\cosh ax$	$\frac{1}{a} \sinh ax$
$\operatorname{sech}^2 ax$	$\frac{1}{a} \tanh ax$
$\sec x$	$\ln \sec x + \tan x $
$\frac{1}{a+bx}$	$\frac{1}{b} \ln(a+bx)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} ax$
$\frac{1}{\sqrt{a^2-x^2}}$	$\frac{1}{a} \sin^{-1} ax$
$\frac{1}{\sqrt{x^2-a^2}}$	$\frac{1}{a} \sec^{-1} ax$

Table 1.3: Table of Integrals

The ugly part of this integral is the $x^2 + 1$ under the square root. So, we let $u = x^2 + 1$. Noting that when $u = f(x)$, we have $du = f'(x) dx$. For our example, $du = 2x dx$. Looking at the integral, part of the integrand can be written as $x dx = \frac{1}{2}u du$. Then, the integral becomes

$$\int \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2} \int \frac{du}{\sqrt{u}}.$$

The substitution has converted our integral into an integral over u . Also, this integral is doable! It is one of the integrals we should know. Namely, we can write it as

$$\frac{1}{2} \int \frac{du}{\sqrt{u}} = \frac{1}{2} \int u^{-1/2} du.$$

This is now easily finished after integrating and using our substitution variable to give

$$\int \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2} \frac{u^{1/2}}{\frac{1}{2}} + C = \sqrt{x^2+1} + C.$$

Note that we have added the required integration constant and that the derivative of the result easily gives the original integrand (after employing the Chain Rule).

Often we are faced with definite integrals, in which we integrate between two limits. There are several ways to use these limits. However, students often forget that a change of variables generally means that the limits have to change.

Example 1.2. Consider the above example with limits added.

$$\int_0^2 \frac{x}{\sqrt{x^2+1}} dx.$$

We proceed as before. We let $u = x^2 + 1$. As x goes from 0 to 2, u takes values from 1 to 5. So, this substitution gives

$$\int_0^2 \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2} \int_1^5 \frac{du}{\sqrt{u}} = \sqrt{u} \Big|_1^5 = \sqrt{5} - 1.$$

When the Method of Substitution fails, there are other methods you can try. One of the most used is the Method of Integration by Parts. Recall the Integration by Parts Formula:

$$\int u dv = uv - \int v du. \quad (1.50)$$

The idea is that you are given the integral on the left and you can relate it to an integral on the right. Hopefully, the new integral is one you can do, or at least it is an easier integral than the one you are trying to evaluate.

However, you are not usually given the functions u and v . You have to determine them. The integral form that you really have is a function of another variable, say x . Another form of the formula can be given as

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx. \quad (1.51)$$

This form is a bit more complicated in appearance, though it is clearer what is happening. The derivative has been moved from one function to the other. Recall that this formula was derived by integrating the product rule for differentiation.

The two formulae are related by using the differential relations

$$\begin{aligned} u = f(x) &\rightarrow du = f'(x) dx, \\ v = g(x) &\rightarrow dv = g'(x) dx. \end{aligned} \quad (1.52)$$

This also gives a method for applying the Integration by Parts Formula.

Example 1.3. Consider the integral $\int x \sin 2x dx$. We choose $u = x$ and $dv = \sin 2x dx$. This gives the correct left side of the Integration by Parts Formula. We next determine v and du :

$$du = \frac{du}{dx} dx = dx,$$

$$v = \int dv = \int \sin 2x dx = -\frac{1}{2} \cos 2x.$$

We note that one usually does not need the integration constant. Inserting these expressions into the Integration by Parts Formula, we have

$$\int x \sin 2x dx = -\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx.$$

Integration by Parts Formula.

Note: Often in physics one needs to move a derivative between functions inside an integrand. The key - use integration by parts to move the derivative from one function to the other under an integral.

We see that the new integral is easier to do than the original integral. Had we picked $u = \sin 2x$ and $dv = x dx$, then the formula still works, but the resulting integral is not easier.

For completeness, we finish the integration. The result is

$$\int x \sin 2x dx = -\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x + C.$$

As always, you can check your answer by differentiating the result, a step students often forget to do. Namely,

$$\begin{aligned} \frac{d}{dx} \left(-\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x + C \right) &= -\frac{1}{2} \cos 2x + x \sin 2x + \frac{1}{4} (2 \cos 2x) \\ &= x \sin 2x. \end{aligned} \quad (1.53)$$

So, we do get back the integrand in the original integral.

We can also perform integration by parts on definite integrals. The general formula is written as

$$\int_a^b f(x)g'(x) dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x) dx. \quad (1.54)$$

Integration by Parts for Definite Integrals.

Example 1.4. Consider the integral

$$\int_0^\pi x^2 \cos x dx.$$

This will require two integrations by parts. First, we let $u = x^2$ and $dv = \cos x$. Then,

$$du = 2x dx. \quad v = \sin x.$$

Inserting into the Integration by Parts Formula, we have

$$\begin{aligned} \int_0^\pi x^2 \cos x dx &= x^2 \sin x \Big|_0^\pi - 2 \int_0^\pi x \sin x dx \\ &= -2 \int_0^\pi x \sin x dx. \end{aligned} \quad (1.55)$$

We note that the resulting integral is easier than the given integral, but we still cannot do the integral off the top of our head (unless we look at Example 3!). So, we need to integrate by parts again. (Note: In your calculus class you may recall that there is a tabular method for carrying out multiple applications of the formula. We will show this method in the next example.)

We apply integration by parts by letting $U = x$ and $dV = \sin x dx$. This gives $dU = dx$ and $V = -\cos x$. Therefore, we have

$$\begin{aligned} \int_0^\pi x \sin x dx &= -x \cos x \Big|_0^\pi + \int_0^\pi \cos x dx \\ &= \pi + \sin x \Big|_0^\pi \\ &= \pi. \end{aligned} \quad (1.56)$$

The final result is

$$\int_0^\pi x^2 \cos x \, dx = -2\pi.$$

There are other ways to compute integrals of this type. First of all, there is the Tabular Method to perform integration by parts. A second method is to use differentiation of parameters under the integral. We will demonstrate this using examples.

