Appendix A

Sequences and Series

In this chapter we will review and extend some of the concepts and definitions that you might have seen previously related to infinite series. Working with infinite series can be a little tricky and we need to understand some of the basics before moving on to the study of series of trigonometric functions.

For example, one can show that the infinite series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges to $\ln 2$. However, the terms can be rearranged to give

$$1 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{11} - \frac{1}{6} + \frac{1}{13}\right) + \cdots = \frac{3}{2}\ln 2.$$

In fact, other rearrangements can be made to give any desired sum!

Other problems with infinite series can occur. Try to sum the following infinite series to find that

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^2} \sim 0.937548\ldots.$$

A sum of even as many as 10^7 terms only gives convergence to four or five decimal places.

The series

$$\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} - \cdots, \quad x > 0$$

diverges for all x. So, you might think this divergent series is useless. However, truncation of this divergent series leads to and approximation of the integral

$$\int_0^\infty \frac{e^{-t}}{x+t}\,dt, \quad x>0.$$

So, can we make sense out of any of these, or other manipulations of infinite series?

A.1 Sequences Real Numbers

We first begin with the definitions for sequences and series of numbers.

Definition A *sequence* is a function whose domain is the set of positive integers.

Examples are

1. $a(n)=n$ yields the sequence $\{1,2,3,4,5,\ldots\}$
2. $a(n)=3n$ yields the sequence $\{3,6,9,12,\ldots\}$

However, one typically uses subscript notation and not functional notation: $a_n=a(n)$. We then call a_n the nth term of the sequence.

Another way to define a particular sequence is recursively.

Definition A *recursive sequence*is defined in two steps:

1. The value of first term (or first few terms)is given.
2. A rule, or recursion formula, to determine later terms from earlier ones is given.

A typical example is given by the Fibonacci sequence. It can be defined by the recursion formula $a_{n+1}=a_n+a_{n-1}$, $n\geq 2$ and the starting values of

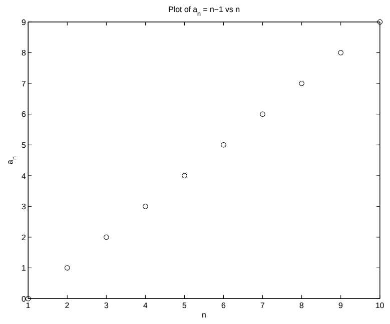

Figure A.1: Plot of $a_n = n - 1$ for $n = 1 \ldots 10$.

$a_1 = 0$ and $a_1 = 1$. The resulting sequence is $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$. Writing the general expression for the nth term is possible, but it is not as simply stated. Recursive definitions are often useful in doing computations for large values of n.

A.2 Convergence of Sequences

Next we are interested in the behavior of the sequence as n gets large. For the sequence defined by $a_n = n - 1$, we find the behavior as shown in Figure A.1. Notice that as n gets large, a_n also gets large. This sequence is said to be divergent.

On the other hand, the sequence defined by $a_n = \frac{1}{2^n}$ approaches a limit as n gets large. This is depicted in Figure A.2. Another related series, $a_n = \frac{(-1)^n}{2^n}$, is shown in Figure A.3. This sequence is the alternating sequence $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$.

Definition The sequence a_n *converges* to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \Rightarrow |a - L| < \epsilon.$$

If no such number exists, then the sequence is said to *diverge.*

In Figures A.4-A.5 we see what this means. For the given sequence, we see that $L = 0$. Given an $\epsilon > 0$, we ask for what value of N the nth terms

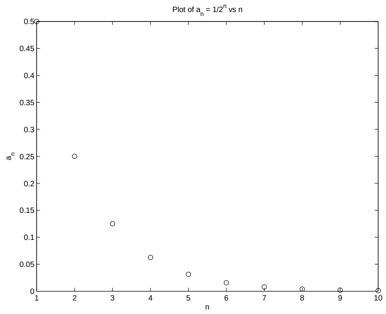

Figure A.2: Plot of $a_n = \frac{1}{2^n}$ for $n = 1 \ldots 10$.

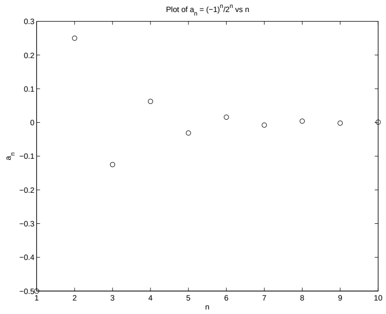

Figure A.3: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$.

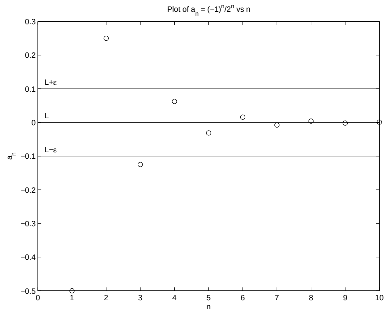

Figure A.4: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$. Picking $\epsilon = 0.1$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 3$.

