

Chapter 4

The Harmonics of Vibrating Strings

4.1 Harmonics and Vibrations

Until now we have studied oscillations in several physical systems. These lead to ordinary differential equations describing the time evolution of the systems and required the solution of initial value problems. In this chapter we will extend our study include oscillations in space. The typical example is the vibrating string.

When one plucks a violin, or guitar, string, the string vibrates exhibiting a variety of sounds. These are enhanced by the violin case, but we will only focus on the simpler vibrations of the string. We will consider the one dimensional wave motion in the string. Physically, the speed of these waves depends on the tension in the string and its mass density. The frequencies we hear are then related to the string shape, or the allowed wavelengths across the string. We will be interested the harmonics, or pure sinusoidal waves, of the vibrating string and how a general wave on a string can be represented as a sum over such harmonics. This will take us into the field of spectral, or Fourier, analysis.

Such systems are governed by partial differential equations. The vibrations of a string are governed by the one dimensional wave equation. Another simple partial differential equation is that of the heat, or diffusion,

equation. This equation governs heat flow. We will consider the flow of heat through a one dimensional rod. The solution of the heat equation also involves the use of Fourier analysis. However, in this case there are no oscillations in time.

There are many applications that are studied using spectral analysis. At the root of these studies is the belief that continuous waveforms are composed of a number of harmonics. Such ideas stretch back to the Pythagoreans' study of the vibrations of strings, which led to their program of a world of harmony. This idea was carried further by Johannes Kepler in his harmony of the spheres approach to planetary orbits. In the 1700's others worked on the superposition theory for vibrating waves on a stretched spring, starting with the wave equation and leading to the superposition of right and left traveling waves. This work was carried out by people such as John Wallis, Brook Taylor and Jean le Rond d'Alembert.

In 1742 d'Alembert solved the wave equation

$$c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0,$$

where $y(x, t)$ is the string height and c is the wave speed. However, his solution led himself and others, like Leonhard Euler and Daniel Bernoulli, to investigate what "functions" could be the solutions of this equation. In fact, this led to a more rigorous approach in the study of analysis by first coming to grips with the concept of a function. For example, in 1749 Euler sought the solution for a plucked string in which case the initial condition $y(x, 0) = h(x)$ has a discontinuous derivative!

In 1753 Daniel Bernoulli viewed the solutions as a superposition of simple vibrations, or harmonics. Such superpositions amounted to looking at solutions of the form

$$y(x, t) = \sum_k a_k \sin \frac{k\pi x}{L} \cos \frac{k\pi ct}{L},$$

where the string extended over a finite interval, $[0, L]$, with fixed ends at $x = 0$ and $x = L$. However, the initial conditions for such superpositions are represented as a sum of sinusoidal functions,

$$y(x, 0) = \sum_k a_k \sin \frac{k\pi x}{L}.$$

It was determined that many functions could not be represented by a finite number of harmonics, even for the simply plucked string given by an initial condition of the form

$$y(x, 0) = \begin{cases} cx, & 0 \leq x \leq L/2 \\ c(L - x), & L/2 \leq x \leq L \end{cases}$$

Thus, the solution consists generally of an infinite series of trigonometric functions.

Such series expansions were also of importance in Joseph Fourier's solution of the heat equation. The use of Fourier expansions became an important tool in the solution of linear partial differential equations, such as the wave equation and the heat equation. More generally, using a technique called the Separation of Variables, higher dimensional problems could be reduced to one dimensional boundary value problems. However, these studies led to very important questions, which in turn opened the doors to whole fields of analysis. Some of the problems raised were:

1. What functions can be represented as the sum of trigonometric functions?
2. How can a function with discontinuous derivatives like the plucked string be represented by a sum of smooth functions?
3. Do such infinite sums of trigonometric functions actually converge to the functions they represent?

There are many other systems in which it makes sense to interpret the solutions as sums of sinusoids of particular frequencies. For example, we can consider ocean waves. Ocean waves are affected by the gravitational pull of the moon and the sun and other numerous forces. These lead to the tides, which in turn have their own periods of motion. In an analysis of wave heights, one can separate out the tidal components by making use of Fourier analysis.

4.2 Boundary Value Problems

Until this point we have solved initial value problems. For an initial value problem one has to solve a differential equation subject to conditions on

the unknown function and its derivatives at one value of the independent variable. For example, for $x = x(t)$ we could have the initial value problem

$$x'' + x = 2, \quad x(0) = 1, x'(0) = 0. \quad (4.1)$$

In the next sections we will study boundary value problems and various tools for solving such problems. For an initial value problem one has to solve a differential equation subject to conditions on the unknown function or its derivatives at more than one value of the independent variable. Typically, initial value problems involve time dependent functions and boundary value problems are spatial. So, with an initial value problem one knows how a system evolves in terms of the differential equation and the state of the system at some fixed time. Then one seeks to determine the state of the system at a later time.

For boundary values problems, one knows how each point responds to its neighbors, but there are conditions that have to be satisfied at the endpoints. An example would be a horizontal beam supported at the ends, like a bridge. The shape of the beam under the influence of gravity, or other forces, would lead to a differential equation and the boundary conditions at the beam ends would affect the solution of the problem. There are also a variety of types of boundary conditions. In the case of a beam, one end could be fixed and the other end could be free to move. We will explore the effects of different boundary value conditions in our discussions and exercises.

As an example, we have a slight modification of the above problem: Find the solution $x = x(t)$ for $0 \leq t \leq 1$ that satisfies the problem

$$x'' + x = 2, \quad x(0) = 1, x(1) = 0. \quad (4.2)$$

As with initial value problems, we need to find the general solution and then apply any conditions that we may have. This is a nonhomogeneous differential equation, so we have that the solution is the sum of a solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, $x(t) = x_h(t) + x_p(t)$. The solution of $x'' + x = 0$ is easily found as

$$x_h(t) = c_1 \cos t + c_2 \sin t.$$

The particular solution is easily found using the Method of Undetermined Coefficients,

$$x_p(t) = 2.$$

Thus, the general solution is

$$x(t) = 2 + c_1 \cos t + c_2 \sin t.$$

We now apply the boundary conditions and see if there are values of c_1 and c_2 that yield a solution to our problem. The first condition, $x(0) = 0$, gives

$$0 = 2 + c_1.$$

Thus, $c_1 = -2$. Using this value for c_1 , the second condition, $x(1) = 1$, gives

$$0 = 2 - 2 \cos 1 + c_2 \sin 1.$$

This yields

$$c_2 = \frac{2(\cos 1 - 1)}{\sin 1}.$$

We have found that there is a solution to the boundary value problem and it is given by

$$x(t) = 2 \left(1 - \cos t \frac{(\cos 1 - 1)}{\sin 1} \sin t \right).$$

Boundary value problems arise in many physical systems, just as the initial value problems we have seen earlier. We will see in the next sections that boundary value problems for ordinary differential equations often appear in the solutions of partial differential equations. However, there is no guarantee that we will have unique solutions of our boundary value problems as we had in our example above.

4.3 Partial Differential Equations

In this section we will introduce some generic partial differential equations and see how the discussion of such equations leads naturally to the study of boundary value problems for ordinary differential equations. However, we will not derive the particular equations at this time.

For ordinary differential equations, the unknown functions are functions of a single variable, e.g., $y = y(x)$. Partial differential equations involve a function and its derivatives, but the unknown function is a function of

Name	2 Vars	3 D
Heat Equation	$u_t = ku_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2 \nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x, y)$	$\nabla^2 u = F(x, y, z)$
Schrödinger's Equation	$i u_t = u_{xx} + F(x, t)u$	$i u_t = \nabla^2 u + F(x, y, z, t)u$

Table 4.1: List of generic partial differential equations.

several variables, $u = u(x, y)$, $u = u(x, y, z, t)$, etc. Therefore, the derivatives are partial derivatives. We will use the standard notations $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc.

There are a few generic equations that one encounters. These can be studied in one to three dimensions and are all linear differential equations. A list is provided in Table 4.1. Here we have introduced the Laplacian operator, $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. Depending on the types of boundary conditions imposed and on the geometry of the system (rectangular, cylindrical, spherical, etc.), one encounters many interesting boundary value problems for ordinary differential equations. These in turn will lead us to the study of eigenvalue problems in function spaces.

Let's look at the heat equation in one dimension. This could describe the heat conduction in a thin insulated rod of length L . It could also describe the diffusion of pollutant in a long narrow stream, or the flow of traffic down a road. In problems involving diffusion processes, one instead calls this equation the diffusion equation.

We now formulate a typical initial-boundary value problem for the heat equation. We begin with a thin rod of length L . We assume that it is laterally insulated so that no heat flows from the middle of the rod. The rod initially has a temperature distribution $u(x, 0) = f(x)$. Placing the rod in an ice bath and assuming the heat flow is only through the ends, one has the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$. Of course, we are dealing with Celsius temperatures and we assume there is plenty of ice to keep that temperature fixed at each end. So, the problem one would need

to solve is given as

$$\begin{array}{ll}
 \text{PDE} & u_t = ku_{xx} \quad 0 < t, \quad 0 \leq x \leq L \\
 \text{IC} & u(x, 0) = f(x) \quad 0 < x < L \\
 \text{BC} & u(0, t) = 0 \quad t > 0 \\
 & u(L, t) = 0 \quad t > 0
 \end{array} \tag{4.3}$$

Here, k is the heat conduction constant and is determined using properties of the bar.

Another problem that will come up in later discussions is that of the vibrating string. A string of length L is stretched out horizontally with both ends fixed. Think of a violin string or a guitar string. Then the string is plucked, giving the string an initial profile. Let $u(x, t)$ be the vertical displacement of the string at position x and time t . The motion of the string is governed by the one dimensional wave equation. The initial-boundary value problem for this problem is given as

$$\begin{array}{ll}
 \text{PDE} & u_{tt} = c^2 u_{xx} \quad 0 < t, \quad 0 \leq x \leq L \\
 \text{IC} & u(x, 0) = f(x) \quad 0 < x < L \\
 \text{BC} & u(0, t) = 0 \quad t > 0 \\
 & u(L, t) = 0 \quad t > 0
 \end{array} \tag{4.4}$$

In this problem c is the wave speed in the string. It depends on the mass per unit length of the string and the tension placed on the string.

