Chapter 8

Problems in Higher Dimensions

In this chapter we will explore several generic examples of the solution of initial-boundary value problems involving higher spatial dimensions. These are described by higher dimensional partial differential equations. The method of solution will be the method of separation of variables. This will result in a system of ordinary differential equations for each problem.

Of these equations, several are boundary value problems, which are eigenvalue problems. We solve the eigenvalue problems for the eigenvalues and eigenfunctions, leading to a set of product solutions satisfying the partial differential equation and the boundary conditions. The general solution can be written as a linear superposition of the product solutions.

As you go through these examples, you will see some common features. For example, the main equation that we have seen are the heat equation and the wave equation. For higher dimensional problems these take the form

$$\begin{aligned} u_t &= k\nabla^2 u, && (8.1) \\ u_{tt} &= c^2\nabla^2 u. && (8.2) \end{aligned}$$

One can first separate out the time dependence. Let $u(\mathbf{r},t)=\phi(\mathbf{r})T(t)$. For these two equations we have

$$\begin{aligned} T'\phi &= kT\nabla^2\phi, && (8.3) \\ T''\phi &= c^2T\nabla^2\phi. && (8.4) \end{aligned}$$

Separating out the time and space dependence, we find

$$\frac{1}{k}\frac{T'}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda, \tag{8.5}$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda. \tag{8.6}$$

Note that in each case we have a function of time equals a function of spatial variables. Thus, they must be a constant, $-\lambda < 0$. The sign of λ is chosen because we expect decaying solutions in time for the heat equation and oscillations in time for the wave equation.

This leads to the respective set of equations for $T(t)$:

$$T' = -\lambda k T, \tag{8.7}$$

$$T'' + c^2\lambda T = 0. \tag{8.8}$$

These are easily solved. We have

$$T(t) = T(0)e^{-\lambda k t}, \tag{8.9}$$

$$T(t) = a\cos\omega t + b\sin\omega t \quad \omega = c\sqrt{\lambda}. \tag{8.10}$$

In both cases the spatial equation becomes

$$\nabla^2\phi + \lambda\phi = 0. \tag{8.11}$$

This is called the Helmholtz equation.

For one dimensional problems, which we have already solved, the Helmholtz equation takes the form $\phi'' + \lambda\phi = 0$. We had to impose the boundary conditions and found that there were a discrete set of eigenvalues, λ_n, and associated eigenfunctions, ϕ_n.

In higher dimensional problems we need to further separate out the space dependence. We will again use the boundary conditions and find the eigenvalues and eigenfunctions, though they will be labelled with more than one index. The resulting boundary value problems belong to a class of eigenvalue problems called Sturm-Liouville problems. We will explore the properties of Sturm-Liouville eigenvalue problems later in the chapter.

Also, in the solution of the ordinary differential equations we will find solutions other than the sines and cosines that we have seen in previous problems, such as the vibrations of a string. The special functions also form a set of orthogonal functions, leading to general Fourier-type series expansions. A more general discussion of these special functions is provided in the next chapter. We will refer to sections of that chapter for properties of the special functions that are encountered in this chapter.

8.1 Vibrations of Rectangular Membranes

Our first example will be the study of the vibrations of a rectangular membrane. You can think of this as a drum with a rectangular cross section as shown in Figure 8.1. We stretch the membrane over the drumhead and fasten the material to the boundaries of the rectangle. The hight of the vibrating membrane is described by its hight from equilibrium, $u(x, y, t)$. This problem is a much simpler example of higher dimensional vibrations than that possed by the oscillating electric and magnetic fields in the last chapter.

The problem is given by a partial differential equation,

$$u_{tt} = c^2 \nabla^2 u = c^2 (u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H. \tag{8.12}$$

a set of boundary conditions,

$$\begin{aligned} &u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad t > 0, \quad 0 < y < H, \\ &u(x, 0, t) = 0, \quad u(x, H, t) = 0, \quad t > 0, \quad 0 < x < L, \end{aligned} \tag{8.13}$$

and a pair of initial conditions (since the equation is second order in time),

$$u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = g(x, y). \tag{8.14}$$

The first step is to separate the variables: $u(x, y, t) = X(x)Y(y)T(t)$. Inserting into the wave equation, we have

$$X(x)Y(y)T''(t) = c^2 \left(X''(x)Y(y)T(t) + X(x)Y''(y)T(t) \right).$$

Dividing by both $u(x, y, t)$ and c^2, we obtain

$$\underbrace{\frac{1}{c^2} \frac{T''}{T}}_{\text{Function of } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{Function of } x \text{ and } y} = -\lambda. \tag{8.15}$$

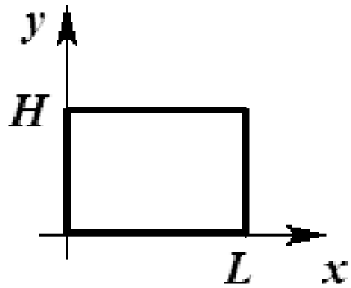

Figure 8.1: The rectangular membrane of length L and width H. There are fixed boundary conditions along the edges.

We see that we have a function of t equals a function of x and y. Thus, both expressions are constant. We expect oscillations in time, so we chose the constant to be $\lambda > 0$. (Note: As usual, the primes mean differentiation with respect to the specific dependent variable. So, there should be no ambiguity.)

These lead to two equations:

$$T'' + c^2\lambda T = 0, \tag{8.16}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda. \tag{8.17}$$

The first equation is easily solved. We have

$$T(t) = a\cos\omega t + b\sin\omega t, \tag{8.18}$$

where

$$\omega = c\sqrt{\lambda}. \tag{8.19}$$

This is the angular frequency in terms of the separation constant, or eigenvalue. It leads to the frequency of oscillations for the various harmonics of the vibrating membrane as

$$\nu = \frac{\omega}{2\pi} = \frac{c}{2\pi}\sqrt{\lambda}. \tag{8.20}$$

Once we know we can compute these frequencies.

Now we solve the spatial equation. Again, we need to do a separation of variables. Rearranging the spatial equation, we have

$$\underbrace{\frac{X''}{X}}_{\text{Function of } x} = \underbrace{-\frac{Y''}{Y} - \lambda}_{\text{Function of } y} = -\mu. \tag{8.21}$$

Here we have a function of x equals a function of y. So, the two expressions are constant, which we indicate with a second separation constant, $-\mu < 0$. We pick the sign in this way because we expect oscillatory solutions for $X(x)$. This leads to two equations:

$$\begin{aligned} X'' + \mu X &= 0, \\ Y'' + (\lambda - \mu)Y &= 0. \end{aligned} \tag{8.22}$$

We now need to use the boundary conditions. We have $u(0, y, t) = 0$ for all $t > 0$ and $0 < y < H$. This implies that $X(0)Y(y)T(t) = 0$ for all t and y in the domain. This is only true if $X(0) = 0$. Similarly, we find that $X(L) = 0$, $Y(0) = 0$, and $Y(H) = 0$. We note that homogeneous boundary conditions are important in carrying out this process. Nonhomogeneous boundary conditions could be imposed, but the techniques are a bit more complicated and we will not discuss these techniques here.