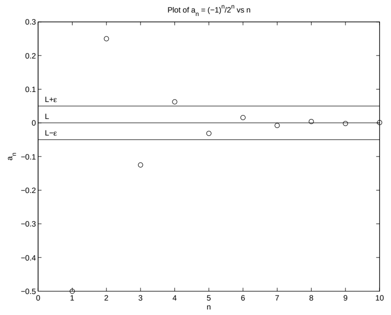

Figure A.5: Plot of $a_n = \frac{(-1)^n}{2^n}$ for $n = 1 \ldots 10$. Picking $\epsilon = 0.05$, one sees that the tail of the sequence lies between $L + \epsilon$ and $L - \epsilon$ for $n > 4$.

$(n > N)$ lie in the interval $[L - \epsilon, L + \epsilon]$. In these figures this interval is depicted by a horizontal band. We see that for convergence, sooner, or later, the tail of the sequence ends up entirely within this band.

If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit L, then we write either $a_n \to \infty$ or $\lim_{n\to\infty} a_n = L$. For example, we have already seen that $\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$.

A.3 Limits Theorems

Once we have defined the notion of convergence of a sequence to some limit, then we can investigate certain properties of limits of sequences.

Here we list a few results on limits theorems and some special limits, which arise often.

Theorem Consider two convergent sequences $\{a_n\}$ and $\{b_n\}$ and a number k. Assume that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then we have

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$.
2. $\lim_{n\to\infty}(kb_n) = kB$.
3. $\lim_{n\to\infty}(a_n b_n) = AB$.
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}, \quad B \neq 0$.

Some special limits are given next. These are generally first encountered in a second course in calculus.

Theorem The following are special cases:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.
2. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.
3. $\lim_{n\to\infty} x^{\frac{1}{n}} = 1, \quad x > 0$.
4. $\lim_{n\to\infty} x^n = 0, \quad |x| < 1$.
5. $\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$.
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

The proofs generally are straight forward, depending upon L'Hopital's Rule and other manipulations. For example, one can prove the first limit by first realizing that $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x}$. This limit is indeterminate as $x \to \infty$ in its current form since the numerator and the denominator get large for large x. In such cases one employs L'Hopital's Rule. Thus, $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$.

The second limit in the list can be proven by first looking at

$$\lim_{n\to\infty} \ln n^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Now, if $\lim_{n\to\infty} \ln f(n) = 0$, then $\lim_{n\to\infty} f(n) = e^0 = 1$. Thus proving the second limit.

The third limit can be done similarly. The reader is left to confirm the other limits.

Example 1. $\lim_{n\to\infty} \frac{n^2+2n+3}{n^3+n}$
Divide the numerator and denominator by n^2. Then
$\lim_{n\to\infty} \frac{n^2+2n+3}{n^3+n} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{3}{n^2}}{n+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n}$.

Example 2. $\lim_{n\to\infty} \frac{\ln(n^2)}{n}$
Rewriting $\frac{\ln(n^2)}{n} = \frac{2\ln(n)}{n}$, we find that
$\lim_{n\to\infty} \frac{\ln(n^2)}{n} = 2\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

Example 3. $\lim_{n\to\infty} (n^2)^{\frac{1}{n}}$
To compute this limit, we rewrite
$\lim_{n\to\infty} (n^2)^{\frac{1}{n}} = \lim_{n\to\infty} (n)^{\frac{1}{n}} (n)^{\frac{1}{n}} = 1$.

Example 4. $\lim_{n\to\infty} (\frac{(n-2)}{n})^n$
This limit can be written as
$\lim_{n\to\infty} (\frac{(n-2)}{n})^n = \lim_{n\to\infty} (1 + \frac{(-2)}{n})^n = e^{-2}$.

A.4 Infinite Series

In this section we investigate the meaning of series, which are infinite sums of the form

$$a_1 + a_2 + a_2 + \ldots. \tag{A.1}$$

A typical example is the infinite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots. \tag{A.2}$$

How would one evaluate this sum? We begin by just adding the terms. For example,

$$1 + \frac{1}{2} = \frac{3}{2},$$

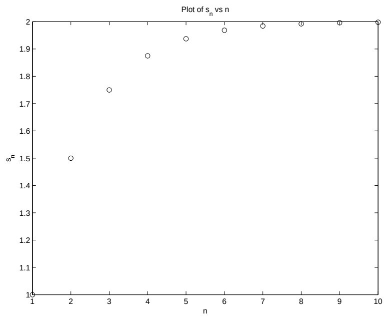

Figure A.6: Plot of $s_n = \sum_{k=1}^{n} \frac{1}{2^{k-1}}$ for $n = 1 \ldots 10$.

