

2

Free Fall and Harmonic Oscillators

“Mathematics began to seem too much like puzzle solving. Physics is puzzle solving, too, but of puzzles created by nature, not by the mind of man.” Maria Goeppert-Mayer (1906-1972)

2.1 Free Fall and Terminal Velocity

IN THIS CHAPTER we will study some common differential equations that appear in physics. We will begin with the simplest types of equations and standard techniques for solving them. We will end this part of the discussion by returning to the problem of free fall with air resistance. We will then turn to the study of oscillations, which are modeled by second order differential equations.

Let us begin with a simple example from introductory physics. Recall that free fall is the vertical motion of an object solely under the force of gravity. It has been experimentally determined that an object at near the surface of the Earth falls at a constant acceleration in the absence of other forces, such as air resistance. This constant acceleration is denoted by $-g$, where g is called the acceleration due to gravity. The negative sign is an indication that we have chosen a coordinate system in which up is positive.

We are interested in determining the position, $y(t)$, of the falling body as a function of time. From the definition of free fall, we have

$$\ddot{y}(t) = -g. \quad (2.1)$$

Note that we will occasionally use a dot to indicate time differentiation. This notation is standard in physics and we will begin to introduce you to this notation, though at times we might use the more familiar prime notation to indicate spatial differentiation, or general differentiation.

In Equation (2.1) we know g . It is a constant. Near the Earth's surface it is about 9.81 m/s^2 or 32.2 ft/s^2 . What we do not know is $y(t)$. This is our first differential equation. In fact it is natural to

Free fall example.

Differentiation with respect to time is often denoted by dots instead of primes.

see differential equations appear in physics often through Newton's Second Law, $F = ma$, as it plays an important role in classical physics. We will return to this point later.

So, how does one solve the differential equation in (2.1)? We can do so by using what we know about calculus. It might be easier to see when we put in a particular number instead of g . You might still be getting used to the fact that some letters are used to represent constants. We will come back to the more general form after we see how to solve the differential equation.

Consider

$$\ddot{y}(t) = 5. \quad (2.2)$$

Recalling that the second derivative is just the derivative of a derivative, we can rewrite this equation as

$$\frac{d}{dt} \left(\frac{dy}{dt} \right) = 5. \quad (2.3)$$

This tells us that the derivative of dy/dt is 5. Can you think of a function whose derivative is 5? (Do not forget that the independent variable is t .) Yes, the derivative of $5t$ with respect to t is 5. Is this the only function whose derivative is 5? No! You can also differentiate $5t + 1$, $5t + \pi$, $5t - 6$, etc. In general, the derivative of $5t + C$ is 5.

So, the equation can be reduced to

$$\frac{dy}{dt} = 5t + C. \quad (2.4)$$

Now we ask if you know a function whose derivative is $5t + C$. Well, you might be able to do this one in your head, but we just need to recall the Fundamental Theorem of Calculus, which relates integrals and derivatives. Thus, we have

$$y(t) = \frac{5}{2}t^2 + Ct + D,$$

where D is a second integration constant.

This is a solution to the original equation. That means the solution is a function that when placed into the differential equation makes both sides of the equal sign the same. You can always check your answer by showing that the solution satisfies the equation. In this case we have

$$\ddot{y}(t) = \frac{d^2}{dt^2} \left(\frac{5}{2}t^2 + Ct + D \right) = \frac{d}{dt}(5t + C) = 5.$$

So, it is a solution.

We also see that there are two arbitrary constants, C and D . Picking any values for these gives a whole family of solutions. As we will see, the equation $\ddot{y}(t) = 5$ is a linear second order ordinary differential

equation. The general solution of such an equation always has two arbitrary constants.

Let's return to the free fall problem. We solve it the same way. The only difference is that we can replace the constant 5 with the constant $-g$. So, we find that

$$\frac{dy}{dt} = -gt + C, \quad (2.5)$$

and

$$y(t) = -\frac{1}{2}gt^2 + Ct + D. \quad (2.6)$$

Once you get down the process, it only takes a line or two to solve.

There seems to be a problem. Imagine dropping a ball that then undergoes free fall. We just determined that there are an infinite number of solutions to where the ball is at any time! Well, that is not possible. Experience tells us that if you drop a ball you expect it to behave the same way every time. Or does it? Actually, you could drop the ball from anywhere. You could also toss it up or throw it down. So, there are many ways you can release the ball before it is in free fall. That is where the constants come in. They have physical meanings.

If you set $t = 0$ in the equation, then you have that $y(0) = D$. Thus, D gives the initial position of the ball. Typically, we denote initial values with a subscript. So, we will write $y(0) = y_0$. Thus, $D = y_0$.

That leaves us to determine C . It appears at first in Equation (2.5). Recall that $\frac{dy}{dt}$, the derivative of the position, is the vertical velocity, $v(t)$. It is positive when the ball moves upward. We will denote the initial velocity $v(0) = v_0$. Inserting $t = 0$ in Equation (2.5), we find that $\dot{y}(0) = C$. This implies that $C = v(0) = v_0$.

Putting this all together, we have the physical form of the solution for free fall as

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0. \quad (2.7)$$

Doesn't this equation look familiar? Now we see that the infinite family of solutions consists of free fall resulting from initially dropping a ball at position y_0 with initial velocity v_0 . The conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$ are called the initial conditions. A solution of a differential equation satisfying a set of initial conditions is often called a particular solution.

So, we have solved the free fall equation. Along the way we have begun to see some of the features that will appear in the solutions of other problems that are modeled with differential equation. Throughout the book we will see several applications of differential equations. We will extend our analysis to higher dimensions, in which we case will be faced with so-called partial differential equations, which involve the partial derivatives of functions of more than one variable.

But are we done with free fall? Not at all! We can relax some of the conditions that we have imposed. We can add air resistance. We will

visit this problem later in this chapter after introducing some more techniques.

Finally, we should also note that free fall at constant g only takes place near the surface of the Earth. What if a tile falls off the shuttle far from the surface? It will also fall to the Earth. Actually, it may undergo projectile motion, which you may recall is a combination of horizontal motion and free fall.

To look at this problem we need to go to the origins of the acceleration due to gravity. This comes out of Newton's Law of Gravitation. Consider a mass m at some distance $h(t)$ from the surface of the (spherical) Earth. Letting M and R be the Earth's mass and radius, respectively, Newton's Law of Gravitation states that Newton's Law of Gravitation

$$ma = F$$

$$m \frac{d^2 h(t)}{dt^2} = G \frac{mM}{(R + h(t))^2}. \quad (2.8)$$

Thus, we arrive at a differential equation

$$\frac{d^2 h(t)}{dt^2} = \frac{GM}{(R + h(t))^2}. \quad (2.9)$$

This equation is not as easy to solve. We will leave it as a homework exercise for the reader.

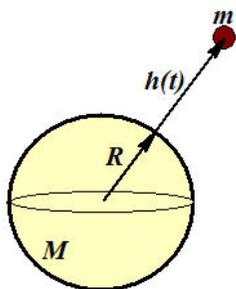


Figure 2.1: Free fall far from the Earth from a height $h(t)$ from the surface.

2.2 First Order Differential Equations

BEFORE MOVING ON, we first define an n -th order ordinary differential equation is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F(y^{(n)}(x), y^{(n-1)}(x), \dots, y'(x), y(x), x) = 0. \quad (2.10)$$

Here $y^{(n)}(x)$ represents the n th derivative of $y(x)$.

An *initial value problem* consists of the differential equation plus the values of the first $n - 1$ derivatives at a particular value of the independent variable, say x_0 :

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \dots, \quad y(x_0) = y_0. \quad (2.11)$$

A *linear n th order differential equation* takes the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \dots + a_1(x)y'(x) + a_0(x)y(x) = f(x). \quad (2.12)$$

If $f(x) \equiv 0$, then the equation is said to be *homogeneous*, otherwise it is *nonhomogeneous*.

Typically, the first differential equations encountered are first order equations. A *first order differential equation* takes the form

$$F(y', y, x) = 0. \quad (2.13)$$

There are two general forms for which one can formally obtain a solution. The first is the separable case and the second is a first order equation. We indicate that we can formally obtain solutions, as one can display the needed integration that leads to a solution. However, the resulting integrals are not always reducible to elementary functions nor does one obtain explicit solutions when the integrals are doable.

2.2.1 Separable Equations

A FIRST ORDER EQUATION IS SEPARABLE if it can be written the form

$$\frac{dy}{dx} = f(x)g(y). \quad (2.14)$$

Special cases result when either $f(x) = 1$ or $g(y) = 1$. In the first case the equation is said to be *autonomous*.

The *general solution* to equation (2.14) is obtained in terms of two integrals:

$$\int \frac{dy}{g(y)} = \int f(x) dx + C, \quad (2.15)$$

where C is an integration constant. This yields a *1-parameter family of solutions* to the differential equation corresponding to different values of C . If one can solve (2.15) for $y(x)$, then one obtains an explicit solution. Otherwise, one has a family of implicit solutions. If an initial condition is given as well, then one might be able to find a member of the family that satisfies this condition, which is often called a *particular solution*.

Separable equations.

Example 2.1. $y' = 2xy$, $y(0) = 2$.

Applying (2.15), one has

$$\int \frac{dy}{y} = \int 2x dx + C.$$

Integrating yields

$$\ln |y| = x^2 + C.$$

Exponentiating, one obtains the general solution,

$$y(x) = \pm e^{x^2+C} = Ae^{x^2}.$$

Here we have defined $A = \pm e^C$. Since C is an arbitrary constant, A is an arbitrary constant. Several solutions in this 1-parameter family are shown in Figure 2.2.

Next, one seeks a particular solution satisfying the initial condition. For $y(0) = 2$, one finds that $A = 2$. So, the particular solution satisfying the initial condition is $y(x) = 2e^{x^2}$.

Example 2.2. $yy' = -x$.

Following the same procedure as in the last example, one obtains:

$$\int y dy = - \int x dx + C \Rightarrow y^2 = -x^2 + A, \quad \text{where } A = 2C.$$

Thus, we obtain an implicit solution. Writing the solution as $x^2 + y^2 = A$, we see that this is a family of circles for $A > 0$ and the origin for $A = 0$. Plots of some solutions in this family are shown in Figure 2.3.

2.2.2 Linear First Order Equations

THE SECOND TYPE OF FIRST ORDER EQUATION encountered is the linear first order differential equation in the standard form

$$y'(x) + p(x)y(x) = q(x). \quad (2.16)$$

In this case one seeks an *integrating factor*, $\mu(x)$, which is a function that one can multiply through the equation making the left side a perfect derivative. Thus, obtaining,

$$\frac{d}{dx} [\mu(x)y(x)] = \mu(x)q(x). \quad (2.17)$$

The integrating factor that works is $\mu(x) = \exp(\int^x p(\xi) d\xi)$. One can derive $\mu(x)$ by expanding the derivative in Equation (2.17),

$$\mu(x)y'(x) + \mu'(x)y(x) = \mu(x)q(x), \quad (2.18)$$

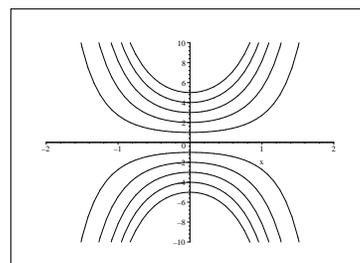


Figure 2.2: Plots of solutions from the 1-parameter family of solutions of Example 2.1 for several initial conditions.

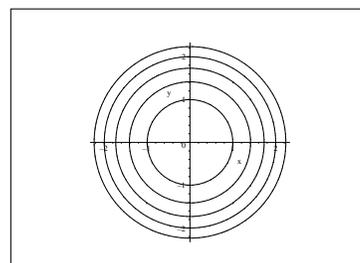


Figure 2.3: Plots of solutions of Example 2.2 for several initial conditions. Linear first order differential equations.

and comparing this equation to the one obtained from multiplying (2.16) by $\mu(x)$:

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x). \quad (2.19)$$

Note that these last two equations would be the same if

$$\frac{d\mu(x)}{dx} = \mu(x)p(x).$$

This is a separable first order equation whose solution is the above given form for the integrating factor,

Integrating factor.

$$\mu(x) = \exp\left(\int^x p(\xi) d\xi\right). \quad (2.20)$$

Equation (2.17) is now easily integrated to obtain the solution

$$y(x) = \frac{1}{\mu(x)} \left[\int^x \mu(\xi)q(\xi) d\xi + C \right]. \quad (2.21)$$

Example 2.3. $xy' + y = x$, $x > 0$, $y(1) = 0$.

One first notes that this is a linear first order differential equation. Solving for y' , one can see that the equation is not separable. Furthermore, it is not in the standard form (2.16). So, we first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 1. \quad (2.22)$$

Noting that $p(x) = \frac{1}{x}$, we determine the integrating factor

$$\mu(x) = \exp\left[\int^x \frac{d\xi}{\xi}\right] = e^{\ln x} = x.$$

Multiplying equation (2.22) by $\mu(x) = x$, we actually get back the original equation! In this case we have found that $xy' + y$ must have been the derivative of something to start. In fact, $(xy)' = xy' + x$. Therefore, the differential equation becomes

$$(xy)' = x.$$

Integrating, one obtains

$$xy = \frac{1}{2}x^2 + C,$$

or

$$y(x) = \frac{1}{2}x + \frac{C}{x}.$$

Inserting the initial condition into this solution, we have $0 = \frac{1}{2} + C$. Therefore, $C = -\frac{1}{2}$. Thus, the solution of the initial value problem is

$$y(x) = \frac{1}{2}\left(x - \frac{1}{x}\right).$$

Example 2.4. $(\sin x)y' + (\cos x)y = x^2$.

Actually, this problem is easy if you realize that

$$\frac{d}{dx}((\sin x)y) = (\sin x)y' + (\cos x)y.$$

But, we will go through the process of finding the integrating factor for practice.

First, rewrite the original differential equation in standard form:

$$y' + (\cot x)y = x^2 \csc x.$$

Then, compute the integrating factor as

$$\mu(x) = \exp\left(\int^x \cot \zeta d\zeta\right) = e^{\ln(\sin x)} = \sin x.$$

Using the integrating factor, the original equation becomes

$$\frac{d}{dx}((\sin x)y) = x^2.$$

Integrating, we have

$$y \sin x = \frac{1}{3}x^3 + C.$$

So, the solution is

$$y = \left(\frac{1}{3}x^3 + C\right) \csc x.$$

There are other first order equations that one can solve for closed form solutions. However, many equations are not solvable, or one is simply interested in the behavior of solutions. In such cases one turns to direction fields. We will return to a discussion of the qualitative behavior of differential equations later.

2.2.3 Terminal Velocity

NOW LET'S RETURN to free fall. What if there is air resistance? We first need to model the air resistance. As an object falls faster and faster, the drag force becomes greater. So, this resistive force is a function of the velocity. There are a couple of standard models that people use to test this. The idea is to write $F = ma$ in the form

$$m\ddot{y} = -mg + f(v), \quad (2.23)$$

where $f(v)$ gives the resistive force and mg is the weight. Recall that this applies to free fall near the Earth's surface. Also, for it to be resistive, $f(v)$ should oppose the motion. If the body is falling, then $f(v)$ should be positive. If it is rising, then $f(v)$ would have to be negative to indicate the opposition to the motion.

One common determination derives from the drag force on an object moving through a fluid. This force is given by

$$f(v) = \frac{1}{2}CA\rho v^2, \quad (2.24)$$

where C is the drag coefficient, A is the cross sectional area and ρ is the fluid density. For laminar flow the drag coefficient is constant.

Unless you are into aerodynamics, you do not need to get into the details of the constants. So, it is best to absorb all of the constants into one to simplify the computation. So, we will write $f(v) = bv^2$. The differential equation including drag can then be rewritten as

$$\dot{v} = kv^2 - g, \quad (2.25)$$

where $k = b/m$. Note that this is a first order equation for $v(t)$. It is separable too!

Formally, we can separate the variables and integrate over time to obtain

$$t + K = \int^v \frac{dz}{kz^2 - g}. \quad (2.26)$$

(Note: We used an integration constant of K since C is the drag coefficient in this problem.) If we can do the integral, then we have a solution for v . In fact, we can do this integral. You need to recall another common method of integration, which we have not reviewed yet. Do you remember Partial Fraction Decomposition? It involves factoring the denominator in the integral. Of course, this is ugly because the constants are represented by letters and are not specific numbers. Letting $\alpha^2 = g/k$, we can write the integrand as

$$\frac{1}{kz^2 - g} = \frac{1}{k} \frac{1}{z^2 - \alpha^2} = \frac{1}{2\alpha k} \left[\frac{1}{z - \alpha} - \frac{1}{z + \alpha} \right]. \quad (2.27)$$

Now, the integrand can be easily integrated giving

$$t + K = \frac{1}{2\alpha k} \ln \left| \frac{v - \alpha}{v + \alpha} \right|. \quad (2.28)$$

Solving for v , we have

$$v(t) = \frac{1 - Be^{2\alpha kt}}{1 + Be^{2\alpha kt}} \alpha, \quad (2.29)$$

where $B \equiv e^K$. B can be determined using the initial velocity.