The boundary values problems we need to solve are:

$$\begin{aligned} X'' + \mu X = 0, &\quad X(0) = 0, X(L) = 0. \\ Y'' + (\lambda - \mu)Y = 0, &\quad Y(0) = 0, Y(H) = 0. \end{aligned} \tag{8.23}$$

We have seen the first of these problems before, except with a λ instead of a μ. The solution is

$$X(x) = \sin\frac{n\pi x}{L}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The second equation is solved in the same way. The differences are that our "eigenvalue" is $\lambda - \mu$, the independent variable is y, and the interval is $[0, H]$. Thus, we can quickly write down the solution as λ instead of a μ. The solution is

$$Y(y) = \sin\frac{m\pi x}{H}, \quad \lambda - \mu = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, 3, \ldots.$$

We have successfully carried out the separation of variables for the wave equation for the vibrating rectangular membrane. The product solutions can be written as

$$u_{nm} = (a\cos\omega_{nm}t + b\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{8.24}$$

Recall that ω is given in terms of λ. We have that

$$\lambda_{mn} - \mu_n = \left(\frac{m\pi}{H}\right)^2$$

and

$$\mu_n = \left(\frac{n\pi}{L}\right)^2.$$

Therefore,

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2. \tag{8.25}$$

So,

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{8.26}$$

The most general solution can now be written as a linear combination of the product solutions and we can solve for the expansion coefficients that will lead to a solution satisfying he initial conditions. However, we will first concentrate on the two dimensional harmonics of this membrane.

For the vibrating string the nth harmonic corresponded to the function $\sin\frac{n\pi x}{L}$. The various harmonics corresponded to the pure tones supported by the string. These then lead to the corresponding frequencies that one would hear. The actual shapes of the harmonics could be sketched by locating the nodes, or places on the string that did not move.

In the same way, we can explore the shapes of the harmonics of the vibrating membrane. These are given by the spatial functions

$$\phi_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{8.27}$$

Instead of nodes, we will look for the *nodal curves*, or *nodal lines*. These are the points (x, y) at which $\phi(x, y) = 0$. Of course, these depend on the indices, n and m.

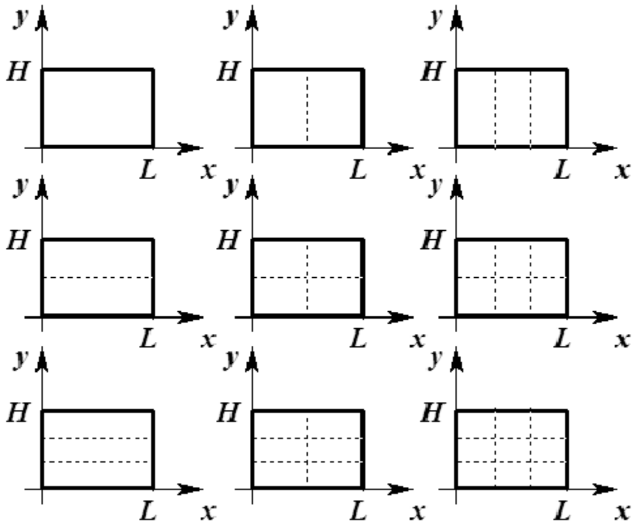

Figure 8.2: The first few modes of the vibrating rectangular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode.

For example, when $n = 1$ and $m = 1$, we have

$$\sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} = 0.$$

These are zero when either

$$\sin \frac{n\pi x}{L} = 0, \quad \text{or} \quad \sin \frac{m\pi y}{H} = 0.$$

Of course, this can only happen for $x = 0, L$ and $y = 0, H$. Thus, there are no interior nodal lines.

When $n = 2$ and $m = 1$, we have $y = 0, H$ and

$$\sin \frac{2\pi x}{L} = 0,$$

or, $x = 0, \frac{L}{2}, L$. Thus, there is one interior nodal line at $x = \frac{L}{2}$. These points stay fixed during the oscillation and all other points oscillate on either side of this line. A similar solution shape results for the (1,2)-mode; i.e., $n = 1$ and $m = 2$.

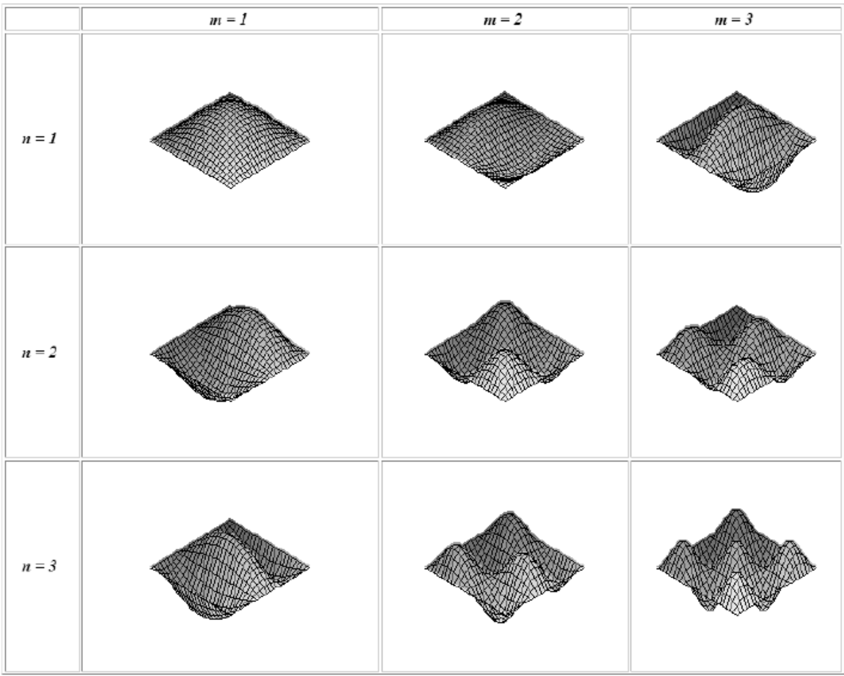

Figure 8.3: A three dimensional view of the vibrating rectangular membrane for the lowest modes.