4.4 The 1D Heat Equation

We would like to see how the solution of such problems involving partial differential equations will lead naturally to studying boundary value problems for ordinary differential equations. We will see this as we solve the initial-boundary value problem given in Equation (4.3). We will employ a method typically used in studying linear partial differential equations, called *the method of separation of variables*.

We assume that u can be written as a product of single variable functions of each independent variable,

$$u(x, t) = X(x)T(t).$$

Substituting this guess into the heat equation, we find that

$$XT' = kX''T.$$

Dividing both sides by k and $u = XT$, we then get

$$\frac{1}{k} \frac{T'}{T} = \frac{X''}{X}.$$

We have separated the functions of time on one side and space on the other side. The only way that a function of t equals a function of x is if the functions are constant functions. Therefore, we set each function equal to a constant, λ :

$$\underbrace{\frac{1}{k} \frac{T'}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda}_{\text{constant}}.$$

This leads to two equations:

$$T' = k\lambda T, \tag{4.5}$$

$$X'' = \lambda X. \tag{4.6}$$

These are ordinary differential equations. The general solutions to these equations are readily found

$$T(t) = Ae^{k\lambda t}, \tag{4.7}$$

$$X(x) = c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x}. \tag{4.8}$$

We need to be a little careful at this point. The aim is to force our product solutions to satisfy the boundary conditions and initial conditions. Also, we should note that λ is arbitrary and may be positive, zero, or negative. We first look at how the boundary conditions on u lead to conditions on X .

The first condition is $u(0, t) = 0$. This implies that

$$X(0)T(t) = 0$$

for all t . The only way that this is true is if $X(0) = 0$. Similarly, $u(L, t) = 0$ implies that $X(L) = 0$. So, we have to solve the boundary value problem

$$X'' - \lambda X = 0, \quad X(0) = 0 = X(L). \tag{4.9}$$

We are seeking nonzero solutions, as $X(x) = 0$ for all x is an obvious and uninteresting solution. We call such solutions *trivial solutions*.

There are three cases to consider in solving the boundary value problem (4.9), depending on the sign of λ .

Case I. $\lambda > 0$

In this case we have the exponential solutions

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}. \quad (4.10)$$

For $X(0) = 0$, we have

$$0 = c_1 + c_2.$$

We will take $c_2 = -c_1$. Then,

$$X(x) = c_1(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}) = 2c_1 \sinh \sqrt{\lambda}x.$$

Applying the second condition, $X(L) = 0$ yields

$$2c_1 \sinh \sqrt{\lambda}L = 0.$$

This will be true only if $c_1 = 0$ or $\sinh \sqrt{\lambda}L = 0$. The latter equation is true only if $\sqrt{\lambda} = 0$. However, in this case $\lambda > 0$. Therefore, $c_1 = 0$. Thus, the only solution in this case is $X(x) = 0$. This leads to a trivial solution, $u(x, t) = 0$.

Case II. $\lambda = 0$

For this case it is easier to set λ to zero in the differential equation from the start. So, $X'' = 0$. Integrating twice, one finds

$$X(x) = c_1 x + c_2.$$

Setting $x = 0$, we have $c_2 = 0$, leaving $X(x) = c_1 x$. Setting $x = L$, we find $c_1 L = 0$. So, $c_1 = 0$ and we are once again left with a trivial solution.

Case III. $\lambda < 0$

In this case it would be simpler to write $\lambda = -\mu^2 < 0$. Then the differential equation is

$$X'' + \mu^2 X = 0.$$

The general solution of this equation is

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

At $x = 0$ we get $0 = c_1$. This leaves $X(x) = c_2 \sin \mu x$. At $x = L$, we find

$$0 = c_2 \sin \mu L.$$

So, either $c_2 = 0$ or $\sin \mu L = 0$. $c_2 = 0$ leads to a trivial solution again. But, there are cases when the sine is zero. Namely,

$$\mu L = n\pi, \quad n = 1, 2, \dots$$

Note that $n = 0$ is not included since this leads to a trivial solution. Also, negative values of n are redundant, since the sine function is an odd function.

In summary, we have found solutions to our boundary value problem (4.9) for particular values of λ . The solutions are

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

for

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots$$

Product solutions of the heat equation (4.3) satisfying the boundary conditions are therefore

$$u_n(x, t) = b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots,$$

where b_n is an arbitrary constant. However, these do not necessarily satisfy the initial condition $u(x, 0) = f(x)$. What we do get is

$$u_n(x, 0) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

So, if our initial condition is in one of these forms, we can pick out the right n and we are done.

For other initial conditions, we have to do more work. Note, since the heat equation is linear, we can write a linear combination of our solutions and have a solution. Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L}.$$

We note that this equation satisfies the boundary conditions. So, the only thing to check is the initial condition:

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, if we are given $f(x)$, can we find the constants b_n ? If we can, then we will have the solution to the full initial-boundary value problem. This will be the subject of the next chapter. However, first we will look at the general form of our boundary value problem and then relate what we have done to the theory of infinite dimensional vector spaces.

Before moving on to the wave equation, we should note that (4.9) is an eigenvalue problem. We can recast the differential equation as

$$LX = \lambda X,$$

where

$$L = D^2 = \frac{d^2}{dx^2}$$

is a linear differential operator. The solutions, $X_n(x)$, are called eigenfunctions and the λ_n 's are the eigenvalues. We will elaborate more on this characterization later in the book.

4.5 The 1D Wave Equation

In this section we will apply the method of separation of variables to the one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (4.11)$$

and subject to the conditions

$$\begin{aligned} u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x), \\ u(0, t) &= 0, \\ u(L, t) &= 0. \end{aligned} \quad (4.12)$$

This problem applies to the propagation of waves on a string of length L with both ends fixed so that they do not move. $u(x, t)$ represents the

vertical displacement of the string over time. The derivation of the wave equation assumes that the vertical displacement is small and the string is uniform. The constant c is the wave speed, given by

$$c = \sqrt{\frac{T}{\mu}},$$

where T is the tension in the string and μ is the mass per unit length. We can understand this in terms of string instruments. The tension can be adjusted to produce different tones and the makeup of the string (nylon or steel, thick or thin) also has an effect. In some cases the mass density is changed simply by using thicker strings. Thus, the thicker strings in a piano produce lower frequency notes.

The u_{tt} term gives the acceleration of a piece of the string. The u_{xx} is the concavity of the string. Thus, for a positive concavity the string is curved upward near the point of interest. Thus, neighboring points tend to pull upward towards the equilibrium position. If the concavity is negative, it would cause a negative acceleration.

The solution of this problem is easily found using separation of variables. We let $u(x, t) = X(x)T(t)$. Then we find

$$XT'' = c^2 X''T,$$

which can be rewritten as

$$\frac{1}{c^2} \frac{T''}{T} = \frac{X''}{X}.$$

Again, we have separated the functions of time on one side and space on the other side. Therefore, we set each function equal to a constant. λ :

$$\underbrace{\frac{1}{c^2} \frac{T''}{T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda}_{\text{constant}}.$$

This leads to two equations:

$$T'' = c^2 \lambda T, \tag{4.13}$$

$$X'' = \lambda X. \tag{4.14}$$

As before, we have the boundary conditions on $X(x)$:

$$X(0) = 0, \quad \text{and} \quad X(L) = 0.$$

Again, this gives us that

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

The main difference from the solution of the heat equation is the form of the time function. Namely, from Equation (4.13) we have to solve

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0. \quad (4.15)$$

This equation takes a familiar form. We let

$$\omega_n = \frac{n\pi c}{L},$$

then we have

$$T'' + \omega_n^2 T = 0.$$

The solutions are easily found as

$$T(t) = A_n \cos \omega_n t + B_n \sin \omega_n t. \quad (4.16)$$

So, our product solutions are of the form $\sin \frac{n\pi x}{L} \cos \omega_n t$ and $\sin \frac{n\pi x}{L} \sin \omega_n t$. The general solution, a superposition of all product solutions, is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \quad (4.17)$$

This solution satisfies the wave equation and the boundary conditions. We still need to satisfy the initial conditions. Note that there are two initial conditions, since the wave equation is second order in time.

First, we have $u(x, 0) = f(x)$. Thus,

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \quad (4.18)$$

In order to obtain the condition on the initial velocity, $u_t(x, 0) = g(x)$, we need to differentiate the general solution with respect to t :

$$u_t(x, t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left[-A_n \sin \frac{n\pi ct}{L} + B_n \cos \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}. \quad (4.19)$$

Then, we have

$$g(x) = u_t(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L}. \quad (4.20)$$

In both cases we have that the given functions, $f(x)$ and $g(x)$, are represented as Fourier sine series. In order to complete the problem we need to determine the constants A_n and B_n for $n = 1, 2, 3, \dots$. Once we have these, we have the complete solution to the wave equation.

We had seen similar results for the heat equation. In the next section we will find out how to determine the Fourier coefficients for such series of sinusoidal functions.

4.6 Introduction to Fourier Series

In this chapter we will look at trigonometric series. In your calculus courses you have probably seen that many functions could have series representations as expansions in powers of x , or $x - a$. This led to MacLaurin or Taylor series. We have seen from our previous product solutions of the heat and wave equations that series representations might also come in the form of expansions in terms of sine and cosine functions.

When dealing with Taylor series, you often had to determine the expansion coefficients. For example, given an expansion of $f(x)$ about $x = a$, you learned that the Taylor series was given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n,$$

where the expansion coefficients are determined as

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Then you found that the Taylor series converged for a certain range of x values. (We will review Taylor series in the next chapter when we study series representations of complex valued functions.)

