Chapter 9

Special Functions

In this chapter we will look at some additional functions which arise often in physical applications and are eigenfunctions for some Sturm-Liouville boundary value problem. We begin with a collection of special functions, called the classical orthogonal polynomials. These include such polynomial functions as the Legendre polynomials, the Hermite polynomials, the Tchebychef (other transliterations used: Chebyshev, Chebyshov, Tchebycheff or Tschebyscheff)and the Gegenbauer polynomials. Also, Bessel functions occur quite often. We will spend most of our time exploring the Legendre and Bessel functions. These functions are typically found as solutions of differential equations using power series methods in a first course in differential equations. We will leave these techniques for an appendix.

9.1 Classical Orthogonal Polynomials

We begin by noting that the sequence of functions $\{1, x, x^2, \ldots\}$ is a basis of linearly independent functions. In fact, by the Stone-Weierstrass Approximation Theorem in analysis this set is a basis of $L^2_\sigma(a,b)$, the space of square integrable functions over the interval $[a,b]$ relative to weight $\sigma(x)$. We are familiar with being able to expand functions over this basis, since

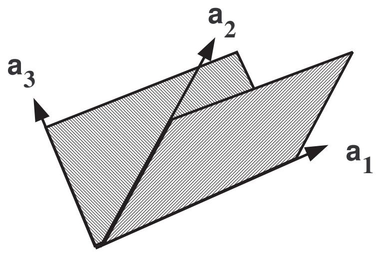

Figure 9.1: The basis $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, of $\mathbf{R}^3$ considered in the text.

the expansions are just Maclaurin series representations of the functions,

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n.$$

However, this basis is not an orthogonal set of basis functions. One can easily see this by integrating the product of two even, or two odd, basis functions with $\sigma(x) = 1$ and $(a,b)=(-1,1)$. For example,

$$\int_{-1}^{1} x^0 x^2 \, dx = \frac{2}{3}.$$

Since we have found that orthogonal bases have been useful in determining the coefficients for expansions of given functions, we might ask if it is possible to obtain an orthogonal basis involving these powers of x. Of course, finite combinations of these basis element are just polynomials!

OK, we will ask. "Given a set of linearly independent basis vectors, can one find an orthogonal basis of the given space?" The answer is yes. We recall from introductory linear algebra, which mostly covers finite dimensional vector spaces, that there is a method for carrying this out called the **Gram-Schmidt Orthogonalization Process.** We will recall this process for finite dimensional vectors and then generalize to function spaces.

Let's assume that we have three vectors that span $\mathbf{R}^3$, given by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ as shown in Figure 9.1. We seek an orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, beginning one vector at a time.

First we take one of the original basis vectors, say $\mathbf{a}_1$, and define

$$\mathbf{e}_1 = \mathbf{a}_1.$$

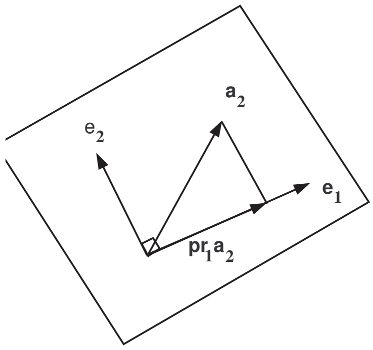

Figure 9.2: A plot of the vectors $\mathbf{e}_1$, $\mathbf{a}_2$, and $\mathbf{e}_2$ needed to find the projection of $\mathbf{a}_2$, on $\mathbf{e}_1$ illustrating the Gram-Schmidt orthogonalization process.

Of course, we might want to normalize our new basis vectors, so we would denote such a normalized vector with a 'hat':

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}_1}{e_1},$$

where $e_1 = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}$.

Next, we want to determine an $\mathbf{e}_2$ that is orthogonal to $\mathbf{e}_1$. We take the next element of the original basis, $\mathbf{a}_2$. In Figure 9.2 we see the orientation of the vectors. Note that the desired orthogonal vector is $\mathbf{e}_2$. $\mathbf{a}_2$ can be written as a sum of $\mathbf{e}_2$ and the projection of $\mathbf{a}_2$ on $\mathbf{e}_1$. Denoting this projection by $\mathbf{pr}_1\mathbf{a}_2$, we then have

$$\mathbf{e}_2 = \mathbf{a}_2 - \mathbf{pr}_1\mathbf{a}_2. \tag{9.1}$$

We recall from our vector calculus class the projection of one vector onto another,

$$\mathbf{pr}_1\mathbf{a}_2 = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{9.2}$$

Note that this is easily proven. First write the projection as a vector of length $a_2 \cos\theta$ in direction $\hat{\mathbf{e}}_1$, where θ is the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$,

$$\mathbf{pr}_1\mathbf{a}_2 = a_2 \cos\theta \frac{\mathbf{e}_1}{e_1}.$$

Recall that the angle between $\mathbf{e}_1$ and $\mathbf{a}_2$ is obtained from

$$\cos\theta = \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{a_2 e_1}.$$

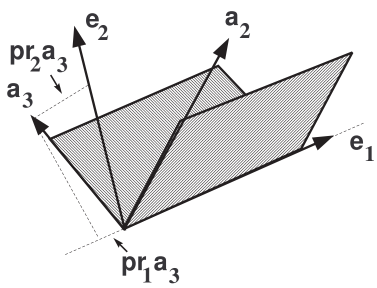

Figure 9.3: A plot of the vectors and their projections for determining $\mathbf{e}_3$.