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{4} &= \frac{7}{4}, \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= \frac{15}{8}, \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= \frac{31}{16}, \end{aligned} \tag{A.3}$$

etc. The values tend to a limit. We can see this graphically in Figure A.6.

In general, we want to make sense out of Equation (A.1). As with the example, we look at a sequence of partial sums. Thus, we consider the sums

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ s_4 &= a_1 + a_2 + a_3 + a_4, \end{aligned} \tag{A.4}$$

etc. In general, we define the nth partial sum as

$$s_n = a_1 + a_2 + \ldots + a_n.$$

If the infinite series (A.1) is to make any sense, then the sequence of partial sums should converge to some limit. We define this limit to be the sum of the infinite series, $\lim_{n\to\infty} s_n = S$.

Definition If the sequence of partial sums converges to the limit L as n gets large, then the infinite series is said to have the sum L.

We will use the compact summation notation

$$\sum_{n=1}^{\infty} = a_1 + a_2 + \ldots + a_n + \ldots .$$

Here n will be referred to as the index and it may start at values other than $n = 1$.

A.5 Geometric Series

Example (A.2) is an example of what is known as a geometric series. A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^2 + \ldots + ar^n + \ldots . \tag{A.5}$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is r.

The sum of a geometric series, when it converges, can easily be determined. We consider the nth partial sum:

$$s_n = a + ar + \ldots + ar^{n-2} + ar^{n-1}. \tag{A.6}$$

Now, multiply this equation by r.

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \tag{A.7}$$

Subtracting these two equations, while noting the many cancellations, we have

$$(1 - r)s_n = a - ar^n. \tag{A.8}$$

Thus, the nth partial sums can be written in the compact form

$$s_n = \frac{a(1 - r^n)}{1 - r}. \tag{A.9}$$

Recalling that the sum, if it exists, is given by $S = \lim_{n\to} s_n$. Letting n get large in the partial sum (A.9) we need only evaluate $\lim n \to r^n$. From our special limits we know that this limit is zero for $|r| < 0$. Thus, we have the sum of the geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}. \tag{A.10}$$

The reader should verify that the geometric series diverges for all other values of r. Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example 1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is

$$S = \frac{1}{1 - \frac{1}{2}} = 2.$$

This agrees with the plot of the partial sums in Figure A.6.

Example 2. $\sum_{k=2}^{\infty} \frac{4}{3^k}$

In this example we note that the first term occurs for $k = 2$. So, $a = \frac{4}{9}$. Also, $r = \frac{1}{3}$. So,

$$S = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

Example 3. $\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n})$

Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed. Thus, we have

$$\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n}) = \sum_{n=1}^{\infty} \frac{3}{2^n} - \sum_{n=1}^{\infty} \frac{2}{5^n}.$$

Now we can add both geometric series:

$$\sum_{n=1}^{\infty} (\frac{3}{2^n} - \frac{2}{5^n}) = \frac{\frac{3}{2}}{1 - \frac{1}{2}} - \frac{\frac{2}{5}}{1 - \frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

A.6 Convergence Tests

Given a general infinite series, it would be nice to know if it converges, or not. Often, we are only interested in the convergence and not the actual sum. It is often difficult to determine the sum, when the series does converge. In this section we will review some of the standard tests for convergence.

First, we have the nth term divergence test. This can be motivated by two examples:

1. $\sum_{n=1}^{\infty} 2^n = 1 + 2 + 4 + 8 + \dots$.
2. $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \dots$.

In the first example it is easy to see that each term is getting larger and larger, and thus the partial sums will grow without bound. In the second case, each term is bigger than one. Thus, the series will be bigger than adding the same number of ones as there are terms in the sum. Obviously, this series will also diverge.

This leads to the nth term divergence test:

Theorem If $\lim a_n \neq 0$ or if this limit does not exist, then $\sum_n a_a$ diverges.

This theorem does not imply that just because the terms are getting smaller, the series will converge. Otherwise, we would not need any other convergence theorems.

For the next theorems, we will assume that the series has a nonnegative terms.

1. **Comparison Test**

   The series $\sum a_n$ converges if there is a convergent series $\sum c_n$ such that $a_n \leq c_n$ for all $n > N$ for some N. The series $\sum a_n$ diverges if there is a divergent series $\sum d_n$ such that $d_n \leq a_n$ for all $n > N$ for some N.

   For this test one has to dream up a second series for comparison. Typically, this requires some experience with convergent series. Often it is better to use other tests first if possible.

2. **Limit Comparison Test**

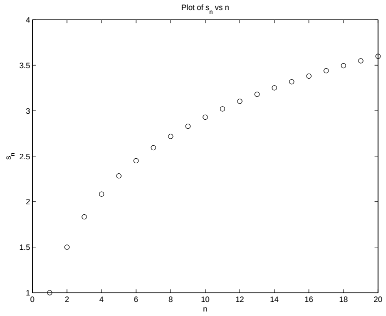

Figure A.7: Plot of the partial sums for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite then $\sum a_n$ and $\sum b_n$ converge together or diverge together.