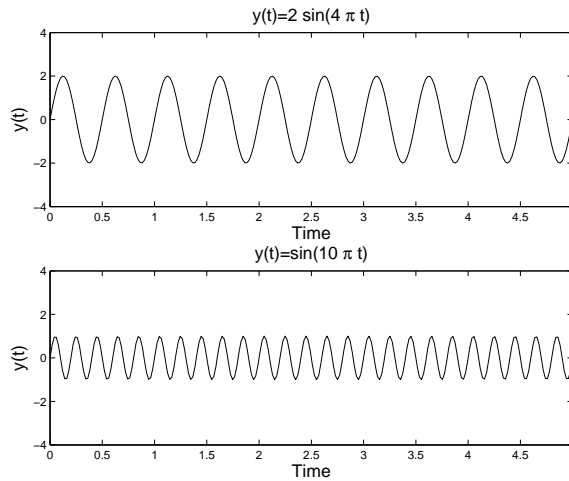


Figure 4.1: Plots of $y(t) = A \sin(2\pi ft)$ on $[0, 5]$ for $f = 2$ Hz and $f = 5$ Hz.

In a similar way, we will investigate the Fourier trigonometric series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}.$$

We will find expressions useful for determining the Fourier coefficients $\{a_n, b_n\}$ given a function $f(x)$ defined on $[-L, L]$. We will also see if the resulting infinite series reproduces $f(x)$. However, we first begin with some basic ideas involving simple sums of sinusoidal functions.

The natural appearance of such sums over sinusoidal functions is in music, or sound. A pure note can be represented as

$$y(t) = A \sin(2\pi ft),$$

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume of the sound. The larger the amplitude, the louder the sound. In Figure 4.1 we show plots of two such tones with $f = 2$ Hz in the top plot and $f = 5$ Hz in the bottom one.

In these plots you should notice the difference due to the amplitudes and the frequencies. You can easily reproduce these plots and others in your favorite plotting utility.

As an aside, you should be cautious when plotting functions, or sampling

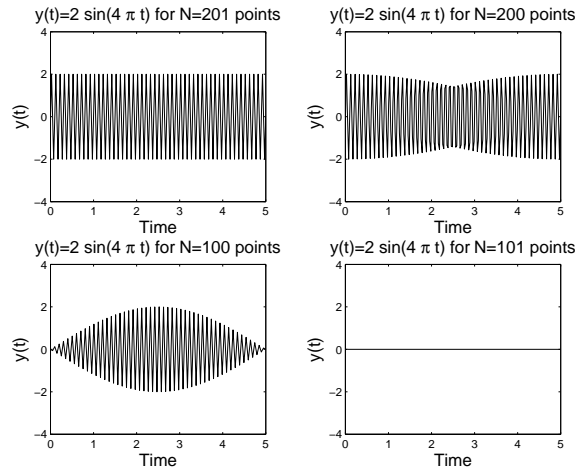


Figure 4.2: Problems can occur while plotting. Here we plot the function $y(t) = 2 \sin 4\pi t$ using $N = 201, 200, 100, 101$ points.

data. The plots you get might not be what you expect, even for a simple sine function. In Figure 4.2 we show four plots of the function $y(t) = 2 \sin(4\pi t)$. In the top left you see a proper rendering of this function. However, if you use a different number of points to plot this function, the results may be surprising. In this example we show what happens if you use $N = 200, 100, 101$ points instead of the 201 points used in the first plot. Such disparities are not only possible when plotting functions, but are also present when collecting data. Typically, when you sample a set of data, you only gather a finite amount of information at a fixed rate. This could happen when getting data on ocean wave heights, digitizing music and other audio to put on your computer, or any other process when you attempt to analyze a continuous signal.

Next, we want to consider what happens when we add several pure tones. After all, most of the sounds that we hear are in fact a combination of pure tones with different amplitudes and frequencies. In Figure 4.3 we see what happens when we add several sinusoids. One notes that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. Looking at the superpositions in Figure 4.3, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest

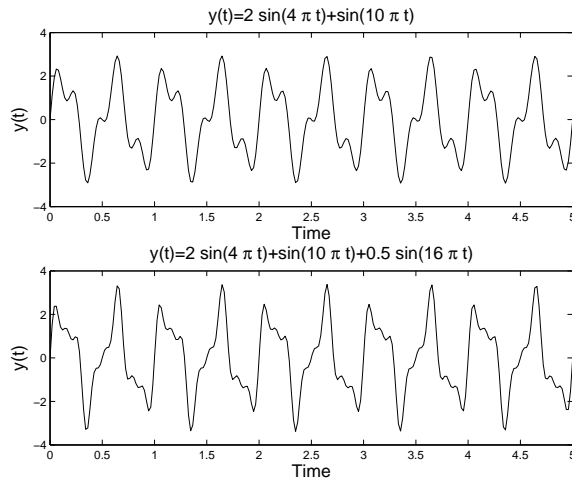


Figure 4.3: Superposition of several sinusoids. Top: Sum of signals with $f = 2$ Hz and $f = 5$ Hz. Bottom: Sum of signals with $f = 2$ Hz, $f = 5$ Hz, and $f = 8$ Hz.

part of the domain which repeats is called the period, T . (We defined this more precisely in an earlier chapter.)

For example, we consider the functions used in Figure 4.3. We began with $y(t) = 2 \sin(4\pi t)$. Recall that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 4.1 we can verify this result. (You can count the full number of cycles in the graph and divide this into the total time to get a more accurate value than just looking at one period.)

In general, if $y(t) = A \sin(2\pi f t)$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1} , or cycles per second.

Returning to the superpositions in Figure 4.3, we have that $\sin(10\pi t)$ has a period of 0.2 Hz and $\sin(16\pi t)$ has a period of 0.125 Hz. The two

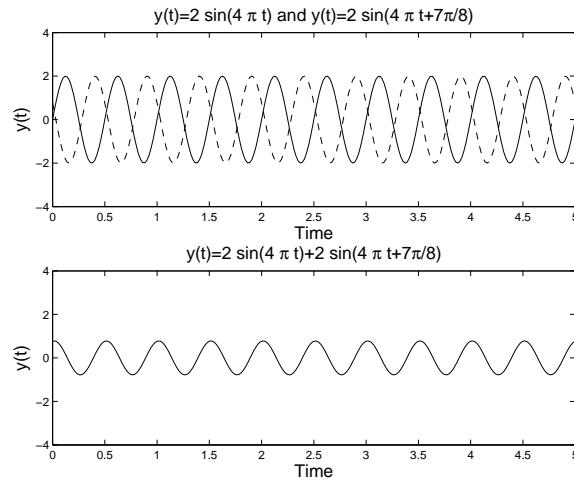


Figure 4.4: Plot of the functions $y(t) = 2 \sin(4\pi t)$ and $y(t) = 2 \sin(4\pi t + 7\pi/8)$ and their sum.

superpositions retain the largest period of the signals added, which is 0.5 Hz from the $\sin(4\pi t)$ terms.

Our goal will be to start with a function and then determine the amplitudes of the simple sinusoids needed to sum to that function. First of all, this might involve an infinite number of such terms. Thus, we will be studying infinite series of functions. Secondly, we will show that using just sine functions will not be enough. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We need only consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like "Row, Row, Row your Boat" starting at slightly different times.

We can easily add shifted sine functions. In Figure 4.4 we show the functions $y(t) = 2 \sin(4\pi t)$ and $y(t) = 2 \sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be written as $y(t) = 2 \sin(4\pi(t + 7/32))$. Thus, this corresponds to a time shift of $-7/8$.

So, we should account for shifted sine functions in our general sum. Of course, we would then need to determine the unknown time shift as well as the amplitudes of the sinusoidal functions that make up our signal, $f(t)$. While this is one approach that some researchers use to analyze signals,

there is a more common approach. This results from a reworking of the shifted function. Consider the general shifted function

$$y(t) = A \sin(2\pi ft + \phi).$$

Note that $2\pi ft + \phi$ is called the *phase* of our sine function and ϕ is called the *phase shift*. We can use the trigonometric identity for the sine of the sum of two angles to obtain

$$y(t) = A \sin(2\pi ft + \phi) = A \sin(\phi) \cos(2\pi ft) + A \cos(\phi) \sin(2\pi ft).$$

Defining $a = A \sin(\phi)$ and $b = A \cos(\phi)$, we can rewrite this as

$$y(t) = a \cos(2\pi ft) + b \sin(2\pi ft).$$

Thus, we see that our signal is a sum of sine and cosine functions with the same frequency and different amplitudes. If we can find a and b , then we can easily determine A and ϕ :

$$A = \sqrt{a^2 + b^2} \quad \tan \phi = \frac{b}{a}.$$

We are now in a position to formally state our goal.

Goal: Given a function $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function.

There are still a few problems in practice. First of all, there are a continuum of frequencies to add over. This typically means our sum is an integral! In practice, our measuring devices do not sample frequently enough to get at high frequency content. So, we have to settle for a discrete set of frequencies. Then our sum would lead to an infinite series, the subject of this chapter. If we only have frequencies $f_n = nf_s$, $n = 1, 2, \dots$ for f_s the sampling rate of our device, then our phases in the trigonometric series are

$$2\pi ft = 2\pi nf_s t.$$

Defining $x = 2\pi f_s t$, we then are lead to sums over

$$y_n(t) = a_n \cos nx + b_n \sin nx.$$

Actually, we will also need a zero frequency term, corresponding to $n = 0$, to account for any vertical shift in our function. In circuit analysis this

corresponds to a background DC signal, or offset. In general, this term represents the average of our signal, or function.

