Poisson's Equation in a Disk

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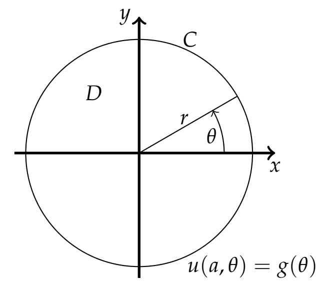

Figure 1: We solve Poisson's equation on a disk of radius a.

We want to use Green's functions to solve Poisson's equation with boundary conditions. After all, Poisson's equation is a nonhomogeneous Laplace equation.

$$\begin{aligned} \nabla^2 u &= f \quad \text{in } D, \\ u &= g \quad \text{on } C, \end{aligned} \tag{1}$$

where D is the disk shown in Figure 1. The Green's function should satisfy

$$\begin{aligned} \nabla^2 G &= \delta(\xi - x, \eta - y) \quad \text{in } D, \\ G &\equiv 0 \quad \text{on } C. \end{aligned} \tag{2}$$

Recall that in the Method of Images $G(x, y; \xi, \eta)$ represents a point charge (or impulse) at (x, y) giving a response at (ξ, η). For each point charge we employ the two-dimensional infinite space Green's function in Cartesian coordinates,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}'| = \frac{1}{4\pi} \ln \left[(x - \xi)^2 + (y - \eta)^2 \right]. \tag{3}$$

We introduce a negative point charge (the mirror charge) at (x', y') for each positive charge at $(x, y) \in D$ so as to make the Green's function vanish on C. We depict these charges in Figure 2. The points (x, y) and (ξ, η) have radial coordinates $\rho = \sqrt{x^2 + y^2}$, and $\rho' = \sqrt{\xi^2 + \eta^2}$. [Later we show that the mirror charge is a distance a^2/ρ from the origin and has charge $Q = -q$.]

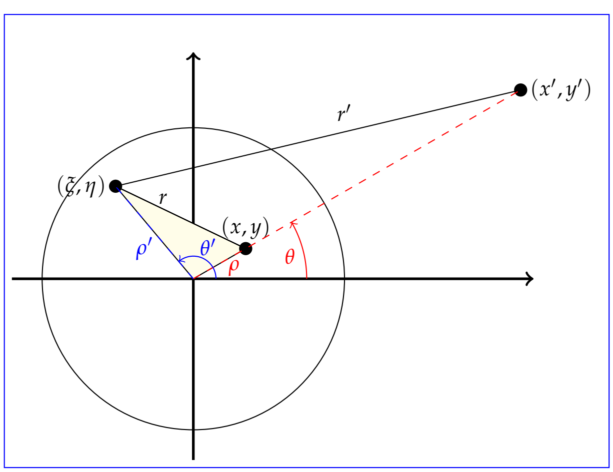

Figure 2: Applying the Method of Images in the construction of the Green's function.

We now construct the Green's function using the infinite space Green's function (3) for the contributions from the two charges,

$$G_+(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right] = \frac{1}{2\pi}\ln r$$
$$G_-(x,y;\xi,\eta) = -\frac{1}{4\pi}\ln\left[(x'-\xi)^2+(y'-\eta)^2\right] = -\frac{1}{2\pi}\ln r'$$

Here we set $q=1$ and $Q=-a/\rho$.

The distances between the charges and (ξ,η) are given by $r = \sqrt{(\xi-x)^2+(\eta-y)^2}$ and $r' = \sqrt{(\xi-x')^2+(\eta-y')^2}$. We can rewrite r and r' in the polar coordinates ρ and ρ' using the Law of Cosines on the small shaded triangle and the larger triangle with sides r and r' opposite the angle $\theta'-\theta$ [See Figures 2-3]:

$$\begin{aligned} r^2 &= \rho^2+\rho'^2-2\rho\rho'\cos(\theta'-\theta), \\ r'^2 &= d^2+\rho'^2-2\rho d\cos(\theta'-\theta). \end{aligned}$$