In Figure 8.2 we show the nodal lines for several modes for $n, m = 1, 2, 3$ The blocked regions appear to vibrate independently. A better view is the three dimensional view depicted in Figure 8.3 . The frequencies of vibration are easily computed using the formula for ω_{nm}.

For completeness, we now see how one satisfies the initial conditions. The general solution is given by a linear superposition of the product solutions. There are two indices to sum over. Thus, the general solution is

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}.) \quad (8.28)$$

The first initial condition is $u(x, y, 0) = f(x, y)$. Setting $t = 0$ in the general solution, we obtain

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \quad (8.29)$$

This is a double sine series. The goal is to find the unknown coefficients a_{nm}. This can be done knowing what we already know about Fourier sine series. We can write this as the single sum

$$f(x, y) = \sum_{n=1}^{\infty} A_n(y) \sin \frac{n\pi x}{L}, \quad (8.30)$$

where

$$A_n(y) = \sum_{m=1}^{\infty} a_{nm} \sin \frac{m\pi y}{H}. \quad (8.31)$$

These are two sine series. Recalling that the coefficients of sine series can be computed as integrals, we have

$$\begin{aligned} A_n(y) &= \frac{2}{L} \int_0^L f(x, y) \sin \frac{n\pi x}{L}\, dx, \\ a_{nm} &= \frac{2}{H} \int_0^H A_n(y) \sin \frac{m\pi y}{H}\, dy. \end{aligned} \quad (8.32)$$

Inserting the first result into the second,

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x, y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dx dy. \quad (8.33)$$

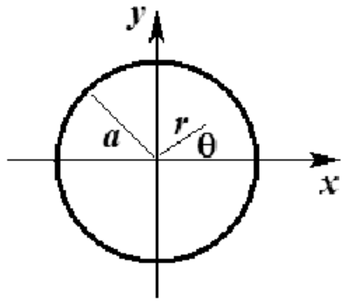

Figure 8.4: The circular membrane of radius a. A general point on the membrane is given by the distance from the center, r, and the angle, . There are fixed boundary conditions along the edge at $r = a$.

We can carry out the same process for satisfying the second initial condition, $u_t(x, y, 0) = g(x, y)$ for the initial velocity of each point. Inserting this into the general solution, we have

$$g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\omega_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{8.34}$$

Again, we have a double sine series. But, now we can write down Fourier coefficients quickly:

$$b_{nm} = \frac{4}{\omega_{nm}LH}\int_0^H\int_0^L g(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dxdy. \tag{8.35}$$

This completes the full solution of the vibrating rectangular membrane problem.

8.2 Vibrations of a Kettle Drum

In this section we consider the vibrations of a circular membrane of radius a as shown in Figure 8.4. Again we are looking for the harmonics of the vibrating membrane, but with the membrane fixed around the circular boundary given by $x^2 + y^2 = a^2$. However, expressing the boundary condition in Cartesian coordinates would be awkward. It would be more natural to use polar coordinates as indicated in the figure. In this case the boundary condition would be $u = 0$ at $r = a$.

Before solving the initial-boundary value problem, we have to cast it in polar coordinates. This means that we need to rewrite the Laplacian in r and θ. To do so would require that we know how to transform derives in x and y into derivatives with respect to r and θ.

First recall that transformations

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}.$$

Now, consider a function $f = f(x(r,\theta), y(r,\theta)) = g(r,\theta)$. (Technically, once we transform a given function of Cartesian coordinates we obtain a new function g of the polar coordinates. Many texts do not rigourously distinguish between the two which is fine when this point is clear.) Thinking of $x = x(r,\theta)$ and $y = y(r,\theta)$, we have from the chain rule for functions of two variables:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \frac{\partial g}{\partial r}\frac{x}{r} - \frac{\partial g}{\partial \theta}\frac{y}{r^2} \\ &= \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}. \end{aligned} \tag{8.36}$$

Here we have used

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r};$$

and

$$\frac{\partial \theta}{\partial x} = \frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right) = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{r^2}.$$

Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \frac{\partial g}{\partial r}\frac{y}{r} + \frac{\partial g}{\partial \theta}\frac{x}{r^2} \\ &= \sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}. \end{aligned} \tag{8.37}$$

The 2D Laplacian can now be computed as

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right)+\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)\\
&= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial g}{\partial r}+\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial r}+\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&= \cos\theta\left(\cos\theta\frac{\partial^2 g}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial g}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}\right)\\
&\quad-\frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 g}{\partial\theta\partial r}-\frac{\sin\theta}{r}\frac{\partial^2 g}{\partial\theta^2}-\sin\theta\frac{\partial g}{\partial r}-\frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\sin\theta\left(\sin\theta\frac{\partial^2 g}{\partial r^2}+\frac{\cos\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}-\frac{\cos\theta}{r^2}\frac{\partial g}{\partial\theta}\right)\\
&\quad+\frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 g}{\partial\theta\partial r}+\frac{\cos\theta}{r}\frac{\partial^2 g}{\partial\theta^2}+\cos\theta\frac{\partial g}{\partial r}-\frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}\right)\\
&= \frac{\partial^2 g}{\partial r^2}+\frac{1}{r}\frac{\partial g}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}. \qquad (8.38)
\end{aligned}$$

The last form often occurs in texts because it is in the form of a Sturm-Liouville operator.

Now that we have written the Laplacian in polar coordinates we can pose the problem of a vibrating circular membrane. It is given by a partial differential equation,

$$u_{tt}=c^2\nabla^2 u=c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right], \qquad (8.39)$$

$$t>0,\quad 0<r<a,\quad -\pi<\theta<\pi,$$

the boundary condition,

$$u(a,\theta,t)=0,\quad t>0,\quad -\pi<\ <\pi, \qquad (8.40)$$

and the initial conditions,

$$u(r,\theta,0)=f(r,\theta),\quad u_t(r,\theta,0)=g(r,\theta). \qquad (8.41)$$

Now were are ready to solve this problem using separation of variables. As before, we can separate out the time dependence. Let $u(r,\theta,t) = T(t)\phi(r,\theta)$. As usual, $T(t)$ can be written in terms of sines and cosines. This leaves the Helmhotz equation,

$$\nabla^2\phi + \lambda\phi = 0.$$

We can separate this equation by letting $\phi(r,\theta) = R(r)\Theta(\theta)$. This gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 R\Theta}{\partial\theta^2} + \lambda R\Theta = 0. \tag{8.42}$$

Dividing by $u = R\Theta$, as usual, leads to

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \lambda = 0. \tag{8.43}$$

The last term is a constant. The first term is a function of r. However, the middle term involves both r and θ. This can be fixed by multiplying the equation by r^2. Rearranging the equation, we can separate out the θ-dependence from the radial dependence. Letting μ be the separation constant, we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu. \tag{8.44}$$

This gives us two ordinary differential equations:

$$\begin{aligned} \frac{d^2\Theta}{d\theta^2} + \mu\Theta &= 0, \\ r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - \mu)R = 0. \end{aligned} \tag{8.45}$$

Let's consider the first of these equations. It should look familiar. If $\mu > 0$, then the general solution is

$$\Theta(\theta) = a\cos\sqrt{\mu}\theta + b\sin\sqrt{\mu}\theta.$$

Now we apply the boundary condition in θ. Looking at our boundary conditions in the problem, we do not see anything involving θ. This is a case where the boundary conditions needed are implied and not stated outright.