For example, consider the infinite series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$. Then, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{(n+1)^2} = 2$. Thus, these two series both converge, or both diverge. If we knew the behavior of the second series, then we could draw a conclusion. Using the next test, we will prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, therefore $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ diverges.

3. **Integral Test**

Consider the infinite series $\sum_{n-1}^{\infty} a_n$. Let $f(n) = a_n$. Then, $\sum_{n-1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge. Here we mean that the integral converges or diverges as an improper integral.

We are interested in the convergence or divergence of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we saw in the Limit Comparison Test example. This infinite series is famous and is called the harmonic series. The plot of the partial sums is given in Figure A.7. It appears that the series could possibly converge or diverge. It is hard to tell graphically.

In this case we can use the Integral Test. In Figure A.8 we plot $f(x) = \frac{1}{x}$ and at each integer n we plot a box from n to $n+1$ of height $\frac{1}{n}$. We can see from the figure that the total area of the boxes is greater than the area under the curve. Since the area of each box is $\frac{1}{n}$, then we have that

$$\int_1^{\infty} \frac{dx}{x} < \sum_{n=1}^{\infty} \frac{1}{n}.$$

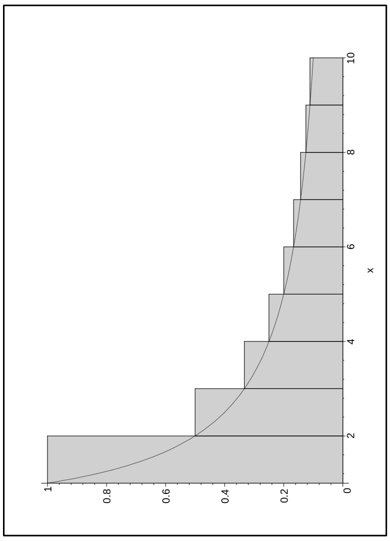

Figure A.8: Plot of $f(x) = x$ and boxes of height $\frac{1}{n}$ and width 1. *Figure needs to be rotated!*

But, we can compute the integral.

$$\int_1^\infty \frac{dx}{x} = \lim_{x\to\infty} (\ln x) = \infty.$$

Thus, the integral diverges and the infinite series is larger than this! So, the harmonic series diverges.

The Integral Test provides us with the convergence behavior for a class of infinite series called p-series. These series are of the form $\sum_{n=1}^\infty \frac{1}{n^p}$. Recalling that the improper integrals $\int_1^\infty \frac{dx}{x^p}$ converge for $p > 1$ and diverge otherwise, we have the p-test:

$$\sum_{n=1}^\infty \frac{1}{n^p} \text{ converges for } p > 1$$

and diverges otherwise.

Example $\sum_{n=1}^\infty \frac{n+1}{n^3-2}$.
We first note that as n gets large, the general term behaves like $\frac{1}{n^2}$ since the numerator behaves like n and the denominator behaves like n^3. So, we expect that this series behaves like the series $\sum_{n=1}^\infty \frac{1}{n^2}$. Thus, by the limit comparison test,

$\lim_{n\to\infty} \frac{n+1}{n^3-2}(n^2) = 1$. These series both converge, or both diverge. However, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test since $p = 2$.

4. **Ratio Test**

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Then the behavior of the infinite series can be determined from

$$\begin{array}{ll} \rho < 1, & \text{converges} \\ \rho > 1, & \text{diverges} \end{array}$$

Example 1. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$.

We compute

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{(n+1)^{10}}{n^{10}} \frac{10^n}{10^{n+1}} \\ &= \lim_{n\to\infty} (1+\frac{1}{n})^{10} \frac{1}{10} \\ &= \frac{1}{10} < 1. \end{aligned} \tag{A.11}$$

Therefore, the series converges by the ratio test.

Example 2. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

In this case we make use of the fact that $(n+1)! = (n+1)n!$. We compute

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1 \end{aligned} \tag{A.12}$$

This series also converges by the ratio test.

5. **Root Test**

Consider the series $\sum_{n=1}^{\infty} a_n$ for $a_n > 0$. Let $\rho = \lim_{n\to\infty} {a_n}^{1/n}$. Then the behavior of the infinite series can be determined from

$$\begin{array}{ll} \rho < 1, & \text{converges} \\ \rho > 1, & \text{diverges} \end{array}$$

Example 1. $\sum_{n=0}^{\infty} e^{-n}$.
We use the nth root test:
$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{-1} = e^{-1} < 1$. Thus, this series converges by the nth root test.

Example 2. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$.
This series also converges by the nth root test.
$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n^n}{2^{n^2}}\right)^{1/n} = \lim_{n\to\infty} \frac{n}{2^n} = 0 < 1$.