Example 1.5. Compute the integral $\int_0^\pi x^2 \cos x \, dx$ using the Tabular Method.

First we identify the two functions under the integral, x^2 and $\cos x$. We then write the two functions and list the derivatives and integrals of each, respectively. This is shown in Table 1.5. Note that we stopped when we reached 0 in the left column.

Using the Tabular Method.

D		I
x^2		$\cos x$
	$\searrow +$	
$2x$		$\sin x$
	$\searrow -$	
2		$-\cos x$
	$\searrow +$	
0		$-\sin x$

Table 1.4: Tabular Method

Next, one draws diagonal arrows, as indicated, with alternating signs attached, starting with +. The indefinite integral is then obtained by summing the products of the functions at the ends of the arrows along with the signs on each arrow:

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

To find the definite integral, one evaluates the antiderivative at the given limits.

$$\begin{aligned} \int_0^\pi x^2 \cos x \, dx &= \left[x^2 \sin x + 2x \cos x - 2 \sin x \right]_0^\pi \\ &= (\pi^2 \sin \pi + 2\pi \cos \pi - 2 \sin \pi) - 0 \\ &= -2\pi. \end{aligned} \tag{1.57}$$

Actually, the Tabular Method works even if a 0 does not appear on the left side. One can go as far as possible, and if a 0 does not appear, then one needs only integrate, if possible, the product of the functions in the last row, adding the next sign in the alternating sign progression. The next example shows how this works.

Example 1.6. Use the Tabular Method to compute $\int e^{2x} \sin 3x \, dx$.

As before, we first set up the table.

D	I
$\sin 3x$	e^{2x}
$3 \cos 3x$	$\frac{1}{2}e^{2x}$
$-9 \sin 3x$	$\frac{1}{4}e^{2x}$

Table 1.5: Tabular Method - Non-terminating Example.

Putting together the pieces, noting that the derivatives in the left column will never vanish, we have

$$\int e^{2x} \sin 3x \, dx = \left(\frac{1}{2} \sin 3x - \frac{3}{4} \cos 3x \right) e^{2x} + \int (-9 \sin 3x) \left(\frac{1}{4} e^{2x} \right) dx.$$

The integral on the right is a multiple of the one on the left, so we can combine them,

$$\frac{13}{4} \int e^{2x} \sin 3x \, dx = \left(\frac{1}{2} \sin 3x - \frac{3}{4} \cos 3x \right) e^{2x},$$

or

$$\int e^{2x} \sin 3x \, dx = \left(\frac{2}{13} \sin 3x - \frac{3}{13} \cos 3x \right) e^{2x}.$$

Another method that one can use to evaluate this integral is to differentiate under the integral sign. This is mentioned in the Richard Feynman's memoir *Surely You're Joking, Mr. Feynman!*. In the book Feynman recounts using this "trick" to be able to do integrals that his MIT classmates could not do. This is based on a theorem in Advanced Calculus.

Differentiation Under the Integral Sign and Feynman's trick.

Theorem 1.1. Let the functions $f(x, t)$ and $\frac{\partial f(x, t)}{\partial x}$ be continuous in both t , and x , in the region of the (t, x) plane which includes $a(x) \leq t \leq b(x)$, $x_0 \leq x \leq x_1$, where the functions $a(x)$ and $b(x)$ are continuous and have continuous derivatives for $x_0 \leq x \leq x_1$. Defining

$$F(x) \equiv \int_{a(x)}^{b(x)} f(x, t) \, dt,$$

then

$$\begin{aligned} \frac{dF(x)}{dx} &= \left(\frac{\partial F}{\partial b} \right) \frac{db}{dx} + \left(\frac{\partial F}{\partial a} \right) \frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) \, dt \\ &= f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) \, dt. \end{aligned} \tag{1.58}$$

for $x_0 \leq x \leq x_1$. This is a generalized version of the Fundamental Theorem of Calculus.

In the next examples we show how we can use this theorem to bypass integration by parts.

Example 1.7. Use differentiation under the integral sign to evaluate $\int xe^x dx$. First, consider the integral

$$I(x, a) = \int e^{ax} dx.$$

Then

$$\frac{\partial I(x, a)}{\partial a} = \int xe^{ax} dx.$$

So,

$$\begin{aligned} \int xe^{ax} dx &= \frac{\partial I(x, a)}{\partial a} \\ &= \frac{\partial}{\partial a} \left(\int e^{ax} dx \right) \\ &= \frac{\partial}{\partial a} \left(\frac{e^{ax}}{a} \right) \\ &= \left(\frac{x}{a} - \frac{1}{a^2} \right) e^{ax} \end{aligned} \quad (1.59)$$

Evaluating this result at $a = 1$, we have

$$\int xe^x dx = (x - 1)e^x.$$

Example 1.8. We will do the integral $\int_0^\pi x^2 \cos x dx$ once more. First, consider the integral

$$\begin{aligned} I(a) &\equiv \int_0^\pi \cos ax dx \\ &= \frac{\sin ax}{a} \Big|_0^\pi \\ &= \frac{\sin a\pi}{a}. \end{aligned} \quad (1.60)$$

Differentiating the integral with respect to a twice gives

$$\frac{d^2 I(a)}{da^2} = - \int_0^\pi x^2 \cos ax dx. \quad (1.61)$$

Evaluation of this result at $a = 1$ leads to the desired result. Thus,

$$\begin{aligned} \int_0^\pi x^2 \cos x dx &= - \frac{d^2 I(a)}{da^2} \Big|_{a=1} \\ &= - \frac{d^2}{da^2} \left(\frac{\sin a\pi}{a} \right) \Big|_{a=1} \\ &= - \frac{d}{da} \left(\frac{a\pi \cos a\pi - \sin a\pi}{a^2} \right) \Big|_{a=1} \\ &= - \left(\frac{a^2 \pi^2 \sin a\pi + 2a\pi \cos a\pi - 2 \sin a\pi}{a^3} \right) \Big|_{a=1} \\ &= -2\pi. \end{aligned} \quad (1.62)$$

Other types of integrals that you will see often are trigonometric integrals. In particular, integrals involving powers of sines and cosines. For odd powers, a simple substitution will turn the integrals into simple powers.