There are other forms for the solution in terms of a tanh function, which the reader can determine as an exercise. One important conclusion is that for large times, the ratio in the solution approaches -1 . Thus, $v \rightarrow -\alpha = -\sqrt{\frac{g}{k}}$. This means that the falling object will reach a terminal velocity.

This is the first use of Partial Fraction Decomposition. We will explore this method further in the section on Laplace Transforms.

As a simple computation, we can determine the terminal velocity. We will take an 80 kg skydiver with a cross sectional area of about 0.093 m^2 . (The skydiver is falling head first.) Assume that the air density is a constant 1.2 kg/m^3 and the drag coefficient is $C = 2.0$. We first note that

$$v_{\text{terminal}} = -\sqrt{\frac{g}{k}} = -\sqrt{\frac{2mg}{CA\rho}}.$$

So,

$$v_{\text{terminal}} = -\sqrt{\frac{2(70)(9.8)}{(2.0)(0.093)(1.2)}} = 78 \text{ m/s}.$$

This is about 175 mph, which is slightly higher than the actual terminal velocity of a sky diver. One would need a more accurate determination of C for a more realistic answer.

2.3 The Simple Harmonic Oscillator

THE NEXT PHYSICAL PROBLEM of interest is that of simple harmonic motion. Such motion comes up in many places in physics and provides a generic first approximation to models of oscillatory motion. This is the beginning of a major thread running throughout this course. You have seen simple harmonic motion in your introductory physics class. We will review SHM (or SHO in some texts) by looking at springs and pendula (the plural of pendulum). We will use this as our jumping board into second order differential equations and later see how such oscillatory motion occurs in AC circuits.

2.3.1 Mass-Spring Systems

WE BEGIN with the case of a single block on a spring as shown in Figure 2.4. The net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant. Here x is the elongation, or displacement of the spring from equilibrium. When the displacement is positive, the spring force is negative and when the displacement is negative the spring force is positive. We have depicted a horizontal system sitting on a frictionless surface. A similar model can be provided for vertically oriented springs. However, you need to account for gravity to determine the location of equilibrium. Otherwise, the oscillatory motion about equilibrium is modeled the same.

From Newton's Second Law, $F = m\ddot{x}$, we obtain the equation for the motion of the mass on the spring:

$$m\ddot{x} + kx = 0.$$

We will later derive solutions of such equations in a methodical way. For now we note that two solutions of this equation are given by

$$\begin{aligned} x(t) &= A \cos \omega t, \\ x(t) &= A \sin \omega t, \end{aligned} \quad (2.30)$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

is the angular frequency, measured in rad/s. It is related to the frequency by

$$\omega = 2\pi f,$$

where f is measured in cycles per second, or Hertz. Furthermore, this is related to the period of oscillation, the time it takes the mass to go through one cycle:

$$T = 1/f.$$

Finally, A is called the amplitude of the oscillation.

2.3.2 The Simple Pendulum

THE SIMPLE PENDULUM consists of a point mass m hanging on a string of length L from some support. [See Figure 2.5.] One pulls the mass back to some starting angle, θ_0 , and releases it. The goal is to find the angular position as a function of time.

There are a couple of possible derivations. We could either use Newton's Second Law of Motion, $F = ma$, or its rotational analogue in terms of torque, $\tau = I\alpha$. We will use the former only to limit the amount of physics background needed.

There are two forces acting on the point mass. The first is gravity. This points downward and has a magnitude of mg , where g is the standard symbol for the acceleration due to gravity. The other force is the tension in the string. In Figure 2.6 these forces and their sum are shown. The magnitude of the sum is easily found as $F = mg \sin \theta$ using the addition of these two vectors.

Now, Newton's Second Law of Motion tells us that the net force is the mass times the acceleration. So, we can write

$$m\ddot{x} = -mg \sin \theta.$$

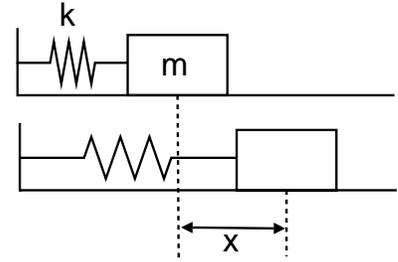


Figure 2.4: Spring-Mass system.

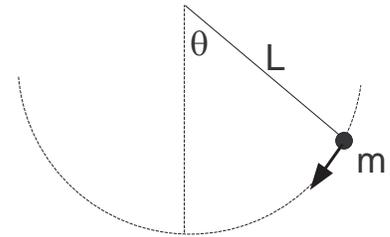


Figure 2.5: A simple pendulum consists of a point mass m attached to a string of length L . It is released from an angle θ_0 .

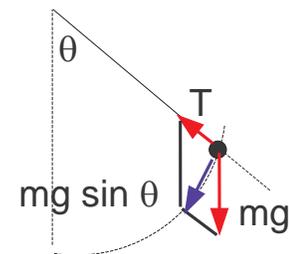


Figure 2.6: There are two forces acting on the mass, the weight mg and the tension T . The net force is found to be $F = mg \sin \theta$.

Next, we need to relate x and θ . x is the distance traveled, which is the length of the arc traced out by the point mass. The arclength is related to the angle, provided the angle is measure in radians. Namely, $x = r\theta$ for $r = L$. Thus, we can write

$$mL\ddot{\theta} = -mg \sin \theta.$$

Canceling the masses, this then gives us the nonlinear pendulum equation

$$L\ddot{\theta} + g \sin \theta = 0. \quad (2.31)$$

There are several variations of Equation (2.31) which will be used in this text. The first one is the linear pendulum. This is obtained by making a small angle approximation. For small angles we know that $\sin \theta \approx \theta$. Under this approximation (2.31) becomes

$$L\ddot{\theta} + g\theta = 0. \quad (2.32)$$

We note that this equation is of the same form as the mass-spring system. We define $\omega = \sqrt{g/L}$ and obtain the equation for simple harmonic motion,

$$\ddot{\theta} + \omega^2\theta = 0.$$

2.4 Second Order Linear Differential Equations

IN THE LAST SECTION we saw how second order differential equations naturally appear in the derivations for simple oscillating systems. In this section we will look at more general second order linear differential equations.

Second order differential equations are typically harder than first order. In most cases students are only exposed to second order linear differential equations. A general form for a *second order linear differential equation* is given by

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \quad (2.33)$$

One can rewrite this equation using operator terminology. Namely, one first defines the differential operator $L = a(x)D^2 + b(x)D + c(x)$, where $D = \frac{d}{dx}$. Then equation (2.33) becomes

$$Ly = f. \quad (2.34)$$

The solutions of linear differential equations are found by making use of the linearity of L . Namely, we consider the *vector space*¹ consisting of real-valued functions over some domain. Let f and g be vectors in this function space. L is a *linear operator* if for two vectors f and g and scalar a , we have that

Nonlinear pendulum equation.

Linear pendulum equation.

The equation for a compound pendulum takes a similar form. We start with the rotational form of Newton's second law $\tau = I\alpha$. Noting that the torque due to gravity acts at the center of mass position ℓ , the torque is given by $\tau = -mg\ell \sin \theta$. Since $\alpha = \ddot{\theta}$, we have $I\ddot{\theta} = -mg\ell \sin \theta$. Then, for small angles $\ddot{\theta} + \omega^2\theta = 0$, where $\omega = \frac{mg\ell}{I}$. for a point mass, $\ell = L$ and $I = mL^2$, leading to the result in the text.

¹ We assume that the reader has been introduced to concepts in linear algebra. Later in the text we will recall the definition of a vector space and see that linear algebra is in the background of the study of many concepts in the solution of differential equations.

- a. $L(f + g) = Lf + Lg$
 b. $L(af) = aLf$.

One typically solves (2.33) by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (2.33) is simply given as $y = y_h + y_p$. This is true because of the linearity of L . Namely,

$$\begin{aligned} Ly &= L(y_h + y_p) \\ &= Ly_h + Ly_p \\ &= 0 + f = f. \end{aligned} \tag{2.35}$$

There are methods for finding a particular solution of a differential equation. These range from pure guessing to the Method of Undetermined Coefficients, or by making use of the Method of Variation of Parameters. We will review some of these methods later.

Determining solutions to the homogeneous problem, $Ly_h = 0$, is not always easy. However, others have studied a variety of second order linear equations and have saved us the trouble for some of the differential equations that often appear in applications.

Again, linearity is useful in producing the general solution of a homogeneous linear differential equation. If y_1 and y_2 are solutions of the homogeneous equation, then the *linear combination* $y = c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation. In fact, if y_1 and y_2 are *linearly independent*,² then $y = c_1y_1 + c_2y_2$ is the general solution of the homogeneous problem. As you may recall, linear independence is established if the Wronskian of the solutions is not zero. In this case, we have

$$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0. \tag{2.36}$$

2.4.1 Constant Coefficient Equations

THE SIMPLEST AND MOST SEEN second order differential equations are those with constant coefficients. The general form for a homogeneous constant coefficient second order linear differential equation is given as

$$ay''(x) + by'(x) + cy(x) = 0, \tag{2.37}$$

where a , b , and c are constants.

² Recall, a set of functions $\{y_i(x)\}_{i=1}^n$ is a linearly independent set if and only if

$$c_1y_1(x) + \dots + c_ny_n(x) = 0$$

implies $c_i = 0$, for $i = 1, \dots, n$.

Solutions to (2.37) are obtained by making a guess of $y(x) = e^{rx}$. Inserting this guess into (2.37) leads to the *characteristic equation*

$$ar^2 + br + c = 0. \quad (2.38)$$

Namely, we compute the derivatives of $y(x) = e^{rx}$, to get $y'(x) = re^{rx}$, and $y''(x) = r^2e^{rx}$. Inserting into (2.37), we have

$$0 = ay''(x) + by'(x) + cy(x) = (ar^2 + br + c)e^{rx}.$$

Since the exponential is never zero, we find that $ar^2 + br + c = 0$.

The roots of this equation, r_1, r_2 , in turn lead to three types of solution depending upon the nature of the roots. In general, we have two linearly independent solutions, $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$, and the general solution is given by a linear combination of these solutions,

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}.$$

For two real distinct roots, we are done. However, when the roots are real, but equal, or complex conjugate roots, we need to do a little more work to obtain usable solutions.

In the case when there is a repeated real root, one has only one independent solution, $y_1(x) = e^{rx}$. The question is how does one obtain the second solution? Since the solutions are independent, we must have that the ratio $y_2(x)/y_1(x)$ is not a constant. So, we guess the form $y_2(x) = v(x)y_1(x) = v(x)e^{rx}$. For constant coefficient second order equations, we can write the equation as

$$(D - r)^2y = 0,$$

where $D = \frac{d}{dx}$.

We now insert $y_2(x)$ into this equation. First we compute

$$(D - r)v e^{rx} = v' e^{rx}.$$

Then,

$$(D - r)^2v e^{rx} = (D - r)v' e^{rx} = v'' e^{rx}.$$

So, if $y_2(x)$ is to be a solution to the differential equation, $(D - r)^2y_2 = 0$, then $v''(x)e^{rx} = 0$ for all x . So, $v''(x) = 0$, which implies that

$$v(x) = ax + b.$$

So,

$$y_2(x) = (ax + b)e^{rx}.$$

Without loss of generality, we can take $b = 0$ and $a = 1$ to obtain the second linearly independent solution, $y_2(x) = xe^{rx}$.

The characteristic equation for $ay'' + by' + cy = 0$ is $ar^2 + br + c = 0$. Solutions of this quadratic equation lead to solutions of the differential equation.

Two real, distinct roots, r_1 and r_2 , give solutions of the form $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$.

Repeated roots, $r_1 = r_2 = r$, give solutions of the form

$$y(x) = (c_1 + c_2x)e^{rx}.$$

Complex roots, $r = \alpha \pm i\beta$, give solutions of the form

$$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

When one has complex roots in the solution of constant coefficient equations, one needs to look at the solutions

$$y_{1,2}(x) = e^{(\alpha \pm i\beta)x}.$$

We make use of Euler's formula

$$e^{i\beta x} = \cos \beta x + i \sin \beta x. \quad (2.39)$$

Then the linear combination of $y_1(x)$ and $y_2(x)$ becomes

$$\begin{aligned} Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} &= e^{\alpha x} [Ae^{i\beta x} + Be^{-i\beta x}] \\ &= e^{\alpha x} [(A+B) \cos \beta x + i(A-B) \sin \beta x] \\ &\equiv e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x). \end{aligned} \quad (2.40)$$

Thus, we see that we have a linear combination of two real, linearly independent solutions, $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$.

The three cases are summarized below followed by several examples.

**Classification of Roots of the Characteristic Equation
for Second Order Constant Coefficient ODEs**

1. **Real, distinct roots** r_1, r_2 . In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
2. **Real, equal roots** $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the *Method of Reduction of Order*. This gives the second solution as $x e^{rx}$. Therefore, the general solution is found as $y(x) = (c_1 + c_2 x) e^{rx}$. [This is covered in the appendix to this chapter.]
3. **Complex conjugate roots** $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. Making use of Euler's identity, $e^{i\theta} = \cos(\theta) + i \sin(\theta)$, these complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$. [This is covered in the appendix to this chapter.]

Example 2.5. $y'' - y' - 6y = 0$ $y(0) = 2, y'(0) = 0$.

The characteristic equation for this problem is $r^2 - r - 6 = 0$. The roots of this equation are found as $r = -2, 3$. Therefore, the general solution can be

quickly written down:

$$y(x) = c_1 e^{-2x} + c_2 e^{3x}.$$

Note that there are two arbitrary constants in the general solution. Therefore, one needs two pieces of information to find a particular solution. Of course, we have the needed information in the form of the initial conditions.

One also needs to evaluate the first derivative

$$y'(x) = -2c_1 e^{-2x} + 3c_2 e^{3x}$$

in order to attempt to satisfy the initial conditions. Evaluating y and y' at $x = 0$ yields

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 + 3c_2 \end{aligned} \tag{2.41}$$

These two equations in two unknowns can readily be solved to give $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is obtained as $y(x) = \frac{6}{5}e^{-2x} + \frac{4}{5}e^{3x}$.

Example 2.6. $y'' + 6y' + 9y = 0$.

In this example we have $r^2 + 6r + 9 = 0$. There is only one root, $r = -3$. Again, the solution is easily obtained as $y(x) = (c_1 + c_2 x)e^{-3x}$.

Example 2.7. $y'' + 4y = 0$.

The characteristic equation in this case is $r^2 + 4 = 0$. The roots are pure imaginary roots, $r = \pm 2i$ and the general solution consists purely of sinusoidal functions: $y(x) = c_1 \cos(2x) + c_2 \sin(2x)$.

Example 2.8. $y'' + 2y' + 4y = 0$.

The characteristic equation in this case is $r^2 + 2r + 4 = 0$. The roots are complex, $r = -1 \pm \sqrt{3}i$ and the general solution can be written as $y(x) = [c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)] e^{-x}$.

Example 2.9. $y'' + 4y = \sin x$.

This is an example of a nonhomogeneous problem. The homogeneous problem was actually solved in Example 2.7. According to the theory, we need only seek a particular solution to the nonhomogeneous problem and add it to the solution of the last example to get the general solution.

The particular solution can be obtained by purely guessing, making an educated guess, or using the Method of Variation of Parameters. We will not review all of these techniques at this time. Due to the simple form of the driving term, we will make an intelligent guess of $y_p(x) = A \sin x$ and determine what A needs to be. Recall, this is the Method of Undetermined Coefficients which we review in later in the chapter. Inserting our guess in the equation gives $(-A + 4A) \sin x = \sin x$. So, we see that $A = 1/3$ works. The general solution of the nonhomogeneous problem is therefore $y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \sin x$.

As we have seen, one of the most important applications of such equations is in the study of oscillations. Typical systems are a mass on a spring, or a simple pendulum. For a mass m on a spring with spring constant $k > 0$, one has from Hooke's law that the position as a function of time, $x(t)$, satisfies the equation

$$m\ddot{x} + kx = 0.$$

This constant coefficient equation has pure imaginary roots ($\alpha = 0$) and the solutions are pure sines and cosines. This is called simple harmonic motion. Adding a damping term and periodic forcing complicates the dynamics, but is nonetheless solvable. We will return to damped oscillations later and also investigate nonlinear oscillations.

2.5 LRC Circuits

ANOTHER TYPICAL PROBLEM often encountered in a first year physics class is that of an LRC series circuit. This circuit is pictured in Figure 2.7. The resistor is a circuit element satisfying Ohm's Law. The capacitor is a device that stores electrical energy and an inductor, or coil, store magnetic energy.

The physics for this problem stems from Kirchoff's Rules for circuits. Namely, the sum of the drops in electric potential are set equal to the rises in electric potential. The potential drops across each circuit element are given by

1. Resistor: $V = IR$.
2. Capacitor: $V = \frac{q}{C}$.
3. Inductor: $V = L\frac{dI}{dt}$.

Furthermore, we need to define the current as $I = \frac{dq}{dt}$, where q is the charge in the circuit. Adding these potential drops, we set them equal to the voltage supplied by the voltage source, $V(t)$. Thus, we obtain

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t).$$

Since both q and I are unknown, we can replace the current by its expression in terms of the charge to obtain

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t).$$

This is a second order equation for $q(t)$.