We next turn to series which have both positive and negative terms. We can toss out the signs by taking absolute values of each of the terms. We then note that since $a_n \le |a_n|$ we have

$$-\sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} |a_n|.$$

If the sum $\sum_{n=1}^{\infty} |a_n|$converges, then the original series converges. This type of convergence is useful, because we can use the previous tests to establish convergence of such series. Thus, we say that a series *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges. If a series converges, but does not converge absolutely, then it is said to *converge conditionally.*

Example $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^2}$.
This series converges absolutely because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$.

Finally, there is one last test that we recall from your introductory calculus class. We consider the alternating series, given by $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$. The convergence of alternating series is determined from Leibniz's Theorem.

Theorem The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if

1. a_n's are positive.
2. $a_n \geq a_{n+1}$ for all n.
3. $a_n \to 0$.

The first condition guarantees that we have alternating signs in the series. The next conditions say that the magnitude if the terms gets smaller and approaches zero.

Example 1. The alternating harmonic series converges. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Example 2. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ also passes the conditions of Leibniz's Theorem

Note that in Example 2 we can show that the series is absolutely convergent. However, the series of absolute value for Example 1 is the harmonic series, so it is not absolutely convergent. Therefore, we have an example of a series that is conditionally convergent.

A.7 The Order of Sequences and Functions

Often we are interested in comparing the magnitude of sequences or of functions. This is useful in approximation theory.

Definition Let a_n and b_n be two sequences. Then if there is are numbers N and K, independent of N, such that

$$\left|\frac{a_n}{b_n}\right| < K \quad \text{whenever} \quad n > N,$$

then we say that a_n is *of the order of* b_n. We write this as

$$a_n = O(b_n) \quad \text{as} \quad n \to \infty.$$

For example, consider the series given by $a_n = \frac{2n+1}{3n^2+2}$ and $b_n = \frac{1}{n}$. Then,

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{2n+1}{3n^2+2}n\right| = \left|\frac{2+1/n}{3+2/n^2}\right| < \frac{2}{3}\left|\frac{1+1/2n}{1+2/3n^2}\right| < 1.$$

Thus,

$$a_n = O(b_n) = O(\frac{1}{n}).$$

In practice one is given a sequence like a_n, but the second sequence needs to be found by looking at the large n behavior of a_n.

In a similar way, we can compare functions.

Definition $f(x)$ is *of the order of* $g(x)$, or $f(x) = O(g(x))$ as $x \to x_0$ if

$$\lim_{x - x_0}\left|\frac{f(x)}{g(x)}\right| < K$$

for some K independent of x_0.

For example, we recall the Taylor series expansion for $\cos x$ gives us that

$$\cos x = 1 - \frac{x^2}{2} + O(x^4) \text{ as } x \to 0.$$

Similarly, we can make use of the binomial expansion to determine the behavior of functions such as $f(x) = (a+x)^b - a^b$. Recall that the first terms of the binomial expansion can be written

$$(1+x)^b = 1 + bx + O(x^2)$$

as $x \to \infty$. We will review the binomial expansion in the next section. It can also be obtained using Taylor series expansions, which we will cover soon.

Inserting this expression, we have

$$f(x) = (a+x)^b - a^b = a^b(1 + \frac{bx}{a} + O(x^2)) - a^b = O(x) \text{ as } x \to 0.$$

A.8 The Binomial Expansion

One series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
&\cdots
\end{aligned} \tag{A.13}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $k+1$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^51$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not know the numerical coefficient in the expansion.

We now list the coefficients for the above expansions.

$$\begin{array}{ccccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \tag{A.14}$$

This pattern is the famous Pascal's triangle. There are many interesting features of this triangle. But we will first ask how each row can be generated.

We see that each row begins and ends with a one. Next the second term and next to last term has a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have

$$\begin{array}{cccccccccccc}
n=2: & & 1 & & & & 2 & & & & 1 & \\
 & & & \searrow & & \swarrow & & \searrow & & \swarrow & & \\
n=3: & 1 & & & 3 & & & & 3 & & & 1
\end{array} \tag{A.15}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{llllllllllllll}
n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array} \tag{A.16}$$

Of course, it would take a while to compute each row up to the desired n. We need a simple expression for computing a specific coefficient. Consider the kth term in the expansion of $(a+b)^n$. Let $r=k-1$. Then this term is of the form $C_r^n a^{n-r}b^r$. We have seen the the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the coefficients have been found to take a simple form.

$$C_r^n = \frac{n!}{(n-r)!r!} = \left(\begin{array}{c} n \\ r \end{array}\right).$$

This is nothing other than the combinatoric symbol for determining how to choose n things r at a time. In our case, this makes sense. We have to count the number of ways that we can arrange the products of r b's with $n-r$ a's. There are n slots to place the b's. For example, the $r=2$ case for $n=4$ involves the six products: *aabb*, *abab*, *abba*, *baab*, *baba*, and *bbaa*. Thus, it is natural to use this notation. The original problem that concerned Pascal was in gambling.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) a^{n-r}b^r. \tag{A.17}$$