Combining these expressions gives Equation (9.2).

From Equations (9.1)-(9.2), we find that

$$\mathbf{e}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1. \tag{9.3}$$

It is a simple matter to verify that $\mathbf{e}_2$ is orthogonal to $\mathbf{e}_1$:

$$\begin{aligned} \mathbf{e}_2 \cdot \mathbf{e}_1 &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 \cdot \mathbf{e}_1 \\ &= \mathbf{a}_2 \cdot \mathbf{e}_1 - \mathbf{a}_2 \cdot \mathbf{e}_1 = 0. \end{aligned} \tag{9.4}$$

Now, we seek a third vector $\mathbf{e}_3$ that is orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$. Pictorially, we can write the given vector $\mathbf{a}_3$ as a combination of vector projections along $\mathbf{e}_1$ and $\mathbf{e}_2$ and the new vector $\mathbf{e}_3$. This is shown in Figure 9.3. Then we have,

$$\mathbf{e}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_1}{e_1^2}\mathbf{e}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{e}_2}{e_2^2}\mathbf{e}_2. \tag{9.5}$$

Again, it is a simple matter to compute the scalar products with $\mathbf{e}_1$ and $\mathbf{e}_2$ to verify orthogonality.

We can generalize the procedure to the N-dimensional case. Let $\mathbf{a}_n$, $n = 1, ..., N$ be a set of linearly independent vectors in $\mathbf{R}^N$. Then, an orthogonal basis can be found by setting $\mathbf{e}_1 = \mathbf{a}_1$ and for $n > 1$,

$$\mathbf{e}_n = \mathbf{a}_n - \sum_{j=1}^{n-1} \frac{\mathbf{a}_n \cdot \mathbf{e}_j}{e_j^2}\mathbf{e}_j. \tag{9.6}$$

Now, we can generalize this idea to function spaces. Let $f_n(x)$, $n \in N$ and $x \in [a, b]$ be a linearly independent sequence of continuous functions. Then, an orthogonal basis of functions, $\phi_n(x)$, $n \in N$ can be found and is given by $\phi_0(x) = f_0(x)$ and

$$\phi_n(x) = f_n(x) - \sum_{j=0}^{n-1} \frac{< f_n, \phi_j >}{\|\phi_j\|^2} \phi_j(x), \quad n = 1, 2, \ldots . \tag{9.7}$$

Here we are using inner products of real valued functions relative to weight $\sigma(x)$,

$$< f, g >= \int_a^b f(x)g(x)\sigma(x)\, dx \tag{9.8}$$

and $||f||^2 =< f, f >$. Note the similarity between this expression and the expression for the finite dimensional case in Equation (9.6).

Example Apply the Gram-Schmidt Orthogonalization process to the set $f_n(x) = x^n$, $n \in N$, when $x \in (-1, 1)$ and $\sigma(x) = 1$.

First, we have $\phi_0(x) = f_0(x) = 1$. Note that

$$\int_{-1}^{1} \phi_0^2(x)\, dx = \frac{1}{2}.$$

We could use this result to fix the noramlization of our new basis, but we will hold off on doing that for now.

Now, we compute the second basis element:

$$\begin{aligned} \phi_1(x) &= f_1(x) - \frac{< f_1, \phi_0 >}{\|\phi_0\|^2} \phi_0(x) \\ &= x - \frac{< x, 1 >}{\|1\|^2} 1 = x, \end{aligned} \tag{9.9}$$

since $< x, 1 >$ is the integral of an odd function over a symmetric interval.

For $\phi_2(x)$, we have

$$\begin{aligned} \phi_2(x) &= f_2(x) - \frac{< f_2, \phi_0 >}{\|\phi_0\|^2} \phi_0(x) - \frac{< f_2, \phi_1 >}{\|\phi_1\|^2} \phi_1(x) \\ &= x^2 - \frac{< x^2, 1 >}{\|1\|^2} 1 - \frac{< x^2, x >}{\|x\|^2} x \end{aligned}$$

Polynomial	Symbol	Interval	$\sigma(x)$
Hermite	$H_n(x)$	$(-\infty, \infty)$	e^{-x^2}
Laguerre	$L_n^\alpha(x)$	$[0, \infty)$	e^{-x}
Legendre	$P_n(x)$	(-1,1)	1
Gegenbauer	$C_n^\lambda(x)$	(-1,1)	$(1-x^2)^{\lambda-1/2}$
Tchebychef of the 1st kind	$T_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Tchebychef of the 2nd kind	$U_n(x)$	(-1,1)	$(1-x^2)^{-1/2}$
Jacobi	$P_n^{(\nu,\mu)}(x)$	(-1,1)	$(1-x)^\nu(1-x)^\mu$

Table 9.1: Common classical orthogonal polynomials with the interval and weight function used to define them.

$$\begin{aligned} &= x^2 - \frac{\int_{-1}^1 x^2\,dx}{\int_{-1}^1 dx} \\ &= x^2 - \frac{1}{3}. \end{aligned} \tag{9.10}$$