We can derive the boundary conditions by making some observations. Let's consider the solution corresponding to the endpoints $\theta = \pm\pi$, noting that at these values for any $r < a$ we are at the same physical point. So, we would expect the solution to have the same value at $\theta = -\pi$ as it has at $\theta = -\pi$. Namely, the solution is continuous at these physical points. Similarly, we expect the slope of the solution to be the same at these points. This tells us that

$$\Theta(\pi) = \Theta(-\pi) \quad \Theta'(\pi) = \Theta'(-\pi).$$

Such boundary conditions are called *periodic boundary conditions.*

Let's apply these conditions to the general solution for $\Theta(\theta)$ First, we set $\Theta(\pi) = \Theta(-\pi)$:

$$a\cos\sqrt{\mu}\pi + b\sin\sqrt{\mu}\pi = a\cos\sqrt{\mu}\pi - b\sin\sqrt{\mu}\pi.$$

This implies that

$$\sin\sqrt{\mu}\pi = 0.$$

So, $\sqrt{\mu} = m$ for $m = 0, 1, 2, 3, \ldots$. For $\Theta'(\pi) = \Theta'(-\pi)$, we have

$$-am\sin m\pi + bm\cos m\pi = am\sin m\pi + bm\cos m\pi.$$

But, this gives no new information.

To summarize so far, we have found the general solutions to the temporal and angular equations. The product solutions will have various products of $\{\cos\omega t, \sin\omega t\}$ and $\{\cos m\theta, \sin m\theta\}_{m=0}^{\infty}$. We also know that $\mu = m^2$ and $\omega = c\sqrt{\lambda}$.

That leaves us with the radial equation. Inserting $\mu = m^2$, we have

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - m^2)R = 0. \tag{8.46}$$

This is not an equation we have encountered before (unless you took a course in differential equations). We need to find solutions to this equation. It turns out that under a simple transformation it becomes an equation whose solutions are well known.

Let $z = \sqrt{\lambda} r$ and $w(z) = R(r)$. We have changed the name of the dependent variable since inserting the transformation into $R(r)$ leads to a function of z that is not the same function.

We need to change derivatives with respect to r into derivative with respect to z. We use the Chain Rule to obtain

$$\frac{dR}{dr} = \frac{dw}{dz}\frac{dz}{dr} = \sqrt{\lambda}\frac{dw}{dz}.$$

Thus, the derivatives transform as

$$r\frac{d}{dr} = z\frac{d}{dz}.$$

Inserting the transformation into the differential equation, we have

$$\begin{aligned} 0 &= r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - m^2)R, \\ 0 &= z\frac{d}{dz}\left(z\frac{dw}{dz}\right) + (z^2 - m^2)w. \end{aligned} \tag{8.47}$$

Expanding the derivative terms, we obtain *Bessel's equation*:

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - m^2)w = 0. \tag{8.48}$$

This equation has well known solutions and they are discussed in Section 1.5. The history of the solutions of this equation, called Bessel functions, does not originate in the study of partial differential equations. These solutions originally came up in the study of the Kepler problem, describing planetary motion. According to Watson in his Treatise on Bessel Functions, the formulation and solution of Kepler's Problem was discovered by Lagrange in 1770. Namely, the problem was to express the radial coordinate and what is called the eccentric anomaly, E, as functions of time. Lagrange found expressions for the coefficients in the expansions of r and E in trigonometric functions of time. However, he only computed the fist few coefficients. In 1816 Bessel had shown that the coefficients in the expansion for r could be given an integral representation. In 1824 he presented a thorough study of these functions, which are now called Bessel functions.

There are two linearly independent solutions of this second order equation: $J_m(z)$, the Bessel function of the first kind of order m,and $N_m(z)$, the

Bessel function of the second kind of order m. Plots of these functions are shown in Figures 1.7 and 1.8. Sometimes the N_m's are called Neumann functions. Also, you might find the notation is given as $Y_m(z)$ in some books. So, we have the general solution of our transformed equation is

$$w(z) = c_1 J_m(z) + c_2 N_m(z).$$

Transforming back into r variables, this becomes

$$R(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 N_m(\sqrt{\lambda} r).$$

Now we are ready to apply the boundary conditions to our last factor in our product solutions. Looking at the original problem we find only one condition: $u(a, \theta, t) = 0$ for $t > 0$ and $-\pi < \; < \pi$. This implies that $R(0) = 0$. But where is our second condition?

This is another unstated boundary condition. Look again at the plots of the Bessel functions. Notice that the Neumann functions are not well behaved at the origin. Do you expect that our solution will become infinite at the center of our drum? No, the solutions should be finite at the center. So, this is the second boundary condition. Namely, $|R(0)| < \infty$. This implies that $c^2 = 0$.

Now we are left with

$$R(r) = J_m(\sqrt{\lambda} r).$$

We have set $c_1 = 1$ for simplicity. We can apply the vanishing condition at $r = a$. This gives

$$J_m(\sqrt{\lambda} a) = 0.$$

Looking again at the plots of $J_m(z)$, we see that there are an infinite number of zeros, but they are not as easy as π! In Table 1.3 we list the nth zeros of J_p. Let's denote the nth zero of $J_m(z)$ by z_{mn}. Then our boundary condition tells us that

$$\sqrt{\lambda} a = z_{mn}.$$

This gives use our eigenvalue as

$$\lambda_{mn} = \left(\frac{z_{mn}}{a} \right)^2.$$

Thus, our radial function, satisfying the boundary conditions, is

$$R(r) = J_m(\frac{z_{mn}}{a} r).$$

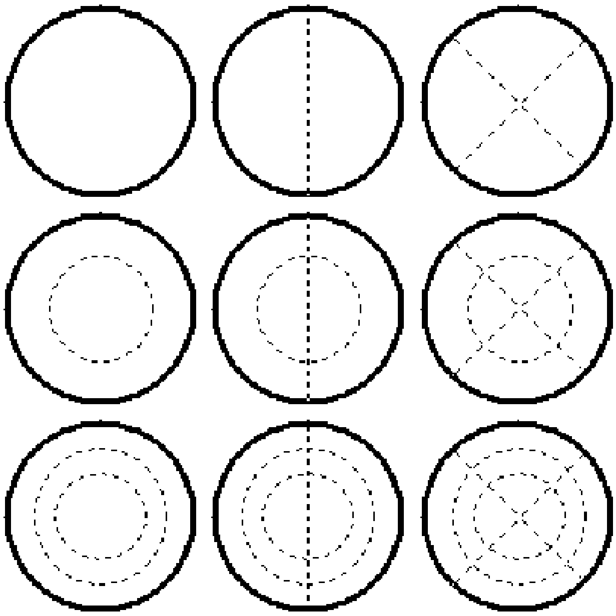

Figure 8.5: The first few modes of the vibrating circular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode.