What if $a \gg b$? Can we use this to get an approximation to $(a+b)^n$? If we neglect b then $(a+b)^n \simeq a^n$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion, which we could state using big O notation. In order to do this we first divide out a as

$$(a+b)^n = a^n(1+\frac{b}{a})^n.$$

Now we have a small parameter, $\frac{b}{a}$. According to what we have seen above, we can use the binomial expansion to write

$$(1+\frac{b}{a})^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) \left(\frac{b}{a}\right)^r. \tag{A.18}$$

Thus, we have a finite sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write

$$(1+\frac{b}{a})^n = 1 + n\frac{b}{a} + O(\left(\frac{b}{a}\right)^2).$$

note that we have used the observation that the second coefficient in the nth row of Pascal's triangle is n.

Summarizing, this then gives

$$\begin{aligned} (a+b)^n &= a^n(1+\frac{b}{a})^n \\ &= a^n(1+n\frac{b}{a}+O(\left(\frac{b}{a}\right)^2)) \\ &= a^n + na^n\frac{b}{a} + a^nO(\left(\frac{b}{a}\right)^2). \end{aligned} \tag{A.19}$$

Therefore, we can approximate $(a+b)^n \simeq a^n + nba^{n-1}$, with an error on the order of ba^{n-2}. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a+b)^n \simeq a^n$, but it is not as good because the error in this case is of the order ba^{n-1}.

We have seen that

$$\frac{1}{1-x} = 1 + x + x^2 + \dots.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$. This is again a binomial to a power, but the power is not a nonnegative integer. It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation (A.17).

This example suggests that our sum may no longer be finite. So, for p a real number, we write

$$(1+x)^p = \sum_{r=0}^{\infty} \begin{pmatrix} p \\ r \end{pmatrix} x^r. \tag{A.20}$$

However, we quickly run into problems with this form. Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned}\begin{pmatrix} p \\ r \end{pmatrix} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \tag{A.21}$$

With this in mind we now state the theorem:

General Binomial Expansion The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (A.17). For p real, we have that

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r. \tag{A.22}$$

Often we need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3). \tag{A.23}$$

A.9 Series of Functions

Our immediate goal is to provide a preparation useful for studying Fourier series, which are series whose terms are functions. So, in this section we begin to discuss series of functions and the convergence of such series. Once more we will need to resort to the convergence of the sequence of partial sums. This means we really need to start with sequences of functions.

A *sequence of functions* is simply a set of functions $f_n(x)$, $n = 1, 2, \dots$ defined on a common domain D. A frequently used example is the sequence of functions $\{1, x, x^2, \dots\}$.

An infinite *series of functions* is given by $\sum_{n=1}^{\infty} f_n(x), \quad x \in D$. Using powers of x again, and example would be $\sum_{n=1}^{\infty} x^n, \quad x \in [-1, 1]$. In order to investigate the convergence of this series, we really mean substitute values for x and determine if the resulting real series of number converges. This means that we would need to consider the Nth partial sums

$$s_N(x) = \sum_{n=1}^{N} f_n(x).$$

Does this sequence of functions converge?

We say that a sequence of functions f_n converge pointwise on D to a limit g if

$$\lim_{n\to\infty} f_n(x) = g(x)$$

for each $x \in D$. More formally, we write that

$$\lim_{n\to\infty} f_n = g \text{ (pointwise on } D)$$

if given $x \in D$ and $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N.$$

> **Example** Consider the sequence of functions $f(x) = \frac{1}{1+nx}, \quad |x| < \infty, \quad n = 1, 2, 3, \dots$. The limits depends on the value of x.
>
> (a) $x = 0$. Here $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1 = 1$.
>
> (b) $x \neq 0$. Here $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = 0$.

Therefore, we can say that $f_n \to g$ pointwise for $|x| < \infty$, where

$$g(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{A.24}$$

We also note that in general N depends on both x and ϵ.

> **Example** We consider the functions
> $f_n(x) = x^n, \quad x \in [0,1], \quad n = 1, 2, \ldots$. We recall that the above definition suggests that for each x we seek an N such that $|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N$. Here are two examples:
>
> i. $x = 0$. Here we have $f_n(0) = 0$ for all n. So, given $\epsilon > 0$ we seek an N such that $|f_n(0) - 0| < \epsilon, \quad \forall n \geq N$, or $0 < \epsilon$. But all n work, so we can pick $N = 1$.
>
> ii. $x = \frac{1}{2}$. In this case we have $f_n(\frac{1}{2}) = \frac{1}{2^n}$, for $n = 1, 2, \ldots$. As n gets large, $f_n \to 0$. So, given $\epsilon > 0$, we seek N such that $|\frac{1}{2^n} - 0| < \epsilon, \quad \forall n \geq N$. This means that $\frac{1}{2^n} < \epsilon$, or $n > -\frac{\ln \epsilon}{\ln 2} \geq N$. Thus, our choice of N depends on ϵ.