Example 1.9. For example, consider

$$\int \cos^3 x \, dx.$$

This can be rewritten as

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx.$$

Let $u = \sin x$. Then $du = \cos x \, dx$. Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\begin{aligned} \int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\ &= \int (1 - u^2) \, du \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C. \end{aligned} \quad (1.63)$$

A quick check confirms the answer:

$$\frac{d}{dx} \left(\sin x - \frac{1}{3}\sin^3 x + C \right) = \cos x - \sin^2 x \cos x = \cos x(1 - \sin^2 x) = \cos^3 x.$$

Even powers of sines and cosines are a little more complicated, but doable. In these cases we need the half angle formulae:

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}, \quad (1.64)$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}. \quad (1.65)$$

Example 1.10. As an example, we will compute

$$\int_0^{2\pi} \cos^2 x \, dx.$$

Substituting the half angle formula for $\cos^2 x$, we have

$$\begin{aligned} \int_0^{2\pi} \cos^2 x \, dx &= \frac{1}{2} \int_0^{2\pi} (1 + \cos 2x) \, dx \\ &= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right)_0^{2\pi} \\ &= \pi. \end{aligned} \quad (1.66)$$

We note that this result appears often in physics. When looking at root mean square averages of sinusoidal waves, one needs the average

Integration of odd powers of sine and cosine.

Integration of even powers of sine and cosine.

of the square of sines and cosines. Recall that the average of a function on interval $[a, b]$ is given as

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx. \quad (1.67)$$

So, the average of $\cos^2 x$ over one period is

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2 x dx = \frac{1}{2}. \quad (1.68)$$

The root mean square is then $\frac{1}{\sqrt{2}}$.

1.1.6 Geometric Series

INFINITE SERIES OCCUR often in mathematics and physics. Two series which occur often are the geometric series and the binomial series. We will discuss these in the next two sections.

A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots + ar^n + \dots \quad (1.69)$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is r .

Example 1.11. For example,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

is an example of a geometric series. We can write this using summation notation,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{n=0}^{\infty} 1 \left(\frac{1}{2}\right)^n.$$

Thus, $a = 1$ is the first term and $r = \frac{1}{2}$ is the common ratio of successive terms. Next, we seek the sum of this infinite series, if it exists.

The sum of a geometric series, when it converges, can easily be determined. We consider the n th partial sum:

$$s_n = a + ar + \dots + ar^{n-2} + ar^{n-1}. \quad (1.70)$$

Now, multiply this equation by r .

$$rs_n = ar + ar^2 + \dots + ar^{n-1} + ar^n. \quad (1.71)$$

Subtracting these two equations, while noting the many cancellations, we have

$$\begin{aligned} (1-r)s_n &= (a + ar + \dots + ar^{n-2} + ar^{n-1}) \\ &\quad - (ar + ar^2 + \dots + ar^{n-1} + ar^n) \\ &= a - ar^n \\ &= a(1 - r^n). \end{aligned} \quad (1.72)$$

Geometric series are fairly common and will be used throughout the book. You should learn to recognize them and work with them.

Thus, the n th partial sums can be written in the compact form

$$s_n = \frac{a(1 - r^n)}{1 - r}. \quad (1.73)$$

Recall that the sum, if it exists, is given by $S = \lim_{n \rightarrow \infty} s_n$. Letting n get large in the partial sum (1.73), we need only evaluate $\lim_{n \rightarrow \infty} r^n$. From our special limits we know that this limit is zero for $|r| < 1$. Thus, we have

Geometric Series

The sum of the geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad |r| < 1. \quad (1.74)$$

The reader should verify that the geometric series diverges for all other values of r . Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example 1.12. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is

$$S = \frac{1}{1 - \frac{1}{2}} = 2.$$

This agrees with the plot of the partial sums in Figure A.6.

Example 1.13. $\sum_{k=2}^{\infty} \frac{4}{3^k}$

In this example we note that the first term occurs for $k = 2$. So, $a = \frac{4}{9}$. Also, $r = \frac{1}{3}$. So,

$$S = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

Example 1.14. $\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n} \right)$

Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed¹. Thus, we have

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n} \right) = \sum_{n=1}^{\infty} \frac{3}{2^n} - \sum_{n=1}^{\infty} \frac{2}{5^n}.$$

Now we can add both geometric series:

$$\sum_{n=1}^{\infty} \left(\frac{3}{2^n} - \frac{2}{5^n} \right) = \frac{\frac{3}{2}}{1 - \frac{1}{2}} - \frac{\frac{2}{5}}{1 - \frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

¹ A rearrangement of terms in an infinite series is allowed when the series is absolutely convergent.

Geometric series are important because they are easily recognized and summed. Other series, which can be summed, are special cases of Taylor series, as we will see later. Another type of series that can be summed is a *telescoping series* as seen in the next example.