We are finally ready to write out the product solutions. They are given by

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos\omega_{mn}t \\ \sin\omega_{mn}t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m\left(\frac{z_{mn}}{a}r\right). \tag{8.49}$$

Here we have indicated choices with the braces, leading to four different types of product solutions. Also, $m = 0, 1, 2, \ldots$, and

$$\omega_{mn} = \frac{z_{mn}}{a}c.$$

As with the rectangular membrane, we are interested in the shapes of the harmonics. So, we consider the spatial solution ($t = 0$) $\phi(r,\theta) = \cos m\theta J_m(\frac{z_{mn}}{a}r)$. Adding the solutions involving $\sin m\theta$ will only rotate these modes. The nodal curves are given by $\phi(r,\theta) = 0$, or $\cos m\theta = 0$, or $J_m(\frac{z_{mn}}{a}r) = 0$. The various nodal curves are shown in Figure 8.5.

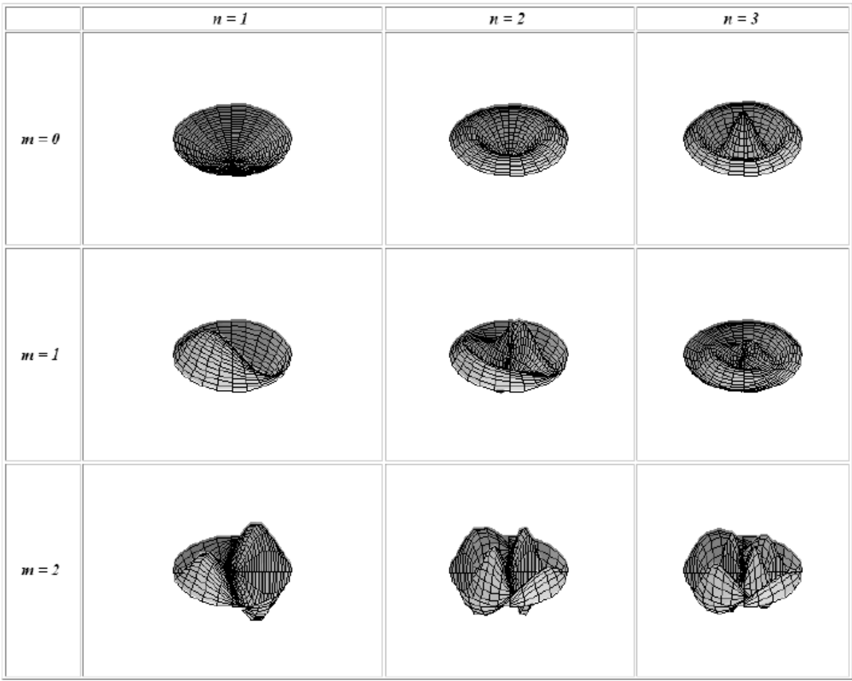

Figure 8.6: A three dimensional view of the vibrating circular membrane for the lowest modes.

For the angular part, we easily see that the nodal curves are radial lines. For $m = 0$, there are no solutions. For $m = 1$, we have $\cos\theta = 0$ implies that $\theta = \pm\frac{\pi}{2}$. These values give the same line as shown in Figure 8.5. For $m = 2$, $\cos 2\theta = 0$ implies that $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$.

We can also consider the nodal curves defined by the Bessel functions. We seek values of r for which $\frac{z_{mn}}{a}r$ is a zero of the Bessel function and lies in the interval $[0, a]$. Thus, we have

$$\frac{z_{mn}}{a}r = z_{mj},$$

or

$$r = \frac{z_{mj}}{z_{mn}}a.$$

These will give circles of this radius with $z_{mj} < z_{mn}$. The zeros can be found in Table 1.3. For $m = 0$ and $n = 1$, there is only one zero and $r = a$. For $m = 0$ and $n = 2$, we have two circles, $r = a$ and $r = \frac{2.405}{5.520}a \approx 0.436a$.

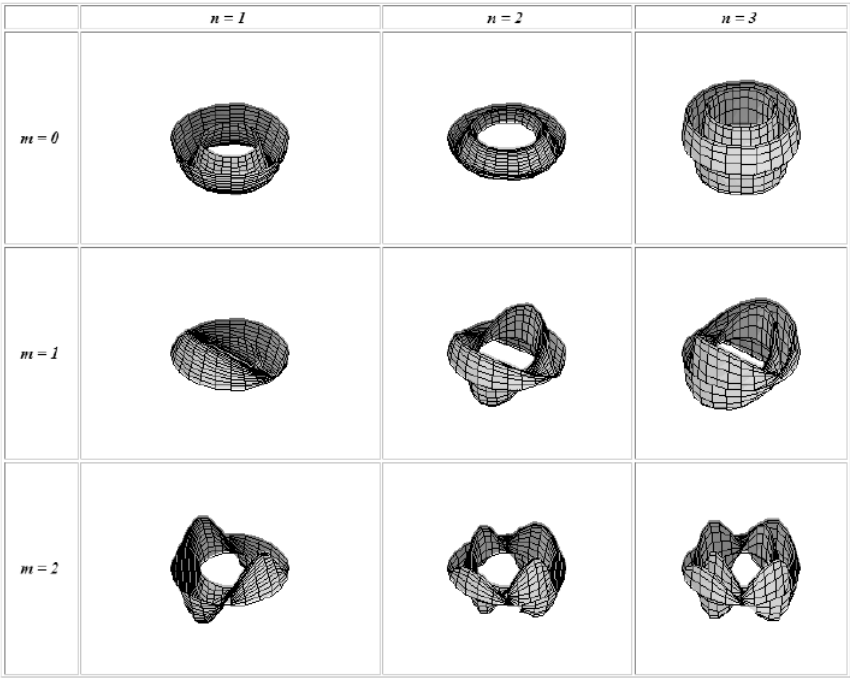

Figure 8.7: A three dimensional view of the vibrating annular membrane for the lowest modes.