More complicated circuits are possible by looking at parallel connections, or other combinations, of resistors, capacitors and inductors.

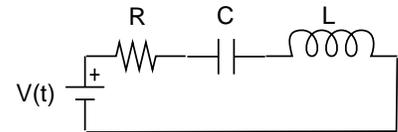


Figure 2.7: Series LRC Circuit.

This will result in several equations for each loop in the circuit, leading to larger systems of differential equations. An example of another circuit setup is shown in Figure 2.8. This is not a problem that can be covered in the first year physics course. One can set up a system of second order equations and proceed to solve them. We will see how to solve such problems later in the text.

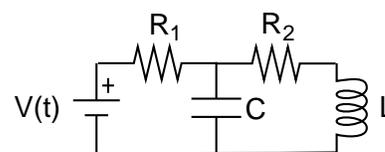


Figure 2.8: Parallel LRC Circuit.

2.5.1 Special Cases

IN THIS SECTION we will look at special cases that arise for the series LRC circuit equation. These include RC circuits, solvable by first order methods and LC circuits, leading to oscillatory behavior.

Case I. RC Circuits

We first consider the case of an RC circuit in which there is no inductor. Also, we will consider what happens when one charges a capacitor with a DC battery ($V(t) = V_0$) and when one discharges a charged capacitor ($V(t) = 0$).

For charging a capacitor, we have the initial value problem

$$R \frac{dq}{dt} + \frac{q}{C} = V_0, \quad q(0) = 0. \quad (2.42)$$

This equation is an example of a linear first order equation for $q(t)$. However, we can also rewrite it and solve it as a separable equation, since V_0 is a constant. We will do the former only as another example of finding the integrating factor.

We first write the equation in standard form:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}. \quad (2.43)$$

The integrating factor is then

$$\mu(t) = e^{\int \frac{dt}{RC}} = e^{t/RC}.$$

Thus,

$$\frac{d}{dt} (qe^{t/RC}) = \frac{V_0}{R} e^{t/RC}. \quad (2.44)$$

Integrating, we have

$$qe^{t/RC} = \frac{V_0}{R} \int e^{t/RC} dt = CV_0 e^{t/RC} + K. \quad (2.45)$$

Note that we introduced the integration constant, K . Now divide out the exponential to get the general solution:

$$q = CV_0 + Ke^{-t/RC}. \quad (2.46)$$

Charging a capacitor.

(If we had forgotten the K , we would not have gotten a correct solution for the differential equation.)

Next, we use the initial condition to get the particular solution. Namely, setting $t = 0$, we have that

$$0 = q(0) = CV_0 + K.$$

So, $K = -CV_0$. Inserting this into the solution, we have

$$q(t) = CV_0(1 - e^{-t/RC}). \quad (2.47)$$

Now we can study the behavior of this solution. For large times the second term goes to zero. Thus, the capacitor charges up, asymptotically, to the final value of $q_0 = CV_0$. This is what we expect, because the current is no longer flowing over R and this just gives the relation between the potential difference across the capacitor plates when a charge of q_0 is established on the plates.

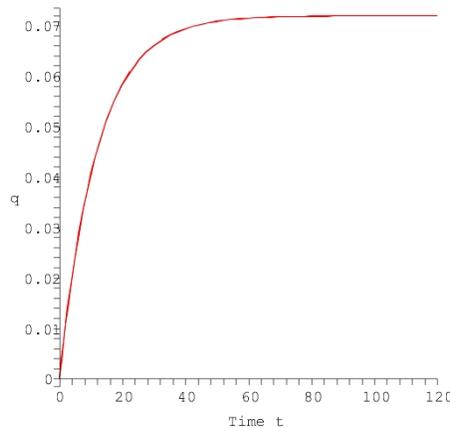


Figure 2.9: The charge as a function of time for a charging capacitor with $R = 2.00 \text{ k}\Omega$, $C = 6.00 \text{ mF}$, and $V_0 = 12 \text{ V}$.

Let's put in some values for the parameters. We let $R = 2.00 \text{ k}\Omega$, $C = 6.00 \text{ mF}$, and $V_0 = 12 \text{ V}$. A plot of the solution is given in Figure 2.9. We see that the charge builds up to the value of $CV_0 = 0.072 \text{ C}$. If we use a smaller resistance, $R = 200 \Omega$, we see in Figure 2.10 that the capacitor charges to the same value, but much faster.

The rate at which a capacitor charges, or discharges, is governed by the time constant, $\tau = RC$. This is the constant factor in the exponential. The larger it is, the slower the exponential term decays. If we set $t = \tau$, we find that

$$q(\tau) = CV_0(1 - e^{-1}) = (1 - 0.3678794412 \dots)q_0 \approx 0.63q_0.$$

Thus, at time $t = \tau$, the capacitor has almost charged to two thirds of its final value. For the first set of parameters, $\tau = 12 \text{ s}$. For the second set, $\tau = 1.2 \text{ s}$.

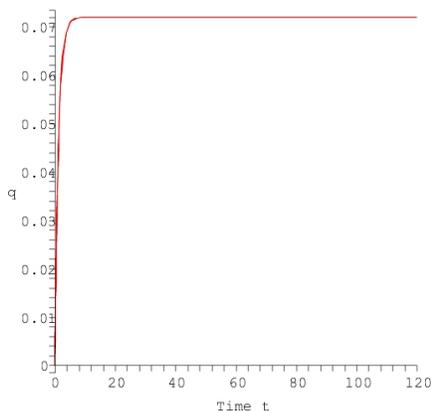


Figure 2.10: The charge as a function of time for a charging capacitor with $R = 200 \Omega$, $C = 6.00 \text{ mF}$, and $V_0 = 12 \text{ V}$.

Now, let's assume the capacitor is charged with charge $\pm q_0$ on its plates. If we disconnect the battery and reconnect the wires to complete the circuit, the charge will then move off the plates, discharging the capacitor. The relevant form of the initial value problem becomes

$$R \frac{dq}{dt} + \frac{q}{C} = 0, \quad q(0) = q_0. \quad (2.48)$$

This equation is simpler to solve. Rearranging, we have

$$\frac{dq}{dt} = -\frac{q}{RC}. \quad (2.49)$$

This is a simple exponential decay problem, which you can solve using separation of variables. However, by now you should know how to immediately write down the solution to such problems of the form $y' = ky$. The solution is

$$q(t) = q_0 e^{-t/\tau}, \quad \tau = RC.$$

We see that the charge decays exponentially. In principle, the capacitor never fully discharges. That is why you are often instructed to place a shunt across a discharged capacitor to fully discharge it.

In Figure 2.11 we show the discharging of the two previous RC circuits. Once again, $\tau = RC$ determines the behavior. At $t = \tau$ we have

$$q(\tau) = q_0 e^{-1} = (0.3678794412 \dots) q_0 \approx 0.37 q_0.$$

So, at this time the capacitor only has about a third of its original value.

Case II. LC Circuits

Another simple result comes from studying LC circuits. We will now connect a charged capacitor to an inductor. In this case, we consider the initial value problem

$$L \ddot{q} + \frac{1}{C} q = 0, \quad q(0) = q_0, \dot{q}(0) = I(0) = 0. \quad (2.50)$$

Discharging a capacitor.

LC Oscillators.

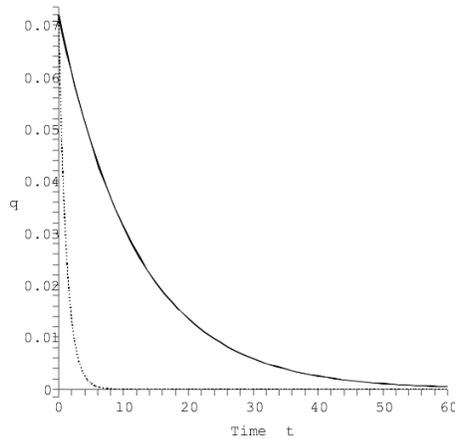


Figure 2.11: The charge as a function of time for a discharging capacitor with $R = 2.00 \text{ k}\Omega$ or $R = 200 \Omega$, and $C = 6.00 \text{ mF}$, and $q_0 = 0.072 \text{ C}$.

Dividing out the inductance, we have

$$\ddot{q} + \frac{1}{LC}q = 0. \quad (2.51)$$

This equation is a second order, constant coefficient equation. It is of the same form as the ones for simple harmonic motion of a mass on a spring or the linear pendulum. So, we expect oscillatory behavior. The characteristic equation is

$$r^2 + \frac{1}{LC} = 0.$$

The solutions are

$$r_{1,2} = \pm \frac{i}{\sqrt{LC}}.$$

Thus, the solution of (2.51) is of the form

$$q(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad \omega = (LC)^{-1/2}. \quad (2.52)$$

Inserting the initial conditions yields

$$q(t) = q_0 \cos(\omega t). \quad (2.53)$$

The oscillations that result are understandable. As the charge leaves the plates, the changing current induces a changing magnetic field in the inductor. The stored electrical energy in the capacitor changes to stored magnetic energy in the inductor. However, the process continues until the plates are charged with opposite polarity and then the process begins in reverse. The charged capacitor then discharges and the capacitor eventually returns to its original state and the whole system repeats this over and over.

The frequency of this simple harmonic motion is easily found. It is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \frac{1}{\sqrt{LC}}. \quad (2.54)$$

This is called the tuning frequency because of its role in tuning circuits.

Of course, this is an ideal situation. There is always resistance in the circuit, even if only a small amount from the wires. So, we really need to account for resistance, or even add a resistor. This leads to a slightly more complicated system in which damping will be present.

2.6 Damped Oscillations

AS WE HAVE INDICATED, simple harmonic motion is an ideal situation. In real systems we often have to contend with some energy loss in the system. This leads to the damping of the oscillations. This energy loss could be in the spring, in the way a pendulum is attached to its support, or in the resistance to the flow of current in an LC circuit. The simplest models of resistance are the addition of a term in first derivative of the dependent variable. Thus, our three main examples with damping added look like:

$$m\ddot{x} + b\dot{x} + kx = 0. \quad (2.55)$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \quad (2.56)$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0. \quad (2.57)$$

These are all examples of the general constant coefficient equation

$$ay''(x) + by'(x) + cy(x) = 0. \quad (2.58)$$

We have seen that solutions are obtained by looking at the characteristic equation $ar^2 + br + c = 0$. This leads to three different behaviors depending on the discriminant in the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (2.59)$$

We will consider the example of the damped spring. Then we have

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}. \quad (2.60)$$

For $b > 0$, there are three types of damping.

Damped oscillator cases.

I. Overdamped, $b^2 > 4mk$

In this case we obtain two real roots. Since this is Case I for constant coefficient equations, we have that

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

We note that $b^2 - 4mk < b^2$. Thus, the roots are both negative. So, both terms in the solution exponentially decay. The damping is so strong that there is no oscillation in the system.

II. Critically Damped, $b^2 = 4mk$

In this case we obtain one real root. This is Case II for constant coefficient equations and the solution is given by

$$x(t) = (c_1 + c_2 t)e^{rt},$$

where $r = -b/2m$. Once again, the solution decays exponentially. The damping is just strong enough to hinder any oscillation. If it were any weaker the discriminant would be negative and we would need the third case.

III. Underdamped, $b^2 < 4mk$

In this case we have complex conjugate roots. We can write $\alpha = -b/2m$ and $\beta = \sqrt{4mk - b^2}/2m$. Then the solution is

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

These solutions exhibit oscillations due to the trigonometric functions, but we see that the amplitude may decay in time due to the overall factor of $e^{\alpha t}$ when $\alpha < 0$. Consider the case that the initial conditions give $c_1 = A$ and $c_2 = 0$. (When is this?) Then, the solution, $x(t) = Ae^{\alpha t} \cos \beta t$, looks like the plot in Figure 2.12.

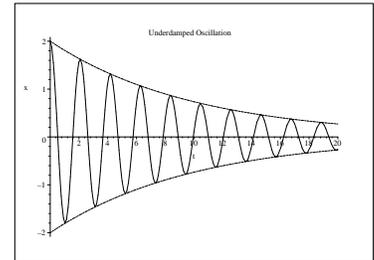


Figure 2.12: A plot of underdamped oscillation given by $x(t) = 2e^{0.1t} \cos 3t$. The dashed lines are given by $x(t) = \pm 2e^{0.1t}$, indicating the bounds on the amplitude of the motion.

2.7 Forced Systems

ALL OF THE SYSTEMS presented at the beginning of the last section exhibit the same general behavior when a damping term is present. An additional term can be added that can cause even more complicated behavior. In the case of LRC circuits, we have seen that the voltage source makes the system nonhomogeneous. It provides what is called a source term. Such terms can also arise in the mass-spring and pendulum systems. One can drive such systems by periodically pushing the mass, or having the entire system moved, or impacted by an outside force. Such systems are called forced, or driven.

Typical systems in physics can be modeled by nonhomogeneous second order equations. Thus, we want to find solutions of equations of the form

$$Ly(x) = a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \quad (2.61)$$

Recall, that one solves this equation by finding the general solution of the homogeneous problem,

$$Ly_h = 0$$

and a particular solution of the nonhomogeneous problem,

$$Ly_p = f.$$

Then the general solution of (2.33) is simply given as $y = y_h + y_p$.

To date, we only know how to solve constant coefficient, homogeneous equations. So, by adding a nonhomogeneous to such equations we need to figure out what to do with the extra term. In other words, how does one find the particular solution?

You could guess a solution, but that is not usually possible without a little bit of experience. So we need some other methods. There are two main methods. In the first case, the Method of Undetermined Coefficients, one makes an intelligent guess based on the form of $f(x)$. In the second method, one can systematically develop the particular solution. We will come back to this method the Method of Variation of Parameters, later in this section.

2.7.1 Method of Undetermined Coefficients

LET'S SOLVE a simple differential equation highlighting how we can handle nonhomogeneous equations.

Example 2.10. Consider the equation

$$y'' + 2y' - 3y = 4. \quad (2.62)$$

The first step is to determine the solution of the homogeneous equation. Thus, we solve

$$y_h'' + 2y_h' - 3y_h = 0. \quad (2.63)$$

The characteristic equation is $r^2 + 2r - 3 = 0$. The roots are $r = 1, -3$. So, we can immediately write the solution

$$y_h(x) = c_1 e^x + c_2 e^{-3x}.$$

The second step is to find a particular solution of (2.62). What possible function can we insert into this equation such that only a 4 remains? If we try something proportional to x , then we are left with a linear function after inserting x and its derivatives. Perhaps a constant function you might think. $y = 4$ does not work. But, we could try an arbitrary constant, $y = A$.

Let's see. Inserting $y = A$ into (2.62), we obtain

$$-3A = 4.$$

Ah ha! We see that we can choose $A = -\frac{4}{3}$ and this works. So, we have a particular solution, $y_p(x) = -\frac{4}{3}$. This step is done.

Combining the two solutions, we have the general solution to the original nonhomogeneous equation (2.62). Namely,

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}.$$

Insert this solution into the equation and verify that it is indeed a solution. If we had been given initial conditions, we could now use them to determine the arbitrary constants.

Example 2.11. What if we had a different source term? Consider the equation

$$y'' + 2y' - 3y = 4x. \quad (2.64)$$

The only thing that would change is the particular solution. So, we need a guess.

We know a constant function does not work by the last example. So, let's try $y_p = Ax$. Inserting this function into Equation (2.64), we obtain

$$2A - 3Ax = 4x.$$

Picking $A = -4/3$ would get rid of the x terms, but will not cancel everything. We still have a constant left. So, we need something more general.

Let's try a linear function, $y_p(x) = Ax + B$. Then we get after substitution into (2.64)

$$2A - 3(Ax + B) = 4x.$$

Equating the coefficients of the different powers of x on both sides, we find a system of equations for the undetermined coefficients:

$$\begin{aligned} 2A - 3B &= 0 \\ -3A &= 4. \end{aligned} \quad (2.65)$$

These are easily solved to obtain

$$\begin{aligned} A &= -\frac{4}{3} \\ B &= \frac{2}{3}A = -\frac{8}{9}. \end{aligned} \quad (2.66)$$

So, the particular solution is

$$y_p(x) = -\frac{4}{3}x - \frac{8}{9}.$$

This gives the general solution to the nonhomogeneous problem as

$$y(x) = y_h(x) + y_p(x) = c_1 e^x + c_2 e^{-3x} - \frac{4}{3}x - \frac{8}{9}.$$

There are general forms that you can guess based upon the form of the driving term, $f(x)$. Some examples are given in Table 2.7.1.

More general applications are covered in a standard text on differential equations. However, the procedure is simple. Given $f(x)$ in a particular form, you make an appropriate guess up to some unknown parameters, or coefficients. Inserting the guess leads to a system of equations for the unknown coefficients. Solve the system and you have the solution. This solution is then added to the general solution of the homogeneous differential equation.