This is enough to get started. However, our discrete sampling will also mean that we do not gather data for a continuous range of times or even for all times. This may be discussed further in a later section of the book. When considering that both t and f are continuous, we are led to Fourier transforms. This is the case for analog signals. When these variables are both discrete, we will be led to discrete transforms. Since this would be the subject of texts on signal and image analysis, we will only consider the case that t is continuous and the frequencies are discrete.

4.6.1 Trigonometric Series

As we have seen in the last section, we are interested in the frequency content of a given function. Thus, we seek representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]. \quad (4.21)$$

Notice that we have opted to drop reference to the frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi f_n t$ when applying these ideas to applications. We have also chosen to use x as an independent variable, instead of t . This is consistent with our solutions of the heat and wave equations earlier in the chapter. Also, note that we have used ‘ \sim ’ to indicate that equality may not hold for all x . At times we may formally use an equal sign, ignoring the convergence properties of the infinite series.

The series representation in Equation (4.21) is called a *Fourier series*. The set of constants $a_0, a_n, b_n, n = 1, 2, \dots$ are called the *Fourier coefficients*. The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0 . Our goal is to find the Fourier series representation given $f(x)$. Later we will be interested in knowing what functions have such a representation and when the Fourier series converges and to what function it converges.

From our discussion in the last section, we see that the infinite series is periodic. The largest period of the trigonometric terms comes from the

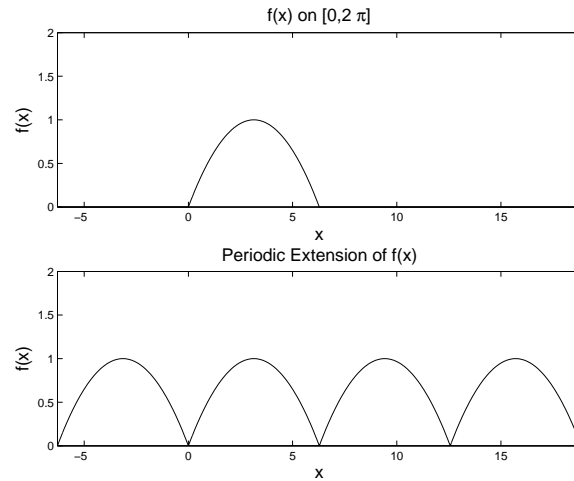


Figure 4.5: Plot of the function $f(t)$ defined on $[0, 2\pi]$ and its periodic extension.

$n = 1$ terms. The periods of $\cos x$ and $\sin x$ are $T = 2\pi$. Thus, the Fourier series has period 2π . This means that the series should be able to represent functions that are periodic of period 2π .

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 4.5 we show a function defined on $[0, 2\pi]$. In the same figure, we should see its periodic extension. It is built from copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to $[0, 2\pi]$ will lead to a representation of the original function. Therefore, we will consider Fourier series representations of functions defined on this interval. Note that we could just as easily consider a function defined on $[-\pi, \pi]$ or any interval of length 2π .

Theorem The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$ when it exists, is given by (4.21) with Fourier coefficients

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx, \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \dots \end{aligned} \quad (4.22)$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will look at the derivations of the a_n 's. First we obtain a_0 .

We begin by integrating the Fourier series in Equation (4.21).

$$\int_0^{2\pi} f(x) dx = \int_0^{2\pi} \frac{a_0}{2} dx + \int_0^{2\pi} \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] dx. \quad (4.23)$$

We will assume that we can integrate the infinite sum term by term. Then we will need to compute

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2} dx &= \frac{a_0}{2} (2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos nx dx &= \left[\frac{\sin nx}{n} \right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin nx dx &= \left[-\frac{\cos nx}{n} \right]_0^{2\pi} = 0. \end{aligned} \quad (4.24)$$

From these results we see that only one term in the integrated sum does not vanish, leaving

$$\int_0^{2\pi} f(x) dx = \pi a_0.$$

This confirms the value for a_0 .

Next, we need to find a_n . We will multiply the Fourier series (4.21) by $\cos mx$ for some positive integer m . This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions at the same time. We will see that this will allow us to solve for the a_n 's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\begin{aligned} \int_0^{2\pi} f(x) \cos mx dx &= \int_0^{2\pi} \frac{a_0}{2} \cos mx dx \\ &+ \int_0^{2\pi} \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \cos mx dx. \end{aligned} \quad (4.25)$$

Integrating term by term, the right side becomes

$$\frac{a_0}{2} \int_0^{2\pi} \cos mx dx + \sum_{n=1}^{\infty} \left[a_n \int_0^{2\pi} \cos nx \cos mx dx + b_n \int_0^{2\pi} \sin nx \cos mx dx \right].$$

(4.26)

We have already established that $\int_0^{2\pi} \cos mx \, dx = 0$.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some trigonometric identities. While you have seen such integrals before in your calculus class, we will review how to carry out such integrals.

We first want to evaluate $\int_0^{2\pi} \cos nx \cos mx \, dx$. We do this by using the product identity for cosines:

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)].$$

Setting $A = mx$ and $B = nx$ in the identity, we can finish the integration.

$$\begin{aligned} \int_0^{2\pi} \cos nx \cos mx \, dx &= \frac{1}{2} \int_0^{2\pi} [\cos(m + n)x + \cos(m - n)x] \, dx \\ &= \frac{1}{2} \left[\frac{\sin(m + n)x}{m + n} + \frac{\sin(m - n)x}{m - n} \right]_0^{2\pi} \\ &= 0. \end{aligned} \tag{4.27}$$

There is a caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For our problem $m + n \neq 0$, since both m and n are positive integers. However, it is possible for $m = n$. This means that the vanishing of our integral can only happen when $m \neq n$. So, what can we do about the $m = n$ case? One way is to start from scratch with our integration. (Another way is to carefully compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

So, for $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx \, dx$. This can also be handled using another trigonometric identity,

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta.)$$

Inserting this into our integral, we find

$$\begin{aligned} \int_0^{2\pi} \cos^2 mx \, dx &= \frac{1}{2} \int_0^{2\pi} (1 + \cos^2 2mx) \, dx \\ &= \frac{1}{2} \left[x + \frac{1}{2m} \sin 2mx \right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi. \end{aligned} \tag{4.28}$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases} \quad (4.29)$$

This holds true for $m, n = 0, 1, \dots$

When we have such a set of functions, they are said to be an orthogonal set over the integration interval.

Definition A set of (real) functions $\{\phi_n(x)\}$ is said to be *orthogonal* on $[a, b]$ if $\int_a^b \phi_n(x)\phi_m(x) \, dx = 0$ when $n \neq m$. Furthermore, if we also have that $\int_a^b \phi_n^2(x) \, dx = 1$, these functions are called *orthonormal*.

From the above computations, we have shown that the set of functions $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π . We can make them orthonormal by dividing each function by $\sqrt{\pi}$.

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)g(x) \, dx$ is the generalization of the dot product, and is called the scalar product (or, inner product) of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. This was discussed in the last section of the previous chapter. The reader may want to go back and read that section at this time.

Returning to the evaluation of the integrals in equation (4.26), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx \, dx$. This can also be evaluated using trigonometric identities. In this case, we need the identity involving products of sines and cosines. It is given by

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

Setting $A = nx$ and $B = mx$, we find that

$$\begin{aligned} \int_0^{2\pi} \sin nx \cos mx \, dx &= \frac{1}{2} \int_0^{2\pi} [\sin(n + m)x + \sin(n - m)x] \, dx \\ &= \frac{1}{2} \left[\frac{-\cos(n + m)x}{n + m} + \frac{-\cos(n - m)x}{n - m} \right]_0^{2\pi} \\ &= (-1 + 1) + (-1 + 1) = 0. \end{aligned} \quad (4.30)$$

For these integrals we also should be careful about setting $n = m$. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx \, dx = \frac{1}{2} \int_0^{2\pi} \sin 2mx \, dx = \frac{1}{2} \left[\frac{-\cos 2mx}{2m} \right]_0^{2\pi} = 0.$$

Finally, we can finish our evaluation of (4.26). We have determined that all but one integral vanishes. In that case, $n = m$. This leaves us with

$$\int_0^{2\pi} f(x) \cos mx \, dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx.$$

Since this is true for all $m = 1, 2, \dots$, we have proven this part of the theorem. The only part left is finding the b'_n 's. However, the procedure is similar and the derivation will be left as an exercise for the reader.

We now consider some examples of finding Fourier coefficients. In all cases we define $f(x)$ on $[0, 2\pi]$. We will explore Fourier series over other intervals in the next section.

Example 1. $f(x) = 3 \cos 2x$.

We first compute the integrals for the Fourier coefficients.

$$a_n = \frac{1}{\pi} \int_0^{2\pi} 3 \cos 2x \cos nx \, dx = 0, \quad n \neq 2.$$

$$a_2 = \frac{1}{\pi} \int_0^{2\pi} 3 \cos^2 2x \, dx = 3.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} 3 \cos 2x \sin nx \, dx = 0, \quad \forall n.$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3 \cos 2x$. Well, we should have known this before doing all of these integrals. So, if we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work.

Example 2. $f(x) = \sin^2 x$.

We could integrate as in the last example, but it is easier to use identities. We know that

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2} \cos 2x.$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \dots$. There is a constant term, so $a_0 = 0$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$.

Example 3. $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$.

