

Chapter 1

Introduction

Before we begin our study of mathematical physics, perhaps we should review some things from your past classes. You definitely need to know something before taking this class. It is assumed that you have taken Calculus and are comfortable with differentiation and integration. You should also have taken some introductory physics class, preferably the calculus based course. Of course, you are not expected to know every detail from these courses. However, there are some topics and methods that will come up and it would be useful to have a handy reference to what it is you should know, especially when it comes to exams.

Most importantly, you should still have your physics and calculus texts to which you can refer throughout the course. Looking back on that old material, you will find that it appears easier than when you first encountered the material. That is the nature of learning mathematics and physics. Your understanding is continually evolving as you explore topics more in depth. It does not always sink in the first time you see it.

In this chapter we will give a quick review of these topics. We will also mention a few new things that might be interesting. This review is meant to make sure that everyone is at the same level.

1.1 What Do I Need To Know From Calculus?

1.1.1 Introduction

There are two main topics in calculus: derivatives and integrals. You learned that derivatives are useful in providing rates of change in either time or space. Integrals provide areas under curves, but also are useful in providing other types of sums over continuous bodies, such as lengths, areas, volumes, moments of inertia, or flux integrals. In physics, one can look at graphs of position versus time and the slope (derivative) of such a function gives the velocity. Then plotting velocity versus time you can either look at the derivative to obtain acceleration, or you could look at the area under the curve and get the displacement:

$$x = \int_{t_0}^t v dt. \quad (1.1)$$

Of course, you need to know how to differentiate and integrate given functions. Even before getting into differentiation and integration, you need to have a bag of functions useful in physics. Common functions are the polynomial and rational functions. You should be fairly familiar with these. Polynomial functions take the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad (1.2)$$

where $a_n \neq 0$. This is the form of a polynomial of degree n . Rational functions consist of ratios of polynomials. Their graphs can exhibit asymptotes.

Next are the exponential and logarithmic functions. The most common are the natural exponential and the natural logarithm. The natural exponential is given by $f(x) = e^x$, where $e \approx 2.718281828 \dots$. The natural logarithm is the inverse to the exponential, denoted by $\ln x$. (One needs to be careful, because some mathematics and physics books use log to mean natural exponential, whereas many of us were first trained to use it to mean the common logarithm, which is the ‘log base 10’.)

The properties of the exponential function follow from our basic properties for exponents. Namely, we have:

$$e^0 = 1, \quad (1.3)$$

$$e^{-a} = \frac{1}{e^a} \quad (1.4)$$

$$e^a e^b = e^{a+b}, \quad (1.5)$$

$$(e^a)^b = e^{ab}. \quad (1.6)$$

The relation between the natural logarithm and natural exponential is given by

$$y = e^x \Leftrightarrow x = \ln y. \quad (1.7)$$

Some common logarithmic properties are

$$\ln 1 = 0, \quad (1.8)$$

$$\ln \frac{1}{a} = -\ln a, \quad (1.9)$$

$$\ln(ab) = \ln a + \ln b, \quad (1.10)$$

$$\ln \frac{a}{b} = \ln a - \ln b, \quad (1.11)$$

$$\ln \frac{1}{b} = -\ln b. \quad (1.12)$$

We will see further applications of these relations as we progress through the course.

1.1.2 Trigonometric Functions

Another set of useful functions are the trigonometric functions. These functions have probably plagued you since high school. They have their origins as far back as the building of the pyramids. Typical applications in your introductory math classes probably have included finding the heights of trees, flag poles, or buildings. It was recognized a long time ago that similar right triangles have fixed ratios of any pair of sides of the two similar triangles. These ratios only change when the non-right angles change.

Thus, the ratio of two sides of a right triangle only depends upon the angle. Since there are six possible ratios (think about it!), then there are six possible functions. These are designated as sine, cosine, tangent and their reciprocals (cosecant, secant and cotangent). In your introductory

Table 1.1: Table of Trigonometric Values

θ	$\cos \theta$	$\sin \theta$	$\tan \theta$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{2}$	0	1	undefined

physics class, you really only needed the first three. You also learned that they are represented as the ratios of the opposite to hypotenuse, adjacent to hypotenuse, etc. Hopefully, you have this down by now.

You should also know the exact values for the special angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$, and their corresponding angles in the second, third and fourth quadrants. This becomes internalized after much use, but we provide these values in Table 1.1 just in case you need a reminder.