There are other questions that can be asked about sequences of functions. Let the sequence of functions f_n be continuous on D. If the sequence of functions converges pointwise to g on D then we can ask the following.

(a) Is g continuous on D?

(b) If each f_n is integrable on $[a,b]$, then does $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b g(x)\,dx$?

(c) If each f_n is differentiable at c, then does $\lim_{n\to\infty} f_n'(c) = g'(c)$?

It turns out that pointwise convergence is not enough to provide an affirmative answer to any of these questions. Though we will nit prove it here, what we will need is uniform convergence.

Definition Consider a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ on D. Let $g(x)$ be defined for $x \in D$. Then the sequence *converges uniformly* on D, or

$$\lim_{n\to} \infty f_n = g \text{ uniformly on } D,$$

if given $\epsilon > 0$, there exists an N such that

$$|f_n(x) - g(x)| < \epsilon, \quad \forall n \geq N \text{ and } \forall x \in D.$$

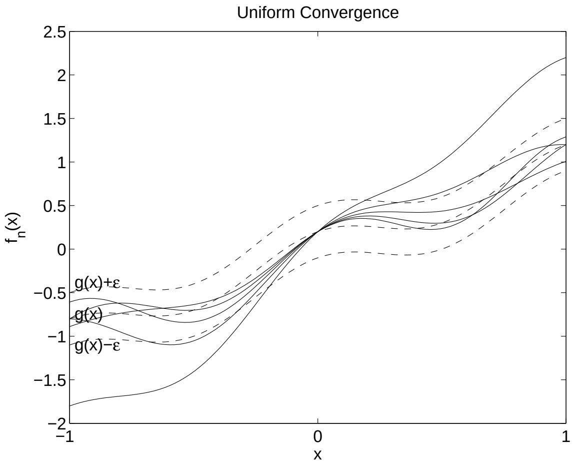

Figure A.9: For uniform convergence, as n gets large, $f_n(x)$ lies in the band $g(x) - \epsilon, g(x) - \epsilon$.

> This definition almost looks like the definition for pointwise convergence. However, the seemingly subtle difference lies in the fact that N does not depend upon x. The sought N works for all x in the domain. As seen in Figure A.9 as n gets large, $f_n(x)$ lies in the band $g(x) - \epsilon, g(x) - \epsilon$.
>
> > **Example** $f_n(x) = x^n$, for $x \in [0, 1]$. Note that in this case as n gets large, $f_n(x)$ does not lie in the band $(g(x) - \epsilon, g(x) - \epsilon)$. This is displayed in Figure A.10.
> >
> > **Example** $f_n(x) == \cos(nx)/n^2$ on $[-1, 1]$. For this example we plot the first several members of the sequence in Figure A.11. We can see that eventually $(n \geq N)$ members of this sequence do lie inside a band of width ϵ about the limit $g(x) = 0$ for all values of x. Thus, this sequence of functions will converge uniformly to the limit.

Finally, we should note that if a sequence of functions is uniformly convergent then it converges pointwise. However, the examples should bear out that the converse is not true.

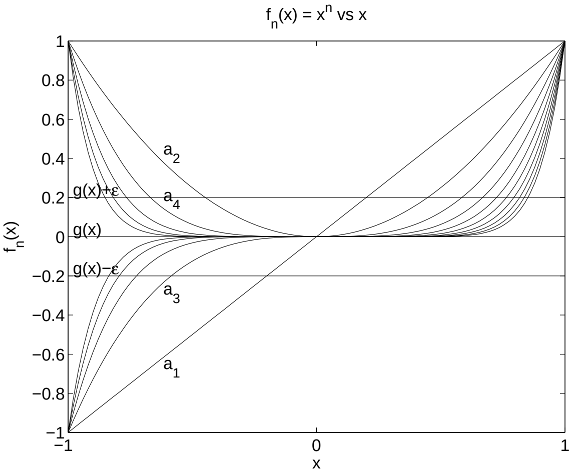

Figure A.10: Plot of $a_n = x^n$ on $[-1, 1]$ for $n = 1 \dots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

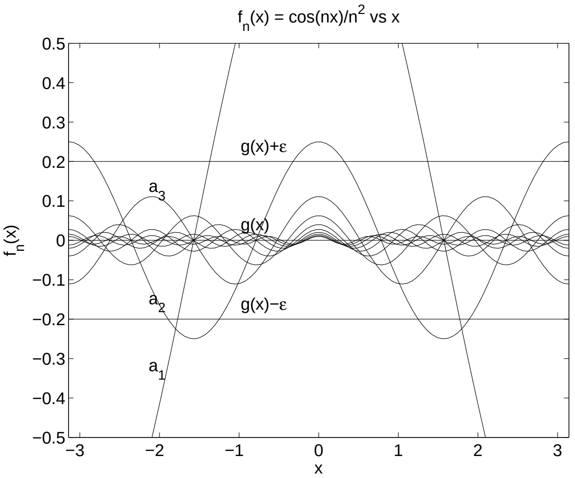

Figure A.11: Plot of $a_n = \cos(nx)/n^2$ on $[-\pi, \pi]$ for $n = 1 \dots 10$ and $g(x) \pm \epsilon$ for $\epsilon = 0.2$.

