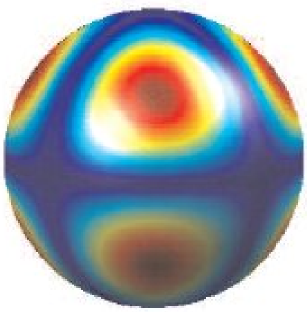
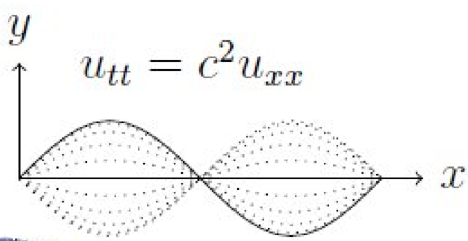
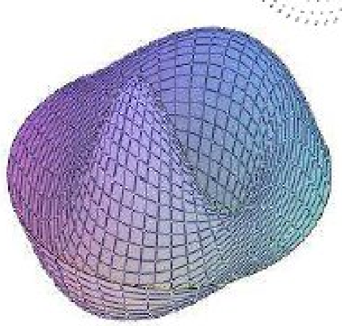
RUSSELL L. HERMAN

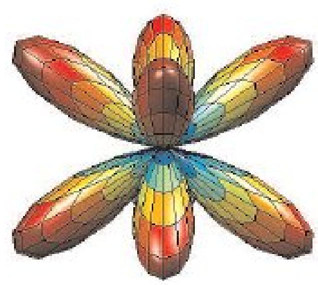
A FIRST COURSE IN PARTIAL DIFFERENTIAL EQUATIONS

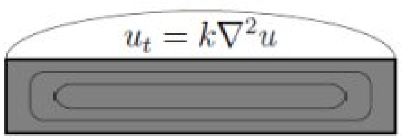


$$\int_V (\varphi \nabla^2 \chi - \chi \nabla^2 \varphi)\, dV = \oint_S (\varphi \nabla \chi - \chi \nabla \varphi) \cdot \hat{\mathbf{n}}\, dS$$









$$T(r,z,t) = T_b + \frac{8(T_i - T_b)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{(2n-1)\pi z}{Z}}{(2n-1)} \frac{J_0(\frac{r}{a} j_{0m}) e^{-\lambda_{nm} k t}}{j_{0m} J_1(j_{0m})}$$

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INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS by Russell Herman. Versions of these notes have been posted online since Spring 2005.
HTTP://PEOPLE.UNCW.EDU/HERMANR/PDE1/PDEBOOK

January 2021

Contents

Dedicated to those students who have endured previous versions of my notes.

Prologue

"How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?." - Albert Einstein (1879-1955)

Introduction

This set of notes was compiled for use in a one semester course on mathematical methods for the solution of partial differential equations typically taken by majors in mathematics, the physical sciences, and engineering. Partial differential equations often arise in the study of problems in applied mathematics, mathematical physics, physical oceanography, meteorology, engineering, and biology, economics, and just about everything else. However, many of the key methods for studying such equations extend back to problems in physics and geometry. In this course we will investigate analytical, graphical, and approximate solutions of some standard partial differential equations. We will study the theory, methods of solution and applications of partial differential equations.

We will first introduce partial differential equations and a few models. A PDE, for short, is an equation involving the derivatives of some unknown multivariable function. It is a natural extenson of ordinary differential equations (ODEs), which are differential equations for an unknown function one one variable. We will begin by classifying some of these equations.

We begin the study of PDEs with the one dimensional heat and wave equations and the two-dimensional Laplace equation on a rectangle. As we progress through the course, we will introduce standard numerical methods since knowing how to numerically solve differential equations can be useful in research. We will also look into the standard solutions, including separation of variables, starting in one dimension and then proceeding to higher dimensions. This naturally leads to finding solutions as Fourier series and special functions, such as Legendre polynomials and Bessel functions. We will end with a short study of first order evolution equations.

The specific topics to be studied and approximate number of lectures will include

First Semester: (26 lectures)

- Introduction (1)

- Derivation of Generic Equations (1)
- Separation of Variables (Heat and Wave Equations) (2)
- 1D Wave Equation - d'Alembert Solution (2)
- Classification of Second Order Equations (1)
- Nonhomogeneous Heat Equation (1)
- Separation of Variables (2D Laplace Equation) (2)
- Fourier Series (4)
- Finite Difference Method (2)
- Sturm-Liouville Theory (3)
- Special Functions (3)
- Equations in 2D - Laplace's Equation, Vibrating Membranes (3)
- 3D Problems and Spherical Harmonics (2)
- First Order PDEs (2)
- Conservation Laws and Shocks (1)

Acknowledgments

MOST, IF NOT ALL, OF THE IDEAS AND EXAMPLES are not my own. These notes are a compendium of topics and examples that I have used in teaching not only differential equations, but also in teaching numerous courses in physics and applied mathematics. Some of the notions even extend back to when I first learned them in courses I had taken.

I would also like to express my gratitude to the many students who have found typos, or suggested sections needing more clarity in the core set of notes upon which this book was based. This applies to the set of notes used in my mathematical physics course, applied mathematics course, and previous differential equations courses.