The problems using trigonometric functions in later courses stem from using identities. We will have many an occasion to do so in this class as well. What is an identity? It is a relation that holds true all of the time. For example, the most common identity for trigonometric functions is

$$\sin^2 \theta + \cos^2 \theta = 1. \quad (1.13)$$

This hold true for every angle θ ! An even simpler identity is

$$\tan \theta = \frac{\sin \theta}{\cos \theta}. \quad (1.14)$$

Other simple identities can be derive from this one. Dividing the equation by $\cos^2 \theta$ or $\sin^2 \theta$ yields

$$\tan^2 \theta + 1 = \sec^2 \theta, \quad (1.15)$$

$$1 + \cot^2 \theta = \csc^2 \theta. \quad (1.16)$$

Other useful identities stem from the use of the sine and cosine of the sum and difference of two angles. Namely, we have that

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A, \quad (1.17)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B. \quad (1.18)$$

Note that the upper (lower) signs are taken together.

The double angle formulae are found by setting $A = B$:

$$\sin(2A) = 2 \sin A \cos B, \quad (1.19)$$

$$\cos(2A) = \cos^2 A - \sin^2 A. \quad (1.20)$$

Using Equation (1.13), we can rewrite (1.20) as

$$\cos(2A) = 2 \cos^2 A - 1, \quad (1.21)$$

$$= 1 - 2 \sin^2 A. \quad (1.22)$$

These, in turn, lead to the half angle formulae. Using $A = 2\alpha$, we find that

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}, \quad (1.23)$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}. \quad (1.24)$$

Finally, another useful set of identities are the product identities. For example, if we add the identities for $\sin(A + B)$ and $\sin(A - B)$, the second terms cancel and we have

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$$

Thus, we have that

$$\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B)). \quad (1.25)$$

Similarly, we have

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B)). \quad (1.26)$$

and

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B)). \quad (1.27)$$

These are the most common trigonometric identities. They appear often and should just roll off of your tongue.

We will also need to understand the behaviors of trigonometric functions. In particular, we know that the sine and cosine functions are periodic.

They are not the only periodic functions, as we shall see. [Just visualize the teeth on a carpenter's saw.] However, they are the most common periodic functions.

A periodic function $f(x)$ satisfies the relation

$$f(x + p) = f(x), \quad \text{for all } x$$

for some constant p . If p is the smallest such number, then p is called the period. Both the sine and cosine functions have period 2π . This means that the graph repeats its form every 2π units. Similarly, $\sin bx$ and $\cos bx$ have the common period $p = \frac{2\pi}{b}$. We will make use of this fact in later chapters.

1.1.3 Other Elementary Functions

So, are there any other functions that are useful in physics? Actually, there are many more. However, you have probably not see many of them to date. We will see by the end of the semester that there are many important functions that arise as solutions of some fairly generic, but important, physics problems. In your calculus classes you have also seen that some relations are represented in parametric form. However, there is at least one other set of elementary functions, which you should know about. These are the hyperbolic functions. Such functions are useful in representing hanging cables, unbounded orbits, and special traveling waves called solitons. They also play a role in special and general relativity.

Hyperbolic functions are actually related to the trigonometric functions, as we shall see after a little bit of complex function theory. For now, we just want to recall a few definitions and an identity. Just as all of the trigonometric functions can be built from the sine and the cosine, the hyperbolic functions can be defined in terms of the hyperbolic sine and hyperbolic cosine:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad (1.28)$$

$$\cosh x = \frac{e^x + e^{-x}}{2}. \quad (1.29)$$

There are four other hyperbolic functions. These are defined in terms of the above functions similar to the relations between the trigonometric

Table 1.2: Table of Derivatives

Function	Derivative
a	0
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a \sec^2 ax$
$\csc ax$	$-a \csc ax \cot ax$
$\sec ax$	$a \sec ax \tan ax$
$\cot ax$	$-a \csc^2 ax$
$\sinh ax$	$a \cosh ax$
$\cosh ax$	$a \sinh ax$
$\tanh ax$	$a \operatorname{sech}^2 ax$
$\operatorname{csch} ax$	$-a \operatorname{csch} ax \operatorname{coth} ax$
$\operatorname{sech} ax$	$-a \operatorname{sech} ax \tanh ax$
$\operatorname{coth} ax$	$-a \operatorname{csch}^2 ax$

functions. For example, we have

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

There are also a whole set of identities, similar to those for the trigonometric functions. Some of these are given by the following:

$$\cosh^2 x - \sinh^2 x = 1, \quad (1.30)$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B \quad (1.31)$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \sinh B \cosh A. \quad (1.32)$$

Others can be derived from these.

1.1.4 Derivatives

Now that we know our elementary functions, we can seek their derivatives. We will not spend time exploring the appropriate limits in any rigorous

way. We are only interested in the results. We provide these in Table 1.2. We expect that you know the meaning of the derivative and all of the usual rules, such as the product and quotient rules.