So far, we have the orthogonal set $\{1, x, x^2 - \frac{1}{3}\}$. If one chooses to normalize these by forcing $\phi_n(1) = 1$, then one obtains the classical Legendre polynomials, $P_n(x) = \phi_n(x)$. This is not the typical normalization. Also, it might not be clear where the normalization constant is. The ϕ_n's can be multiplied by any constant and this will only affect the "length", $||\phi_n||^2$. Thus, we have so far that $\phi_2(x) = C(x^2 - \frac{1}{3})$. Setting $x = 1$, $\phi_2(1) = \frac{2}{3}$. Therefore, one obtains

$$P_2(x) = \phi_2(x) = \frac{1}{2}(3x^2 - 1).$$

The set of Legendre polynomials is just one set of classical orthogonal polynomials that can be obtained in this way. Many had originally appeared as solutions of important boundary value problems in physics. They all have similar properties and we will just elaborate some of these for the Legendre functions in the next section. Others in this group are shown in Table 9.1.

n	$(x^2-1)^n$	$\frac{d^n}{dx^n}(x^2-1)^n$	$\frac{1}{2^n n!}$	$P_n(x)$
0	1	1	1	1
1	x^2-1	$2x$	$\frac{1}{2}$	x
2	x^4-2x^2+1	$12x^2-4$	$\frac{1}{8}$	$\frac{1}{2}(3x^2-1)$
3	$x^6-3x^4+3x^2+1$	$12x^2-4$	$120x^3-72x$	$\frac{1}{2}(5x^3-3x)$

Table 9.2: Tabular computation of the Legendre polynomials using the Rodrigues formula.

9.2 Legendre Polynomials

In the last section we saw the Legendre polynomials in the context of orthogonal bases for a set of square integrable functions in $L^2(-1,1)$. In your first course in differential equations, you saw these polynomials as one of the solutions of the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n \in N. \tag{9.11}$$

Recall that these were obtained by using power series expansion methods. In this section we will explore a few of the properties of these functions.

First, there is the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \in N. \tag{9.12}$$

From this, one can see that $P_n(x)$ is an nth degree polynomial. Also, for n odd, the polynomial is an odd function and for n even, the polynomial is an even function.

One can systematically generate the Legendre polynomials in tabular form as shown in Table 9.2. Note that we get the same result as we found in the last section using orthogonalization. In Figure 9.4 we show a few Legnedre polynomials.

The classical orthogonal polynomials also satisfy three term recursion formulae. In the case of the Legendre polynomials, we have

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad n = 1, 2, \ldots.$$

This can also be rewritten by replacing n with $n-1$ as

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x), \quad n = 1, 2, \ldots.$$

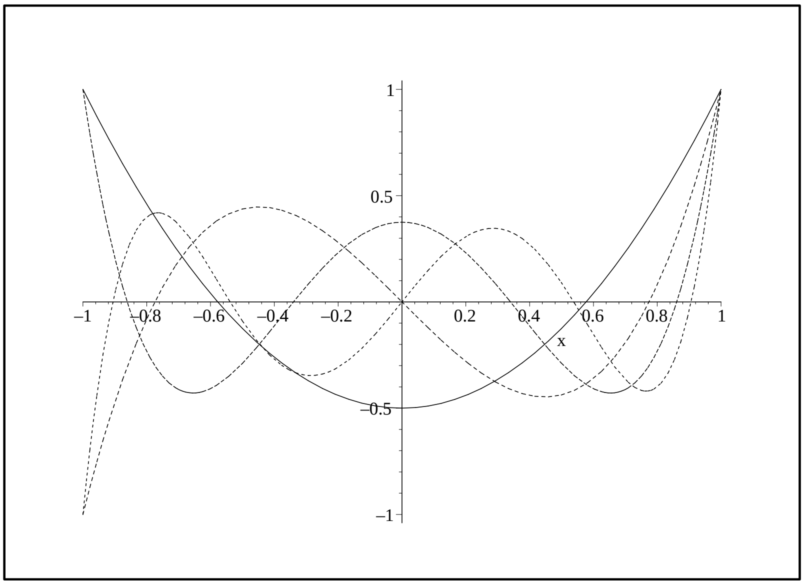

Figure 9.4: Plots of the Legendre polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

We will prove this two different ways.

First, we use the orthogonality properties of Legendre polynomials and the fact that the coefficient of x^n in $P_n(x)$ is $\frac{1}{2^n n!}\frac{(2n)!}{n!}$. This last fact can be obtained from Rodrigues formula. We see this by focussing on the leading coefficient of $(x^2-1)^n$, which is x^{2n}. The first derivative is $2nx^{2n-1}$. The second derivative is $2n(2n-1)x^{2n-2}$. The jth derivative is $[2n(2n-1)\dots(2n-j+1)]x^{2n-j}$. Thus, the nth derivative is $[2n(2n-1)\dots(n+1)]x^n$. This proves that $P_n(x)$ has degree n. The leading coefficient of $P_n(x)$ can now be written as

$$\begin{aligned}\frac{1}{2^n n!}[2n(2n-1)\dots(n+1)] &= \frac{1}{2^n n!}[2n(2n-1)\dots(n+1)]\frac{n(n-1)\dots1}{n(n-1)\dots1}\\ &= \frac{1}{2^n n!}\frac{(2n)!}{n!}. \qquad (9.13)\end{aligned}$$