7
First Order Partial Differential Equations

"The profound study of nature is the most fertile source of mathematical discoveries." - Joseph Fourier (1768-1830)

7.1 Introduction

We begin our study of partial differential equations with *first order partial differential equations.* Before doing so, we need to define a few terms.

Recall (see the appendix on differential equations) that an n-th order ordinary differential equation is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

n-th order ordinary differential equation

$$F(y^{(n)}(x), y^{(n-1)}(x), \ldots, y'(x), y(x), x) = 0. \tag{7.1}$$

Here $y^{(n)}(x)$ represents the nth derivative of $y(x)$. Furthermore, and initial value problem consists of the differential equation plus the values of the first $n-1$ derivatives at a particular value of the independent variable, say x_0:

Initial value problem.

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \ldots, \quad y(x_0) = y_0. \tag{7.2}$$

If conditions are instead provided at more than one value of the independent variable, then we have a boundary value problem. .

If the unknown function is a function of several variables, then the derivatives are partial derivatives and the resulting equation is a partial differential equation. Thus, if $u = u(x, y, \ldots)$, a general partial differential equation might take the form

$$F\left(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{7.3}$$

Since the notation can get cumbersome, there are different ways to write the partial derivatives. First order derivatives could be written as

$$\frac{\partial u}{\partial x}, u_x, \partial_x u, D_x u.$$

Second order partial derivatives could be written in the forms

$$\frac{\partial^2 u}{\partial x^2}, u_{xx}, \partial_{xx} u, D_x^2 u.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, u_{xy}, \partial_{xy} u, D_y D_x u.$$

Note, we are assuming that $u(x, y, \ldots)$ has continuous partial derivatives. Then, according to Clairaut's Theorem (Alexis Claude Clairaut, 1713-1765) , mixed partial derivatives are the same.

Examples of some of the partial differential equation treated in this book are shown in Table 1.1. However, being that the highest order derivatives in these equation are of second order, these are second order partial differential equations. In this chapter we will focus on first order partial differential equations. Examples are given by

$$\begin{aligned} u_t + u_x &= 0. \\ u_t + u u_x &= 0. \\ u_t + u u_x &= u. \\ 3u_x - 2u_y + u &= x. \end{aligned}$$

For function of two variables, which the above are examples, a general first order partial differential equation for $u = u(x, y)$ is given as

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in D \subset R^2. \tag{7.4}$$

This equation is too general. So, restrictions can be placed on the form, leading to a classification of first order equations. A linear first order partial differential equation is of the form

Linear first order partial differential equation.

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y). \tag{7.5}$$

Note that all of the coefficients are independent of u and its derivatives and each term in linear in u, u_x, or u_y.

We can relax the conditions on the coefficients a bit. Namely, we could assume that the equation is linear only in u_x and u_y. This gives the quasilinear first order partial differential equation in the form

Quasilinear first order partial differential equation.

$$a(x, y, u)u_x + b(x, y, u)u_y = f(x, y, u). \tag{7.6}$$

Note that the u-term was absorbed by $f(x, y, u)$.

In between these two forms we have the semilinear first order partial differential equation in the form

Semilinear first order partial differential equation.

$$a(x, y)u_x + b(x, y)u_y = f(x, y, u). \tag{7.7}$$

Here the left side of the equation is linear in u, u_x and u_y. However, the right hand side can be nonlinear in u.

For the most part, we will introduce the Method of Characteristics for solving quasilinear equations. But, let us first consider the simpler case of linear first order constant coefficient partial differential equations.

7.2 *Linear Constant Coefficient Equations*

LET'S CONSIDER THE LINEAR FIRST ORDER CONSTANT COEFFICIENT partial differential equation

$$au_x + bu_y + cu = f(x,y), \tag{7.8}$$

for a, b, and c constants with $a^2 + b^2 > 0$. We will consider how such equations might be solved. We do this by considering two cases, $b = 0$ and $b \neq 0$.

For the first case, $b = 0$, we have the equation

$$au_x + cu = f.$$

We can view this as a first order linear (ordinary) differential equation with y a parameter. Recall that the solution of such equations can be obtained using an integrating factor. [See the discussion after Equation (B.7).] First rewrite the equation as

$$u_x + \frac{c}{a}u = \frac{f}{a}.$$

Introducing the integrating factor

$$\mu(x) = \exp\left(\int^x \frac{c}{a}\,d\xi\right) = e^{\frac{c}{a}x},$$

the differential equation can be written as

$$(\mu u)_x = \frac{f}{a}\mu.$$

Integrating this equation and solving for $u(x,y)$, we have

$$\begin{aligned}
\mu(x)u(x,y) &= \frac{1}{a}\int f(\xi,y)\mu(\xi)\,d\xi + g(y) \\
e^{\frac{c}{a}x}u(x,y) &= \frac{1}{a}\int f(\xi,y)e^{\frac{c}{a}\xi}\,d\xi + g(y) \\
u(x,y) &= \frac{1}{a}\int f(\xi,y)e^{\frac{c}{a}(\xi-x)}\,d\xi + g(y)e^{-\frac{c}{a}x}.
\end{aligned} \tag{7.9}$$

Here $g(y)$ is an arbitrary function of y.