Example 1.15. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ The first few terms of this series are

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots$$

It does not appear that we can sum this infinite series. However, if we used the partial fraction expansion

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

then we find the partial sums can be written as

$$\begin{aligned} s_k &= \sum_{n=1}^k \frac{1}{n(n+1)} \\ &= \sum_{n=1}^k \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{k} - \frac{1}{k+1} \right). \quad (1.75) \end{aligned}$$

We see that there are many cancellations of neighboring terms, leading to the series collapsing (like a telescope) to something manageable:

$$s_k = 1 - \frac{1}{k+1}.$$

Taking the limit as $k \rightarrow \infty$, we find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Example 1.16. The Partition Function

A common occurrence of geometric series is a series of exponentials. An example of this occurs in statistical mechanics. Statistical mechanics is the branch of physics which explores the thermodynamic behavior of systems containing a large number of particles. An important tool is the partition function, Z . This function is the sum of terms, $e^{-\epsilon_n/kT}$, over all possible quantum states of the system. Here ϵ_n is the energy of the n th state, T the temperature, and k is Boltzmann's constant. Given Z , one can compute macroscopic quantities, such as the average energy,

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta},$$

where $\beta = 1/kT$.

For the case of the quantum harmonic oscillator, the energy states are given by $\epsilon_n = \left(n + \frac{1}{2} \right) \hbar\omega$. The partition function is then

$$Z = \sum_{n=0}^{\infty} e^{-\beta\epsilon_n}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} \\
&= e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega}. \tag{1.76}
\end{aligned}$$

The terms in the last sum are really powers of an exponential,

$$e^{-\beta n\hbar\omega} = \left(e^{-\beta\hbar\omega}\right)^n.$$

So,

$$Z = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n.$$

This is a geometric series, which can be summed as long as $e^{-\beta\hbar\omega} < 1$. Thus,

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}.$$

Multiplying the numerator and denominator by $e^{\beta\hbar\omega/2}$, we have

$$Z = \frac{1}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} = (2 \sinh \beta\hbar\omega/2)^{-1}.$$

1.1.7 The Binomial Expansion

ONE SERIES EXPANSION which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a + b)^p$ in powers of a and b . We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p . While the binomial expansion can be obtained using Taylor series, we will provide a more interesting derivation here to show that

$$(a + b)^p = \sum_{r=0}^{\infty} C_p^r a^{n-r} b^r, \tag{1.77}$$

where the C_p^r are called the *binomial coefficients*.

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a + b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p .

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned}
(a + b)^0 &= 1 \\
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2
\end{aligned}$$

The binomial expansion is a special series expansion used to approximate expressions of the form $(a + b)^p$ for $b \ll a$, or $(1 + x)^p$ for $|x| \ll 1$.

Let $r = k - 1$. Then this term is of the form $C_r^n a^{n-r} b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the binomial coefficients have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \binom{n}{r}.$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count the number of ways that we can arrange r products of b with $n - r$ products of a . There are n slots to place the b 's. For example, the $r = 2$ case for $n = 4$ involves the six products: $aabb$, $abab$, $abba$, $baab$, $baba$, and $baaa$. Thus, it is natural to use this notation.

So, we have found that

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r. \quad (1.82)$$

Now consider the geometric series $1 + x + x^2 + \dots$. We have seen that such a series converges for $|x| < 1$, giving

$$1 + x + x^2 + \dots = \frac{1}{1-x}.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$.

This is again a binomial to a power, but the power is not an integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (A.35).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$(1+x)^p = \sum_{r=0}^{\infty} \binom{p}{r} x^r. \quad (1.83)$$

However, we quickly run into problems with this form. Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\binom{-1}{1} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\dots$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned} \binom{p}{r} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \quad (1.84)$$

With this in mind we now state the theorem:

General Binomial Expansion	
<p>The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (A.35). For p real, we have the following binomial series:</p> $(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r, \quad x < 1. \quad (1.85)$	

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3). \quad (1.86)$$

Example 1.17. Approximate $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ for $v \ll c$. This can be rewritten as

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}.$$

Using the binomial expansion for $p = -1/2$, we have

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \left(-\frac{1}{2}\right) \left(-\frac{v^2}{c^2}\right) = 1 + \frac{v^2}{2c^2}.$$

Example 1.18. Small differences in large numbers.

As an example, we could compute $f(R, h) = \sqrt{R^2 + h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m. Inserting these values into a scientific calculator, one finds that

$$f(6378164, 1) = \sqrt{6378164^2 + 1} - 6378164 = 1 \times 10^{-7} \text{ m}.$$

In some calculators one might obtain 0, in other calculators, or computer algebra systems like Maple, one might obtain other answers. What answer do you get and how accurate is your answer?

The problem with this computation is that $R \gg h$. Therefore, the computation of $f(R, h)$ depends on how many digits the computing device can handle.

The factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is important in special relativity. Namely, this is the factor relating differences in time and length measurements by observers moving relative inertial frames. For celestial speeds, this is an appropriate approximation.

The best way to get an answer is to use the binomial approximation. Writing $x = \frac{h}{R}$, we have

$$\begin{aligned}
 f(R, h) &= \sqrt{R^2 + h^2} - R \\
 &= R\sqrt{1 + x^2} - R \\
 &\simeq R\left[1 + \frac{1}{2}x^2\right] - R \\
 &= \frac{1}{2}Rx^2 \\
 &= \frac{1}{2}\frac{h}{R^2} = 7.83926 \times 10^{-8} \text{ m.} \quad (1.87)
 \end{aligned}$$

Of course, you should verify how many digits should be kept in reporting the result.

In the next examples, we show how computations taking a more general form can be handled. Such general computations appear in proofs involving general expansions without specific numerical values given.

Example 1.19. Obtain an approximation to $(a + b)^p$ when a is much larger than b , denoted by $a \gg b$.

If we neglect b then $(a + b)^p \simeq a^p$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion. Namely, what is the power of b/a of the first neglected term in this expansion?