This example will take a little more work. We cannot bypass evaluating any integrals this time. This function is discontinuous, so we will have to compute each integral by breaking up the integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx \\ &= \frac{1}{\pi} \int_0^{\pi} dx + \frac{1}{\pi} \int_{\pi}^{2\pi} (-1) dx \\ &= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi + \pi) = 0. \end{aligned} \tag{4.31}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \\ &= \frac{1}{\pi} \left[\int_0^{\pi} \cos nx dx - \int_{\pi}^{2\pi} \cos nx dx \right] \\ &= \frac{1}{\pi} \left[\left(\frac{1}{n} \sin nx \right)_0^{\pi} - \left(\frac{1}{n} \sin nx \right)_{\pi}^{2\pi} \right] \\ &= 0. \end{aligned} \tag{4.32}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \\ &= \frac{1}{\pi} \left[\int_0^{\pi} \sin nx dx - \int_{\pi}^{2\pi} \sin nx dx \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \left[\left(-\frac{1}{n} \cos nx \right)_0^\pi + \left(\frac{1}{n} \cos nx \right)_\pi^{2\pi} \right] \\
&= \frac{1}{\pi} \left[-\frac{1}{n} \cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n} \cos n\pi \right] \\
&= \frac{2}{n\pi} (1 - \cos n\pi). \tag{4.33}
\end{aligned}$$

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (4.21), we note that $\cos n\pi = (-1)^n$. Therefore,

$$1 - \cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases} \tag{4.34}$$

So, half of the b_n 's are zero. While we could write the Fourier series representation as

$$f(x) \sim \frac{4}{\pi} \sum_{n=1, \text{ odd}}^{\infty} \frac{1}{n} \sin nx,$$

we could let $n = 2k - 1$ and write

$$f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k - 1} \sin(2k - 1)x,$$

But does this series converge? Does it converge to $f(x)$? We will answer this question by looking at several examples in the next section.

4.6.2 Fourier Series Over Other Intervals

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$, or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such

intervals arise often in applications. For example, one can study vibrations of a one dimensional string of length L and set up the axes with the left end at $x = 0$ and the right end at $x = L$. Another problem would be to study the temperature distribution along a one dimensional rod of length L . Such problems lead to the original studies of Fourier series. As we will see later, symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling our interval. Then we could apply this transformation to our Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

Define $x \in [0, 2\pi]$ and $t \in [0, L]$. A transformation relating these intervals is simply $x = \frac{2\pi t}{L}$. Given $f(x)$, this transformation would result in a new function $g(t) = f(x(t))$. This function is defined on $[0, L]$. We would like to determine the Fourier series representation of this function. We begin with the representation for $f(x)$.

Recall the form of the Fourier representation for $f(x)$ in Equation (4.21):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]. \quad (4.35)$$

Inserting the transformation relating x and t , we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L} \right]. \quad (4.36)$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to determine the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

We need to make a substitution in the integral of $x = \frac{2\pi t}{L}$. We also will need to transform the differential, $dx = \frac{2\pi}{L} dt$. Thus, the resulting form for our coefficient is

$$a_n = \frac{2}{L} \int_0^L g(t) \cos \frac{2n\pi t}{L} \, dt. \quad (4.37)$$

Similarly, we find that

$$b_n = \frac{2}{L} \int_0^L g(t) \sin \frac{2n\pi t}{L} \, dt. \quad (4.38)$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos \frac{2n\pi t}{L}$ is L/n , which means that the representation for $g(t)$ has a period of L .

At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader. In Table 4.2 we summarize some commonly used Fourier series representations.

We will end our discussion for now with some special cases and an example for a function defined on $[-\pi, \pi]$.

Example Let $f(x) = |x|$ on $[-\pi, \pi]$. We compute the coefficients, beginning as usual with a_0 . We have

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx \\ &= \frac{2}{\pi} \int_0^{\pi} |x| dx = \pi \end{aligned} \quad (4.45)$$

At this point we need to remind the reader about the integration of even and odd functions.

1. **Even Functions:** In this evaluation we made use of the fact that the integrand is an even function. Recall that $f(x)$ is an *even function* if $f(-x) = f(x)$ for all x . One can recognize even functions as they are symmetric with respect to the y -axis. If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx. \quad (4.46)$$

One can prove this by splitting off the integration over negative values of x , using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned} \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= - \int_a^0 f(-y) dy + \int_0^a f(x) dx \\ &= \int_0^a f(y) dy + \int_0^a f(x) dx \\ &= 2 \int_0^a f(x) dx. \end{aligned} \quad (4.47)$$

Table 4.2: Special Fourier Series Representations on Different Intervals

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \quad (4.39)$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L} dx. \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} dx. \quad n = 1, 2, \dots \end{aligned} \quad (4.40)$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \quad (4.41)$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \frac{2n\pi x}{L} dx. \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \frac{2n\pi x}{L} dx. \quad n = 1, 2, \dots \end{aligned} \quad (4.42)$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]. \quad (4.43)$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx. \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx. \quad n = 1, 2, \dots \end{aligned} \quad (4.44)$$

2. **Odd Functions:** A similar computation could be done for odd functions. $f(x)$ is an *odd function* if $f(-x) = -f(x)$ for all x . The graphs of such functions are symmetric with respect to the origin. If one integrates an odd function over a symmetric interval, then one has that

$$\int_{-a}^a f(x) dx = 0. \quad (4.48)$$

We now continue with our computation of the Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx. \quad (4.49)$$

Here we have made use of the fact that $|x| \cos nx$ is an even function. In order to compute the resulting integral, we need to use integration by parts,

$$\int_a^b u dv = uv|_a^b - \int_a^b v du,$$

by letting $u = x$ and $dv = \cos nx dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n} \sin nx$. Continuing with the computation, we have

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx \\ &= \frac{2}{\pi} \left[\frac{1}{n} x \sin nx \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx dx \right] \\ &= -\frac{2}{n\pi} \left[-\frac{1}{n} \cos nx \right]_0^{\pi} \\ &= -\frac{2}{\pi n^2} (1 - (-1)^n). \end{aligned} \quad (4.50)$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n . This lead to a factor $(1 - (-1)^n)$. This factor can be simplified as

$$1 - (-1)^n = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \quad (4.51)$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n 's is simpler. We note that we have to integrate $|x| \sin nx$ from $x = -\pi$ to π . The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n .

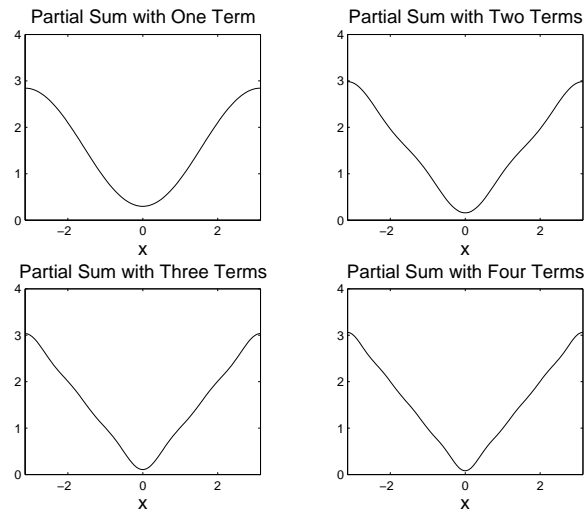


Figure 4.6: Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

Putting this all together, the Fourier series representation of $f(x) = |x|$ on $[-\pi, \pi]$ is given as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1, \text{ odd}}^{\infty} \frac{\cos nx}{n^2}. \quad (4.52)$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let $n = 2k - 1$ for $k = 1, 2, 3, \dots$. Then we only get the odd integers. The series can then be written as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}. \quad (4.53)$$

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. Here is an example of an infinite series of functions. What does this series sum to? We show in Figure 4.6 the first few partial sums. They appear to be converging to $f(x) = |x|$ fairly quickly.

Even though $f(x)$ was defined on $[-\pi, \pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 4.7, we see that the

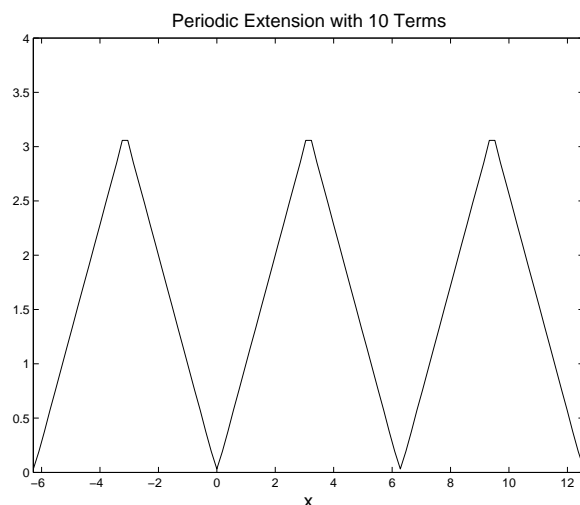


Figure 4.7: Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

representation agrees with $f(x)$ on the interval $[-\pi, \pi]$. Outside this interval we have a periodic extension of $f(x)$ with period 2π .

Another example is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$. This is determined to be

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \quad (4.54)$$

As seen in Figure 4.8 we again obtain the periodic extension of our function. In this case we needed many more terms. Also, the vertical parts of the first plot are nonexistent. In the second plot we only plot the points and not the typical connected points that most software packages plot as the default style.