In order to prove the three term recursion formula we consider the expression $nP_n(x)-(2n-1)xP_{n-1}(x)$. While each term is a polynomial of degree n, the leading order terms cancel. We first look at the coefficient of the leading order term in the second term. It is

$$(2n-1)\frac{1}{2^{n-1}(n-1)!}\frac{(2n-2)!}{(n-1)!}=\frac{1}{2^{n-1}(n-1)!}\frac{(2n-1)!}{(n-1)!}$$

The coefficient of the leading term for $nP_n(x)$ can be written as

$$n\frac{1}{2^n n!}\frac{(2n)!}{n!}=n[\frac{2n}{2n^2}]\frac{1}{2^{n-1}(n-1)!}\frac{(2n-1)!}{(n-1)!}$$

After some simple cancellations in the first factors, we see that the leading order terms cancel.

The next terms will be of degree $n-2$. This is because the P_n's are either even or odd functions, thus only containing even, or odd, powers of x. We conclude that

$$nP_n(x)-(2n-1)xP_{n-1}(x)=\text{ polynomial of degree } n-2.$$

Therefore, since the Legendre polynomials form a basis, we can write this polynomial as a linear combination of of Legendre polynomials:

$$nP_n(x)-(2n-1)xP_{n-1}(x)=c_0P_0(x)+c_1P_1(x)+\ldots+c_{n-2}P_{n-2}(x).$$

Multiplying by $P_m(x)$ for $m=0,1,\ldots,n-3$, and integrating from -1 to 1, we obtain

$$0=c_m\|P_m\|^2$$

using orthogonality. Thus, all of these c_m's are zero, leaving

$$nP_n(x)-(2n-1)xP_{n-1}(x)=c_{n-2}P_{n-2}(x).$$

The final coefficient can be found by using the normalization condition, $P_n(1)=1$. Thus, $c_{n-2}=n-(2n-1)=-(n-1)$.

A second proof of the three term recursion formula can be obtained from the *generating function* of the Legendre polynomials. Many special functions have such generating functions. For Legendre polynomials the generating function is given by

$$g(x,t)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n,\quad |x|<1,|t|<1. \tag{9.14}$$

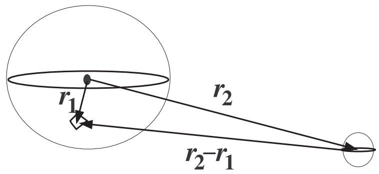

Figure 9.5: The position vectors used to describe the tidal force on the Earth due to the moon.

This generating function occurs often in applications. In particular, it arises in potential theory, such as electromagnetic or gravitational potentials. These potential functions are $\frac{1}{r}$ type functions. For example, the gravitational potential between the Earth and the moon is proportional to the reciprocal of the magnitude of the difference between their positions relative to some coordinate system.

An even better example, would be to place the origin at the center of the Earth and consider the forces on the non-pointlike Earth due to the moon. Consider a piece of the Earth at position $\mathbf{r}_1$ and the moon at position $\mathbf{r}_2$ as shown in Figure 9.5. The tidal potential Φ is given as

$$\Phi \propto \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{1}{\sqrt{(\mathbf{r}_2 - \mathbf{r}_1)\cdot(\mathbf{r}_2 - \mathbf{r}_1)}} = \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}},$$

where θ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Typically, one of the position vectors is larger than the other. In this case, we have $r_1 \ll r_2$. So, one can write

$$\Phi \propto \frac{1}{\sqrt{r_1^2 - 2r_1r_2\cos\theta + r_2^2}} = \frac{1}{r_2}\frac{1}{\sqrt{1 - 2\frac{r_1}{r_2}\cos\theta + \left(\frac{r_1}{r_2}\right)^2}}.$$

Now, define $x = \cos\theta$ and $t = \frac{r_1}{r_2}$. We then have the tidal potential is proportional to the generating function for the Legendre polynomials! So, we can write the tidal potential as

$$\Phi \propto \frac{1}{r_2}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r_1}{r_2}\right)^n.$$

The first term in the expansion will give the usual force between the Earth and the moon as point masses, or spheres. The next terms will give expressions for the tidal effects.

Now that we have some idea as to where this generating function might have originated, we now make some use of it. First of all, it can be used to provide values of the Legendre polynomials at key points. Thus, $P_n(0)$ is found by looking at $g(0,t)$,

$$g(0,t) = \frac{1}{\sqrt{1+t^2}} = \sum_{n=0}^{\infty} P_n(0)t^n. \tag{9.15}$$

However, we can use the binomial expansion to find our result. Namely, we have

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 + \dots.$$

Comparing these expansions, we have the $P_n(0) = 0$ for n odd and one can show that $P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$ for n even. [note that the double factorial is defined by $n!! = (n-2)(n-2)!!$. So, $5!! = 5(3)(1)$ and $6!! = 6(4)(2)$.]

A simpler evaluation is to find $P_n(-1)$. In this case we have

$$g(-1,t) = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots.$$

Therefore, $P_n(-1) = (-1)^n$.