$f(x)$	Guess
$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$
$a e^{bx}$	$A e^{bx}$
$a \cos \omega x + b \sin \omega x$	$A \cos \omega x + B \sin \omega x$

Example 2.12. As a final example, let's consider the equation

$$y'' + 2y' - 3y = 2e^{-3x}. \quad (2.67)$$

According to the above, we would guess a solution of the form $y_p = Ae^{-3x}$. Inserting our guess, we find

$$0 = 2e^{-3x}.$$

Oops! The coefficient, A , disappeared! We cannot solve for it. What went wrong?

The answer lies in the general solution of the homogeneous problem. Note that e^x and e^{-3x} are solutions to the homogeneous problem. So, a multiple of e^{-3x} will not get us anywhere. It turns out that there is one further modification of the method. If the driving term contains terms that are solutions of the homogeneous problem, then we need to make a guess consisting of the smallest possible power of x times the function which is no longer a solution of the homogeneous problem. Namely, we guess $y_p(x) = Axe^{-3x}$. We compute the derivative of our guess, $y'_p = A(1 - 3x)e^{-3x}$ and $y''_p = A(9x - 6)e^{-3x}$. Inserting these into the equation, we obtain

$$[(9x - 6) + 2(1 - 3x) - 3x]Ae^{-3x} = 2e^{-3x},$$

or

$$-4A = 2.$$

So, $A = -1/2$ and $y_p(x) = -\frac{1}{2}xe^{-3x}$.

Modified Method of Undetermined Coefficients

In general, if any term in the guess $y_p(x)$ is a solution of the homogeneous equation, then multiply the guess by x^k , where k is the smallest positive integer such that no term in $x^k y_p(x)$ is a solution of the homogeneous problem.

2.7.2 Forced Oscillations

As an example of a simple forced system, we can consider forced linear oscillations. For example one can force the mass-spring system. In general, such a system would satisfy the equation

$$m\ddot{x} + b\dot{x} + kx = F(t), \quad (2.68)$$

where m is the mass, b is the damping constant, k is the spring constant, and $F(t)$ is the driving force. If $F(t)$ is of simple form, then we can employ the Method of Undetermined Coefficients. As the damping term only complicates the solution we will assume that $b = 0$. Furthermore, we will introduce a sinusoidal driving force, $F(t) = F_0 \cos \omega t$. Then, the simple driven system becomes

$$m\ddot{x} + kx = F_0 \cos \omega t. \quad (2.69)$$

As we have seen, one first determines the solution to the homogeneous problem,

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. In order to apply the Method of Undetermined Coefficients, one has to make a guess which is not a solution of the homogeneous solution. The first guess would be to use $x_p = A \cos \omega t$. This is fine if $\omega \neq \omega_0$. Otherwise, one would need to use the Modified Method of Undetermined Coefficients as described in the last section. The details will be left to the reader.

The general solution to the problem is thus

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \begin{cases} \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t, & \omega \neq \omega_0, \\ \frac{F_0}{2m\omega_0} t \sin \omega_0 t, & \omega = \omega_0. \end{cases} \quad (2.70)$$

Special cases of these solutions provide interesting physics, which can be explored by the reader in the homework. In the case that $\omega = \omega_0$, we see that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency. As the system is moving to the left, one pushes it to the left. If it is moving to the right, one is adding energy in that direction. This forces the amplitude of oscillation to continue to grow until the system breaks.

In the case that $\omega \neq \omega_0$, one can rewrite the solution in a simple form. Let's choose the initial conditions as $x(0) = 0$, $\dot{x}(0) = 0$. Then one has (see Problem 13)

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}. \quad (2.71)$$

The case of resonance.

For values of ω near ω_0 , one finds the solution consists of a rapid oscillation, due to the $\sin \frac{(\omega_0 + \omega)t}{2}$ factor, with a slowly varying amplitude, $\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}$. The reader can investigate this solution. This leads to what are called beats.

2.7.3 Cauchy-Euler Equations

ANOTHER CLASS OF SOLVABLE linear differential equations that is of interest are the Cauchy-Euler type of equations. These are given by

$$ax^2y''(x) + bxy'(x) + cy(x) = 0. \quad (2.72)$$

Note that in such equations the power of x in each of the coefficients matches the order of the derivative in that term. These equations are solved in a manner similar to the constant coefficient equations.

One begins by making the guess $y(x) = x^r$. Inserting this function and its derivatives,

$$y'(x) = rx^{r-1}, \quad y''(x) = r(r-1)x^{r-2},$$

into Equation (2.72), we have

$$[ar(r-1) + br + c]x^r = 0.$$

Since this has to be true for all x in the problem domain, we obtain the characteristic equation

$$ar(r-1) + br + c = 0. \quad (2.73)$$

Just like the constant coefficient differential equation, we have a quadratic equation and the nature of the roots again leads to three classes of solutions. If there are two real, distinct roots, then the general solution takes the form $y(x) = c_1x^{r_1} + c_2x^{r_2}$.

Deriving the solution for Case 2 for the Cauchy-Euler equations works in the same way as the second for constant coefficient equations, but it is a bit messier. First note that for the real root, $r = r_1$, the characteristic equation has to factor as $(r - r_1)^2 = 0$. Expanding, we have

$$r^2 - 2r_1r + r_1^2 = 0.$$

The general characteristic equation is

$$ar(r-1) + br + c = 0.$$

Rewriting this, we have

$$r^2 + \left(\frac{b}{a} - 1\right)r + \frac{c}{a} = 0.$$

The solutions of Cauchy-Euler equations can be found using the characteristic equation $ar(r-1) + br + c = 0$.

For two real, distinct roots, the general solution takes the form

$$y(x) = c_1x^{r_1} + c_2x^{r_2}.$$

For one root, $r_1 = r_2 = r$, the general solution is of the form

$$y(x) = (c_1 + c_2 \ln |x|)x^r.$$

Comparing equations, we find

$$\frac{b}{a} = 1 - 2r_1, \quad \frac{c}{a} = r_1^2.$$

So, the general Cauchy-Euler equation in this case takes the form

$$x^2 y'' + (1 - 2r_1)xy' + r_1^2 y = 0.$$

Now we seek the second linearly independent solution in the form $y_2(x) = v(x)x^{r_1}$. We first list this function and its derivatives,

$$\begin{aligned} y_2(x) &= vx^{r_1}, \\ y_2'(x) &= (xv' + r_1v)x^{r_1-1}, \\ y_2''(x) &= (x^2v'' + 2r_1xv' + r_1(r_1 - 1)v)x^{r_1-2}. \end{aligned} \tag{2.74}$$

Inserting these forms into the differential equation, we have

$$\begin{aligned} 0 &= x^2 y_2'' + (1 - 2r_1)xy_2' + r_1^2 y_2 \\ &= (xv'' + v')x^{r_1+1}. \end{aligned} \tag{2.75}$$

Thus, we need to solve the equation

$$xv'' + v' = 0,$$

or

$$\frac{v''}{v'} = -\frac{1}{x}.$$

Integrating, we have

$$\ln |v'| = -\ln |x| + C.$$

Exponentiating, we have one last differential equation to solve,

$$v' = \frac{A}{x}.$$

Thus,

$$v(x) = A \ln |x| + k.$$

So, we have found that the second linearly independent equation can be written as

$$y_2(x) = x^{r_1} \ln |x|.$$

We now turn to the case of complex conjugate roots, $r = \alpha \pm i\beta$. When dealing with the Cauchy-Euler equations, we have solutions of the form $y(x) = x^{\alpha+i\beta}$. The key to obtaining real solutions is to first recall that

$$x^y = e^{\ln x^y} = e^{y \ln x}.$$

For complex conjugate roots, $r = \alpha \pm i\beta$, the general solution takes the form

$$y(x) = x^\alpha (c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|)).$$

Thus, a power can be written as an exponential and the solution can be written as

$$y(x) = x^{\alpha+i\beta} = x^{\alpha}e^{i\beta \ln x}, \quad x > 0.$$

We can now find two real, linearly independent solutions, $x^{\alpha} \cos(\beta \ln |x|)$ and $x^{\alpha} \sin(\beta \ln |x|)$ following the same steps as earlier for the constant coefficient case.

The results are summarized in the table below followed by examples.

Classification of Roots of the Characteristic Equation for Cauchy-Euler Differential Equations
1. Real, distinct roots r_1, r_2 . In this case the solutions corresponding to each root are linearly independent. Therefore, the general solution is simply $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.
2. Real, equal roots $r_1 = r_2 = r$. In this case the solutions corresponding to each root are linearly dependent. To find a second linearly independent solution, one uses the Method of Reduction of Order. This gives the second solution as $x^r \ln x $. Therefore, the general solution is found as $y(x) = (c_1 + c_2 \ln x) x^r$.
3. Complex conjugate roots $r_1, r_2 = \alpha \pm i\beta$. In this case the solutions corresponding to each root are linearly independent. These complex exponentials can be rewritten in terms of trigonometric functions. Namely, one has that $x^{\alpha} \cos(\beta \ln x)$ and $x^{\alpha} \sin(\beta \ln x)$ are two linearly independent solutions. Therefore, the general solution becomes $y(x) = x^{\alpha} (c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$.

Example 2.13. $x^2 y'' + 5xy' + 12y = 0$

As with the constant coefficient equations, we begin by writing down the characteristic equation. Doing a simple computation,

$$\begin{aligned} 0 &= r(r-1) + 5r + 12 \\ &= r^2 + 4r + 12 \\ &= (r+2)^2 + 8, \\ -8 &= (r+2)^2, \end{aligned} \tag{2.76}$$

one determines the roots are $r = -2 \pm 2\sqrt{2}i$. Therefore, the general solution is $y(x) = [c_1 \cos(2\sqrt{2} \ln |x|) + c_2 \sin(2\sqrt{2} \ln |x|)] x^{-2}$

Example 2.14. $t^2 y'' + 3ty' + y = 0, \quad y(1) = 0, y'(1) = 1.$

For this example the characteristic equation takes the form

$$r(r-1) + 3r + 1 = 0,$$

or

$$r^2 + 2r + 1 = 0.$$

There is only one real root, $r = -1$. Therefore, the general solution is

$$y(t) = (c_1 + c_2 \ln |t|)t^{-1}.$$

However, this problem is an initial value problem. At $t = 1$ we know the values of y and y' . Using the general solution, we first have that

$$0 = y(1) = c_1.$$

Thus, we have so far that $y(t) = c_2 \ln |t|t^{-1}$. Now, using the second condition and

$$y'(t) = c_2(1 - \ln |t|)t^{-2},$$

we have

$$1 = y(1) = c_2.$$

Therefore, the solution of the initial value problem is $y(t) = \ln |t|t^{-1}$.

Nonhomogeneous Cauchy-Euler Equations We can also solve some nonhomogeneous Cauchy-Euler equations using the Method of Undetermined Coefficients. We will demonstrate this with a couple of examples.

Example 2.15. Find the solution of $x^2y'' - xy' - 3y = 2x^2$.

First we find the solution of the homogeneous equation. The characteristic equation is $r^2 - 2r - 3 = 0$. So, the roots are $r = -1, 3$ and the solution is $y_h(x) = c_1x^{-1} + c_2x^3$.

We next need a particular solution. Let's guess $y_p(x) = Ax^2$. Inserting the guess into the nonhomogeneous differential equation, we have

$$\begin{aligned} 2x^2 &= x^2y'' - xy' - 3y = 2x^2 \\ &= 2Ax^2 - 2Ax^2 - 3Ax^2 \\ &= -3Ax^2. \end{aligned} \tag{2.77}$$

So, $A = -2/3$. Therefore, the general solution of the problem is

$$y(x) = c_1x^{-1} + c_2x^3 - \frac{2}{3}x^2.$$

Example 2.16. Find the solution of $x^2y'' - xy' - 3y = 2x^3$.

In this case the nonhomogeneous term is a solution of the homogeneous problem, which we solved in the last example. So, we will need a modification of the method. We have a problem of the form

$$ax^2y'' + bxy' + cy = dx^r,$$

where r is a solution of $ar(r-1) + br + c = 0$. Let's guess a solution of the form $y = Ax^r \ln x$. Then one finds that the differential equation reduces to $Ax^r(2ar - a + b) = dx^r$. [You should verify this for yourself.]

With this in mind, we can now solve the problem at hand. Let $y_p = Ax^3 \ln x$. Inserting into the equation, we obtain $4Ax^3 = 2x^3$, or $A = 1/2$. The general solution of the problem can now be written as

$$y(x) = c_1 x^{-1} + c_2 x^3 + \frac{1}{2} x^3 \ln x.$$

2.7.4 Method of Variation of Parameters

A MORE SYSTEMATIC way to find particular solutions is through the use of the Method of Variation of Parameters. The derivation is a little detailed and the solution is sometimes messy, but the application of the method is straight forward if you can do the required integrals. We will first derive the needed equations and then do some examples.

We begin with the nonhomogeneous equation. Let's assume it is of the standard form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x). \quad (2.78)$$

We know that the solution of the homogeneous equation can be written in terms of two linearly independent solutions, which we will call $y_1(x)$ and $y_2(x)$:

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x).$$

Replacing the constants with functions, then we now longer have a solution to the homogeneous equation. Is it possible that we could stumble across the right functions with which to replace the constants and somehow end up with $f(x)$ when inserted into the left side of the differential equation? It turns out that we can.

So, let's assume that the constants are replaced with two unknown functions, which we will call $c_1(x)$ and $c_2(x)$. This change of the parameters is where the name of the method derives. Thus, we are assuming that a particular solution takes the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x). \quad (2.79)$$

If this is to be a solution, then insertion into the differential equation should make it true. To do this we will first need to compute some derivatives.

The first derivative is given by

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x). \quad (2.80)$$

Next we will need the second derivative. But, this will give use eight terms. So, we will first make an assumption. Let's assume that the last two terms add to zero:

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \quad (2.81)$$

We assume the nonhomogeneous equation has a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

It turns out that we will get the same results in the end if we did not assume this. The important thing is that it works!

So, we now have the first derivative as

$$y'_p(x) = c_1(x)y'_1(x) + c_2(x)y'_2(x). \quad (2.82)$$

The second derivative is then only four terms:

$$y''_p(x) = c_1(x)y''_1(x) + c_2(x)y''_2(x) + c'_1(x)y'_1(x) + c'_2(x)y'_2(x). \quad (2.83)$$

Now that we have the derivatives, we can insert the guess into the differential equation. Thus, we have

$$\begin{aligned} f(x) &= a(x)(c_1(x)y''_1(x) + c_2(x)y''_2(x) + c'_1(x)y'_1(x) + c'_2(x)y'_2(x)) \\ &\quad + b(x)(c_1(x)y'_1(x) + c_2(x)y'_2(x)) \\ &\quad + c(x)(c_1(x)y_1(x) + c_2(x)y_2(x)). \end{aligned} \quad (2.84)$$

Regrouping the terms, we obtain

$$\begin{aligned} f(x) &= c_1(x)(a(x)y''_1(x) + b(x)y'_1(x) + c(x)y_1(x)) \\ &\quad + c_2(x)(a(x)y''_2(x) + b(x)y'_2(x) + c(x)y_2(x)) \\ &\quad + a(x)(c'_1(x)y'_1(x) + c'_2(x)y'_2(x)). \end{aligned} \quad (2.85)$$

Note that the first two rows vanish since y_1 and y_2 are solutions of the homogeneous problem. This leaves the equation

$$c'_1(x)y'_1(x) + c'_2(x)y'_2(x) = \frac{f(x)}{a(x)}. \quad (2.86)$$

In summary, we have assumed a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

This is only possible if the unknown functions $c_1(x)$ and $c_2(x)$ satisfy the system of equations

$$\begin{aligned} c'_1(x)y_1(x) + c'_2(x)y_2(x) &= 0 \\ c'_1(x)y'_1(x) + c'_2(x)y'_2(x) &= \frac{f(x)}{a(x)}. \end{aligned} \quad (2.87)$$

It is standard to solve this system for the derivatives of the unknown functions and then present the integrated forms. However, one could just start from here.

Example 2.17. Consider the problem: $y'' - y = e^{2x}$. We want the general solution of this nonhomogeneous problem.

To solve the differential equation $Ly = f$, we assume $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, for $Ly_{1,2} = 0$. Then, one need only solve a simple system of equations.

System (2.87) can be solved as

$$\begin{aligned} c'_1(x) &= -\frac{fy_2}{aW(y_1, y_2)}, \\ c'_1(x) &= \frac{fy_1}{aW(y_1, y_2)}, \end{aligned}$$

where $W(y_1, y_2) = y_1y'_2 - y'_1y_2$ is the Wronskian.