In order to do this we first divide out a as

$$(a + b)^p = a^p \left(1 + \frac{b}{a}\right)^p.$$

Now we have a small parameter, $\frac{b}{a}$. According to what we have seen earlier, we can use the binomial expansion to write

$$\left(1 + \frac{b}{a}\right)^n = \sum_{r=0}^{\infty} \binom{p}{r} \left(\frac{b}{a}\right)^r. \quad (1.88)$$

Thus, we have a sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write

$$\left(1 + \frac{b}{a}\right)^p = 1 + p\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

Here we used $O()$, big-Oh notation, to indicate the size of the first neglected term. (This notation is formally defined in another section.)

Summarizing, this then gives

$$(a + b)^p = a^p \left(1 + \frac{b}{a}\right)^p$$

$$\begin{aligned}
&= a^p \left(1 + p \frac{b}{a} + O \left(\left(\frac{b}{a} \right)^2 \right) \right) \\
&= a^p + pa^p \frac{b}{a} + a^p O \left(\left(\frac{b}{a} \right)^2 \right). \quad (1.89)
\end{aligned}$$

Therefore, we can approximate $(a + b)^p \simeq a^p + pba^{p-1}$, with an error on the order of b^2a^{p-2} . Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a + b)^p \simeq a^p$, but it is not typically good enough in applications because the error in this case is of the order ba^{p-1} .

Example 1.20. Approximate $f(x) = (a + x)^p - a^p$ for $x \ll a$.

In an earlier example we computed $f(R, h) = \sqrt{R^2 + h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m. We can make use of the binomial expansion to determine the behavior of similar functions in the form $f(x) = (a + x)^p - a^p$. Inserting the binomial expression into $f(x)$, we have as $\frac{x}{a} \rightarrow 0$ that

$$\begin{aligned}
f(x) &= (a + x)^p - a^p \\
&= a^p \left[\left(1 + \frac{x}{a} \right)^p - 1 \right] \\
&= a^p \left[\frac{px}{a} + O \left(\left(\frac{x}{a} \right)^2 \right) \right] \\
&= O \left(\frac{x}{a} \right) \quad \text{as } \frac{x}{a} \rightarrow 0. \quad (1.90)
\end{aligned}$$

This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as

$$(a + x)^p - a^p = a^{p-1}px + O \left(\left(\frac{x}{a} \right)^2 \right) \quad \text{as } \frac{x}{a} \rightarrow 0.$$

We could use this approximation to answer the original question by letting $a = R^2$, $x = 1$ and $p = \frac{1}{2}$. Then, our approximation would be of order

$$O \left(\left(\frac{x}{a} \right)^2 \right) = O \left(\left(\frac{1}{6378164^2} \right)^2 \right) \sim 2.4 \times 10^{-14}.$$

Thus, we have

$$\sqrt{6378164^2 + 1} - 6378164 \approx a^{p-1}px$$

where

$$a^{p-1}px = (6378164^2)^{-1/2}(0.5)1 = 7.83926 \times 10^{-8}.$$

This is the same result we had obtained before.

So far, this is enough to get started in the course. We will recall other topics as we need them. For example, we will discuss the method of partial fraction decomposition when we discuss terminal velocity in the next chapter and when we cover applications of the Laplace transform later in the book.

1.2 *What I Need From My Intro Physics Class?*

SO, WHAT DO WE NEED to know about physics? You should be comfortable with common terms from mechanics and electromagnetism. In some cases, we will review specific topics. However, it would be helpful to review some topics from your introductory and modern physics texts.

As you may recall, your study of physics began with the simplest systems. You first studied motion for point masses. You were then introduced to the concepts of position, displacement, velocity and acceleration. You studied motion first in one dimension and even then can only do problems in which the acceleration is constant, or piecewise constant. You looked at horizontal motion and then vertical motion, in terms of free fall. Finally, you moved into two dimensions and considered projectile motion. Some calculus was introduced and you learned how to represent vector quantities.

You then asked, "What causes a change in the state of motion of a body?" We are led to a discussion of forces. The types of forces encountered are the weight, the normal force, tension, the force of gravity and then centripetal forces. You might have also seen spring forces, which we will see shortly, lead to oscillatory motion - the underlying theme of this book.

Next, you found out that there are well known conservation principles for energy and momentum. In these cases you were led to the concepts of work, kinetic energy and potential energy. You found out that even when mechanical energy is not conserved, you could account for the missing energy as the work done by nonconservative forces. Momentum becomes important in collision problems or when looking at impulses.

With these basic ideas under your belt, you proceeded to study more complicated systems. Looking at extended bodies, most notably rigid bodies, led to the study of rotational motion. you found out that there are analogues to all of the previously discussed concepts for point masses. For example, there are the natural analogues of rotational velocity and acceleration. The cause of rotational acceleration is the torque. The analogue to mass is the moment of inertia.

The next level of complication, which sometimes is not covered, are bulk systems. One can study fluids, solids and gases. These can be investigated by looking at things like mass density, pressure, volume and temperature. This leads to the study of thermodynamics in which one studies the transfer of energy between a system and its surroundings. This involves the relationship between the work done on the system, the heat energy added to a systems and its change in internal

energy.

Bulk systems can also suffer deformations when a force per area is applied. This can lead to the idea that small deformations can lead to the propagation of energy throughout the system in the form of waves. We will later explore this wave motion in several systems.

The second course in physics is spent on electricity and magnetism, leading to electromagnetic waves. You first learned about charges and charge distributions, electric fields, electric potentials. Then you found out that moving charges produce magnetic fields and are affected by external magnetic fields. Furthermore, changing magnetic fields produce currents. This can all be summarized by Maxwell's equations, which we will recall later in the course. These equations, in turn, predict the existence of electromagnetic waves.

Depending how far you delved into the book, you may have seen excursions into optics and the impact that trying to understand the existence of electromagnetic waves has had on the development of so-called "modern physics". For example, in trying to understand what medium electromagnetic waves might propagate through, Einstein proposed an answer that completely changed the way we understand the nature of space and time. In trying to understand how ionized gases radiate and interact with matter, Einstein and others were lead down a path that has lead to quantum mechanics and further challenges to our understanding of reality.