We can also use the generating function to find recursion relations. To prove the three term recursion that we introduced above, we need only differentiate the generating function with respect to t in Equation (9.14) and rearrange the result. First note that

$$\frac{\partial g}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{1-2xt+t^2} g(x,t).$$

Combining this with

$$\frac{\partial g}{\partial t} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1},$$

we have

$$(x-t)g(x,t) = (1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Inserting the series expression for $g(x,t)$ and distributing the sum on the right side, we obtain

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}.$$

Rearranging leads to three separate sums:

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1} = 0. \quad (9.16)$$

Each term contains powers of t that we would like to combine into a single sum. This is done by reindexing. For the first sum, we could use the new index $k = n-1$. Then,

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = \sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k.$$

These different indices are just another way of writing out the terms. Note that

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

and

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k = 0 + P_1(x) + 2P_2(x)t + 3P_3(x)t^2 + \ldots$$

actually give the same sum. The indices are sometimes referred to dummy indices because they do not show up in the expanded expressions and can be replaced with another letter.

If we want to do so, we could now replace all of the k's with n's. The second sum in Equation (9.16) just needs the replacement $n = k$ and the last sum we reindex using $k = n+1$. Therefore, Equation (9.16) becomes

$$\sum_{k=-1}^{\infty}(k+1)P_{k+1}(x)t^k - \sum_{k=0}^{\infty}(2k+1)xP_k(x)t^k + \sum_{k=1}^{\infty}kP_{k-1}(x)t^k = 0. \quad (9.17)$$

We can now combine all of the terms, noting the $k=-1$ term is zero and the $k=0$ terms give

$$P_1(x) - xP_0(x) = 0. \quad (9.18)$$

Therefore, for $k > 0$,

$$\sum_{k=1}^{\infty}\left[(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x)\right]t^k = 0. \quad (9.19)$$

Since this is true for all t, the coefficients of the t^k's are zero, or

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0, \quad k = 1, 2, \ldots.$$

There are other recursion relations. For example,

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{9.20}$$

This can be proven using the generating function by differentiating $g(x,t)$ with respect to x and rearranging the resulting infinite series just as in this last manipulation.

Another use of the generating function is to obtain the normalization constant. Namely, $\|P_n\|^2$. Squaring the generating function, we have

$$\frac{1}{1-2xt+t^2} = \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \tag{9.21}$$

Integrating from -1 to 1 and using orthogonality, we have

$$\begin{aligned} \int_{-1}^{1} \frac{dx}{1-2xt+t^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_{-1}^{1} t^{n+m} P_n(x)P_m(x)\, dx \\ &= \sum_{n=0}^{\infty} \int_{-1}^{1} t^{2n} P_n^2(x)\, dx. \end{aligned} \tag{9.22}$$

However,

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right).$$

One can expand this expression about $t = 0$ to obtain

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n}.$$

Comparing this result with Equation (9.22), we find that

$$\|P_n\|^2 = \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}. \tag{9.23}$$

Finally, we can expand functions in this orthogonal basis. This is just a generalized Fourier series. A *Fourier-Legendre series* expansion for $f(x)$ on $(-1,1)$ takes the form

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x). \tag{9.24}$$

As with Fourier trigonometric series, we can determine the coefficients by multiplying both sides by $P_m(x)$ and integrating. Orthogonality give the usual form for the generalized Fourier coefficients. In this case, we have

$$c_n = \frac{< f, P_n >}{\|P_n\|^2}.$$

We have just found $\|P_n\|^2 = \frac{2}{2n+1}$. Therefore, the Fourier-Legendre coefficients are

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{9.25}$$

Example 1 Expand $f(x) = x^3$ in a Fourier-Legendre Series.

We need to compute

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} x^3 P_n(x)\, dx. \tag{9.26}$$

We first note that

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \quad \text{for } m < n.$$

This is proven using Rodrigues formula in Equation (9.12). We have

$$\int_{-1}^{1} x^m P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} x^m \frac{d^n}{dx^n} (x^2 - 1)^n\, dx.$$

For $m < n$, we integrate by parts m-times and use the facts that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. The right hand side vanishes. As a result, we will have that $c_n = 0$ for $n > 3$ in this example.

This leaves the computation of $c_0, c_1, c_2$and c_3. Since x^3 is an odd function and P_0 and P_2 are even functions, $c_0 = 0$ and $c_2 = 0$. This leaves us with only two coefficients to compute. These are

$$c_1 = \frac{3}{2} \int_{-1}^{1} x^4\, dx = \frac{3}{5}$$

$$c_3 = \frac{7}{2} \int_{-1}^{1} x^3 \left[\frac{1}{2}(5x^3 - 3x)\right] dx = \frac{2}{5}.$$

Thus,

$$x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x).$$

Of course, this is simple to check:

$$\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x) = \frac{3}{5}x + \frac{2}{5}\left[\frac{1}{2}(5x^3 - 3x)\right] = x^3.$$

Well, maybe we could have guessed this without doing any integration. Let's see. Since $f(x) = x^3$ has degree three, we do not expect the expansion in Legendre polynomials for $f(x)$ to have polynomials of order greater than three. So, we assume from the beginning that

$$f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x).$$

Then,

$$x^3 = c_0 + c_1x + \frac{1}{2}c_2(3x^2 - 1) + \frac{1}{2}c_2(5x^3 - 3x).$$

However, there are no quadratic terms on the left side, so $c_2 = 0$. Then, there are no constant terms left except c_0. So, $c_0 = 0$. This leaves

$$\begin{aligned} x^3 &= c_1x + \frac{1}{2}c_2(5x^3 - 3x) \\ &= (c_1 - \frac{3}{2}c_2)x + \frac{5}{2}c_2x^3. \end{aligned} \tag{9.27}$$

Equating coefficients of the remaining like terms, we have that $c_2 = \frac{2}{5}$ and $c_1 = \frac{3}{2}c_2 = \frac{3}{5}$.