The general solution to the homogeneous problem $y_h'' - y_h = 0$ is

$$y_h(x) = c_1 e^x + c_2 e^{-x}.$$

In order to use the Method of Variation of Parameters, we seek a solution of the form

$$y_p(x) = c_1(x)e^x + c_2(x)e^{-x}.$$

We find the unknown functions by solving the system in (2.87), which in this case becomes

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= e^{2x}. \end{aligned} \quad (2.88)$$

Adding these equations we find that

$$2c_1'e^x = e^{2x} \rightarrow c_1' = \frac{1}{2}e^x.$$

Solving for $c_1(x)$ we find

$$c_1(x) = \frac{1}{2} \int e^x dx = \frac{1}{2}e^x.$$

Subtracting the equations in the system yields

$$2c_2'e^{-x} = -e^{2x} \rightarrow c_2' = -\frac{1}{2}e^{3x}.$$

Thus,

$$c_2(x) = -\frac{1}{2} \int e^{3x} dx = -\frac{1}{6}e^{3x}.$$

The particular solution is found by inserting these results into y_p :

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= \left(\frac{1}{2}e^x\right)e^x + \left(-\frac{1}{6}e^{3x}\right)e^{-x} \\ &= \frac{1}{3}e^{2x}. \end{aligned} \quad (2.89)$$

Thus, we have the general solution of the nonhomogeneous problem as

$$y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{3}e^{2x}.$$

Example 2.18. Now consider the problem: $y'' + 4y = \sin x$.

The solution to the homogeneous problem is

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x. \quad (2.90)$$

We now seek a particular solution of the form

$$y_h(x) = c_1(x) \cos 2x + c_2(x) \sin 2x.$$

We let $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$, $a(x) = 1$, $f(x) = \sin x$ in system (2.87):

$$\begin{aligned} c_1'(x) \cos 2x + c_2'(x) \sin 2x &= 0 \\ -2c_1'(x) \sin 2x + 2c_2'(x) \cos 2x &= \sin x. \end{aligned} \quad (2.91)$$

Now, use your favorite method for solving a system of two equations and two unknowns. In this case, we can multiply the first equation by $2 \sin 2x$ and the second equation by $\cos 2x$. Adding the resulting equations will eliminate the c_1' terms. Thus, we have

$$c_2'(x) = \frac{1}{2} \sin x \cos 2x = \frac{1}{2} (2 \cos^2 x - 1) \sin x.$$

Inserting this into the first equation of the system, we have

$$c_1'(x) = -c_2'(x) \frac{\sin 2x}{\cos 2x} = -\frac{1}{2} \sin x \sin 2x = -\sin^2 x \cos x.$$

These can easily be solved:

$$c_2(x) = \frac{1}{2} \int (2 \cos^2 x - 1) \sin x \, dx = \frac{1}{2} (\cos x - \frac{2}{3} \cos^3 x).$$

$$c_1(x) = - \int \sin^2 x \cos x \, dx = -\frac{1}{3} \sin^3 x.$$

The final step in getting the particular solution is to insert these functions into $y_p(x)$. This gives

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= \left(-\frac{1}{3} \sin^3 x\right) \cos 2x + \left(\frac{1}{2} \cos x - \frac{1}{3} \cos^3 x\right) \sin x \\ &= \frac{1}{3} \sin x. \end{aligned} \quad (2.92)$$

So, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \sin x. \quad (2.93)$$

2.8 Numerical Solutions of ODEs

SO FAR WE HAVE SEEN some of the standard methods for solving first and second order differential equations. However, we have had to restrict ourselves to very special cases in order to get nice analytical solutions to our initial value problems. While these are not the only equations for which we can get exact results (see Section 2.7.3 for another common class of second order differential equations), there are many cases in which exact solutions are not possible. In such cases

we have to rely on approximation techniques, including the numerical solution of the equation at hand.

The use of numerical methods to obtain approximate solutions of differential equations and systems of differential equations has been known for some time. However, with the advent of powerful computers and desktop computers, we can now solve many of these problems with relative ease. The simple ideas used to solve first order differential equations can be extended to the solutions of more complicated systems of partial differential equations, such as the large scale problems of modeling ocean dynamics, weather systems and even cosmological problems stemming from general relativity.

In this section we will look at the simplest method for solving first order equations, Euler's Method. While it is not the most efficient method, it does provide us with a picture of how one proceeds and can be improved by introducing better techniques, which are typically covered in a numerical analysis text.

Let's consider the class of first order initial value problems of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \quad (2.94)$$

We are interested in finding the solution $y(x)$ of this equation which passes through the initial point (x_0, y_0) in the xy -plane for values of x in the interval $[a, b]$, where $a = x_0$. We will seek approximations of the solution at N points, labeled x_n for $n = 1, \dots, N$. For equally spaced points we have $\Delta x = x_1 - x_0 = x_2 - x_1$, etc. Then, $x_n = x_0 + n\Delta x$. In Figure 2.13 we show three such points on the x -axis.

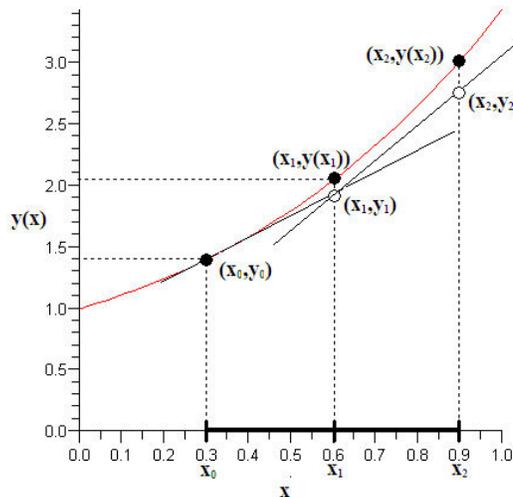


Figure 2.13: The basics of Euler's Method are shown. An interval of the x axis is broken into N subintervals. The approximations to the solutions are found using the slope of the tangent to the solution, given by $f(x, y)$. Knowing previous approximations at (x_{n-1}, y_{n-1}) , one can determine the next approximation, y_n .

We will develop a simple numerical method, called Euler's Method. We rely on Figure 2.13 to do this. As already noted, we first break the interval of interest into N subintervals with $N + 1$ points x_n . We

already know a point on the solution $(x_0, y(x_0)) = (x_0, y_0)$. How do we find the solution for other x values?

We first note that the differential equation gives us the slope of the tangent line at $(x, y(x))$ of the solution $y(x)$. The slope is $f(x, y(x))$. Referring to Figure 2.13, we see the tangent line drawn at (x_0, y_0) . We look now at $x = x_1$. A vertical line intersects both the solution curve and the tangent line. While we do not know the solution, we can determine the tangent line and find the intersection point. As seen in the figure, this intersection point is in theory close to the point on the solution curve. So, we will designate y_1 as the approximation of the solution $y(x_1)$. We just need to determine y_1 .

The idea is simple. We approximate the derivative in the differential equation by its difference quotient:

$$\frac{dy}{dx} \approx \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{\Delta x}. \quad (2.95)$$

But, we have by the differential equation that the slope of the tangent to the curve at (x_0, y_0) is

$$y'(x_0) = f(x_0, y_0).$$

Thus,

$$\frac{y_1 - y_0}{\Delta x} \approx f(x_0, y_0). \quad (2.96)$$

So, we can solve this equation for y_1 to obtain

$$y_1 = y_0 + \Delta x f(x_0, y_0). \quad (2.97)$$

This give y_1 in terms of quantities that we know.

We now proceed to approximate $y(x_2)$. Referring to Figure 2.13, we see that this can be done by using the slope of the solution curve at (x_1, y_1) . The corresponding tangent line is shown passing through (x_1, y_1) and we can then get the value of y_2 . Following the previous argument, we find that

$$y_2 = y_1 + \Delta x f(x_1, y_1). \quad (2.98)$$

Continuing this procedure for all x_n , we arrive at the following numerical scheme for determining a numerical solution to Euler's equation:

$$\begin{aligned} y_0 &= y(x_0), \\ y_n &= y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}), \quad n = 1, \dots, N. \end{aligned} \quad (2.99)$$

Example 2.19. We will consider a standard example for which we know the exact solution. This way we can compare our results. The problem is given that

$$\frac{dy}{dx} = x + y, \quad y(0) = 1, \quad (2.100)$$

find an approximation for $y(1)$.

First, we will do this by hand. We will break up the interval $[0, 1]$, since we want the solution at $x = 1$ and the initial value is at $x = 0$. Let $\Delta x = 0.50$. Then, $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1.0$. Note that $N = \frac{b-a}{\Delta x} = 2$.

We can carry out Euler's Method systematically. We set up a table for the needed values. Such a table is shown in Table 2.1.

n	x_n	$y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}) = 0.5x_{n-1} + 1.5y_{n-1}$
0	0	1
1	0.5	$0.5(0) + 1.5(1.0) = 1.5$
2	1.0	$0.5(0.5) + 1.5(1.5) = 2.5$

Table 2.1: Application of Euler's Method for $y' = x + y$, $y(0) = 1$ and $\Delta x = 0.5$.

Note how the table is set up. There is a column for each x_n and y_n . The first row is the initial condition. We also made use of the function $f(x, y)$ in computing the y_n 's. This sometimes makes the computation easier. As a result, we find that the desired approximation is given as $y_2 = 2.5$.

Is this a good result? Well, we could make the spatial increments smaller. Let's repeat the procedure for $\Delta x = 0.2$, or $N = 5$. The results are in Table 2.2.

n	x_n	$y_n = 0.2x_{n-1} + 1.2y_{n-1}$
0	0	1
1	0.2	$0.2(0) + 1.2(1.0) = 1.2$
2	0.4	$0.2(0.2) + 1.2(1.2) = 1.48$
3	0.6	$0.2(0.4) + 1.2(1.48) = 1.856$
4	0.8	$0.2(0.6) + 1.2(1.856) = 2.3472$
5	1.0	$0.2(0.8) + 1.2(2.3472) = 2.97664$

Table 2.2: Application of Euler's Method for $y' = x + y$, $y(0) = 1$ and $\Delta x = 0.2$.

Now we see that our approximation is $y_1 = 2.97664$. So, it looks like the value is near 3, but we cannot say much more. Decreasing Δx more shows that we are beginning to converge to a solution. We see this in Table 2.3.

Δx	$y_N \approx y(1)$
0.5	2.5
0.2	2.97664
0.1	3.187484920
0.01	3.409627659
0.001	3.433847864
0.0001	3.436291854

Table 2.3: Results of Euler's Method for $y' = x + y$, $y(0) = 1$ and varying Δx

Of course, these values were not done by hand. The last computation would have taken 1000 lines in the table, or at least 40 pages! One could use a computer to do this. A simple code in Maple would look like the following:

```

> restart:
> f:=(x,y)->y+x;
> a:=0: b:=1: N:=100: h:=(b-a)/N;
> x[0]:=0: y[0]:=1:
  for i from 1 to N do
  y[i]:=y[i-1]+h*f(x[i-1],y[i-1]):
  x[i]:=x[0]+h*(i):
  od:
evalf(y[N]);

```

In this case we could simply use the exact solution. The exact solution is easily found as

$$y(x) = 2e^x - x - 1.$$

(The reader can verify this.) So, the value we are seeking is

$$y(1) = 2e - 2 = 3.4365636\dots$$

Thus, even the last numerical solution was off by about 0.00027.

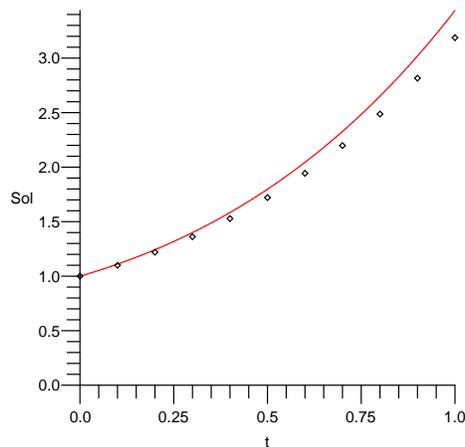


Figure 2.14: A comparison of the results Euler's Method to the exact solution for $y' = x + y$, $y(0) = 1$ and $N = 10$.

Adding a few extra lines for plotting, we can visually see how well the approximations compare to the exact solution. The Maple code for doing such a plot is given below.

```

> with(plots):
> Data:= [seq([x[i],y[i]],i=0..N)]:
> P1:=pointplot(Data,symbol=DIAMOND):
> Sol:=t->-t-1+2*exp(t);
> P2:=plot(Sol(t),t=a..b,Sol=0..Sol(b)):
> display({P1,P2});

```

We show in Figures 2.14-2.15 the results for $N = 10$ and $N = 100$. In Figure 2.14 we can see how quickly the numerical solution diverges from the exact solution. In Figure 2.15 we can see that visually the solutions agree, but we note that from Table 2.3 that for $\Delta x = 0.01$, the solution is still off in the second decimal place with a relative error of about 0.8%.

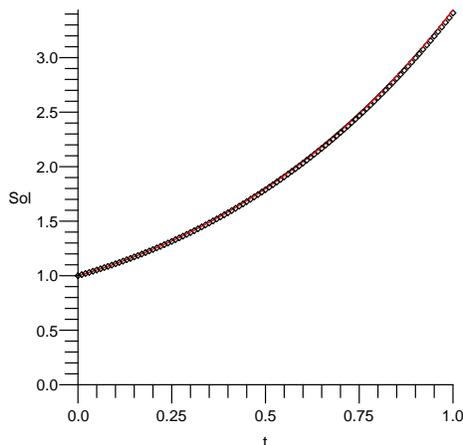


Figure 2.15: A comparison of the results Euler's Method to the exact solution for $y' = x + y$, $y(0) = 1$ and $N = 100$.

Why would we use a numerical method when we have the exact solution? Exact solutions can serve as test cases for our methods. We can make sure our code works before applying them to problems whose solution is not known.

There are many other methods for solving first order equations. One commonly used method is the fourth order Runge-Kutta method. This method has smaller errors at each step as compared to Euler's Method. It is well suited for programming and comes built-in in many packages like Maple and MATLAB. Typically, it is set up to handle systems of first order equations.

In fact, it is well known that n th order equations can be written as a system of n first order equations. Consider the simple second order equation

$$y'' = f(x, y).$$

This is a larger class of equations than the second order constant coefficient equation. We can turn this into a system of two first order differential equations by letting $u = y$ and $v = y' = u'$. Then, $v' = y'' = f(x, u)$. So, we have the first order system

$$\begin{aligned} u' &= v, \\ v' &= f(x, u). \end{aligned} \tag{2.101}$$

We will not go further into the Runge-Kutta Method here. You can find more about it in a numerical analysis text. However, we will

see that systems of differential equations do arise naturally in physics. Such systems are often coupled equations and lead to interesting behaviors.

2.9 Linear Systems

2.9.1 Coupled Oscillators

IN THE LAST SECTION we saw that the numerical solution of second order equations, or higher, can be cast into systems of first order equations. Such systems are typically coupled in the sense that the solution of at least one of the equations in the system depends on knowing one of the other solutions in the system. In many physical systems this coupling takes place naturally. We will introduce a simple model in this section to illustrate the coupling of simple oscillators. However, we will reserve solving the coupled system of oscillators until the next chapter after exploring the needed mathematics.

There are many problems in physics that result in systems of equations. This is because the most basic law of physics is given by Newton's Second Law, which states that if a body experiences a net force, it will accelerate. Thus,

$$\sum \mathbf{F} = m\mathbf{a}.$$

Since $\mathbf{a} = \ddot{\mathbf{x}}$ we have a system of second order differential equations in general for three dimensional problems, or one second order differential equation for one dimensional problems.

We have already seen the simple problem of a mass on a spring as shown in Figure 2.4. Recall that the net force in this case is the restoring force of the spring given by Hooke's Law,

$$F_s = -kx,$$

where $k > 0$ is the spring constant and x is the elongation of the spring. When it is positive, the spring force is negative and when it is negative the spring force is positive. The equation for simple harmonic motion for the mass-spring system was found to be given by

$$m\ddot{x} + kx = 0.$$

This second order equation can be written as a system of two first order equations in terms of the unknown position and velocity. We first set $y = \dot{x}$ and then rewrite the second order equation in terms of x and y . Thus, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \quad (2.102)$$

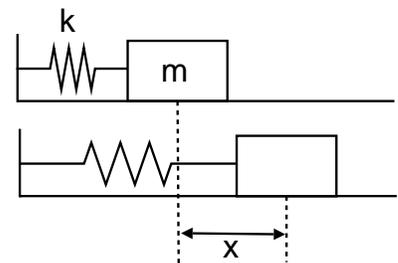


Figure 2.16: Spring-Mass system.

The coefficient matrix for this system is $\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$, where $\omega^2 = \frac{k}{m}$.

One can look at more complicated spring-mass systems. Consider two blocks attached with two springs as in Figure 2.17. In this case we apply Newton's second law for each block. We will designate the elongations of each spring from equilibrium as x_1 and x_2 . These are shown in Figure 2.17.

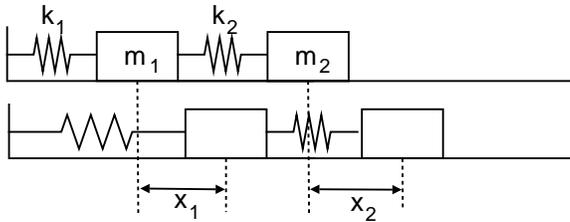


Figure 2.17: Spring-Mass system.