Also, you should be familiar with the Chain Rule. Recall that this rule tells us that if we have a composition of functions, such as the elementary functions above, then we can compute the derivative of the composite function. Namely, if $h(x) = f(g(x))$, then

$$\frac{dh}{dx} = \frac{d}{dx} (f(g(x))) = \frac{df}{dg}|_{g(x)} \frac{dg}{dx} = f'(g(x))g'(x). \quad (1.33)$$

For example, let $H(x) = 5 \cos(\pi \tanh 2x^2)$. This is a composition of three functions, $H(x) = f(g(h(x)))$, where $f(x) = 5 \cos x$, $g(x) = \pi \tanh x$, and $h(x) = 2x^2$. Then the derivative becomes

$$\begin{aligned} H'(x) &= 5 \left(-\sin(\pi \tanh 2x^2) \right) \frac{d}{dx} \left((\pi \tanh 2x^2) \right) \\ &= -5\pi \sin(\pi \tanh 2x^2) \operatorname{sech}^2 2x^2 \frac{d}{dx} (2x^2) \\ &= -20\pi x \sin(\pi \tanh 2x^2) \operatorname{sech}^2 2x^2. \end{aligned} \quad (1.34)$$

1.1.5 Integrals

Integration is typically a bit harder. Imagine being given the result in (1.34) and having to figure out the integral. As you may recall from the Fundamental Theorem of Calculus, the integral is the inverse operation to differentiation:

$$\int \frac{df}{dx} dx = f(x) + C. \quad (1.35)$$

However, it is not always easy to determine a given integral. In fact some integrals are not even doable! However, you learned in calculus that there are some methods that might yield an answer. While you might be happier using a computer with a computer algebra systems, such as Maple, you should know a few basic integrals and know how to use tables for some of the more complicated ones. In fact, it can be exhilarating when you can do a given integral without reference to a computer or a Table of Integrals. However, you should be prepared to do some integrals using what you have been taught in calculus. We will review a few of these methods and some of the standard integrals in this section.

Table 1.3: Table of Integrals

Function	Indefinite Integral
a	ax
x^n	$\frac{x^{n+1}}{n+1}$
e^{ax}	$\frac{1}{a}e^{ax}$
$\frac{1}{x}$	$\ln x$
$\sin ax$	$-\frac{1}{a}\cos ax$
$\cos ax$	$\frac{1}{a}\sin ax$
$\sec^2 ax$	$\frac{1}{a}\tan ax$
$\sinh ax$	$\frac{1}{a}\cosh ax$
$\cosh ax$	$\frac{1}{a}\sinh ax$
$\operatorname{sech}^2 ax$	$\frac{1}{a}\tanh ax$
$\frac{1}{a+bx}$	$\frac{1}{b}\ln(a+bx)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1} ax$
$\frac{1}{\sqrt{a^2-x^2}}$	$\frac{1}{a}\sin^{-1} ax$
$\frac{1}{x^2-a^2}$	$\frac{1}{a}\sec^{-1} ax$

First of all, there are some integrals you should be expected to know without any work. These integrals appear often and are just an application of the Fundamental Theorem of Calculus to the previous Table 1.2. The basic integrals that students should know of the top of their heads are given in Table 1.3.

These are not the only integrals you should be able to do. However, we can expand the list by recalling a few of the techniques that you learned in calculus. There are just a few: The Method of Substitution, Integration by Parts, Integration Using Partial Fraction Decomposition, and Trigonometric Integrals.

Example 1 When confronted with an integral, you should first ask if a simple substitution would reduce the integral to one you know how to do. So, as an example, consider the following integral

$$\int \frac{x}{\sqrt{x^2+1}} dx.$$

The ugly part of this integral is the x^2+1 under the square root. So, we

let $u = x^2 + 1$. Noting that when $u = f(x)$, we have $du = f'(x) dx$. For our example, $du = 2x dx$. Looking at the integral, part of the integrand can be written as $x dx = \frac{1}{2}u du$. Then, our integral becomes

$$\int \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2} \int \frac{du}{\sqrt{u}}.$$

The substitution has converted our integral into an integral over u . Also, this integral is doable! It is one of the integrals we should know. Namely, we can write it as

$$\frac{1}{2} \int \frac{du}{\sqrt{u}} = \frac{1}{2} \int u^{-1/2} du.$$

This is now easily finished after integrating and using our substitution variable to give

$$\int \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2} \frac{u^{1/2}}{\frac{1}{2}} = \sqrt{x^2+1} + C.$$

Note that we have added the required integration constant and that the derivative of the result easily gives the original integrand (after employing the Chain Rule).

Often we are faced with definite integrals, in which we integrate between two limits. There are several ways to use these limits. However, students often forget that a change of variables generally means that the limits have to change.