Example 2 Expand the Heaviside function in a Fourier-Legendre Series.

In this case, we cannot find the expansion coefficients without some integration. We have to compute

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{2n+1}{2}\int_0^1 P_n(x)\,dx. \tag{9.28}$$

We can make use of the formula (9.20) for $n > 1$. Then,

$$c_n = \frac{1}{2}\int_0^1 [P'_{n+1}(x) - P'_{n-1}(x)]\,dx = \frac{1}{2}[P_{n-1}(0) - P_{n+1}(0)].$$

For $n = 0$, we have

$$c_0 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

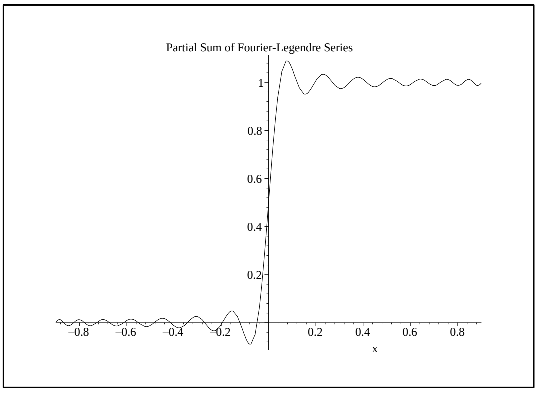

Figure 9.6: Sum of first 21 terms for Fourier-Legendre series expansion of Heaviside function.

Then we have the expansion

$$f(x) \sim \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} [P_{n-1}(0) - P_{n+1}(0)] P_n(x),$$

which can be written as

$$\begin{aligned}
f(x) &\sim \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} [P_{2n-2}(0) - P_{2n}(0)] P_{2n-1}(x) \\
&= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left[(-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!!} - (-1)^n \frac{(2n-1)!!}{(2n)!!} \right] P_{2n-1}(x) \\
&= \frac{1}{2} - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n-2)!!} \left[1 + \frac{2n-1}{2n} \right] P_{2n-1}(x) \\
&= \frac{1}{2} - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n-2)!!} \frac{4n-1}{2n} P_{2n-1}(x). \qquad (9.29)
\end{aligned}$$

The sum of the first 21 terms are shown in Figure 9.6.

9.3 Spherical Harmonics

9.4 Gamma Function

Another function that often occurs in the study of special functions is the Gamma function. We will see that the Gamma function is the natural generalization of the factorial function. Also, we will need the Gamma function in the next section on Bessel functions.

For $x > 0$ we define the Gamma function as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0. \tag{9.30}$$

We first show that the Gamma function generalizes the factorial function. In fact, we have

$$\Gamma(1) = 1$$

and

$$\Gamma(x+1) = x\Gamma(x).$$

The reader can prove the second equation by simply performing an integration by parts. For n an integer, we can iterate the second expression to obtain

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = n(n-1)\cdots 2\Gamma(1) = n!.$$

This can also be written as

$$\Gamma(n) = (n-1)!.$$

We can also define the Gamma function for negative, non-integer values of x. We first note that by iteration on $n \in Z^+$, we have

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\Gamma(x), \quad x < 0, x+n > 0.$$

Solving for $\Gamma(x)$, we then find

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)\cdots(x+1)x}, \quad x < 0, x+n > 0.$$

Note that the Gamma function is undefined at zero and the negative integers.

Another useful formula is
$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$
It is simply found as
$$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt.$$
Letting $t = z^2$, we have
$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-z^2}\, dz.$$

Due to the symmetry of the integrand, we obtain the classic integral
$$\Gamma(\frac{1}{2}) = \int_{-\infty}^\infty e^{-z^2}\, dz,$$
which we had integrated when we computed the Fourier transform of a Gaussian. Recall that
$$\int_{-\infty}^\infty e^{-z^2}\, dz = \sqrt{\pi}.$$
Therefore, we have confirmed that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

We have seen that the factorial function can be written in terms of Gamma functions. One can also relate the odd double factorials in terms of the Gamma function. First, we note that
$$(2n)!! = 2^n n!, \quad (2n+1)!! = \frac{(2n+1)!}{2^n n!}.$$
In particular, one can prove
$$\Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}.$$
Formally, this gives
$$\left(\frac{1}{2}\right)! = \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}.$$

Another useful relation is
$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$
This result can be proven using complex variable methods. However, one needs to be able to integrate around branch cuts, which we have not covered in this book.

9.5 Bessel Functions

Another important differential equation that arises in many physics applications is

$$x^2y'' + xy' + (x^2 - p^2)y = 0. \tag{9.31}$$

This equation is readily put into self-adjoint form as

$$(xy')' + (x - \frac{p^2}{x})y = 0. \tag{9.32}$$

This equation was solved in the first course on differential equations using power series methods, namely by using the Frobenius Method. One assumes a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s},$$

where one seeks allowed values of the constant s and a recursion relation for the coefficients, a_n. One finds that $s = \pm p$ and

$$a_n = -\frac{a_{n-2}}{(n+s)^2 - p^2}, \quad n \geq 2.$$

One solution is the Bessel function of the first kind of order p, given as

$$y(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}. \tag{9.33}$$

In Figure 9.7 we display the first few Bessel functions of the first kind of integer order. Note that these functions can be described as decaying oscillatory functions.