The location of the mirror charge is shown later to be

$$d = \sqrt{x'^2+y'^2} = \frac{a^2}{\rho}.$$

Therefore,

$$\begin{aligned} r'^2 &= \frac{a^4}{\rho^2}+\rho'^2-2\frac{a^2\rho'}{\rho}\cos(\theta'-\theta) \\ &= \frac{1}{\rho^2}\Big(a^4+\rho^2\rho'^2-2\rho\rho' a^2\cos(\theta'-\theta)\Big) \end{aligned}$$

Note that for a fixed point (x,y),

$$\frac{r^2}{r'^2}\bigg|_{\rho'=a} = \frac{\rho^2+a^2-2\rho a\cos(\theta'-\theta)}{\frac{1}{\rho^2}\left(a^4+\rho^2a^2-2\rho a^3\cos(\theta'-\theta)\right)} = \frac{\rho^2}{a^2}$$

is constant on the boundary $\rho' = a$. Then, we can take the Green's function as

$$\begin{aligned} G(x,y;\xi,\eta) &= \frac{1}{2\pi}\ln\frac{ra}{r'\rho} \\ &= \frac{1}{4\pi}\ln\frac{a^2(\rho^2+\rho'^2-2\rho\rho'\cos(\theta'-\theta))}{a^4+\rho^2\rho'^2-2\rho\rho' a^2\cos(\theta'-\theta)}. \end{aligned}$$

Note that G is symmetric in $\rho=\sqrt{x^2+y^2}$, and $\rho'=\sqrt{\xi^2+\eta^2}$.

Once we have the Green's function, then using Green's Second Theorem, we obtain the solution of Poisson's equation as

$$u(x,y) = \int_D G(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta + \int_C (u\nabla_{r'}G - G\nabla_{r'}u)\cdot\mathbf{n}\,ds'.$$

For the special case that the domain is a unit disk, $a=1$, then

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\frac{\rho^2+\rho'^2-2\rho\rho'\cos(\theta'-\theta)}{1+\rho^2\rho'^2-2\rho\rho'\cos(\theta'-\theta)}$$

and

$$\nabla_{r'} G \cdot \mathbf{n}\Big|_{\rho'=1} = \frac{1}{2\pi} \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(\theta-\theta')}.$$

Then, the solution on the unit disk is given by

$$\begin{aligned} u(x,y) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^1 \ln \frac{\rho^2+\rho'^2-2\rho\rho'\cos(\theta'-\theta)}{1+\rho^2\rho'^2-2\rho\rho'\cos(\theta'-\theta)} f(\theta')\,\rho'\,d\rho'\,d\theta' \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\theta')} g(\theta')\,d\theta'. \end{aligned}$$

We see that the boundary contribution contains the Poisson kernel.

For $a \neq 1$,

$$G(x,y;\xi,\eta) = \frac{1}{4\pi} \ln \frac{a^2(\rho^2+\rho'^2-2\rho\rho'\cos(\theta'-\theta))}{a^4+\rho^2\rho'^2-2a^2\rho\rho'\cos(\theta'-\theta)}$$

and

$$\nabla_{r'} G \cdot \mathbf{n}\Big|_{\rho'=a} = \frac{1}{2\pi a} \frac{a^2-\rho^2}{a^2+\rho^2 - 2\rho a\cos(\theta-\theta')}.$$

Then, the solution on the unit disk is given by

$$\begin{aligned} u(x,y) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^a \ln \frac{a^2(\rho^2+\rho'^2-2\rho\rho'\cos(\theta'-\theta))}{a^4+\rho^2\rho'^2-2a^2\rho\rho'\cos(\theta'-\theta)} f(\theta')\,\rho'\,d\rho'\,d\theta' \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2-\rho^2}{a^2+\rho^2-2\rho a\cos(\theta-\theta')} g(\theta')\,d\theta'. \end{aligned}$$

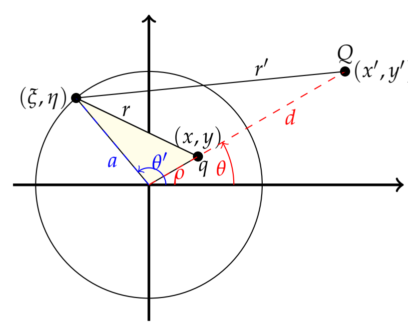

Figure 3: Locating the mirror charge Q a distance d from the origin along the line connecting both charges.