Example 2 Consider the above example with limits added.

$$\int_0^2 \frac{x}{\sqrt{x^2+1}} dx.$$

We proceed as before. We let $u = x^2 + 1$. As x goes from 0 to 2, u takes values from 1 to 5. So, our substitution gives

$$\int_0^2 \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2} \int_1^5 \frac{du}{\sqrt{u}} = \sqrt{u} \Big|_1^5 = \sqrt{5} - 1.$$

When the Method of substitution fails, there are other methods you can try. One of the most used is the Method of Integration by Parts. Recall the Integration by Parts Formula:

$$\int u dv = uv - \int v du. \tag{1.36}$$

The idea is that you are given the integral on the left and you can relate it to an integral on the right. Hopefully, the new integral is one you can do, or at least it is an easier integral than the one you are trying to evaluate.

However, you are not usually given the functions u and v . You have to determine them. The integral form that you really have is a function of another variable, say x . Another form of the formula can be given as

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx. \quad (1.37)$$

This form is a bit more complicated in appearance, though it is clearer what is happening. The derivative has been moved from one function to the other. Recall that this formula was derived by integrating the product rule for differentiation.

The two formulae are related by using the relations

$$\begin{aligned} u = f(x) &\rightarrow du = f'(x) dx, \\ v = g(x) &\rightarrow dv = g'(x) dx. \end{aligned} \quad (1.38)$$

This also gives a method for applying the Integration by Parts Formula.

Example 3 Consider the integral $\int x \sin 2x dx$. We choose $u = x$ and $dv = \sin 2x dx$. This gives the correct left side of the formula. We next determine v and du :

$$\begin{aligned} du &= \frac{du}{dx} dx = dx, \\ v &= \int dv = \int \sin 2x dx = -\frac{1}{2} \cos 2x. \end{aligned}$$

We note that one usually does not need the integration constant. Inserting these expressions into the Integration by Parts Formula, we have

$$\int x \sin 2x dx = -\frac{1}{2}x \cos 2x + \frac{1}{2} \int \cos 2x dx.$$

We see that the new integral is easier to do than the original integral. Had we picked $u = \sin 2x$ and $dv = x dx$, then the formula still works, but the resulting integral is not easier.

For completeness, we can finish the integration. The result is

$$\int x \sin 2x dx = -\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x + C.$$

As always, you can check your answer by differentiating the result, a step students often forget to do. Namely,

$$\begin{aligned} \frac{d}{dx} \left(-\frac{1}{2}x \cos 2x + \frac{1}{4} \sin 2x + C \right) &= -\frac{1}{2} \cos 2x + x \sin 2x + \frac{1}{4}(2 \cos 2x) \\ &= x \sin 2x. \end{aligned} \quad (1.39)$$

So, we do get back the integrand in the original integral.

We can also perform integration by parts on definite integrals. The general formula is written as

$$\int_a^b f(x)g'(x) dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x) dx. \quad (1.40)$$

Example 4 Consider the integral

$$\int_0^\pi x^2 \cos x dx.$$

This will require two integrations by parts. First, we let $u = x^2$ and $dv = \cos x$. Then,

$$du = 2x dx. \quad v = \sin x.$$

Inserting into the Integration by Parts Formula, we have

$$\begin{aligned} \int_0^\pi x^2 \cos x dx &= x^2 \sin x|_0^\pi - 2 \int_0^\pi x \sin x dx \\ &= -2 \int_0^\pi x \sin x dx. \end{aligned} \quad (1.41)$$

We note that the resulting integral is easier than the given integral, but we still cannot do the integral off the top of our head (unless we look at Example 3!). So, we need to integrate by parts again. (Note: In your calculus class you may recall that there is a tabular method for carrying out multiple applications of the formula. However, we will leave that to the reader and proceed with the brute force computation.)

We apply integration by parts by letting $U = x$ and $dV = \sin x dx$. This gives that $dU = dx$ and $V = -\cos x$. Therefore, we have

$$\begin{aligned} \int_0^\pi x \sin x dx &= -x \cos x|_0^\pi + \int_0^\pi \cos x dx \\ &= \pi + \sin x|_0^\pi \\ &= \pi. \end{aligned} \quad (1.42)$$

The final result is

$$\int_0^\pi x^2 \cos x \, dx = -2\pi.$$

Other types of integrals that you will see often are trigonometric integrals. In particular, integrals involving powers of sines and cosines. For odd powers, a simple substitution will turn the integrals into simple powers.