Fourier Series on $[a, b]$ *Optional*

A Fourier series representation is also possible for a general interval, $t \in [a, b]$. As before, we just need to transform this interval to $[0, 2\pi]$. Let

$$x = 2\pi \frac{t - a}{b - a}.$$

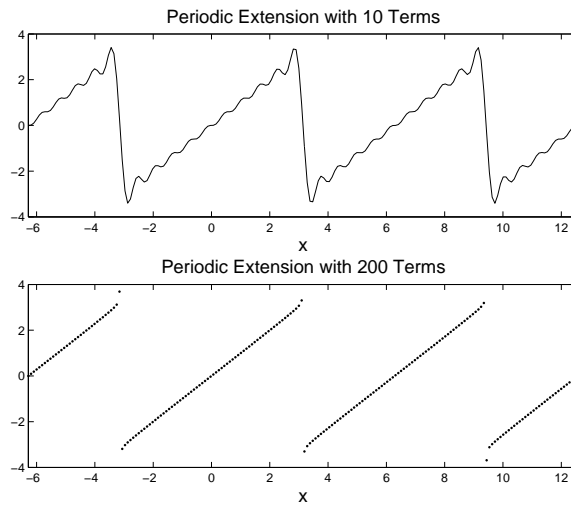


Figure 4.8: Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

Inserting this into the Fourier series (4.21) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right]. \quad (4.55)$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, mathematics students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned} g(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a} \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\cos \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} + \sin \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right. \\ &\quad \left. + b_n \left(\sin \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} - \cos \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\cos \frac{2n\pi t}{b-a} \left(a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \right) \right. \\
&\quad \left. + \sin \frac{2n\pi t}{b-a} \left(a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a} \right) \right]. \quad (4.56)
\end{aligned}$$

Defining $A_0 = a_0$ and

$$\begin{aligned}
A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\
B_n &\equiv a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a}, \quad (4.57)
\end{aligned}$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a, b]$.

$$g(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos \frac{2n\pi t}{b-a} + B_n \sin \frac{2n\pi t}{b-a} \right]. \quad (4.58)$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi \frac{t-a}{b-a}$ we have

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\
&= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi(t-a)}{b-a} \, dt, \quad (4.59)
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \\
&= \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi(t-a)}{b-a} \, dt. \quad (4.60)
\end{aligned}$$

Then, inserting these integrals in A_n , combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$\begin{aligned}
A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\
&= \frac{2}{b-a} \int_a^b g(t) \left[\cos \frac{2n\pi(t-a)}{b-a} \cos \frac{2n\pi a}{b-a} - \sin \frac{2n\pi(t-a)}{b-a} \sin \frac{2n\pi a}{b-a} \right] dt \\
&= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi t}{b-a} \, dt. \quad (4.61)
\end{aligned}$$

A similar computation gives

$$B_n = \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi t}{b-a} dt. \quad (4.62)$$

Summarizing, we have shown that:

Theorem The Fourier series representation of $f(x)$ defined on $[a, b]$ when it exists, is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right]. \quad (4.63)$$

with Fourier coefficients

$$\begin{aligned} a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a} dx. \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a} dx. \quad n = 1, 2, \dots \end{aligned} \quad (4.64)$$

4.6.3 Sine and Cosine Series

In the last two examples we have seen Fourier series representations that contain only sine or cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called sine series and those involving just cosines (and the constant term) are called cosine series.

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0, \pi]$. Note from Figures 4.6-4.8 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.

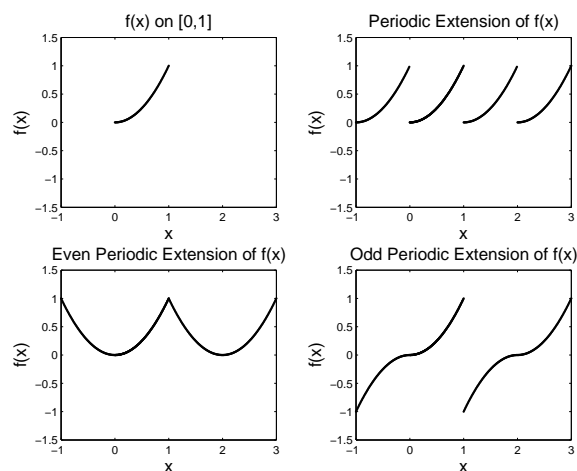


Figure 4.9: This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0, 1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0, L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

In Figure 4.9 we show a function defined on $[0, 1]$. To the right is its periodic extension to the whole real axis. This representation has a period of $L = 1$. The bottom left plot is obtained by first reflecting f about the y -axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the periodic, even and odd extensions. Now, starting with $f(x)$ defined on $[0, L]$, we would like to

determine the Fourier series representations leading to these extensions. [For easy reference, the results are summarized in Table 4.3] We have already seen that the periodic extension of $f(x)$ is obtained through the Fourier series representation in Equation (4.73).

Given $f(x)$ defined on $[0, L]$, the *even periodic extension* is obtained by simply computing the Fourier series representation for the even function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ f(-x) & -L < x < 0. \end{cases} \quad (4.65)$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L, L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (4.63) with $a = -L$ and $b = L$.]:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \quad (4.66)$$

with Fourier coefficients

$$a_n = \frac{1}{L} \int_{-L}^L f_e(x) \cos \frac{n\pi x}{L} dx. \quad n = 0, 1, 2, \dots \quad (4.67)$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the *Cosine Series Representation* of $f(x)$ for $x \in [0, L]$ is given as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \quad (4.68)$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \quad n = 0, 1, 2, \dots \quad (4.69)$$

Similarly, given $f(x)$ defined on $[0, L]$, the *odd periodic extension* is obtained by simply computing the Fourier series representation for the odd function

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \quad (4.70)$$

The resulting series expansion leads to defining the *Sine Series Representation* of $f(x)$ for $x \in [0, L]$ as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \quad (4.71)$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \quad n = 1, 2, \dots \quad (4.72)$$

1. **Example** In Figure 4.9 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.

For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is as follows:

```
> restart:
> L:=1:
> f:=x^2:
> assume(n, integer):
> a0:=2/L*int(f,x=0..L);

a0 := 2/3

> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);

an := -----
      2  2
     n~ Pi

> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);

bn := - -----
      n~ Pi

> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
```

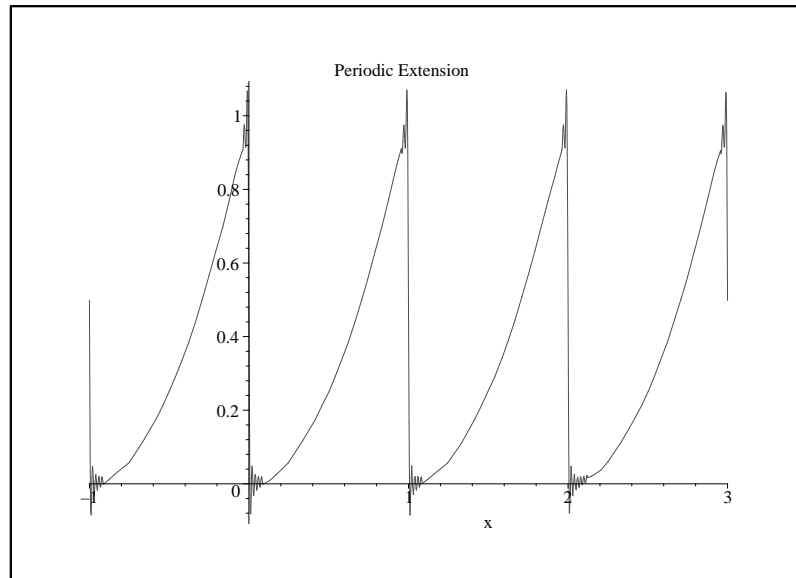


Figure 4.10: The periodic extension of $f(x) = x^2$ on $[0, 1]$.

```

-1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title='Periodic Extension',
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);

```

(a) **Periodic Extension - Trigonometric Fourier Series**

Using the above code, we have that $a_0 = \frac{2}{3}$, $a_n = \frac{1}{n^2\pi^2}$ and $b_n = -\frac{1}{n\pi}$. Thus, the resulting series is given as

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi^2} \cos 2n\pi x - \frac{1}{n\pi} \sin 2n\pi x \right].$$

In Figure 4.10 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f . There appear to be some problems with the convergence around integer values of x . We will later see that this is because of the discontinuities in the periodic extension.

(b) **Even Periodic Extension - Cosine Series**

In this case we compute $a_0 = \frac{2}{3}$ and $a_n = \frac{4(-1)^n}{n^2\pi^2}$. Therefore, we have

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x.$$

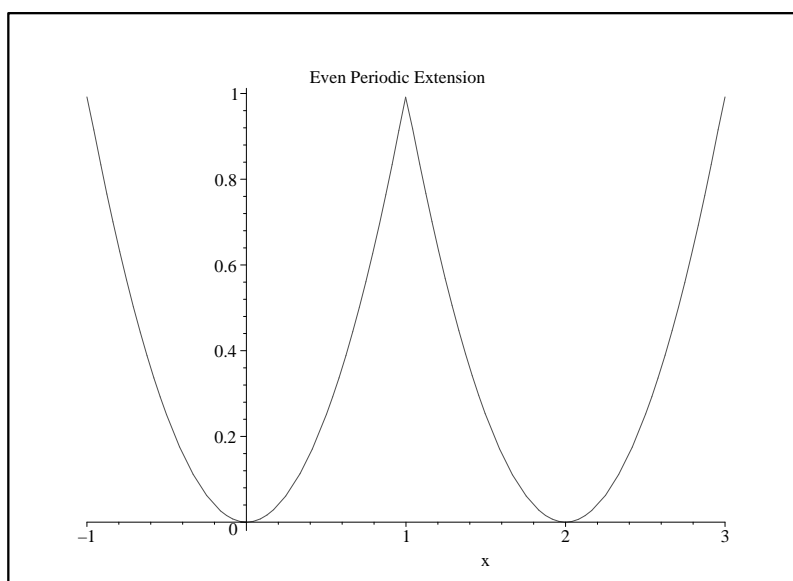


Figure 4.11: The even periodic extension of $f(x) = x^2$ on $[0, 1]$.

In Figure 4.11 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

(c) **Odd Periodic Extension - Sine Series**

Finally, we look at the sine series for this function. We find that $b_n = -\frac{2}{n^3\pi^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2)$. Therefore,

$$f(x) \sim -\frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} (n^2\pi^2(-1)^n - 2(-1)^n + 2) \sin n\pi x.$$

Once again we see discontinuities in the extension as seen in Figure 4.12. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 4.9.