A second linearly independent solution is obtained for p not an integer as $J_{-p}(x)$. However, for p an integer, the $\Gamma(n+p+1)$ factor leads to evaluations of the Gamma function at zero, or negative integers, when p is negative. Thus, the above series is not defined in these cases.

Another method for obtaining a second linearly independent solution is through a linear combination of $J_p(x)$ and $J_{-p}(x)$ as

$$N_p(x) = \frac{\cos \pi p J_p(x) - J_{-p}(x)}{\sin \pi p}. \tag{9.34}$$

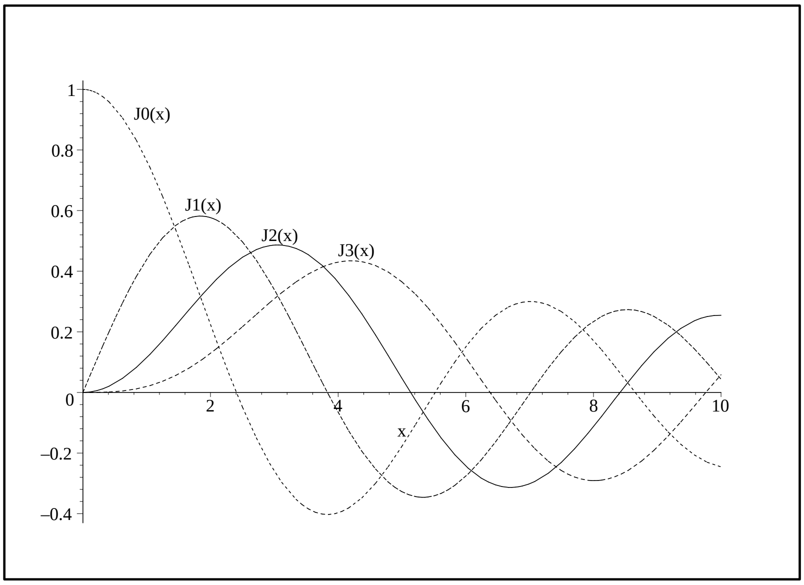

Figure 9.7: Plots of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$.

These functions are called the Neumann functions, or Bessel functions of the second kind of order p.

In Figure 9.8 we display the first few Bessel functions of the second kind of integer order. Note that these functions are also decaying oscillatory functions. However, they are singular at $x = 0$. In many applications these functions do not satisfy the boudary condition that one desires a bounded solution at $x = 0$. For example, one standard problem is to describe the oscillations of a circular drumhead. In this case the Bessel functions describe the radial part of the solution and one does not expect a singular solution at the center of the drum.

Bessel functions satisfy a variety of properties, which we will only list at this time for Bessel functions of the first kind.

Derivative Identities

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x). \tag{9.35}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = x^{-p} J_{p+1}(x). \tag{9.36}$$

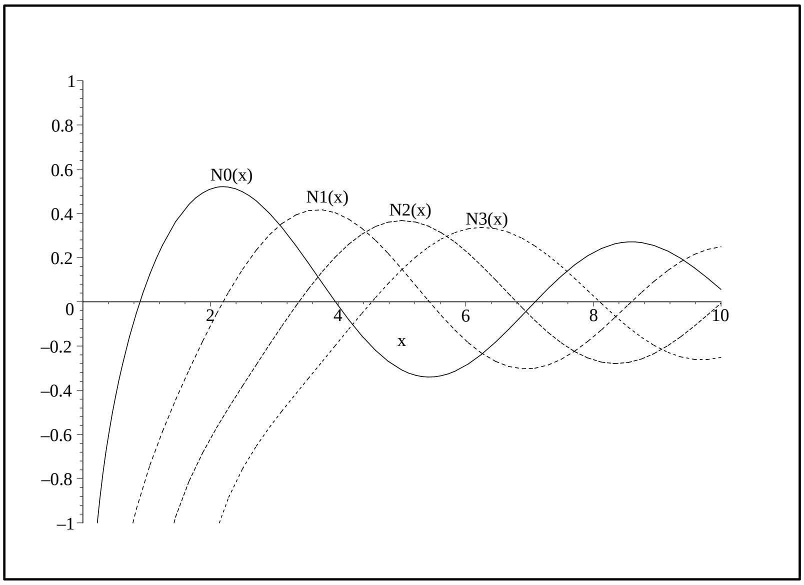

Figure 9.8: Plots of the Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$, and $N_3(x)$.

Recursion Formulae

$$J_{p-1}(x) - J_{p+1}(x) = \frac{2p}{x} J_{p-1}(x). \tag{9.37}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x). \tag{9.38}$$

Orthogonality

$$\int_0^a xJ(j_{pn}x)J_p(j_{pm}x)\,dx = \frac{a^2}{2}\left[J_{p+1}(j_{pn}a)\right]^2 \delta_{n,m} \tag{9.39}$$

where j_{pn} is the nth root of $J_p(x)$, $J_p(j_{pn}) = 0$. A list of some of these roots are provided in Table 9.3.