For mass m_1 , the forces acting on it are due to each spring. The first spring with spring constant k_1 provides a force on m_1 of $-k_1x_1$. The second spring is stretched, or compressed, based upon the relative locations of the two masses. So, it will exert a force on m_1 of $k_2(x_2 - x_1)$.

Similarly, the only force acting directly on mass m_2 is provided by the restoring force from spring 2. So, that force is given by $-k_2(x_2 - x_1)$. The reader should think about the signs in each case.

Putting this all together, we apply Newton's Second Law to both masses. We obtain the two equations

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1). \end{aligned} \quad (2.103)$$

Thus, we see that we have a coupled system of two second order differential equations.

One can rewrite this system of two second order equations as a system of four first order equations by letting $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$. This leads to the system

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) \\ \dot{x}_4 &= -\frac{k_2}{m_2}(x_2 - x_1). \end{aligned} \quad (2.104)$$

As we will see, this system can be written more compactly in matrix

form:

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad (2.105)$$

However, before we can solve this system of first order equations, we need to recall a few things from linear algebra. This will be done in the next chapter. For now, we will return to simpler systems and explore the behavior of typical solutions in these planar systems.

2.9.2 Planar Systems

WE NOW CONSIDER examples of solving a coupled system of first order differential equations in the plane. We will focus on the theory of linear systems with constant coefficients. Understanding these simple systems helps in future studies of nonlinear systems, which contain much more interesting behaviors, such as the onset of chaos. In the next chapter we will return to these systems and describe a matrix approach to obtaining the solutions.

A general form for first order systems in the plane is given by a system of two equations for unknowns $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= P(x, y, t) \\ y'(t) &= Q(x, y, t). \end{aligned} \quad (2.106)$$

An *autonomous* system is one in which there is no explicit time dependence:

$$\begin{aligned} x'(t) &= P(x, y) \\ y'(t) &= Q(x, y). \end{aligned} \quad (2.107)$$

Otherwise the system is called *nonautonomous*.

A *linear system* takes the form

$$\begin{aligned} x' &= a(t)x + b(t)y + e(t) \\ y' &= c(t)x + d(t)y + f(t). \end{aligned} \quad (2.108)$$

A *homogeneous* linear system results when $e(t) = 0$ and $f(t) = 0$.

A *linear, constant coefficient system* of first order differential equations is given by

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f. \end{aligned} \quad (2.109)$$

We will focus on linear, homogeneous systems of constant coefficient first order differential equations:

$$\begin{aligned}x' &= ax + by \\y' &= cx + dy.\end{aligned}\tag{2.110}$$

As we will see later, such systems can result by a simple translation of the unknown functions. These equations are said to be coupled if either $b \neq 0$ or $c \neq 0$.

We begin by noting that the system (2.110) can be rewritten as a second order constant coefficient linear differential equation, which we already know how to solve. We differentiate the first equation in system (2.110) and systematically replace occurrences of y and y' , since we also know from the first equation that $y = \frac{1}{b}(x' - ax)$. Thus, we have

$$\begin{aligned}x'' &= ax' + by' \\&= ax' + b(cx + dy) \\&= ax' + bcx + d(x' - ax).\end{aligned}\tag{2.111}$$

Rewriting the last line, we have

$$x'' - (a + d)x' + (ad - bc)x = 0.\tag{2.112}$$

This is a linear, homogeneous, constant coefficient ordinary differential equation. We know that we can solve this by first looking at the roots of the characteristic equation

$$r^2 - (a + d)r + ad - bc = 0\tag{2.113}$$

and writing down the appropriate general solution for $x(t)$. Then we can find $y(t)$ using Equation (2.110):

$$y = \frac{1}{b}(x' - ax).$$

We now demonstrate this for a specific example.

Example 2.20. Consider the system of differential equations

$$\begin{aligned}x' &= -x + 6y \\y' &= x - 2y.\end{aligned}\tag{2.114}$$

Carrying out the above outlined steps, we have that $x'' + 3x' - 4x = 0$. This can be shown as follows:

$$\begin{aligned}x'' &= -x' + 6y' \\&= -x' + 6(x - 2y) \\&= -x' + 6x - 12\left(\frac{x' + x}{6}\right) \\&= -3x' + 4x\end{aligned}\tag{2.115}$$

The resulting differential equation has a characteristic equation of $r^2 + 3r - 4 = 0$. The roots of this equation are $r = 1, -4$. Therefore, $x(t) = c_1e^t + c_2e^{-4t}$. But, we still need $y(t)$. From the first equation of the system we have

$$y(t) = \frac{1}{6}(x' + x) = \frac{1}{6}(2c_1e^t - 3c_2e^{-4t}).$$

Thus, the solution to the system is

$$\begin{aligned} x(t) &= c_1e^t + c_2e^{-4t}, \\ y(t) &= \frac{1}{3}c_1e^t - \frac{1}{2}c_2e^{-4t}. \end{aligned} \quad (2.116)$$

Sometimes one needs initial conditions. For these systems we would specify conditions like $x(0) = x_0$ and $y(0) = y_0$. These would allow the determination of the arbitrary constants as before.

Example 2.21. Solve

$$\begin{aligned} x' &= -x + 6y \\ y' &= x - 2y. \end{aligned} \quad (2.117)$$

given $x(0) = 2, y(0) = 0$.

We already have the general solution of this system in (2.116). Inserting the initial conditions, we have

$$\begin{aligned} 2 &= c_1 + c_2, \\ 0 &= \frac{1}{3}c_1 - \frac{1}{2}c_2. \end{aligned} \quad (2.118)$$

Solving for c_1 and c_2 gives $c_1 = 6/5$ and $c_2 = 4/5$. Therefore, the solution of the initial value problem is

$$\begin{aligned} x(t) &= \frac{2}{5}(3e^t + 2e^{-4t}), \\ y(t) &= \frac{2}{5}(e^t - e^{-4t}). \end{aligned} \quad (2.119)$$

2.9.3 Equilibrium Solutions and Nearby Behaviors

IN STUDYING SYSTEMS of differential equations, it is often useful to study the behavior of solutions without obtaining an algebraic form for the solution. This is done by exploring equilibrium solutions and solutions nearby equilibrium solutions. Such techniques will be seen to be useful later in studying nonlinear systems.

We begin this section by studying *equilibrium solutions* of system (2.109). For equilibrium solutions the system does not change in time. Therefore, equilibrium solutions satisfy the equations $x' = 0$ and $y' = 0$. Of course, this can only happen for constant solutions. Let x_0 and y_0

be the (constant) equilibrium solutions. Then, x_0 and y_0 must satisfy the system

$$\begin{aligned} 0 &= ax_0 + by_0 + e, \\ 0 &= cx_0 + dy_0 + f. \end{aligned} \quad (2.120)$$

This is a linear system of nonhomogeneous algebraic equations. One only has a unique solution when the determinant of the system is not zero, i.e., $ad - bc \neq 0$. Using Cramer's (determinant) Rule for solving such systems, we have

$$x_0 = -\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y_0 = -\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \quad (2.121)$$

If the system is homogeneous, $e = f = 0$, then we have that the origin is the equilibrium solution; i.e., $(x_0, y_0) = (0, 0)$. Often we will have this case since one can always make a change of coordinates from (x, y) to (u, v) by $u = x - x_0$ and $v = y - y_0$. Then, $u_0 = v_0 = 0$.

Next we are interested in the behavior of solutions near the equilibrium solutions. Later this behavior will be useful in analyzing more complicated nonlinear systems. We will look at some simple systems that are readily solved.

Example 2.22. Stable Node (sink)

Consider the system

$$\begin{aligned} x' &= -2x \\ y' &= -y. \end{aligned} \quad (2.122)$$

This is a simple uncoupled system. Each equation is simply solved to give

$$x(t) = c_1 e^{-2t} \text{ and } y(t) = c_2 e^{-t}.$$

In this case we see that all solutions tend towards the equilibrium point, $(0, 0)$. This will be called a stable node, or a sink.

Before looking at other types of solutions, we will explore the stable node in the above example. There are several methods of looking at the behavior of solutions. We can look at solution plots of the dependent versus the independent variables, or we can look in the xy -plane at the parametric curves $(x(t), y(t))$.

Solution Plots: One can plot each solution as a function of t given a set of initial conditions. Examples are shown in Figure 2.18 for several initial conditions. Note that the solutions decay for large t . Special cases result for various initial conditions. Note that for $t = 0$,

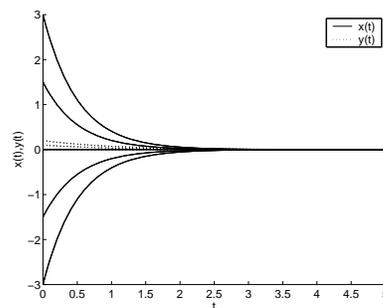


Figure 2.18: Plots of solutions of Example 2.22 for several initial conditions.

$x(0) = c_1$ and $y(0) = c_2$. (Of course, one can provide initial conditions at any $t = t_0$. It is generally easier to pick $t = 0$ in our general explanations.) If we pick an initial condition with $c_1=0$, then $x(t) = 0$ for all t . One obtains similar results when setting $y(0) = 0$.

Phase Portrait: There are other types of plots which can provide additional information about the solutions even if we cannot find the exact solutions as we can for these simple examples. In particular, one can consider the solutions $x(t)$ and $y(t)$ as the coordinates along a parameterized path, or curve, in the plane: $\mathbf{r} = (x(t), y(t))$. Such curves are called *trajectories* or *orbits*. The xy -plane is called the *phase plane* and a collection of such orbits gives a *phase portrait* for the family of solutions of the given system.

One method for determining the equations of the orbits in the phase plane is to eliminate the parameter t between the known solutions to get a relationship between x and y . In the above example we can do this, since the solutions are known. In particular, we have

$$x = c_1 e^{-2t} = c_1 \left(\frac{y}{c_2} \right)^2 \equiv Ay^2.$$

Another way to obtain information about the orbits comes from noting that the slopes of the orbits in the xy -plane are given by dy/dx . For autonomous systems, we can write this slope just in terms of x and y . This leads to a first order differential equation, which possibly could be solved analytically, solved numerically, or just used to produce a *direction field*. We will see that direction fields are useful in determining qualitative behaviors of the solutions without actually finding explicit solutions.

First we will obtain the orbits for Example 2.22 by solving the corresponding slope equation. First, recall that for trajectories defined parametrically by $x = x(t)$ and $y = y(t)$, we have from the Chain Rule for $y = y(x(t))$ that

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{2.123}$$

For the system in (2.122) we use Equation (2.123) to obtain the equation for the slope at a point on the orbit:

$$\frac{dy}{dx} = \frac{y}{2x}.$$

The general solution of this first order differential equation is found using separation of variables as $x = Ay^2$ for A an arbitrary constant.

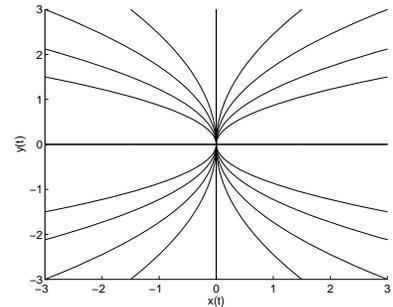


Figure 2.19: Orbits for Example 2.22.

Plots of these solutions in the phase plane are given in Figure 2.19. [Note that this is the same form for the orbits that we had obtained above by eliminating t from the solution of the system.]

Once one has solutions to differential equations, we often are interested in the long time behavior of the solutions. Given a particular initial condition (x_0, y_0) , how does the solution behave as time increases? For orbits near an equilibrium solution, do the solutions tend towards, or away from, the equilibrium point? The answer is obvious when one has the exact solutions $x(t)$ and $y(t)$. However, this is not always the case.

Let's consider the above example for initial conditions in the first quadrant of the phase plane. For a point in the first quadrant we have that

$$dx/dt = -2x < 0,$$

meaning that as $t \rightarrow \infty$, $x(t)$ get more negative. Similarly,

$$dy/dt = -y < 0,$$

indicates that $y(t)$ is also getting smaller for this problem. Thus, these orbits tend towards the origin as $t \rightarrow \infty$. This qualitative information was obtained without relying on the known solutions to the problem.

Direction Fields: Another way to determine the behavior of our system is to draw the direction field. Recall that a direction field is a vector field in which one plots arrows in the direction of tangents to the orbits. This is done because the slopes of the tangent lines are given by dy/dx . For the system (2.110), the slope is

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

In general, for nonautonomous systems, we obtain a first order differential equation of the form

$$\frac{dy}{dx} = F(x, y).$$

This particular equation can be solved by the reader.

Example 2.23. Draw the direction field for Example 2.22.

We can use software to draw direction fields. However, one can sketch these fields by hand. we have that the slope of the tangent at this point is given by

$$\frac{dy}{dx} = \frac{-y}{-2x} = \frac{y}{2x}.$$

For each point in the plane one draws a piece of tangent line with this slope. In Figure 2.20 we show a few of these. For $(x, y) = (1, 1)$ the slope is $dy/dx = 1/2$. So, we draw an arrow with slope $1/2$ at this point. From system (2.122),

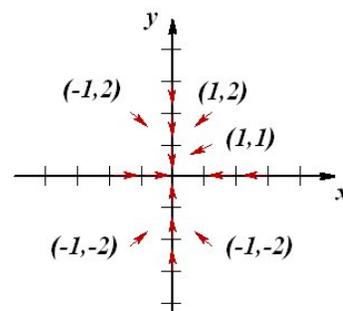


Figure 2.20: Sketch of tangent vectors for Example 2.22.

we have that x' and y' are both negative at this point. Therefore, the vector points down and to the left.

We can do this for several points, as shown in Figure 2.20. Sometimes one can quickly sketch vectors with the same slope. For this example, when $y = 0$, the slope is zero and when $x = 0$ the slope is infinite. So, several vectors can be provided. Such vectors are tangent to curves known as isoclines in which $\frac{dy}{dx} = \text{constant}$.

It is often difficult to provide an accurate sketch of a direction field. Computer software can be used to provide a better rendition. For Example 2.22 the direction field is shown in Figure 2.21. Looking at this direction field, one can begin to “see” the orbits by following the tangent vectors.

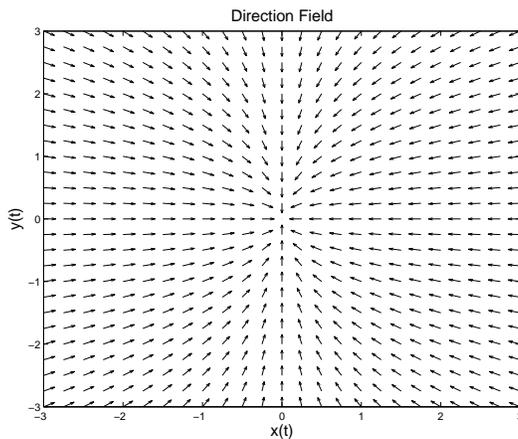


Figure 2.21: Direction field for Example 2.22.

Of course, one can superimpose the orbits on the direction field. This is shown in Figure 2.22. Are these the patterns you saw in Figure 2.21?

In this example we see all orbits “flow” towards the origin, or equilibrium point. Again, this is an example of what is called a stable node or a sink. (Imagine what happens to the water in a sink when the drain is unplugged.)

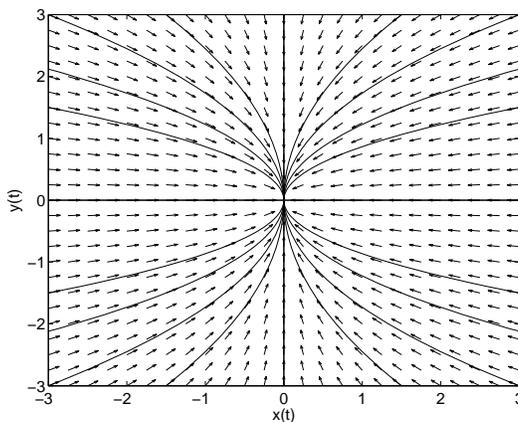


Figure 2.22: Phase portrait for Example 2.22.

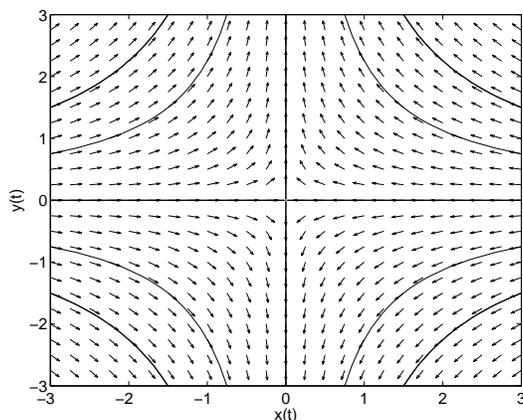
Example 2.24. Saddle

Consider the system

$$\begin{aligned}x' &= -x \\y' &= y.\end{aligned}\tag{2.124}$$

This is another uncoupled system. The solutions are again simply gotten by integration. We have that $x(t) = c_1 e^{-t}$ and $y(t) = c_2 e^t$. Here we have that x decays as t gets large and y increases as t gets large. In particular, if one picks initial conditions with $c_2 = 0$, then orbits follow the x -axis towards the origin. For initial points with $c_1 = 0$, orbits originating on the y -axis will flow away from the origin. Of course, in these cases the origin is an equilibrium point and once at equilibrium, one remains there.