Example 5 For example, consider

$$\int \cos^3 x \, dx.$$

This can be rewritten as

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx.$$

Let $u = \sin x$. Then $du = \cos x \, dx$. Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\begin{aligned} \int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\ &= \int (1 - u^2) \, du \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C. \end{aligned} \tag{1.43}$$

A quick check confirms the answer:

$$\frac{d}{dx} \left(\sin x - \frac{1}{3}\sin^3 x + C \right) = \cos x - \sin^2 x \cos x = \cos x(1 - \sin^2 x) = \cos^3 x.$$

Even powers of sines and cosines are a little more complicated, but doable. In these cases we need the half angle formulae:

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}, \tag{1.44}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}. \tag{1.45}$$

Example 6 As an example, we will compute

$$\int_0^{2\pi} \cos^2 x \, dx.$$

Substituting the half angle formula for $\cos^2 x$, we have

$$\begin{aligned} \int_0^{2\pi} \cos^2 x \, dx &= \frac{1}{2} \int_0^{2\pi} (1 + \cos 2x) \, dx \\ &= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right)_0^{2\pi} \\ &= \pi. \end{aligned} \tag{1.46}$$

We note that this result appears often in physics. When looking at root mean square averages of sinusoidal waves, one needs the average of the square of sines and cosines. Recall that the average of a function on interval $[a, b]$ is given as

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) \, dx. \tag{1.47}$$

So, the average of $\cos^2 x$ over one period is

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2 x \, dx = \frac{1}{2}. \tag{1.48}$$

The root mean square is then $\frac{1}{\sqrt{2}}$.

So far, this is enough to get started in the course. We will recall other topics as we need them. In particular, we discuss the method of partial fraction decomposition when we discuss terminal velocity in the next chapter and applications of the Laplace transform later in the book.

1.2 What I Need From My Intro Physics Class?

So, what do we need to know about physics? You should be comfortable with common terms from mechanics and electromagnetism. In some cases, we will review specific topics. However, it would be helpful to review some topics from your introductory physics text.

As you may recall, your study of physics began with the simplest systems. We first studied motion for point masses. We are introduced to the concepts of position, displacement, velocity and acceleration. We studied motion first in one dimension and even then can only do problems in which

the acceleration is constant, or piecewise constant. We looked at horizontal motion and then vertical motion, in terms of free fall. Finally, we moved into two dimensions and considered projectile motion. Some calculus was introduced and you learned how to represent vector quantities.

We then ask, “What causes a change in the state of motion of a body?” We are lead to a discussion of forces. The types of forces encountered are the weight, the normal force, tension, the force of gravity and then centripetal forces. You might have also seen spring forces, which we will see shortly, lead to oscillatory motion - the underlying theme of this book.

Next, you find out that there are well known conservation principles for energy and momentum. In these cases you are lead to the concepts of work, kinetic energy and potential energy. You find out that even when mechanical energy is not conserved, you can account for the missing energy as the work done by nonconservative forces. Momentum becomes important in collision problems or when looking at impulses.

With these basic ideas under your belt, you proceed to study more complicated systems. We can look at extended bodies, most notably rigid bodies. This lead to the study of rotational motion. One finds out that there are analogues to all of the previously discussed concepts for point masses. For example, we have rotational velocity and acceleration. The cause of rotational acceleration is the torque. The analogue to mass is the moment of inertia.

The next level of complication, which sometimes is not covered, are bulk systems. One can study fluids, solids and gases. These can be investigated by looking at things like mass density, pressure, volume and temperature. This leads to the study of thermodynamics in which one studies the transfer of energy between a system and its surroundings. This involves the relationship between the work done on the system, the heat energy added to a systems and its change in internal energy.

Bulk systems can also suffer deformations when a force per area is applied. This can lead to the idea that small deformations can lead to the propagation of energy throughout the system in the form of waves. We will later explore this wave motion in several systems.

The second course in physics is spent on electricity and magnetism, leading to electromagnetic waves. One first learns about charges and charge

distributions, electric fields, electric potentials. Then we find out that moving charges produce magnetic fields and are affected by external magnetic fields. Furthermore, changing magnetic fields produce currents. This can all be summarized by Maxwell's equations, which we will recall later in the course. These equations, in turn, predict the existence of electromagnetic waves.

Depending how far one delves into the book, one may see excursions into optics and the impact that trying to understand the existence of electromagnetic waves has had on the development of so-called "modern physics". For example, in trying to understand what medium electromagnetic waves might propagate through, Einstein proposed an answer that completely changed the way we understand the nature of space and time. In trying to understand how ionized gases radiate and interact with matter, Einstein and others were lead down a path that has lead to quantum mechanics and further challenges to our understanding of reality.

So, that is the introductory physics course in a nutshell. In fact, that is most of physics. The rest is detail, which you will explore in your other courses as you progress toward a degree in physics.

1.3 Technology and Tables

As we progress through the course, you will often have to compute integrals and derivatives by hand. However, many of you know that some of the tedium can be alleviated by using computers, or even looking up what you need in tables. In some cases you might even find applets online that can quickly give you the answers you seek.