So, that is the introductory physics course in a nutshell. In fact, that is most of physics. The rest is detail, which you will explore in your other courses as you progress toward a degree in physics.

1.3 *Technology and Tables*

AS WE PROGRESS through the course, you will often have to compute integrals and derivatives by hand. However, many of you know that some of the tedium can be alleviated by using computers, or even looking up what you need in tables. In some cases you might even find applets online that can quickly give you the answers you seek.

However, you also need to be comfortable in doing many computations by hand. This is necessary, especially in your early studies, for several reasons. For example, you should try to evaluate integrals by hand when asked to do them. This reinforces the techniques, as outlined earlier. It exercises your brain in much the same way that you might jog daily to exercise your body. Who knows, keeping your brain active this way might even postpone Alzheimer's. The more comfortable you are with derivations and evaluations, the easier it is to follow future lectures without getting bogged down by the details, wonder-

ing how your professor got from step A to step D. You can always use a computer algebra system, or a Table of Integrals, to check on your work.

Problems can arise when depending purely on the output of computers, or other "black boxes". Once you have a firm grasp on the techniques and a feeling as to what answers should look like, then you can feel comfortable with what the computer gives you. Sometimes, Computer Algebra Systems (CAS) like Maple can give you strange looking answers, and sometimes even wrong answers. Also, these programs cannot do every integral, or solve every differential equation, that you ask them to do. Even some of the simplest looking expressions can cause computer algebra systems problems. Other times you might even provide wrong input, leading to erroneous results.

Another source of indefinite integrals, derivatives, series expansions, etc, is a Table of Mathematical Formulae. There are several good books that have been printed. Even some of these have typos in them, so you need to be careful. However, it may be worth the investment to have such a book in your personal library. Go to the library, or the bookstore, and look at some of these tables to see how useful they might be.

There are plenty of online resources as well. For example, there is the Wolfram Integrator at <http://integrals.wolfram.com/> as well as the recent <http://www.wolframalpha.com/>. There is also a wealth of information at the following sites: <http://www.sosmath.com/>, <http://www.math2.org/>, <http://mathworld.wolfram.com/>, and <http://functions.wolfram.com/>.

1.4 Appendix: Dimensional Analysis

IN THE FIRST CHAPTER in your introductory physics text you were introduced to dimensional analysis. Dimensional analysis is useful for recalling particular relationships between variables by looking at the units involved, independent of the system of units employed. Though most of the time you have used SI, or MKS, units in most of your physics problems.

There are certain basic units - length, mass and time. By the second course, you found out that you could add charge to the list. We can represent these as [L], [M], [T] and [C]. Other quantities typically have units that can be expressed in terms of the basic units. These are called derived units. So, we have that the units of acceleration are $[L]/[T]^2$ and units of mass density are $[M]/[L]^3$. Slightly more complicated

units arise for force. Since $F = ma$, the units of force are

$$[F] = [m][a] = [M] \frac{[L]}{[T]^2}.$$

Similarly, units of magnetic field can be found, though with a little more effort. Recall that $F = qvB \sin \theta$ for a charge q moving with speed v through a magnetic field B at an angle of θ . $\sin \theta$ has no units. So,

$$\begin{aligned} [B] &= \frac{[F]}{[q][v]} \\ &= \frac{\frac{[M][L]}{[T]^2}}{[C] \frac{[L]}{[T]}} \\ &= \frac{[M]}{[C][T]}. \end{aligned} \quad (1.91)$$

Now, assume that you do not know how B depended on F , q and v , but you knew the units of all of the quantities. Can you figure out the relationship between them? We could write

$$[B] = [F]^\alpha [q]^\beta [v]^\gamma$$

and solve for the exponents by inserting the dimensions. Thus, we have

$$[M][C]^{-1}[T]^{-1} = ([M][L][T]^{-2})^\alpha [C]^\beta ([L][T]^{-1})^\gamma.$$

Right away we can see that $\alpha = 1$ and $\beta = -1$ by looking at the powers of $[M]$ and $[C]$, respectively. Thus,

$$[M][C]^{-1}[T]^{-1} = [M][L][T]^{-2}[C]^{-1} ([L][T]^{-1})^\gamma = [M][C]^{-1}[L]^{1+\gamma}[T]^{-2-\gamma}.$$

We see that picking $\gamma = -1$ balances the exponents and gives the correct relation

$$[B] = [F][q]^{-1}[v]^{-1}.$$

An important theorem at the heart of dimensional analysis is the Buckingham II Theorem. In essence, this theorem tells us that physically meaningful equations in n variables can be written as an equation involving $n - m$ dimensionless quantities, where m is the number of dimensions used. The importance of this theorem is that one can actually compute useful quantities without even knowing the exact form of the equation!

The Buckingham II Theorem was introduced by Edgar Buckingham (1867-1940) in 1914. Let q_i be n physical variables that are related by

$$f(q_1, q_2, \dots, q_n) = 0. \quad (1.92)$$

Assuming that m dimensions are involved, we let π_i be $k = n - m$ dimensionless variables. Then the equation (1.92) can be rewritten as

The Buckingham II Theorem.

a function of these dimensionless variables as

$$F(\pi_1, \pi_2, \dots, \pi_k) = 0, \quad (1.93)$$

where the π_i 's can be written in terms of the physical variables as

$$\pi_i = q_1^{k_1} q_2^{k_2} \cdots q_n^{k_n}, \quad i = 1, \dots, k. \quad (1.94)$$

Well, this is our first really new concept (apart from some mathematical tricks) and it is probably a mystery as to its importance. It also seems a bit abstract. However, this is the basis for some of the proverbial "back of the envelope calculations" which you might have heard about. So, let's see how it can be used.