Mirror Charge

We need to locate the mirror image in order to have the total "potential" vanish on the boundary. Essentially, the potential outside a point charge q is given by $V = \frac{q}{r}$ for a sphere and $q \ln r$ for a disk. In Figure 3 we place the mirror charge Q a distance d from the origin along the line connecting both charges.

Sphere

For an arbitrary point (ξ,η) on the boundary, $\rho' = a$, the total potential is given by

$$V_{tot} = \frac{q}{\sqrt{\rho^2+a^2-2\rho a\cos(\theta'-\theta)}} + \frac{Q}{\sqrt{d^2+a^2-2ad\cos(\theta'-\theta)}}.$$

These will cancel if at first we let $d = \frac{a^2}{\rho}$. Then the second potential function becomes

$$\begin{aligned} \frac{Q}{\sqrt{d^2+a^2-2ad\cos(\theta'-\theta)}} &= \frac{Q}{\sqrt{\frac{a^4}{\rho^2}+a^2-2\frac{a^3}{\rho}\cos(\theta'-\theta)}} \\ &= \frac{\rho}{a}\frac{Q}{\sqrt{a^2+\rho^2-2a\rho\cos(\theta'-\theta)}} \end{aligned}$$

So, if we set $Q = -\frac{aq}{\rho}$, then the total potential vanishes on the surface of the disk.

Disk

In the case of a disk, the two dimensional potential is of the form

$$V = \frac{q}{2\pi} \ln r = \frac{q}{4\pi} \ln \left[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta' - \theta) \right] + B_1,$$

where B_1 is a constant independent of ρ.

We need a second solution corresponding to the image charge placed along a line connecting the origin with the point (x, y) at a distance d from the origin such that the total potential is a constant for $\rho' = a$. Thus, we consider

$$\begin{aligned} V_{tot}(\rho' = a) &= \frac{q}{4\pi} \ln \left[\rho^2 + a^2 - 2\rho a \cos(\theta' - \theta) \right] - \frac{q}{4\pi} \ln \left[d^2 + a^2 - 2ad \cos(\theta' - \theta) \right] + B_1 - B_2 \\ &= \frac{q}{4\pi} \ln \frac{\rho^2 + a^2 - 2\rho a \cos(\theta' - \theta)}{d^2 + a^2 - 2ad \cos(\theta' - \theta)} + B_1 - B_2. \end{aligned}$$

now, we let $d = \frac{\alpha}{\rho}$, to obtain

$$\begin{aligned} V_{tot}(\rho' = a) &= \frac{q}{4\pi} \ln \frac{\rho^2 + a^2 - 2\rho a \cos(\theta' - \theta)}{d^2 + a^2 - 2ad \cos(\theta' - \theta)} + B_1 - B_2. \\ &= \frac{q}{4\pi} \ln \frac{\rho^2 + a^2 - 2\rho a \cos(\theta' - \theta)}{\frac{\alpha^2}{\rho^2} + a^2 - 2a\frac{\alpha}{\rho} \cos(\theta' - \theta)} + B_1 - B_2. \\ &= \frac{q}{4\pi} \ln \frac{\rho^2(\rho^2 + a^2 - 2\rho a \cos(\theta' - \theta))}{a^2 \left(\frac{\rho^2 + \alpha^2}{a^2} - 2\frac{\alpha}{a}\rho \cos(\theta' - \theta) \right)} + B_1 - B_2. \end{aligned}$$

Letting $\alpha = a^2$ and $q = 1$,

$$V_{tot}(\rho' = a) = \frac{1}{4\pi} \ln \frac{\rho^2}{a^2} + B_1 - B_2.$$

Setting $B_1 = \frac{1}{2\pi} \ln a$ and $B_2 = \frac{1}{2\pi} \ln \rho$, we have $V_{tot}(\rho' = a) = 0$. This gives the Green's function as

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{ra}{r'\rho}.$$