However, you also need to be comfortable in doing many computations by hand. This is necessary, especially in your early studies, for several reasons. For example, you should try to evaluate integrals by hand when asked to do them. This reinforces the techniques, as outlined earlier. It exercises your brain in much the same way that you might jog daily to exercise your body. Who knows, keeping your brain active this way might even postpone Alzheimer's. The more comfortable you are with derivations and evaluations, the easier it is to follow future lectures without getting bogged down by the details, wondering how your professor got from step A to step D. You can always use a computer algebra system, or a Table of

Integrals, to check on your work.

Problems can arise when depending purely on the output of computers, or other "black boxes". Once you have a firm grasp on the techniques and a feeling as to what answers should look like, then you can feel comfortable with what the computer gives you. Sometimes, programs like Maple can give you strange looking answers, and sometimes wrong answers. Also, Maple cannot do every integral, or solve every differential equation, that you ask it to do. Even some of the simplest looking expressions can cause computer algebra systems problems. Other times you might even provide wrong input, leading to erroneous results.

Another source of indefinite integrals, derivatives, series expansions, etc, is a Table of Mathematical Formulae. There are several good books that have been printed. Even some of these have typos in them, so you need to be careful. However, it may be worth the investment to have such a book in your personal library. Go to the library, or the bookstore, and look at some of these tables to see how useful they might be.

There are plenty of online resources as well. For example, there is the Wolfram Integrator at <http://integrals.wolfram.com/>. There is also a wealth of information at the following sites: <http://www.sosmath.com/>, <http://www.math2.org/>, <http://mathworld.wolfram.com/>, and <http://functions.wolfram.com/>.

1.4 Back of the Envelope Computations

In the first chapter in your introductory physics text you were introduced to dimensional analysis. Dimensional analysis is useful for recalling particular relationships between variables by looking at the units involved, independent of the system of units employed. Though most of the time you have used SI, or MKS, units in most of your physics problems.

There are certain basic units - length, mass and time. By the second course, you found out that you could add charge to the list. We can represent these as [L], [M], [T] and [C]. Other quantities typically have units that can be expressed in terms of the basic units. These are called derived units. So, we have that the units of acceleration are $[L]/[T]^2$ and units of mass density are $[M]/[L]^3$. Similarly, units of magnetic field can be

found, though with a little more effort. Recall that $F = qvB \sin \theta$ for a charge q moving with speed v through a magnetic field B at an angle of θ . $\sin \theta$ has no units. So,

$$\begin{aligned} [B] &= \frac{[F]}{[q][v]} \\ &= \frac{\frac{[M][L]}{[T]^2}}{[C] \frac{[L]}{[T]}} \\ &= \frac{[M]}{[C][T]}. \end{aligned} \tag{1.49}$$

Now, assume that you do not know how B depended on F , q and v , but you knew the units of all of the quantities. Can you figure out the relationship between them? We could write

$$[B] = [F]^\alpha [q]^\beta [v]^\gamma$$

and solve for the exponents by inserting the dimensions. Thus, we have

$$[M][C]^{-1}[T]^{-1} = ([M][L][T]^{-2})^\alpha [C]^\beta ([L][T]^{-1})^\gamma.$$

Right away we can see that $\alpha = 1$ and $\beta = -1$ by looking at the powers of $[M]$ and $[C]$, respectively. Thus,

$$[M][C]^{-1}[T]^{-1} = [M][L][T]^{-2}[C]^{-1} ([L][T]^{-1})^\gamma = [M][C]^{-1}[L]^{1+\gamma}[T]^{-2-\gamma}.$$

We see that picking $\gamma = -1$ balances the exponents and gives the correct relation

$$[B] = [F][q]^{-1}[v]^{-1}.$$

An important theorem at the heart of dimensional analysis is the Buckingham Π Theorem. In essence, this theorem tells us that physically meaningful equations in n variables can be written as an equation involving $n - m$ dimensionless quantities, where m is the number of dimensions used. The importance of this theorem is that one can actually compute useful quantities without even knowing the exact form of the equation!

The Buckingham Π Theorem¹ was introduced by E. Buckingham in 1914. Let q_i be n physical variables that are related by

$$f(q_1, q_2, \dots, q_n) = 0. \tag{1.50}$$

¹http://en.wikipedia.org/wiki/Buckingham_Pi_theorem

Assuming that m dimensions are involved, we let π_i be $k = n - m$ dimensionless variables. Then the equation (1.50) can be rewritten as a function of these dimensionless variables as

$$F(\pi_1, \pi_2, \dots, \pi_k) = 0, \quad (1.51)$$

where the π_i 's can be written in terms of the physical variables as

$$\pi_i = q_1^{k_1} q_2^{k_2} \dots q_n^{k_n}, \quad i = 1, \dots, k. \quad (1.52)$$

Well, this is our first new concept and it is probably a mystery as to its importance. It also seems a bit abstract. However, this is the basis for some of the proverbial "back of the envelope calculations" which you might have heard about. So, let's see how it can be used.