For $m = 0$ and $n = 3$ we obtain circles of radii $r = a, \frac{5.520}{8.654}a, \frac{2.405}{8.654}a$. Similar computations result for larger values of m.

For a three dimensional view, one can look at Figure 8.6. Imagine that the various regions are oscillating independently and that the points on the nodal curves are not moving.

More complicated vibrations can be dreamt up for this geometry. We could consider an annulus in which the drum is formed from two concentric circular cylinders and the membrane is stretch between the two with an annular cross section. The separation would follow as before except now the boundary conditions are that the membrane is fixed around the two circular boundaries. In this case we cannot toss out the Neumann functions because the origin is not part of the drum head.

In this case,specifying the radii as a and b with $b < a$, we have to satisfy the conditions:

$$R(a) = c_1 J_m(\sqrt{\lambda}a) + c_2 N_m(\sqrt{\lambda}a) = 0,$$
$$R(b) = c_1 J_m(\sqrt{\lambda}b) + c_2 N_m(\sqrt{\lambda}b) = 0. \tag{8.50}$$

This leads to two homogeneous equations for c_1 and c_2. The determinant has to vanish, giving a nice complicated condition on λ. In Figure 8.7 we show various modes for a particular choice of a and b.

8.3 Sturm-Liouville Problems

In earlier chapters we have explored the solution of boundary value problems that led to trigonometric eigenfunctions. Such functions can be used to represent functions in Fourier series expansions. In this chapter we have seen that expansions in other functions are possible. We would like to generalize some of these techniques in order to solve other boundary value problems. We will see that many physically interesting boundary value problems lead to a class of problems called Sturm-Liouville eigenvalue problems. This includes our previous examples of the wave and heat equations. We will also be introduced to new functions, which are gathered in the next chapter.

We begin by noting that in physics many problems arise in the form of boundary value problems involving second order ordinary differential equations. For example, we might want to solve the equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{8.51}$$

subject to boundary conditions. We can write such an equation in operator form by defining the differential operator

$$L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x).$$

Then, Equation (8.51) takes the form

$$Lu = f.$$

In many of the problems we have encountered, the resulting solutions were written in term of series expansions over a set of eigenfunctions. This indicates that perhaps we might seek solutions to the eigenvalue problem $L\phi = \lambda\phi$ with homogeneous boundary conditions and then seek a solution as an expansion of the eigenfunctions. Formally, we let $u = \sum_{n=1}^{\infty} c_n \phi_n(x)\, dx$. However, we are not guaranteed a nice set of eigenfunctions. We need an appropriate set to form a basis in the function space. Also, it would be nice to have orthogonality so that we can solve for the expansion coefficients as was done in Section **??**. It turns out that any linear second order operator can be turned into an operator that posses just the right properties for us to carry out this procedure. The resulting operator is the Sturm-Liouville operator, which we will now explore.

We define the *Sturm-Liouville operator* as

$$\mathcal{L} = \frac{d}{dx}p(x)\frac{d}{dx} + q(x). \tag{8.52}$$

The *regular Sturm-Liouville eigenvalue problem* is given by the differential equation

$$\mathcal{L}u = -\lambda\sigma(x)u,$$

or

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u + \lambda\sigma(x)u = 0, \tag{8.53}$$

for $x \in [a, b]$ and the set of homogeneous boundary conditions

$$\begin{aligned} \alpha_1 u(a) + \beta_1 u'(a) &= 0, \\ \alpha_2 u(b) + \beta_2 u'(b) &= 0. \end{aligned} \tag{8.54}$$

The functions $p(x)$, $q(x)$ and $\sigma(x)$ are assumed to be continuous on (a, b) and $p(x) > 0$, $q(x) > 0$ on $[a, b]$.

The *alpha*'s and β's are constants. For different values, one has special types of boundary conditions. For $\beta_i = 0$, we have what are called Dirichlet conditions. Namely, $u(a) = 0$ and $u(b) = 0$. For $\alpha_i = 0$, we have Neumann boundary conditions. In this case, $u'(a) = 0$ and $u'(b) = 0$. In terms of the heat equation example, Dirichlet conditions correspond to maintaining a fixed temperature at the ends of the rod. The Neumann boundary conditions would correspond to no heat flow across the ends, as there would be no temperature gradient at those points.

Another type of boundary condition that is often encountered is the periodic boundary condition. Consider the heated rod that has been bent to form a circle. Then the two end points are physically the same. So, we would expect that the temperature and the temperature gradient should agree at those points. For this case we write $u(a) = u(b)$ and $u'(a) = u'(b)$.

Theorem Any second order linear operator can be put into the form of a Sturm-Liouville operator.

The proof of this is straight forward. Consider the equation (8.51). If $a_1(x) = a_2'(x)$, then we can write the equation in the form

$$\begin{aligned} f(x) &= a_2(x)y'' + a_1(x)y' + a_0(x)y \\ &= (a_2(x)y')' + a_0(x)y. \end{aligned} \tag{8.55}$$

This is in the correct form. We just identify $p(x) = a_2(x)$ and $q(x) = a_0(x)$.

However, consider the differential equation

$$x^2y'' + xy' + 2y = 0.$$

In this case $a_2(x) = x^2$ and $a_2'(x) = 2x \neq a_1(x)$. This equation is not of Sturm-Liouville type. But, we can change it to a Sturm Liouville type of equation.

In the Sturm Liouville operator the derivative terms are gather together into one perfect derivative. This is similar to what we saw in the first chapter when we solved linear first order equations. In that case we sought an integrating factor. We can do the same thing here. We seek a multiplicative function $\mu(x)$ that we can multiply through (8.51) so that it can be written in Sturm-Liouville form. We first divide out the $a_2(x)$, giving

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{f(x)}{a_2(x)}.$$

Now, we multiply the differential equation by μ :

$$\mu(x)y'' + \mu(x)\frac{a_1(x)}{a_2(x)}y' + \mu(x)\frac{a_0(x)}{a_2(x)}y = \mu(x)\frac{f(x)}{a_2(x)}.$$

The first two terms can now be combined into an exact derivative $(\mu y')'$ if $\mu(x)$ satisfies

$$\frac{d\mu}{dx} = \mu(x)\frac{a_1(x)}{a_2(x)}.$$

This is easily solved to give

$$\mu(x) = e^{\int \frac{a_1(x)}{a_2(x)}\, dx}.$$

Thus, the original equation can be multiplied by

$$\frac{1}{a_2(x)} e^{\int \frac{a_1(x)}{a_2(x)}\, dx}$$

to turn it into Sturm-Liouville form.