Table 4.3: Fourier Cosine and Sine Series Representations on $[0, L]$ **Fourier Series on $[0, L]$**

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \quad (4.73)$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L} dx. \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L} dx. \quad n = 1, 2, \dots \end{aligned} \quad (4.74)$$

Fourier Cosine Series on $[0, L]$

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \quad (4.75)$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \quad n = 0, 1, 2, \dots \quad (4.76)$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \quad (4.77)$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \quad n = 1, 2, \dots \quad (4.78)$$

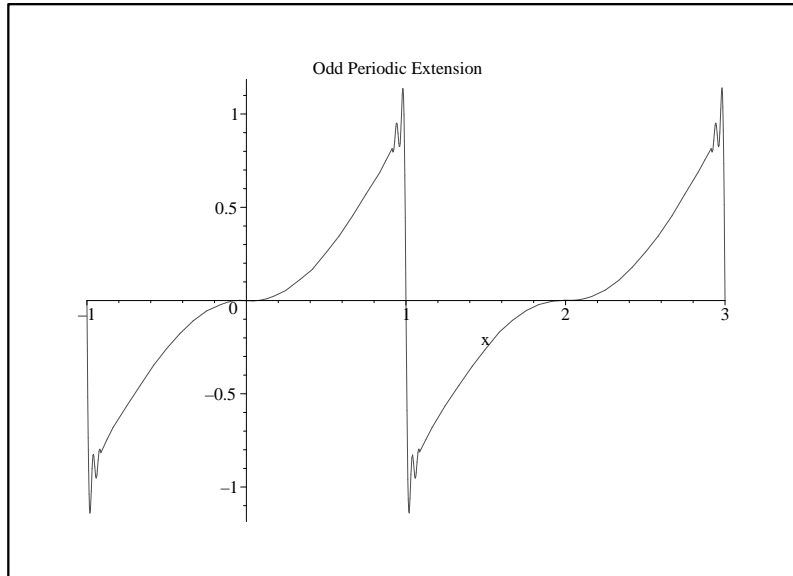


Figure 4.12: The odd periodic extension of $f(x) = x^2$ on $[0, 1]$.

4.7 Solution of the Heat Equation

We started out the chapter seeking the solution of an initial-boundary value problem involving the heat equation and the wave equation. In particular, we found the general solution for the problem of heat flow in a one dimensional rod of length L with fixed zero temperature ends. The problem was given by

$$\begin{array}{ll}
 \text{PDE} & u_t = k u_{xx} \quad 0 < t, \quad 0 \leq x \leq L \\
 \text{IC} & u(x, 0) = f(x) \quad 0 < x < L \\
 \text{BC} & u(0, t) = 0 \quad t > 0 \\
 & u(L, t) = 0 \quad t > 0.
 \end{array} \tag{4.79}$$

We found the solution using separation of variables. This resulted in a sum over various product solutions:

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L},$$

where

$$\lambda_n = - \left(\frac{n\pi}{L} \right)^2.$$

This equation satisfies the boundary conditions. However, we had only gotten to state initial condition using this solution. Namely,

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

We were left with having to determine the constants b_n . Once we know them, we have the solution.

Now we can get the Fourier coefficients when we are given the initial condition, $f(x)$. They are given by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

We consider a couple of examples with different initial conditions.

Example 1 $f(x) = \sin x$ for $L = \pi$.

In this case the solution takes the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin nx,$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

However, the initial condition takes the form of the first term in the expansion; i.e., the $n = 1$ term. So, we need not carry out the integral because we can immediately write $b_1 = 1$ and $b_n = 0$, $n = 2, 3, \dots$. Therefore, the solution consists of just one term,

$$u(x, t) = e^{-kt} \sin x.$$

In Figure 4.13 we see that how this solution behaves for $k = 1$ and $t \in [0, 1]$.

Example 2 $f(x) = x(1 - x)$ for $L = 1$.

This example requires a bit more work. The solution takes the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt} \sin n\pi x,$$

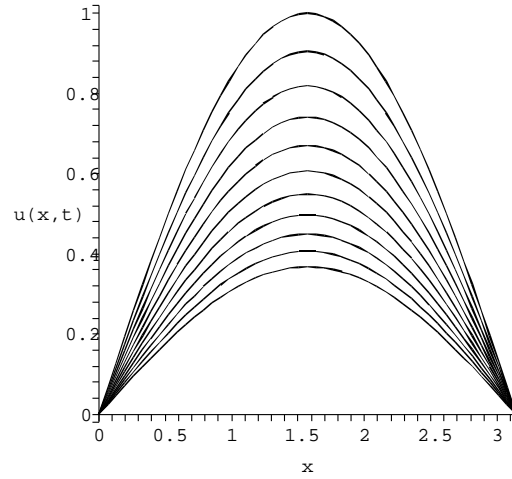


Figure 4.13: The evolution of the initial condition $f(x) = \sin x$ for $L = \pi$ and $k = 1$.

where

$$b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx.$$

This integral is easily computed using integration by parts

$$\begin{aligned}
 b_n &= 2 \int_0^1 x(1-x) \sin n\pi x \, dx \\
 &= \left[2x(1-x) \left(-\frac{1}{n\pi} \cos n\pi x \right) \right]_0^1 + \frac{2}{n\pi} \int_0^1 (1-2x) \cos n\pi x \, dx \\
 &= -\frac{2}{n^2\pi^2} \left\{ [(1-2x) \sin n\pi x]_0^1 + 2 \int_0^1 \sin n\pi x \, dx \right\} \\
 &= \frac{4}{n^3\pi^3} [\cos n\pi x]_0^1 \\
 &= \frac{4}{n^3\pi^3} (\cos n\pi - 1) \\
 &= \begin{cases} 0, & n \text{ even} \\ -\frac{8}{n^3\pi^3}, & n \text{ odd} \end{cases}. \tag{4.80}
 \end{aligned}$$

So, we have that the solution can be written as

$$u(x, t) = \frac{8}{\pi^3} \sum_{\ell=1}^{\infty} \frac{1}{(2\ell-1)^3} e^{-(2\ell-1)^2\pi^2 kt} \sin(2\ell-1)\pi x.$$

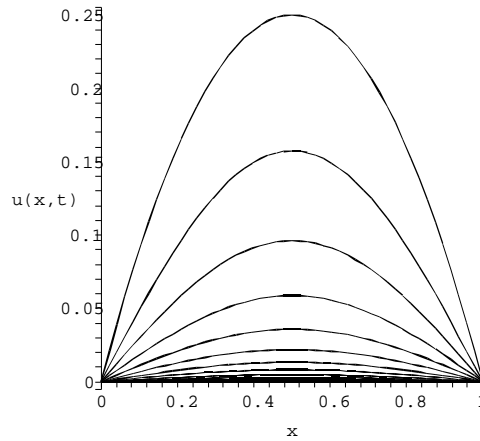


Figure 4.14: The evolution of the initial condition $f(x) = x(1-x)$ for $L = 1$ and $k = 1$.

In Figure 4.13 we see that how this solution behaves for $k = 1$ and $t \in [0, 1]$. Twenty terms were used. We see that this solution diffuses much faster than the last example. Most of the terms damp out quickly as the solution asymptotically approaches the first term.

4.8 Finite Length Strings

We now return to the physical example of wave propagation in a string. We have found that the general solution can be represented as a sum over product solutions. We will restrict our discussion to the special case that the initial velocity is zero and the original profile is given by $u(x, 0) = f(x)$. The solution is then

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \quad (4.81)$$

satisfying

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \quad (4.82)$$

We have learned that the Fourier sine series coefficients are given by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}. \quad (4.83)$$

Note that we are using A_n 's only because of the development of the solution.

We can rewrite this solution in a more compact form. First, we define the wave numbers,

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, \dots,$$

and the angular frequencies,

$$\omega_n = ck_n = \frac{n\pi c}{L}.$$

Then the product solutions take the form

$$\sin k_n x \cos \omega_n t.$$

Using our trigonometric identities, these products can be written as

$$\sin k_n x \cos \omega_n t = \frac{1}{2} [\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t)].$$

Inserting this expression in our solution, we have

$$u(x, t) = \frac{1}{2} \sum_{n=1}^{\infty} A_n [\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t)]. \quad (4.84)$$

Since $\omega_n = ck_n$, we can put this into a more suggestive form:

$$u(x, t) = \frac{1}{2} \left[\sum_{n=1}^{\infty} A_n \sin k_n(x + ct) + \sum_{n=1}^{\infty} A_n \sin k_n(x - ct) \right]. \quad (4.85)$$

We see that each sum is simply the sine series for $f(x)$ but evaluated at either $x + ct$ or $x - ct$. Thus, the solution takes the form

$$u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)]. \quad (4.86)$$

If $t = 0$, then we have $u(x, 0) = \frac{1}{2} [f(x) + f(x)] = f(x)$. So, the solution satisfies the initial condition. At $t = 1$, the sum has a term $f(x - c)$.

Recall from your mathematics classes that this is simply a shifted version of $f(x)$. Namely, it is shifted to the right. For general times, the function is shifted by ct to the right. For larger values of t , this shift is further to the right. The function (wave) shifts to the right with velocity c . Similarly, $f(x + ct)$ is a wave travelling to the left with velocity $-c$.

Thus, the waves on the string consist of waves travelling to the right and to the left. However, the story does not stop here. We have a problem when needing to shift $f(x)$ across the boundaries. The original problem only defines $f(x)$ on $[0, L]$. If we are not careful, we would think that the function leaves the interval leaving nothing left inside. However, we have to recall that our sine series representation for $f(x)$ has a period of $2L$. So, before we apply this shifting, we need to account for its periodicity. In fact, being a sine series, we really have the odd periodic of $f(x)$ being shifted. The details of such analysis would take us too far from our current goal. However, we can illustrate this with a few figures.

We begin by plucking a string of length L . This can be represented by the function

$$f(x) = \begin{cases} \frac{x}{a} & 0 \leq x \leq a \\ \frac{L-x}{L-a} & a \leq x \leq L \end{cases} \quad (4.87)$$

where the string is pulled up one unit at $x = a$. This is shown in Figure 4.15.

Next, we create an odd function by extending the function to a period of $2L$. This is shown in Figure 4.16.