In fact, there is only one line on which to pick initial conditions such that the orbit leads towards the equilibrium point. No matter how small c_2 is, sooner, or later, the exponential growth term will dominate the solution. One can see this behavior in Figure 2.23.



Similar to the first example, we can look at a variety of plots. These are given by Figures 2.23-2.24. The orbits can be obtained from the system as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{y}{x}.$$

The solution is $y = \frac{A}{x}$. For different values of $A \neq 0$ we obtain a family of hyperbolae. These are the same curves one might obtain for the level curves of a surface known as a saddle surface, $z = xy$. Thus, this type of equilibrium point is classified as a *saddle point*. From the phase portrait we can verify that there are many orbits that lead away from the origin (equilibrium point), but there is one line of initial conditions that leads to the origin and that is the x -axis. In this case, the line of initial conditions is given by the x -axis.

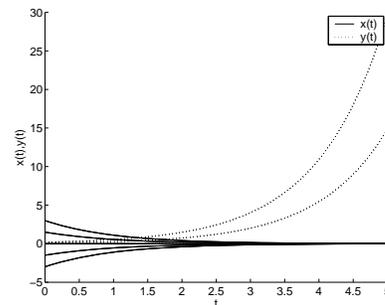


Figure 2.23: Plots of solutions of Example 2.24 for several initial conditions.

Figure 2.24: Phase portrait for Example 2.24, a saddle.

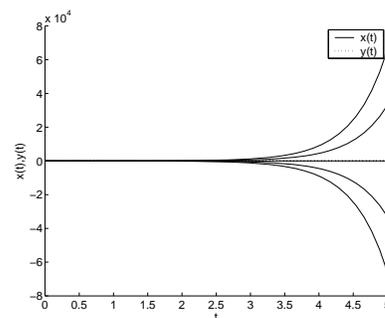


Figure 2.25: Plots of solutions of Example 2.25 for several initial conditions.

Example 2.25. Unstable Node (source)

$$\begin{aligned}x' &= 2x \\y' &= y.\end{aligned}\tag{2.125}$$

This example is similar to Example 2.22. The solutions are obtained by replacing t with $-t$. The solutions, orbits and direction fields can be seen in Figures 2.25-2.26. This is once again a node, but all orbits lead away from the equilibrium point. It is called an unstable node or a source.

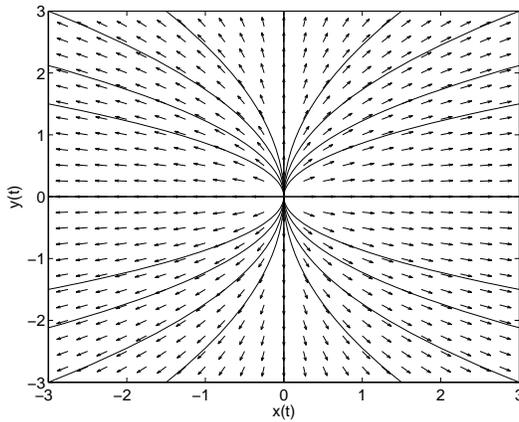


Figure 2.26: Phase portrait for Example 2.25, an unstable node or source.

Example 2.26. Center

$$\begin{aligned}x' &= y \\y' &= -x.\end{aligned}\tag{2.126}$$

This system is a simple, coupled system. Neither equation can be solved without some information about the other unknown function. However, we can differentiate the first equation and use the second equation to obtain

$$x'' + x = 0.$$

We recognize this equation from the last chapter as one that appears in the study of simple harmonic motion. The solutions are pure sinusoidal oscillations:

$$x(t) = c_1 \cos t + c_2 \sin t, \quad y(t) = -c_1 \sin t + c_2 \cos t.$$

In the phase plane the trajectories can be determined either by looking at the direction field, or solving the first order equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

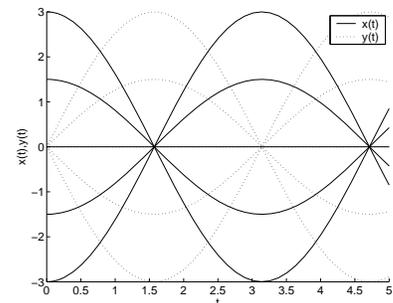


Figure 2.27: Plots of solutions of Example 2.26 for several initial conditions.

Performing a separation of variables and integrating, we find that

$$x^2 + y^2 = C.$$

Thus, we have a family of circles for $C > 0$. (Can you prove this using the general solution?) Looking at the results graphically in Figures 2.27-2.28 confirms this result. This type of point is called a *center*.

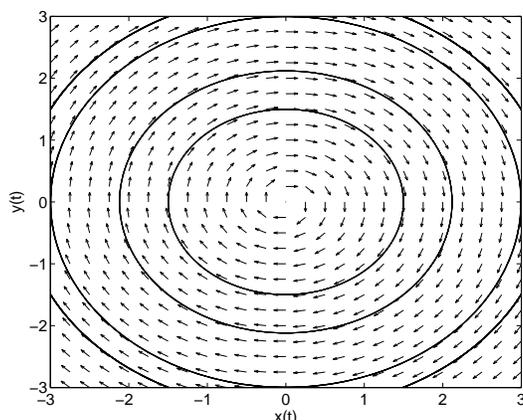


Figure 2.28: Phase portrait for Example 2.26, a center.

Example 2.27. Focus (spiral)

$$\begin{aligned}x' &= \alpha x + y \\y' &= -x.\end{aligned}\tag{2.127}$$

In this example, we will see an additional set of behaviors of equilibrium points in planar systems. We have added one term, αx , to the system in Example 2.26. We will consider the effects for two specific values of the parameter: $\alpha = 0.1, -0.2$. The resulting behaviors are shown in the remaining graphs. We see orbits that look like spirals. These orbits are stable and unstable *spirals* (or *foci*, the plural of focus.)

We can understand these behaviors by once again relating the system of first order differential equations to a second order differential equation. Using the usual method for obtaining a second order equation form a system, we find that $x(t)$ satisfies the differential equation

$$x'' - \alpha x' + x = 0.$$

We recall from our first course that this is a form of *damped simple harmonic motion*. We will explore the different types of solutions that will result for various α 's.

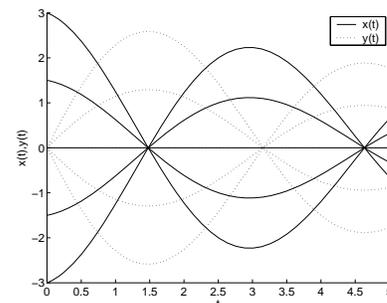


Figure 2.29: Plots of solutions of Example 2.27 for several initial conditions, $\alpha = -0.2$.

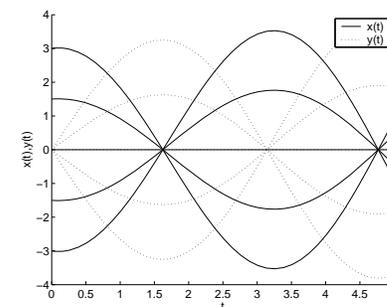


Figure 2.30: Plots of solutions of Example 2.27 for several initial conditions, $\alpha = 0.1$.

The characteristic equation is $r^2 - \alpha r + 1 = 0$. The solution of this quadratic equation is

$$r = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

There are five special cases to consider as shown below.

Classification of Solutions of $x'' - \alpha x' + x = 0$
<p>1. $\alpha = -2$. There is one real solution. This case is called <i>critical damping</i> since the solution $r = -1$ leads to exponential decay. The solution is $x(t) = (c_1 + c_2 t)e^{-t}$.</p>
<p>2. $\alpha < -2$. There are two real, negative solutions, $r = -\mu, -\nu$, $\mu, \nu > 0$. The solution is $x(t) = c_1 e^{-\mu t} + c_2 e^{-\nu t}$. In this case we have what is called <i>overdamped</i> motion. There are no oscillations</p>
<p>3. $-2 < \alpha < 0$. There are two complex conjugate solutions $r = \alpha/2 \pm i\beta$ with real part less than zero and $\beta = \frac{\sqrt{4-\alpha^2}}{2}$. The solution is $x(t) = (c_1 \cos \beta t + c_2 \sin \beta t)e^{\alpha t/2}$. Since $\alpha < 0$, this consists of a decaying exponential times oscillations. This is often called an <i>underdamped</i> oscillation.</p>
<p>4. $\alpha = 0$. This leads to <i>simple harmonic motion</i>.</p>
<p>5. $0 < \alpha < 2$. This is similar to the underdamped case, except $\alpha > 0$. The solutions are growing oscillations.</p>
<p>6. $\alpha = 2$. There is one real solution. The solution is $x(t) = (c_1 + c_2 t)e^t$. It leads to unbounded growth in time.</p>
<p>7. For $\alpha > 2$. There are two real, positive solutions $r = \mu, \nu > 0$. The solution is $x(t) = c_1 e^{\mu t} + c_2 e^{\nu t}$, which grows in time.</p>

For $\alpha < 0$ the solutions are losing energy, so the solutions can oscillate with a diminishing amplitude. (See Figure 2.29.) For $\alpha > 0$, there is a growth in the amplitude, which is not typical. (See Figure 2.30.) Of course, there can be overdamped motion if the magnitude of α is too large.

Example 2.28. Degenerate Node For this example, we will write out the solutions. It is a coupled system for which only the second equation is coupled.

$$\begin{aligned} x' &= -x \\ y' &= -2x - y. \end{aligned} \tag{2.128}$$

There are two possible approaches:

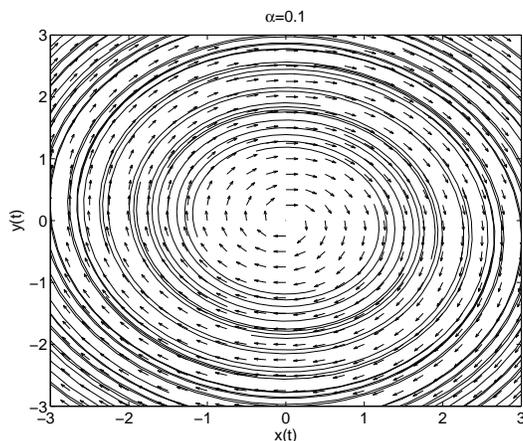


Figure 2.31: Phase portrait for Example 2.27 with $\alpha = 0.1$. This is an unstable focus, or spiral.

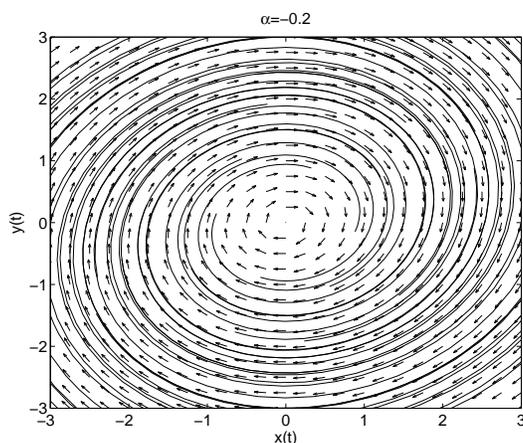


Figure 2.32: Phase portrait for Example 2.27 with $\alpha = -0.2$. This is a stable focus, or spiral.

a. We could solve the first equation to find $x(t) = c_1 e^{-t}$. Inserting this solution into the second equation, we have

$$y' + y = -2c_1 e^{-t}.$$

This is a relatively simple linear first order equation for $y = y(t)$. The integrating factor is $\mu = e^t$. The solution is found as $y(t) = (c_2 - 2c_1 t)e^{-t}$.

b. Another method would be to proceed to rewrite this as a second order equation. Computing x'' does not get us very far. So, we look at

$$\begin{aligned} y'' &= -2x' - y' \\ &= 2x - y' \\ &= -2y' - y. \end{aligned} \tag{2.129}$$

Therefore, y satisfies

$$y'' + 2y' + y = 0.$$

The characteristic equation has one real root, $r = -1$. So, we write

$$y(t) = (k_1 + k_2 t)e^{-t}.$$

This is a stable degenerate node. Combining this with the solution $x(t) = c_1 e^{-t}$, we can show that $y(t) = (c_2 - 2c_1 t)e^{-t}$ as before.

In Figure 2.33 we see several orbits in this system. It differs from the stable node shown in Figure 2.19 in that there is only one direction along which the orbits approach the origin instead of two. If one picks $c_1 = 0$, then $x(t) = 0$ and $y(t) = c_2 e^{-t}$. This leads to orbits running along the y -axis as seen in the figure.

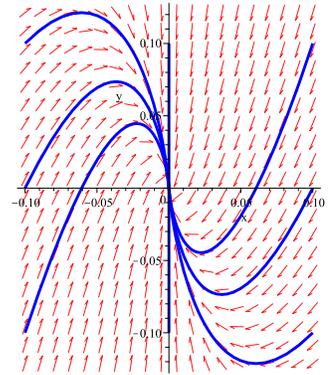


Figure 2.33: Plots of solutions of Example 2.28 for several initial conditions.

Example 2.29. A Line of Equilibria, Zero Root

$$\begin{aligned} x' &= 2x - y \\ y' &= -2x + y. \end{aligned} \tag{2.130}$$

In this last example, we have a coupled set of equations. We rewrite it as a second order differential equation:

$$\begin{aligned} x'' &= 2x' - y' \\ &= 2x' - (-2x + y) \\ &= 2x' + 2x + (x' - 2x) = 3x'. \end{aligned} \tag{2.131}$$

So, the second order equation is

$$x'' - 3x' = 0$$

and the characteristic equation is $0 = r(r - 3)$. This gives the general solution as

$$x(t) = c_1 + c_2 e^{3t}$$

and thus

$$y = 2x - x' = 2(c_1 + c_2^3 t) - (3c_2 e^{3t}) = 2c_1 - c_2 e^{3t}.$$

In Figure 2.34 we show the direction field. The constant slope field seen in this example is confirmed by a simple computation:

$$\frac{dy}{dx} = \frac{-2x + y}{2x - y} = -1.$$

Furthermore, looking at initial conditions with $y = 2x$, we have at $t = 0$,

$$2c_1 - c_2 = 2(c_1 + c_2) \Rightarrow c_2 = 0.$$

Therefore, points on this line remain on this line forever, $(x, y) = (c_1, 2c_1)$. This line of fixed points is called a line of equilibria.

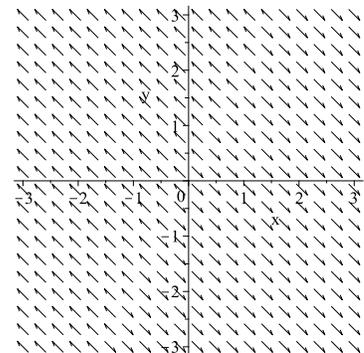


Figure 2.34: Plots of direction field of Example 2.29.

2.9.4 Polar Representation of Spirals

IN THE EXAMPLES with a center or a spiral, one might be able to write the solutions in polar coordinates. Recall that a point in the plane can be described by either Cartesian (x, y) or polar (r, θ) coordinates. Given the polar form, one can find the Cartesian components using

$$x = r \cos \theta \text{ and } y = r \sin \theta.$$

Given the Cartesian coordinates, one can find the polar coordinates using

$$r^2 = x^2 + y^2 \text{ and } \tan \theta = \frac{y}{x}. \quad (2.132)$$

Since x and y are functions of t , then naturally we can think of r and θ as functions of t . The equations that they satisfy are obtained by differentiating the above relations with respect to t .

Differentiating the first equation in (2.132) gives

$$rr' = xx' + yy'.$$

Inserting the expressions for x' and y' from system 2.110, we have

$$rr' = x(ax + by) + y(cx + dy).$$

In some cases this may be written entirely in terms of r 's. Similarly, we have that

$$\theta' = \frac{xy' - yx'}{r^2},$$

which the reader can prove for homework.

In summary, when converting first order equations from rectangular to polar form, one needs the relations below.

Time Derivatives of Polar Variables

$$\begin{aligned} r' &= \frac{xx' + yy'}{r}, \\ \theta' &= \frac{xy' - yx'}{r^2}. \end{aligned} \quad (2.133)$$

Example 2.30. Rewrite the following system in polar form and solve the resulting system.

$$\begin{aligned} x' &= ax + by \\ y' &= -bx + ay. \end{aligned} \quad (2.134)$$

We first compute r' and θ' :

$$rr' = xx' + yy' = x(ax + by) + y(-bx + ay) = ar^2.$$

$$r^2\theta' = xy' - yx' = x(-bx + ay) - y(ax + by) = -br^2.$$

This leads to simpler system

$$\begin{aligned} r' &= ar \\ \theta' &= -b. \end{aligned} \quad (2.135)$$

This system is uncoupled. The second equation in this system indicates that we traverse the orbit at a constant rate in the clockwise direction. Solving these equations, we have that $r(t) = r_0e^{at}$, $\theta(t) = \theta_0 - bt$. Eliminating t between these solutions, we finally find the polar equation of the orbits:

$$r = r_0e^{-a(\theta-\theta_0)t/b}.$$

If you graph this for $a \neq 0$, you will get stable or unstable spirals.