Generating Function

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \quad x > 0, t \neq 0. \tag{9.40}$$

Integral Representation

p	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
1	2.405	3.832	5.135	6.379	7.586	8.780
2	5.520	7.016	8.147	9.760	11.064	12.339
3	8.654	10.173	11.620	13.017	14.373	15.700
4	11.792	13.323	14.796	16.224	17.616	18.982
5	14.931	16.470	17.960	19.410	20.827	22.220
6	18.071	19.616	21.117	22.583	24.018	25.431
7	21.212	22.760	24.270	25.749	27.200	28.628
8	24.353	25.903	27.421	28.909	30.371	31.813
9	27.494	29.047	30.571	32.050	33.512	34.983

Table 9.3: The zeros of Bessel Functions

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - n\theta)\,d\theta, \quad x > 0, n \in \mathrm{Z}. \tag{9.41}$$

Fourier-Bessel Series

Since the Bessel functions are an orthogonal set of eigenfunctions of a Sturm-Liouville problem, we can expand square integrable functions in this basis. In fact, the eigenvalue problem is given in the form

$$x^2y'' + xy' + (\lambda x^2 - p^2)y = 0. \tag{9.42}$$

The solutions are then of the form $J_p(\lambda x)$, as can be shown by making the substitution $t = \lambda x$ in the differential equation. Furthermore, if $0 < x < a$, and one solves the differential equation with boundary conditions that $y(x)$ is bounded at $x = 0$ and $y(a) = 0$, then one can show that

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn}x) \tag{9.43}$$

where the Fourier-Bessel coefficients are found using the orthogonality relation as

$$c_n = \frac{2}{a^2\left[J_{p+1}(j_{pn}a)\right]^2}\int_0^a xf(x)J_p(j_{pn}x)\,dx. \tag{9.44}$$

Example Expand $f(x) = 1$ for $0 < x < 1$ in a Fourier-Bessel series of the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_0(j_{pn}x)$$

.

We need only compute the Fourier-Bessel coefficients in Equation (9.45):

$$c_n = \frac{2}{[J_1(j_{0n})]^2} \int_0^1 x J_0(j_{0n}x)\, dx. \tag{9.45}$$

From Equation (9.35) we have

$$\begin{aligned}
\int_0^1 x J_0(j_{0n}x)\, dx &= \frac{1}{j_{0n}^2} \int_0^{j_{0n}} y J_0(y)\, dy \\
&= \frac{1}{j_{0n}^2} \int_0^{j_{0n}} \frac{d}{dy} \left[y J_1(y) \right] dy \\
&= \frac{1}{j_{0n}^2} \left[y J_1(y) \right]_0^{j_{0n}} \\
&= \frac{1}{j_{0n}} J_1(j_{0n}).
\end{aligned} \tag{9.46}$$

As a result, we have found that the desired Fourier-Bessel expansion is

$$1 = 2 \sum_{n=1}^{\infty} \frac{J_0(j_{0n}x)}{j_{0n} J_1(j_{0n})}, \quad 0 < x < 1. \tag{9.47}$$

In Figure 9.9 we have show the partial sum for the first fifty terms of this series. We note the slow convergence due to the Gibbs phenomenon.

Note: For reference, this was done in Maple using the following code:

```
2*sum(BesselJ(0,BesselJZeros(0,n)*x)
/(BesselJZeros(0,n)*BesselJ(1,BesselJZeros(0,n))),n=1..50)
```

9.6 Hypergeometric Functions

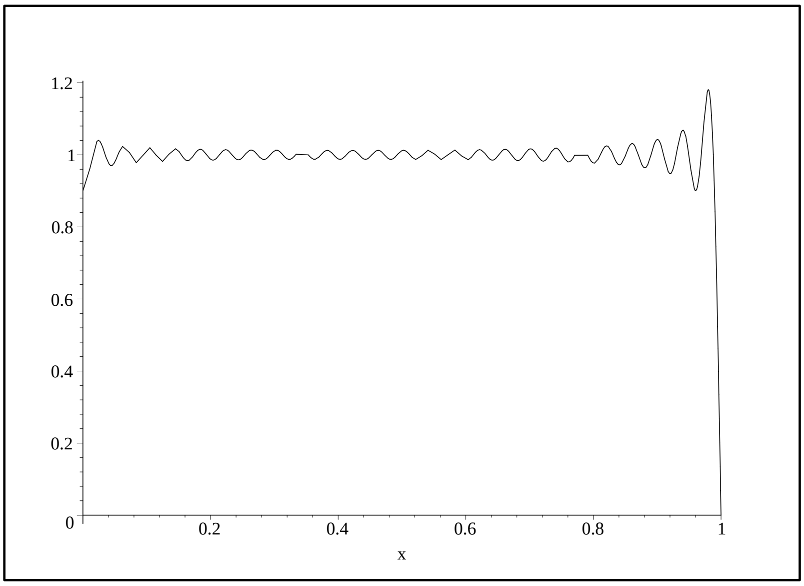

Figure 9.9: Plot of the first 50 terms of the Fourier-Bessel series in Equation (9.47) for $f(x) = 1$ on $0 < x < 1$.