For the example above,

$$x^2 y'' + xy' + 2y = 0,$$

we need only multiply by

$$\frac{1}{x^2} e^{\int \frac{dx}{x}} = \frac{1}{x}.$$

Therefore, we obtain

$$0 = xy'' + y' + \frac{2}{x} y = (xy')' + \frac{2}{x} y.$$

8.4 Properties of Sturm-Liouville Problems

There are several properties that can be proven for the Sturm-Liouville eigenvalue problems. However, we will not prove them all here.

1. The eigenvalues are real, countable, ordered and there is a smallest eigenvalue. Thus, we can write them as $\lambda_1 < \lambda_2 < \dots$. However, there is no largest eigenvalue and $n \to \infty$, $\lambda_n \to \infty$.
2. For each eigenvalue λ_n there exists an eigenfunction ϕ_n with $n-1$ zeros on (a, b).
3. Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function, $\sigma(x)$. Defining the inner product of $f(x)$ and $g(x)$ as

$$< f, g >= \int_a^b f(x) g(x) \sigma(x)\, dx, \tag{8.56}$$

then the orthogonality condition can be written in the form

$$< \phi_n, \phi_m >=< \phi_n, \phi_n > \delta_{nm}. \tag{8.57}$$

4. The set of eigenfunctions is complete, i.e., any piecewise smooth function can be represented by a generalized Fourier series expansion of the eigenfunctions,
$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$
where
$$c_n = \frac{< f, \phi_n >}{< \phi_n, \phi_n >}.$$
Actually, one needs $f(x) \in L^2_\sigma[a,b]$, the set of square integrable functions over $[a,b]$ with weight function $\sigma(x) : < f, f >< \infty$.

5. Multiply the eigenvalue problem
$$\mathcal{L}\phi_n = -\lambda_n \sigma(x)\phi_n$$
by ϕ_n and integrate. Solve this result for λ_n, to find the *Rayleigh Quotient*
$$\lambda_n = \frac{-p\phi_n \frac{d\phi_n}{dx}\big|_a^b - \int_a^b \left[p \left(\frac{d\phi_n}{dx} \right)^2 - q\phi_n^2 \right] dx}{< \phi_n, \phi_n >}$$
The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some of the other properties.

8.4.1 Identities and Adjoint Operators

Before turning to any proofs, we will need two more tools. For the Sturm-Liouville operator, $\mathcal{L}$, we have two identities:

Lagrange's Identity $u\mathcal{L}v - v\mathcal{L}u = [p(uv' - vu')]'$.

The proof follows by a simple manipulations of the operator:

$$\begin{aligned}
u\mathcal{L}v - v\mathcal{L}u &= u\left[\frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv\right] - v\left[\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right] \\
&= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p\frac{du}{dx}\right) \\
&= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + p\frac{du}{dx}\frac{dv}{dx} - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - p\frac{du}{dx}\frac{dv}{dx} \\
&= \frac{d}{dx}\left[pu\frac{dv}{dx} - pv\frac{du}{dx}\right]. \qquad (8.58)
\end{aligned}$$

Green's Identity $\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx = [p(uv' - vu')]|_a^b$.

This identity is simply proven by integrating Lagrange's identity.

Next, we define the domain of an operator and introduce the notion of adjoint operators.

Definition The *domain* of a differential operator L is the set of all $u \in L^2_\sigma[a,b]$ satisfying a given set of homogeneous boundary conditions.

Definition The *adjoint*, $L^\dagger$, of operator L satisfies

$$< u, Lv > = < L^\dagger u, v >$$

for all v in the domain of L and u in the domain of $L^\dagger$.

Example Find the adjoint of $L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$.

We first look at the inner product

$$< u, Lv > = \int_a^b u(a_2 v'' + a_1 v' + a_0 v)\,dx.$$

Then we have to move the operator L from v and determine what operator is acting on u. For a simple operator like $L = \frac{d}{dx}$, this is easily done using integration by parts. For the given operator, we apply several integrations by parts to individual terms. Noting that

$$\int_a^b u(x)a_1(x)v'(x)\,dx = a_1(x)u(x)v(x)|_a^b - \int_a^b (u(x)a_1(x))'v(x)\,dx, \quad (8.59)$$

and

$$\begin{aligned}\int_a^b u(x)a_2(x)v''(x)\,dx &= a_2(x)u(x)v'(x)|_a^b - \int_a^b (u(x)a_2(x))'v(x)'\,dx \\ &= \left[a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]|_a^b + \int_a^b (u(x)a_2(x))''v(x)\,dx, \quad (8.60)\end{aligned}$$

we have

$$\begin{aligned} < u, Lv > &= \int_a^b u(a_2 v'' + a_1 v' + a_0 v)\,dx \\ &= \left[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)\right]|_a^b \\ &\quad + \int_a^b \left[(a_2 u)'' - (a_1 u)' + a_0 u\right] v\,dx. \qquad (8.61)\end{aligned}$$

Inserting the boundary conditions for v, one finds boundary conditions for u such that

$$[a_1(x)u(x)v(x) + a_2(x)u(x)v'(x) - (a_2(x)u(x))'v(x)]\,|_a^b = 0.$$

This leaves

$$< u, Lv >= \int_a^b [(a_2u)'' - (a_1u)' + a_0u]\, v\, dx \equiv < L^\dagger u, v > .$$

Therefore,

$$L^\dagger = \frac{d^2}{dx^2}a_2(x) - \frac{d}{dx}a_1(x) + a_0(x). \tag{8.62}$$

When $L^\dagger = L$, the operator is called *formally self-adjoint*. When the domain of L is the same as the domain of $L^\dagger$, the term *self-adjoint* is used.

Example Determine $L^\dagger$ and its domain for operator $Lu = \frac{du}{dx}$ and u satisfying $u(0) = 2u(1)$ on $[0, 1]$.

We need to find the adjoint operator satisfying $< v, Lu >=< L^\dagger v, u >$. Therefore, we rewrite the integral

$$< v, Lu >= \int_0^1 v\frac{du}{dx}\, dx = uv|_0^1 - \int_0^1 u\frac{dv}{dx}\, dx.$$

From this we have

1. $L^\dagger = -\frac{d}{dx}$.
2. $uv|_0^1 = 0 \Rightarrow 0 = u(1)[v(1) - 2v(0)] \Rightarrow v(1) = 2v(0)$.