Example 2.31. Consider the specific system

$$\begin{aligned} x' &= -y + x \\ y' &= x + y. \end{aligned} \quad (2.136)$$

In order to convert this system into polar form, we compute

$$\begin{aligned} rr' &= xx' + yy' = x(-y + x) + y(x + y) = r^2. \\ r^2\theta' &= xy' - yx' = x(x + y) - y(-y + x) = r^2. \end{aligned}$$

This leads to simpler system

$$\begin{aligned} r' &= r \\ \theta' &= 1. \end{aligned} \quad (2.137)$$

Solving these equations yields

$$r(t) = r_0e^t, \quad \theta(t) = t + \theta_0.$$

Eliminating t from this solution gives the orbits in the phase plane, $r(\theta) = r_0e^{\theta-\theta_0}$.

A more complicated example arises for a nonlinear system of differential equations. Consider the following example.

Example 2.32.

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2) \\ y' &= x + y(1 - x^2 - y^2). \end{aligned} \quad (2.138)$$

Transforming to polar coordinates, one can show that in order to convert this system into polar form, we compute

$$r' = r(1 - r^2), \quad \theta' = 1.$$

This uncoupled system can be solved and this is left to the reader.

2.10 Appendix: The Nonlinear Pendulum

WE CAN ALSO MAKE the simple pendulum more realistic by adding damping. This could be due to energy loss in the way the string is attached to the support or due to the drag on the mass, etc. Assuming that the damping is proportional to the angular velocity, we have equations for the damped nonlinear and damped linear pendula:

$$L\ddot{\theta} + b\dot{\theta} + g \sin \theta = 0. \quad (2.139)$$

$$L\ddot{\theta} + b\dot{\theta} + g\theta = 0. \quad (2.140)$$

Finally, we can add forcing. Imagine that the support is attached to a device to make the system oscillate horizontally at some frequency. Then we could have equations such as

$$L\ddot{\theta} + b\dot{\theta} + g \sin \theta = F \cos \omega t. \quad (2.141)$$

We will look at these and other oscillation problems later in our discussion.

Before returning to studying the equilibrium solutions of the nonlinear pendulum, we will look at how far we can get at obtaining analytical solutions. First, we investigate the simple linear pendulum.

The linear pendulum equation (2.32) is a constant coefficient second order linear differential equation. The roots of the characteristic equations are $r = \pm \sqrt{\frac{g}{L}}i$. Thus, the general solution takes the form

$$\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}}t\right). \quad (2.142)$$

We note that this is usually simplified by introducing the angular frequency

$$\omega \equiv \sqrt{\frac{g}{L}}. \quad (2.143)$$

One consequence of this solution, which is used often in introductory physics, is an expression for the period of oscillation of a simple pendulum. The period is found to be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}. \quad (2.144)$$

As we have seen, this value for the period of a simple pendulum was derived assuming a small angle approximation. How good is this approximation? What is meant by a *small* angle? We could recall from calculus that the Taylor series approximation of $\sin \theta$ about $\theta = 0$:

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \quad (2.145)$$

One can obtain a bound on the error when truncating this series to one term after taking a numerical analysis course. But we can just simply plot the relative error, which is defined as

$$\text{Relative Error} = \frac{\sin \theta - \theta}{\sin \theta}.$$

A plot of the relative error is given in Figure 2.35. Thus for $\theta \approx 0.4$ radians (or, degrees) we have that the relative error is about 4%.

We would like to do better than this. So, we now turn to the non-linear pendulum. We first rewrite Equation (2.141) in the simpler form

$$\ddot{\theta} + \omega^2 \theta = 0. \quad (2.146)$$

We next employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt} \left[\frac{1}{2} \dot{\theta}^2 + \int^{\theta(t)} F(\phi) d\phi \right] = (\ddot{\theta} + F(\theta)) \dot{\theta}.$$

For our problem, we multiply Equation (2.146) by $\dot{\theta}$,

$$\dot{\theta} \ddot{\theta} + \omega^2 \theta \dot{\theta} = 0$$

and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt} \left[\frac{1}{2} \dot{\theta}^2 - \omega^2 \cos \theta \right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2} \dot{\theta}^2 - \omega^2 \cos \theta = c. \quad (2.147)$$

Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \sqrt{2(c + \omega^2 \cos \theta)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$t = \int dt = \int \frac{d\theta}{\sqrt{2(c + \omega^2 \cos \theta)}}.$$

Of course, one needs to be able to do the integral. When one gets a solution in this implicit form, one says that the problem has been solved by quadratures. Namely, the solution is given in terms of some integral.

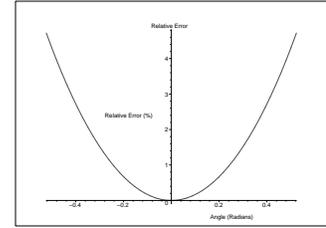


Figure 2.35: The relative error in percent when approximating $\sin \theta$ by θ .

In fact, the above integral can be transformed into what is known as an elliptic integral of the first kind. We will rewrite our result and then use it to obtain an approximation to the period of oscillation of the nonlinear pendulum, leading to corrections to the linear result found earlier.

We will first rewrite the constant found in (2.147). This requires a little physics. The swinging of a mass on a string, assuming no energy loss at the pivot point, is a conservative process. Namely, the total mechanical energy is conserved. Thus, the total of the kinetic and gravitational potential energies is a constant. The kinetic energy of the masses on the string is given as

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2.$$

The potential energy is the gravitational potential energy. If we set the potential energy to zero at the bottom of the swing, then the potential energy is $U = mgh$, where h is the height that the mass is from the bottom of the swing. A little trigonometry gives that $h = L(1 - \cos \theta)$. So,

$$U = mgL(1 - \cos \theta).$$

So, the total mechanical energy is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos \theta). \quad (2.148)$$

We note that a little rearranging shows that we can relate this to Equation (2.147):

$$\frac{1}{2}\dot{\theta}^2 - \omega^2 \cos \theta = \frac{1}{mL^2}E - \omega^2 = c.$$

We can use Equation (2.148) to get a value for the total energy. At the top of the swing the mass is not moving, if only for a moment. Thus, the kinetic energy is zero and the total energy is pure potential energy. Letting θ_0 denote the angle at the highest position, we have that

$$E = mgL(1 - \cos \theta_0) = mL^2\omega^2(1 - \cos \theta_0).$$

Therefore, we have found that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2 \cos \theta = \omega^2(1 - \cos \theta_0). \quad (2.149)$$

Using the half angle formula,

$$\sin^2 \frac{\theta}{2} = \frac{1}{2}(1 - \cos \theta),$$

we can rewrite Equation (2.149) as

$$\frac{1}{2}\dot{\theta}^2 = 2\omega^2 \left[\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right]. \quad (2.150)$$

Solving for θ' , we have

$$\frac{d\theta}{dt} = 2\omega \left[\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right]^{1/2}. \quad (2.151)$$

One can now apply separation of variables and obtain an integral similar to the solution we had obtained previously. Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, we have that

$$T = \frac{2}{\omega} \int_0^{\theta_0} \frac{d\phi}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}}. \quad (2.152)$$

This result is not much different than our previous result, but we can now easily transform the integral into an elliptic integral.³ We define

$$z = \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta_0}{2}}$$

and

$$k = \sin \frac{\theta_0}{2}.$$

Then Equation (2.152) becomes

$$T = \frac{4}{\omega} \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \quad (2.153)$$

This is done by noting that $dz = \frac{1}{2k} \cos \frac{\theta}{2} d\theta = \frac{1}{2k} (1-k^2z^2)^{1/2} d\theta$ and that $\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} = k^2(1-z^2)$. The integral in this result is an elliptic integral of the first kind. In particular, the elliptic integral of the first kind is defined as

$$F(\phi, k) \equiv \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \int_0^{\sin \phi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

In some contexts, this is known as the incomplete elliptic integral of the first kind and $K(k) = F(\frac{\pi}{2}, k)$ is called the complete integral of the first kind.

There are table of values for elliptic integrals and now one can use a computer algebra system to compute values of such integrals. For small angles, we have that k is small. So, we can develop a series expansion for the period, T , for small k . This is simply done by first expanding

$$(1 - k^2z^2)^{-1/2} = 1 + \frac{1}{2}k^2z^2 + \frac{3}{8}k^2z^4 + O((kz)^6)$$

using the binomial expansion which we review later in the text. Inserting the expansion in the integrand and integrating term by term, one finds that

$$T = 2\pi \sqrt{\frac{L}{g}} \left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots \right]. \quad (2.154)$$

³ Elliptic integrals were first studied by Leonhard Euler and Giulio Carlo de' Toschi di Fagnano (1682-1766), who studied the lengths of curves such as the ellipse and the lemniscate, $(x^2 + y^2)^2 = x^2 - y^2$.

This expression gives further corrections to the linear result, which only provides the first term. In Figure 2.36 we show the relative errors incurred when keeping the k^2 and k^4 terms versus not keeping them.

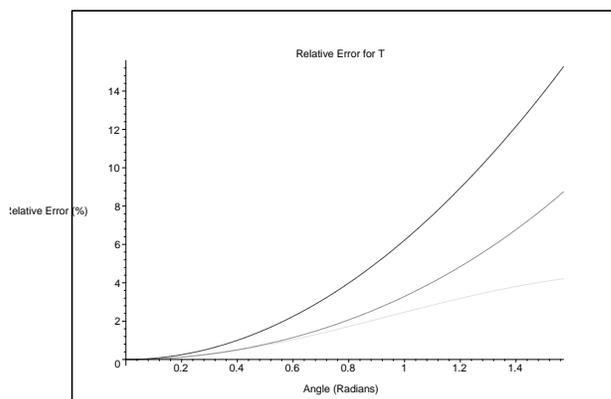


Figure 2.36: The relative error in percent when approximating the exact period of a nonlinear pendulum with one, two, or three terms in Equation (2.154).

Problems

1. Find all of the solutions of the first order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

- a. $\frac{dy}{dx} = \frac{e^x}{2y}$.
- b. $\frac{dy}{dt} = y^2(1 + t^2)$, $y(0) = 1$.
- c. $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{x}$.
- d. $xy' = y(1 - 2y)$, $y(1) = 2$.
- e. $y' - (\sin x)y = \sin x$.
- f. $xy' - 2y = x^2$, $y(1) = 1$.
- g. $\frac{ds}{dt} + 2s = st^2$, $s(0) = 1$.
- h. $x' - 2x = te^{2t}$.
- i. $\frac{dy}{dx} + y = \sin x$, $y(0) = 0$.
- j. $\frac{dy}{dx} - \frac{3}{x}y = x^3$, $y(1) = 4$.

2. Find all of the solutions of the second order differential equations. When an initial condition is given, find the particular solution satisfying that condition.

- a. $y'' - 9y' + 20y = 0$.

- b. $y'' - 3y' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$.
 c. $x^2y'' + 5xy' + 4y = 0$, $x > 0$.
 d. $x^2y'' - 2xy' + 3y = 0$, $x > 0$.

3. Consider the differential equation

$$\frac{dy}{dx} = \frac{x}{y} - \frac{x}{1+y}.$$

- a. Find the 1-parameter family of solutions (general solution) of this equation.
 b. Find the solution of this equation satisfying the initial condition $y(0) = 1$. Is this a member of the 1-parameter family?

4. The initial value problem

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}, \quad y(1) = 1$$

does not fall into the class of problems considered in our review. However, if one substitutes $y(x) = xz(x)$ into the differential equation, one obtains an equation for $z(x)$ which can be solved. Use this substitution to solve the initial value problem for $y(x)$.

5. Consider the nonhomogeneous differential equation $x'' - 3x' + 2x = 6e^{3t}$.

- a. Find the general solution of the homogenous equation.
 b. Find a particular solution using the Method of Undetermined Coefficients by guessing $x_p(t) = Ae^{3t}$.
 c. Use your answers in the previous parts to write down the general solution for this problem.

6. Find the general solution of the given equation by the method given.

- a. $y'' - 3y' + 2y = 10$. Method of Undetermined Coefficients.
 b. $y'' + y' = 3x^2$. Variation of Parameters.

7. Find the general solution of each differential equation. When an initial condition is given, find the particular solution satisfying that condition.

- a. $y'' - 3y' + 2y = 20e^{-2x}$, $y(0) = 0$, $y'(0) = 6$.
 b. $y'' + y = 2 \sin 3x$.
 c. $y'' + y = 1 + 2 \cos x$.
 d. $x^2y'' - 2xy' + 2y = 3x^2 - x$, $x > 0$.

8. Verify that the given function is a solution and use Reduction of Order to find a second linearly independent solution.

a. $x^2y'' - 2xy' - 4y = 0$, $y_1(x) = x^4$.

b. $xy'' - y' + 4x^3y = 0$, $y_1(x) = \sin(x^2)$.

9. A ball is thrown upward with an initial velocity of 49 m/s from 539 m high. How high does the ball get and how long does it take before it hits the ground? [Use results from first problem done in class, free fall, $y'' = -g$.]

10. Consider the solution of a simple growth and decay problem, $y(t) = y_0e^{kt}$, to solve this typical radioactive decay problem: Forty percent of a radioactive substance disappears in 100 years.

- What is the half-life of the substance?
- After how many years will 90% be gone?

11. A spring fixed at its upper end is stretched six inches by a 10-pound weight attached at its lower end. The spring-mass system is suspended in a viscous medium so that the system is subjected to a damping force of $5\frac{dx}{dt}$ lbs. Describe the motion of the system if the weight is drawn down an additional 4 inches and released. What would happen if you changed the coefficient "5" to "4"? [You may need to consult your introductory physics text.]

12. Consider an LRC circuit with $L = 1.00$ H, $R = 1.00 \times 10^2 \Omega$, $C = 1.00 \times 10^{-4}$ f, and $V = 1.00 \times 10^3$ V. Suppose that no charge is present and no current is flowing at time $t = 0$ when a battery of voltage V is inserted. Find the current and the charge on the capacitor as functions of time. Describe how the system behaves over time.

13. Consider the problem of forced oscillations as described in section 2.7.2.

- Derive the general solution in Equation (2.70).
- Use a CAS to plot the general solution in Equation (2.70) for the following cases:
- Derive the form in Equation (2.71).
- Use a CAS to plot the solution in Equation (2.71) for the following cases:

14. A certain model of the motion of a tossed whiffle ball is given by

$$mx'' + cx' + mg = 0, \quad x(0) = 0, \quad x'(0) = v_0.$$

Here m is the mass of the ball, $g=9.8$ m/s² is the acceleration due to gravity and c is a measure of the damping. Since there is no x term, we can write this as a first order equation for the velocity $v(t) = x'(t)$:

$$mv' + cv + mg = 0.$$

- Find the general solution for the velocity $v(t)$ of the linear first order differential equation above.
- Use the solution of part a to find the general solution for the position $x(t)$.
- Find an expression to determine how long it takes for the ball to reach its maximum height?
- Assume that $c/m = 10 \text{ s}^{-1}$. For $v_0 = 5, 10, 15, 20 \text{ m/s}$, plot the solution, $x(t)$, versus the time.
- From your plots and the expression in part c, determine the rise time. Do these answers agree?
- What can you say about the time it takes for the ball to fall as compared to the rise time?

15. Consider the system

$$\begin{aligned}x' &= -4x - y \\y' &= x - 2y.\end{aligned}$$

- Determine the second order differential equation satisfied by $x(t)$.
- Solve the differential equation for $x(t)$.
- Using this solution, find $y(t)$.
- Verify your solutions for $x(t)$ and $y(t)$.
- Find a particular solution to the system given the initial conditions $x(0) = 1$ and $y(0) = 0$.

16. Use the transformations relating polar and Cartesian coordinates to prove that

$$\frac{d\theta}{dt} = \frac{1}{r^2} \left[x \frac{dy}{dt} - y \frac{dx}{dt} \right].$$

17. Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.)

a.

$$\begin{aligned}x' &= 3x \\y' &= -2y.\end{aligned}$$

b.

$$\begin{aligned}x' &= -y \\y' &= -5x.\end{aligned}$$

c.

$$\begin{aligned}x' &= 2y \\y' &= -3x.\end{aligned}$$

d.

$$\begin{aligned}x' &= x - y \\y' &= y.\end{aligned}$$

e.

$$\begin{aligned}x' &= 2x + 3y \\y' &= -3x + 2y.\end{aligned}$$

18. In example 2.32 a conversion to polar coordinates lead to the equation $r' = r(1 - r^2)$. Solve this equation for initial values of $r(0) = 0, 0.5, 1.0, 2.0$. Based upon these solutions, describe the behavior of all solutions to the original system in Cartesian coordinates.