Example 1.21. *Using dimensional analysis to obtain the period of a simple pendulum.*

Let's consider the period of a simple pendulum; e.g., a point mass hanging on a massless string. The period, T , of the pendulum's swing could depend upon the string length, ℓ , the mass of the "pendulum bob", m , and gravity in the form of the acceleration due to gravity, g . These are the q_i 's in the theorem. We have four physical variables. The only units involved are length, mass and time. So, $m = 3$. This means that there are $k = n - m = 1$ dimensionless variables, call it π . So, there must be an equation of the form

$$F(\pi) = 0$$

in terms of the dimensionless variable

$$\pi = \ell^{k_1} m^{k_2} T^{k_3} g^{k_4}.$$

We just need to find the k_i 's. This could be done by inspection, or we could write out the dimensions of each factor and determine how π can be dimensionless. Thus,

$$\begin{aligned} [\pi] &= [\ell]^{k_1} [m]^{k_2} [T]^{k_3} [g]^{k_4} \\ &= [L]^{k_1} [M]^{k_2} [T]^{k_3} \left(\frac{[L]}{[T]^2} \right)^{k_4} \\ &= [L]^{k_1+k_4} [M]^{k_2} [T]^{k_3-2k_4}. \end{aligned} \quad (1.95)$$

π will be dimensionless when

$$\begin{aligned} k_1 + k_4 &= 0, \\ k_2 &= 0, \\ k_3 - 2k_4 &= 0. \end{aligned} \quad (1.96)$$

This is a linear homogeneous system of three equations and four unknowns. We can satisfy these equations by setting $k_1 = -k_4$, $k_2 = 0$, and $k_3 = 2k_4$. Therefore, we have

$$\pi = \ell^{-k_4} T^{2k_4} g^{k_4} = \left(\ell^{-1} T^2 g \right)^{k_4}.$$

Formula for the period of a pendulum.

k_4 is arbitrary, so we can pick the simplest value, $k_4 = 1$. Then,

$$F\left(\frac{T^2 g}{\ell}\right) = 0.$$

Assuming that this equation has one zero, z , which has to be verified by other means, we have that

$$\frac{gT^2}{\ell} = z = \text{const.}$$

Thus, we have determined that the period is independent of the mass and proportional to the square root of the length. The constant can be determined by experiment as $z = 4\pi^2$. Thus,

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

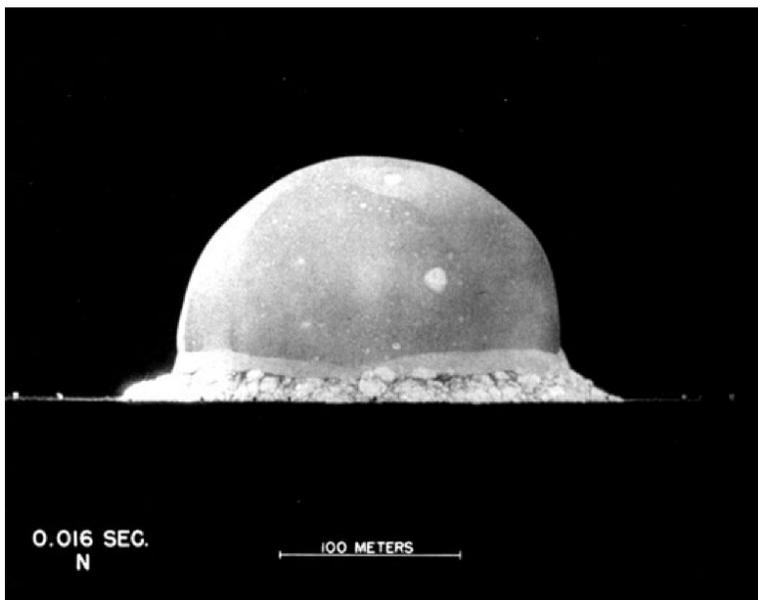


Figure 1.1: A photograph of the first atomic bomb test. This image was found at <http://www.atomicarchive.com>.

Example 1.22. *Estimating the energy of an atomic bomb.*

A more interesting example was provided by Sir Geoffrey Taylor in 1941 for determining the energy release of an atomic bomb. Let's assume that the energy is released in all directions from a single point. Possible physical variables are the time since the blast, t , the energy, E , the distance from the blast, r , the atmospheric density ρ and the atmospheric pressure, p . We have five physical variables and only three units. So, there should be two dimensionless quantities. Let's determine these.

We set

$$\pi = E^{k_1} t^{k_2} r^{k_3} p^{k_4} \rho^{k_5}.$$

Energy release in the first atomic bomb.

Inserting the respective units, we find that

$$\begin{aligned}
 [\pi] &= [E]^{k_1} [t]^{k_2} [r]^{k_3} [p]^{k_4} [\rho]^{k_5} \\
 &= \left([M][L]^2 [T]^{-2} \right)^{k_1} [T]^{k_2} [L]^{k_3} \left([M][L]^{-1} [T]^{-2} \right)^{k_4} \left([M][L]^{-3} \right)^{k_5} \\
 &= [M]^{k_1+k_4+k_5} [L]^{2k_1+k_3-k_4-3k_5} [T]^{-2k_1+k_2-2k_4}. \quad (1.97)
 \end{aligned}$$

Note: You should verify the units used. For example, the units of force can be found using $F = ma$ and work (energy) is force times distance. Similarly, you need to know that pressure is force per area.