8.4.2 Orthogonality of Eigenfunctions

We are now ready to prove That the eigenvalues of a Sturm-Liouville problem are real and the corresponding eigenfunctions are orthogonal.

Theorem The eigenvalues of the Sturm-Liouville problem are real.

Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n:

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n.$$

The complex conjugate of this equation is

$$\mathcal{L}\overline{\phi}_n = -\overline{\lambda}_n \sigma \overline{\phi}_n.$$

Now, multiply the first equation by $\overline{\phi}_n$ and the second equation by ϕ_n and then subtract the results. We obtain

$$\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi}_n = (\overline{\lambda}_n - \lambda_n)\sigma \phi_n \overline{\phi}_n.$$

Integrating both sides and noting that by Green's identity and the boundary conditions for a self-adjoint operator, we have

$$0 = (\overline{\lambda}_n - \lambda_n) \int_a^b \sigma \|\phi_n\|^2 \, dx.$$

The integral is nonnegative, so we have $\overline{\lambda}_n = \lambda_n$. Therefore, the eigenvalues are real.

Theorem The eigenfunctions corresponding to different eigenvalues of the Sturm-Liouville problem are orthogonal.

This is proven similar to the last proof.

Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n,

$$\mathcal{L}\phi_n = -\lambda_n \sigma \phi_n,$$

and let $\phi_m(x)$ be a solution of the eigenvalue problem associated with $\lambda_m \neq \lambda_n$,

$$\mathcal{L}\phi_m = -\lambda_m \sigma \phi_m,$$

Now, multiply the first equation by ϕ_m and the second equation by ϕ_n. Subtracting the results, we obtain

$$\phi_m \mathcal{L}\phi_n - \phi_n \mathcal{L}\phi_m = (\lambda_m - \lambda_n)\sigma \phi_n \phi_m$$

Integrating both sides and using Green's identity and the boundary conditions for a self-adjoint operator, we have

$$0 = (\lambda_m - \lambda_n) \int_a^b \sigma \phi_n \phi_m \, dx.$$

Since the eigenvalues are different, we have that

$$\int_a^b \sigma\phi_n\phi_m\,dx = 0.$$

Therefore, the eigenfunctions are orthogonal with respect to the weight function $\sigma(x)$.

8.5 The Eigenfunction Expansion Method

In section **??** we saw generally how one can use the eigenfunctions of a differential operator to solve a nonhomogeneous boundary value problem. In this chapter we have seen that Sturm-Liouville eigenvalue problems have the requisite set of orthogonal eigenfunctions. In this section we will apply this method to solve a particular nonhomogenous eigenvalue problem.

Recall that one starts with a nonhomogeneous differential equation

$$\mathcal{L}y = f$$

where $y(x)$ is to satisfy given homogeneous boundary conditions. The method makes use of the eigenfunctions satisfying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n\sigma\phi_n$$

subject to the given boundary conditions. Then one assumes that $y(x)$ can be written as an expansion in the eigenfunctions,

$$y(x) = \sum_{n=1}^{\infty} c_n\phi_n(x),$$

and inserts the expansion into the nonhomogeneous equation. This gives

$$f(x) = \mathcal{L}\left(\sum_{n=1}^{\infty} c_n\phi_n(x)\right) = -\sum_{n=1}^{\infty} c_n\lambda_n\sigma(x)\phi_n(x).$$

The expansion coefficients are then found by making use of the orthogonality of the eigenfunctions. Thus,

$$c_m = \frac{\int_a^b f(x)\phi_m(x)\,dx}{\lambda_m\int_a^b \phi_m^2(x)\sigma(x)\,dx}.$$

As an example, we consider the solution of the boundary value problem

$$(xy')' + \frac{y}{x} = \frac{1}{x}, \quad x \in [1, e], \tag{8.63}$$

$$y(1) = 0 = y(e). \tag{8.64}$$

This equation is already in self-adjoint form. So, we know that the associated Sturm-Liouville eigenvalue problem has an orthogonal set of eigenfunctions. We first determine this set. Namely, we need to solve

$$(x\phi')' + \frac{\phi}{x} = -\lambda\sigma\phi, \quad \phi(1) = 0 = \phi(e). \tag{8.65}$$

Rearranging the terms and multiplying by x, we have that

$$x^2\phi'' + x\phi' + (1 + \lambda\sigma x)\phi = 0.$$

This is almost an equation of Cauchy-Euler type. Picking the weight function $\sigma(x) = \frac{1}{x}$, we have

$$x^2\phi'' + x\phi' + (1 + \lambda)\phi = 0.$$

This is easily solved. The characteristic equation is

$$r^2 + (1 + \lambda) = 0.$$

One obtains nontrivial solutions of the eigenvalue problem satisfying the boundary conditions when $\lambda > -1$. The solutions are

$$\phi_n(x) = A \sin(n\pi \ln x), \quad , n = 1, 2, \ldots .$$

where $\lambda_n = n^2\pi^2 - 1$.

It is often useful to normalize the eigenfunctions. This means that one chooses A so that the norm of each eigenfunction is one. Thus, we have

$$\begin{aligned} 1 &= \int_1^e \phi_n(x)^2\sigma(x)\, dx \\ &= A^2 \int_1^e \sin(n\pi \ln x)\frac{1}{x}\, dx \\ &= A^2 \int_0^1 \sin(n\pi y)\, dy = \frac{1}{2}A^2. \end{aligned} \tag{8.66}$$

Thus, $A = \sqrt{2}$.

We now turn towards solving the nonhomogeneous problem, $\mathcal{L}y = \frac{1}{x}$. We first expand the unknown solution in terms of the eigenfunctions,

$$y(x) = \sum n = 1^{\infty} c_n \sqrt{2} \sin(n\pi \ln x).$$

Inserting this solution into the differential equation, we have

$$\frac{1}{x} = \mathcal{L}y = -\sum_{n=1}^{\infty} c_n \lambda_n \sqrt{2} \sin(n\pi \ln x) \frac{1}{x}.$$

Next, we make use of orthogonality. Multiplying both sides by $\phi_m(x) = \sqrt{2} \sin(m\pi \ln x)$ and integrating, gives

$$\lambda_m c_m = \int_1^e \sqrt{2} \sin(m\pi \ln x) \frac{1}{x}\, dx = \frac{\sqrt{2}}{m\pi}[(-1)^m - 1].$$

Solving for c_m, we have

$$c_m = \frac{\sqrt{2}}{m\pi} \frac{[(-1)^m - 1]}{m^2\pi^2 - 1}.$$

Finally, we insert our coefficients into the expansion for $y(x)$. The solution is then

$$y(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{[(-1)^n - 1]}{n^2\pi^2 - 1} \sin(n\pi \ln(x)).$$