Finally, we construct the periodic extension of this to the entire line. In Figure 4.17 we show in the lower part of the figure copies of the periodic extension, one moving to the right and the other moving to the left. (Actually, the copies are $\frac{1}{2}f(x \pm ct)$.) The top plot is the sum of these solutions. The physical string lies in the interval $[0,1]$.

The time evolution for this plucked string is shown for several times in Figure 4.18. This results in a wave that appears to reflect from the ends as time increases.

The relation between the angular frequency and the wave number, $\omega = ck$, is called a dispersion relation. In this case ω depends on k linearly. If one knows the dispersion relation, then one can find the wave speed as $c = \frac{\omega}{k}$. In this case, all of the harmonics travel at the same speed. In cases where

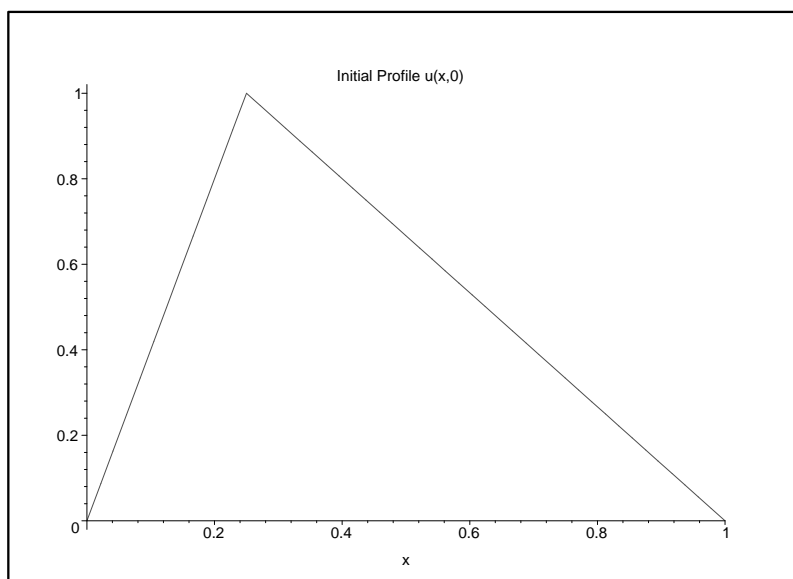


Figure 4.15: The initial profile for a string of length one plucked at $x = 0.25$.

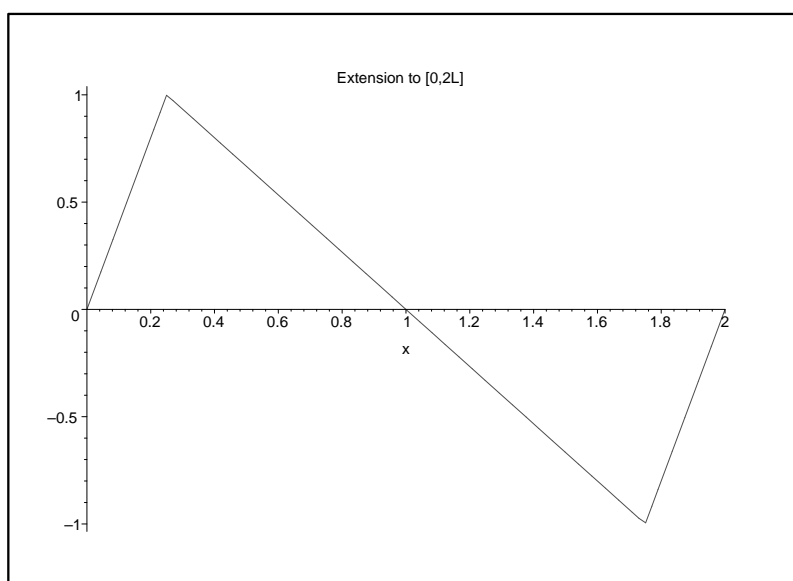


Figure 4.16: The odd extension about the right end of the plucked string.

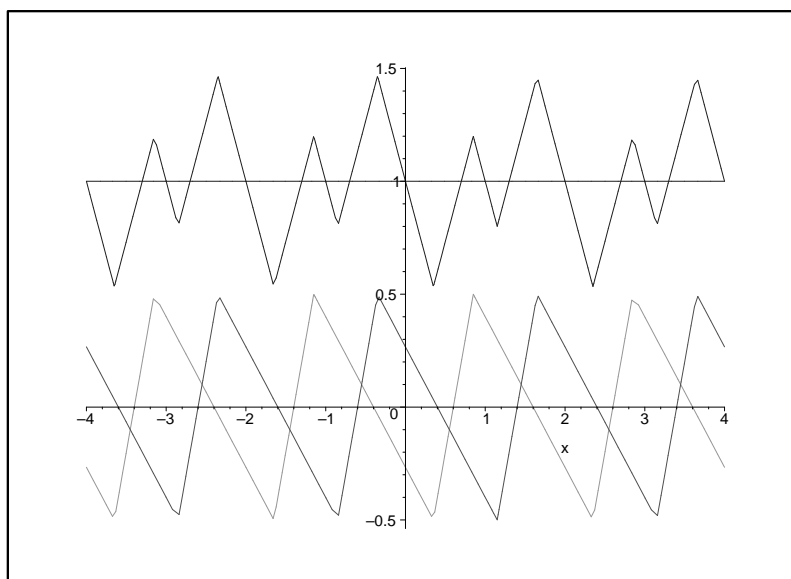
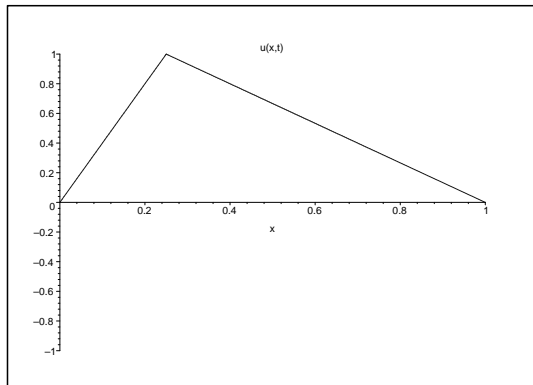
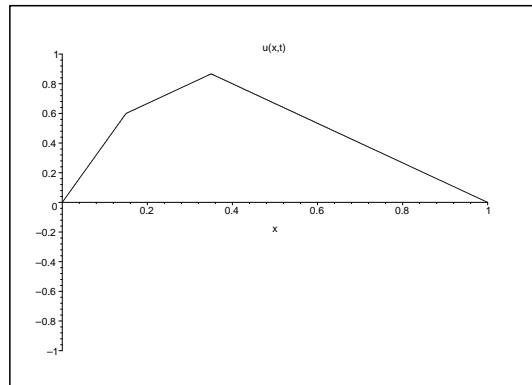


Figure 4.17: Summing the odd periodic extensions. The lower plot shows copies of the periodic extension, one moving to the right and the other moving to the left. The upper plot is the sum.

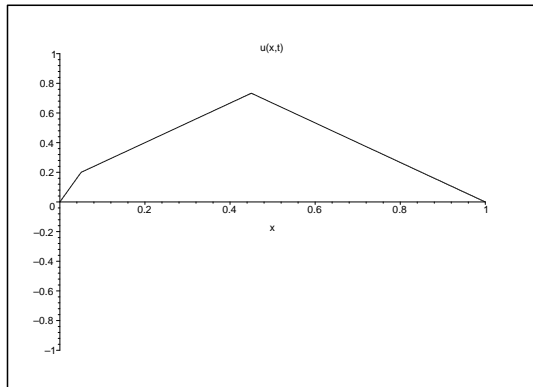
they do not, we have nonlinear dispersion, which we will discuss later.



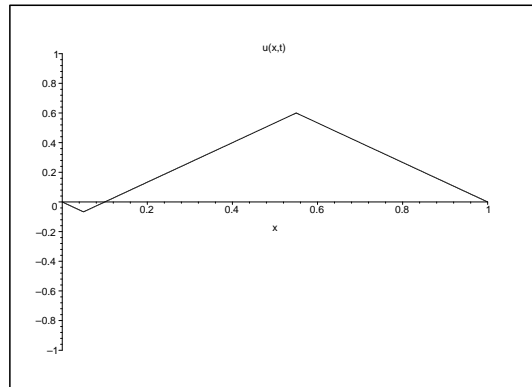
(a)



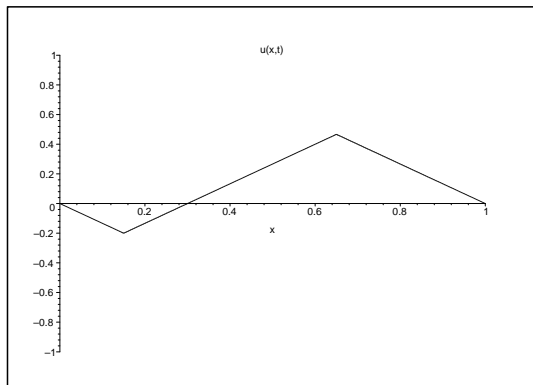
(b)



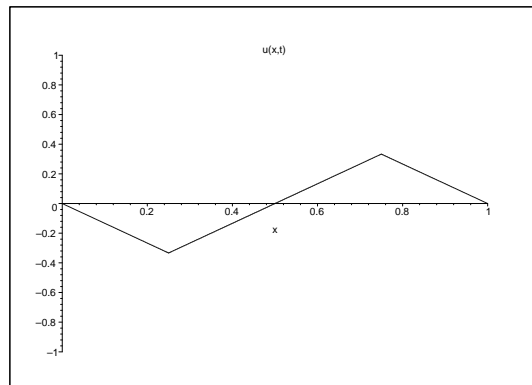
(c)



(d)



(e)



(f)

Figure 4.18: This Figure shows the plucked string at six times in progression from (a) to (f).