A.10 Infinite Series of Functions

We now turn our attention to infinite series of functions, which will form the basis of our study of Fourier series. Recall that we are interested in the convergence of the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n(x)$ for $x \in D$. But the sequence of partial sums is just a sequence of functions So, it is natural to define the convergence of sequences of functions in terms of pointwise and uniform convergence.

We define the sequence of partial sums

$$s_n(x) = \sum_{j=1}^{n} f_j(x).$$

Then the definitions of pointwise and uniform convergence are as follows:

Definition $\sum f_j(x)$ *converges pointwise* to $f(x)$ on D if given $x \in D$, and $\epsilon > 0$, there exists and N such that

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$.

Definition $\sum f_j(x)$ *converges uniformly* to $f(x)$ on D given $\epsilon > 0$, there exists and N such that

$$|f(x) - s_n(x)| < \epsilon$$

for all $n > N$ and all $x \in D$.

Again, we state without proof the following:

1. Uniform convergence implies pointwise convergence.
2. If f_n is continuous on D, and $\sum_{n}^{\infty} f_n$ converges uniformly to f on D, then f is continuous on D.
3. If f_n is continuous on $[a, b] \subset D$, $\sum_{n}^{\infty} f_n$ converges uniformly on D, and $\int_a^b f_n(x)\,dx$ exists, then

$$\sum_{n}^{\infty} \int_a^b f_n(x)\,dx = \int_a^b \sum_{n}^{\infty} f_n(x)\,dx = \int_a^b g(x)\,dx.$$

4. If f_n' is continuous on $[a,b] \subset D$, $\sum_n^\infty f_n$ converges pointwise to g on D, and $\sum_n^\infty f_n'$ converges uniformly on D,, then $\sum_n^\infty f_n'(x) = \frac{d}{dx}(\sum_n^\infty f_n(x)) = g'(x)$ for $x \in (a,b)$.

Since uniform convergence of series gives so much, like term by term integration and differentiation, we would like to be able to recognize when we have a uniformly convergent series. One test for such convergence is the Weierstrass M-Test.

Theorem Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions on D. If $|f_n(x)| \le M_n$, for $x \in D$ and $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty f_n$ converges uniformly of D.

Proof First, we note that for $x \in D$

$$\sum_{n=1}^\infty |f_n(x)| \le \sum_{n=1}^\infty M_n.$$

Thus, since by the assumption that $\sum_{n=1}^\infty M_n$ converges, we have that $\sum_{n=1}^\infty f_n$ converges absolutely on D. Therefore, $\sum_{n=1}^\infty f_n$ converges pointwise on D. So, let $\sum_{n=1}^\infty f_n = g$.

We now want to prove that this convergence is in fact uniform. So, given $\epsilon > 0$ we need to find an N such that

$$|g(x) - \sum_{j=1}^n f_j(x)| < \epsilon$$

if $n \ge N$ for all $x \in D$.

So, for any $x \in D$,

$$\begin{aligned} |g(x) - \sum_{j=1}^n f_j(x)| &= |\sum_{j=1}^\infty f_j(x) - \sum_{j=1}^n f_j(x)| \\ &= |\sum_{j=n+1}^\infty f_j(x)| \\ &\le \sum_{j=n+1}^\infty |f_j(x)|, \quad \text{by the triangle inequality} \\ &\le \sum_{j=n+1}^\infty M_j. \end{aligned} \tag{A.25}$$

Now, the sum over the M_j's is convergent, so we can choose our N such that

$$\sum_{j=n+1}^{\infty} M_j < \epsilon, \quad n \geq N.$$

Then, we have from above that

$$|g(x) - \sum_{j=1}^{n} f_j(x)| \leq \sum_{j=n+1}^{\infty} M_j < \epsilon$$

for all $n \geq N$ and $x \in D$. Thus, $\sum f_j \to g$ uniformly on D. **QED**

We now given an example of how to use the M-Test.

> **Example** We consider the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ defined on $[-\pi, \pi]$. Each term is bounded by $\left|\frac{\cos nx}{n^2}\right| = \frac{1}{n^2} \equiv M_n$. We know that $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Thus, we can conclude that the original series converges uniformly, as it satisfies the conditions of the Weierstrass M-Test.