For π to be dimensionless, we have to solve the system:

$$\begin{aligned}
 k_1 + k_4 + k_5 &= 0, \\
 2k_1 + k_3 - k_4 - 3k_5 &= 0, \\
 -2k_1 + k_2 - 2k_4 &= 0. \quad (1.98)
 \end{aligned}$$

This is a set of three equations and five unknowns. The only way to solve this system is to solve for three unknowns in term of the remaining two. (In linear algebra one learns how to solve this using matrix methods.) Let's solve for k_1, k_2 , and k_5 in terms of k_3 and k_4 . The system can be written as

$$\begin{aligned}
 k_1 + k_5 &= -k_4, \\
 2k_1 - 3k_5 &= k_4 - k_3, \\
 2k_1 - k_2 &= -2k_4. \quad (1.99)
 \end{aligned}$$

These can be solved by solving for k_1 and k_4 using the first two equations and then finding k_2 from the last one. Solving this system yields:

$$k_1 = -\frac{1}{5}(2k_4 + k_3) \quad k_2 = \frac{2}{5}(3k_4 - k_3) \quad k_5 = \frac{1}{5}(k_3 - 3k_4).$$

We have the freedom to pick values for k_3 and k_4 . Two independent sets of simple values are obtained by picking one variable as zero and the other as one. This will give the following two cases:

Case I. $k_3 = 1$ and $k_4 = 0$.

In this case we then have $k_1 = -\frac{1}{5}$, $k_2 = -\frac{2}{5}$, and $k_5 = \frac{1}{5}$. This gives

$$\pi_1 = E^{-1/5} t^{-2/5} r \rho^{1/5} = r \left(\frac{\rho}{E t^2} \right)^{1/5}.$$

Case II. $k_3 = 0$ and $k_4 = 1$.

In this case we then have $k_1 = -\frac{2}{5}$, $k_2 = \frac{6}{5}$, and $k_5 = -\frac{3}{5}$.

$$\pi_2 = E^{-2/5} t^{6/5} p \rho^{-3/5} = p \left(\frac{t^6}{\rho^3 E^2} \right)^{1/5}.$$

Thus, we have that the relation between the energy and the other variables is of the form

$$F\left(r\left(\frac{\rho}{Et^2}\right)^{1/5}, p\left(\frac{t^6}{\rho^3 E^2}\right)^{1/5}\right) = 0.$$

Of course, this is not enough to determine the explicit equation. However, Taylor was able to use this information to get an energy estimate.

Note that π_1 is dimensionless. It can be represented as a function of the dimensionless variable π_2 . So, assuming that $\pi_1 = h(\pi_2)$, we have that

$$h(\pi_2) = r\left(\frac{\rho}{Et^2}\right)^{1/5}.$$

Note that for $t = 1$ second, the energy is expected to be huge, so $\pi_2 \approx 0$. Thus,

$$r\left(\frac{\rho}{Et^2}\right)^{1/5} \approx h(0).$$

Simple experiments suggest that $h(0)$ is of order one, so

$$r \approx \left(\frac{Et^2}{\rho}\right)^{1/5}.$$

In 1947 Taylor applied his earlier analysis to movies of the first atomic bomb test in 1945 and his results were close to the actual values. How can one do this? You can find pictures of the first atomic bomb test with a superimposed length scale [online](#).

We can rewrite the above result to get the energy estimate:

$$E \approx \frac{r^5 \rho}{t^2}.$$

As an exercise, you can estimate the radius of the explosion at the given time and determine the energy of the blast in so many tons of TNT.

Problems

1. Prove the following identities using only the definitions of the trigonometric functions, the Pythagorean identity, or the identities for sines and cosines of sums of angles.

- a. $\cos 2x = 2 \cos^2 x - 1$.
- b. $\sin 3x = A \sin^3 x + B \sin x$, for what values of A and B ?

2. Do the following.

- a. Write $(\cosh x - \sinh x)^6$ in terms of exponentials.
- b. Prove $\cosh 2x = \cosh^2 x + \sinh^2 x$.
- c. If $\cosh x = \frac{13}{12}$ and $x < 0$, find $\sinh x$ and $\tanh x$.

d. Find the exact value of $\sinh(\operatorname{arccosh} 3)$

3. Compute the following integrals

a. $\int xe^{2x^2} dx$.

b. $\int_0^3 \frac{5x}{\sqrt{x^2+16}} dx$.

c. $\int x^3 \sin 3x dx$. (Do this using integration by parts, the Tabular Method, and differentiation under the integral sign.)

d. $\int \cos^4 3x dx$.

e. $\int_0^{\pi/2} \sec^3 x dx$.

f. $\int \sqrt{9-x^2} dx$

g. $\int \frac{dx}{(4-x^2)^2}$, using the substitution $x = 2 \tanh u$.

h. $\int \frac{dx}{(x+4)^{3/2}}$, using the substitutions

- $x = 2 \tan u$ and
- $x = 2 \sinh u$.

4. Find the sum for each of the series:

a. $\sum_{n=0}^{\infty} \frac{(-1)^n 3}{4^n}$.

b. $\sum_{n=2}^{\infty} \frac{2}{5^n}$.

c. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right)$.

d. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$.

5. Evaluate the following expressions at the given point. Use your calculator or your computer (such as Maple). Then use series expansions to find an approximation to the value of the expression to as many places as you trust.

a. $f(x) = \frac{1}{\sqrt{1+2x^2}} - 1 + x^2$ at $x = 5.00 \times 10^{-3}$.

b. $f(R, h) = R - \sqrt{R^2 + h^2}$ for $R = 1.374 \times 10^3$ km and $h = 1.00$ m.

c. $f(x) = 1 - \frac{1}{\sqrt{1-x}}$ for $x = 2.5 \times 10^{-13}$.

6. Use dimensional analysis to derive a possible expression for the drag force F_D on a soccer ball of diameter D moving at speed v through air of density ρ and viscosity μ . [Hint: Assuming viscosity has units $\frac{[M]}{[L][T]}$, there are two possible dimensionless combinations: $\pi_1 = \mu D^\alpha \rho^\beta v^\gamma$ and $\pi_2 = F_D D^\alpha \rho^\beta v^\gamma$. Determine α , β , and γ for each case and interpret your results.]

7. **Challenge:** Read the section on dimensional analysis. In particular, look at the results of Example 1.22. Using measurements in/on Figure 1.1, obtain an estimate of the energy of the blast in tons of TNT. Explain your work. Does your answer make sense? Why?