Example 1 Let's consider the period of a simple pendulum; e.g., a point mass hanging on a massless string. The period, T , of the pendulum's swing could depend upon the the string length, ℓ , the mass of the "pendulum bob", m , and gravity in the form of the acceleration due to gravity, g . These are the q_i 's in the theorem. We have four physical variables. The only units involved are length, mass and time. So, $m = 3$. This means that there are $k = n - m = 1$ dimensionless variables, call it π . So, there must be an equation of the form

$$F(\pi) = 0$$

in terms of the dimensionless variable

$$\pi = \ell^{k_1} m^{k_2} T^{k_3} g^{k_4}.$$

We just need to find the k_i 's. This could be done by inspection, or we could write out the dimensions of each factor and determine how π can be dimensionless. Thus,

$$\begin{aligned} [\pi] &= [\ell]^{k_1} [m]^{k_2} [T]^{k_3} [g]^{k_4} \\ &= [L]^{k_1} [M]^{k_2} [T]^{k_3} \left(\frac{[L]}{[T]^2} \right)^{k_4} \\ &= [L]^{k_1+k_4} [M]^{k_2} [T]^{k_3-2k_4}. \end{aligned} \quad (1.53)$$

π will be dimensionless when

$$\begin{aligned} k_1 + k_4 &= 0, \\ k_2 &= 0, \\ k_3 - 2k_4 &= 0. \end{aligned} \quad (1.54)$$

This is a linear homogeneous system of three equations and four unknowns. We can satisfy these equations by setting $k_1 = -k_4$, $k_2 = 0$, and $k_3 = 2k_4$. Therefore, we have

$$\pi = \ell^{-k_4} T^{2k_4} g^{k_4} = \left(\ell^{-1} T^2 g \right)^{k_4}.$$

k_4 is arbitrary, so we can pick the simplest value, $k_4 = 1$. Then,

$$F\left(\frac{T^2 g}{\ell}\right) = 0.$$

Assuming that this equation has one zero, z , which has to be verified by other means, we have that

$$\frac{gT^2}{\ell} = z = \text{const..}$$

Thus, we have determined that the period is independent of the mass and proportional to the square root of the length. The constant can be determined by experiment as $z = 4\pi^2$.

Example 2 A more interesting example was provided by Sir Geoffrey Taylor in 1941 for determining the energy release of an atomic bomb. Let's assume that the energy is released in all directions from a single point. Possible physical variables are the time since the blast, t , the energy, E , the distance from the blast, r , the atmospheric density ρ and the atmospheric pressure, p . We have five physical variables and only three units. So, there should be two dimensionless quantities. Let's determine these.

We set

$$\pi = E^{k_1} t^{k_2} r^{k_3} p^{k_4} \rho^{k_5}.$$

Inserting the respective units, we find that

$$\begin{aligned} [\pi] &= [E]^{k_1} [t]^{k_2} [r]^{k_3} [p]^{k_4} [\rho]^{k_5} \\ &= \left([M][L]^2 [T]^{-2} \right)^{k_1} [T]^{k_2} [L]^{k_3} \left([M][L]^{-1} [T]^{-2} \right)^{k_4} \left([M][L]^{-3} \right)^{k_5} \\ &= [M]^{k_1+k_4+k_5} [L]^{2k_1+k_3-k_4-3k_5} [T]^{-2k_1+k_2-2k_4}. \end{aligned} \quad (1.55)$$

Note: You should verify the units used. For example, the units of force can be found using $F = ma$ and work (energy) is force times distance. Similarly, you need to know that pressure is force per area.

For π to be dimensionless, we have to solve the system:

$$\begin{aligned} k_1 + k_4 + k_5 &= 0, \\ 2k_1 + k_3 - k_4 - 3k_5 &= 0, \\ -2k_1 + k_2 - 2k_4 &= 0. \end{aligned} \tag{1.56}$$

This is a set of three equations and five unknowns. The only way to solve this system is to solve for three unknowns in term of the remaining two. (In linear algebra one learns how to solve this using matrix methods.) Let's solve for k_1, k_2 , and k_5 in terms of k_3 and k_4 . The system can be written as

$$\begin{aligned} k_1 + k_5 &= -k_4, \\ 2k_1 - 3k_5 &= k_4 - k_3, \\ 2k_1 - k_2 &= -2k_4. \end{aligned} \tag{1.57}$$

These can be solved by solving for k_1 and k_4 using the first two equations and then finding k_2 from the last one. Solving this system yields:

$$k_1 = -\frac{1}{5}(2k_4 + k_3) \quad k_2 = \frac{2}{5}(3k_4 - k_3) \quad k_5 = \frac{1}{5}(k_3 - 3k_4).$$

We have the freedom to pick values for k_3 and k_4 . Two independent set of simple values would be to pick one variable as zero and the other as one. This will give our two dimensionless variables:

Case I. $k_3 = 1$ and $k_4 = 0$.

In this case we then have $k_1 = -\frac{1}{5}$, $k_2 = -\frac{2}{5}$, and $k_5 = \frac{1}{5}$. This gives

$$\pi_1 = E^{-1/5} t^{-2/5} r \rho^{1/5} = r \left(\frac{\rho}{Et^2} \right)^{1/5}.$$

Case II. $k_3 = 0$ and $k_4 = 1$.

In this case we then have $k_1 = -\frac{2}{5}$, $k_2 = \frac{6}{5}$, and $k_5 = -\frac{3}{5}$.

$$\pi_2 = E^{-2/5} t^{6/5} p \rho^{-3/5} = p \left(\frac{t^6}{\rho^3 E^2} \right)^{1/5}.$$

Thus, we have that the relation between the energy and the other variables is of the form

$$F\left(r\left(\frac{\rho}{Et^2}\right)^{1/5}, p\left(\frac{t^6}{\rho^3 E^2}\right)^{1/5}\right) = 0.$$

Of course, this is not enough to determine the explicit equation. However, Taylor was able to use this information to get an energy estimate.

Note that π_1 is dimensionless. It can be represented as a function of the dimensionless variable π_2 . So, assuming that $\pi_1 = h(\pi_2)$, we have that

$$h(\pi_2) = r\left(\frac{\rho}{Et^2}\right)^{1/5}.$$

Note that for $t = 1$ second, the energy is expected to be huge, so $\pi_2 \approx 0$. Thus,

$$r\left(\frac{\rho}{Et^2}\right)^{1/5} \approx h(0).$$

Simple experiments suggest that $h(0)$ is of order one, so

$$r \approx \left(\frac{Et^2}{\rho}\right)^{1/5}.$$

In 1947 Taylor applied his earlier analysis to movies of the first atomic bomb test in 1945 and his results were close to the actual values. How can one do this? You can find pictures of the first atomic bomb test with a superimposed length scale online².

We can rewrite the above result to get the energy estimate:

$$E \approx \frac{r^5 \rho}{t^2}.$$

As an exercise, you can estimate the radius of the explosion at the given time and determine how the energy of the blast in so many tons of TNT.

²<http://www.atomicarchive.com/Photos/Trinity/image7.shtml>

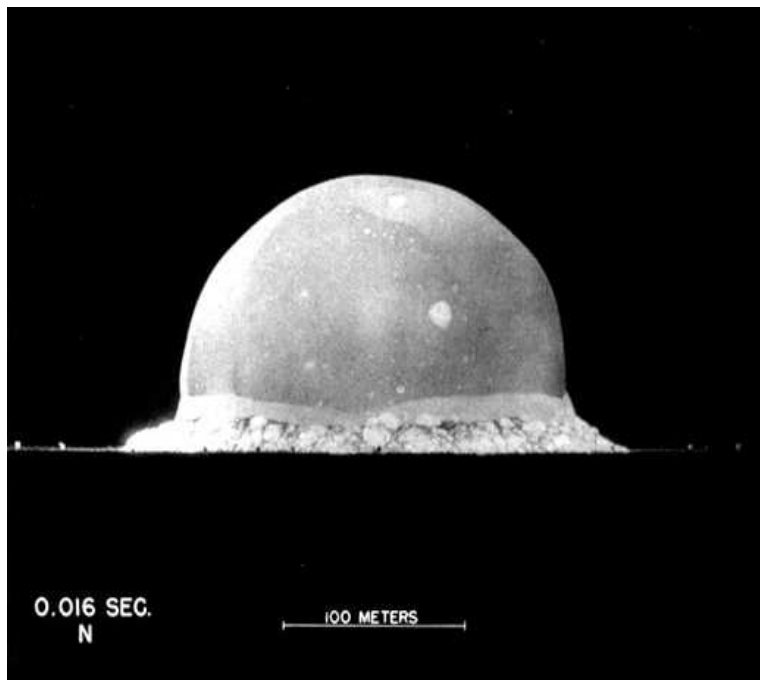


Figure 1.1: A photograph of the first atomic bomb test.
<http://www.atomicarchive.com/Photos/Trinity/image7.shtml>

1.5 Chapter 1 Problems

1. Compute the following integrals

(a) $\int x e^{-x^2} dx$.

(b) $\int_0^3 \frac{5x}{\sqrt{x^2+16}} dx$.

(c) $\int x^2 \sin 3x dx$.

(d) $\int \cos^4 3x dx$.