For the second case, $b \neq 0$, we have to solve the equation

$$au_x + bu_y + cu = f.$$

It would help if we could find a transformation which would eliminate one of the derivative terms reducing this problem to the previous case. That is what we will do.

We first note that

$$\begin{aligned}
au_x + bu_y &= (a\mathbf{i} + b\mathbf{j}) \cdot (u_x\mathbf{i} + u_y\mathbf{j}) \\
&= (a\mathbf{i} + b\mathbf{j}) \cdot \nabla u.
\end{aligned} \tag{7.10}$$

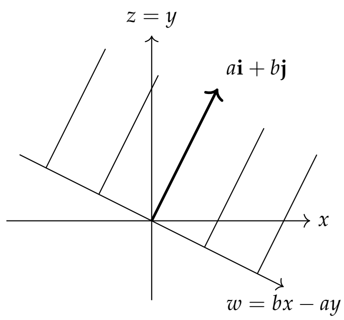

Figure 7.1: Coordinate systems for transforming $au_x + bu_y + cu = f$ into $bv_z + cv = f$ using the transformation $w = bx - ay$ and $z = y$.

Recall from multivariable calculus that the last term is nothing but a directional derivative of $u(x, y)$ in the direction $a\mathbf{i} + b\mathbf{j}$. [Actually, it is proportional to the directional derivative if $a\mathbf{i} + b\mathbf{j}$ is not a unit vector.]

Therefore, we seek to write the partial differential equation as involving a derivative in the direction $a\mathbf{i} + b\mathbf{j}$ but not in a directional orthogonal to this. In Figure 7.1 we depict a new set of coordinates in which the w direction is orthogonal to $a\mathbf{i} + b\mathbf{j}$.

We consider the transformation

$$\begin{aligned} w &= bx - ay, \\ z &= y. \end{aligned} \tag{7.11}$$

We first note that this transformation is invertible,

$$\begin{aligned} x &= \frac{1}{b}(w + az), \\ y &= z. \end{aligned} \tag{7.12}$$

Next we consider how the derivative terms transform. Let $u(x, y) = v(w, z)$. Then, we have

$$\begin{aligned} au_x + bu_y &= a\frac{\partial}{\partial x}v(w, z) + b\frac{\partial}{\partial y}v(w, z), \\ &= a\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] \\ &\quad +b\left[\frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] \\ &= a[bv_w + 0 \cdot v_z] + b[-av_w + v_z] \\ &= bv_z. \end{aligned} \tag{7.13}$$

Therefore, the partial differential equation becomes

$$bv_z + cv = f\left(\frac{1}{b}(w + az), z\right).$$

This is now in the same form as in the first case and can be solved using an integrating factor.

Example 7.1. Find the general solution of the equation $3u_x - 2u_y + u = x$.

First, we transform the equation into new coordinates.

$$w = bx - ay = -2x - 3y,$$

and $z = y$. The,

$$\begin{aligned} u_x - 2u_y &= 3[-2v_w + 0 \cdot v_z] - 2[-3v_w + v_z] \\ &= -2v_z. \end{aligned} \tag{7.14}$$

The new partial differential equation for $v(w, z)$ is

$$-2\frac{\partial v}{\partial z} + v = x = -\frac{1}{2}(w + 3z).$$

Rewriting this equation,

$$\frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{1}{4}(w+3z),$$

we identify the integrating factor

$$\mu(z) = \exp\left[-\int^z \frac{1}{2}\, d\zeta\right] = e^{-z/2}.$$

Using this integrating factor, we can solve the differential equation for $v(w,z)$.

$$\begin{aligned}
\frac{\partial}{\partial z}\left(e^{-z/2}v\right) &= \frac{1}{4}(w+3z)e^{-z/2}, \\
e^{-z/2}v(w,z) &= \frac{1}{4}\int^z (w+3\zeta)e^{-\zeta/2}\, d\zeta \\
&= -\frac{1}{2}(w+6+3z)e^{-z/2} + c(w) \\
v(w,z) &= -\frac{1}{2}(w+6+3z) + c(w)e^{z/2} \\
u(x,y) &= x - 3 + c(-2x-3y)e^{y/2}.
\end{aligned} \tag{7.15}$$

7.3 Quasilinear Equations: The Method of Characteristics

7.3.1 Geometric Interpretation

We consider the quasilinear partial differential equation in two independent variables,

$$a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0. \tag{7.16}$$

Let $u = u(x,y)$ be a solution of this equation. Then,

$$f(x,y,u) = u(x,y) - u = 0$$

describes the solution surface, or integral surface,

Integral surface.

We recall from multivariable, or vector, calculus that the normal to the integral surface is given by the gradient function,

$$\nabla f = (u_x, u_y, -1).$$

Now consider the vector of coefficients, $\mathbf{v} = (a,b,c)$ and the dot product with the gradient above:

$$\mathbf{v}\cdot\nabla f = au_x + bu_y - c.$$

This is the left hand side of the partial differential equation. Therefore, for the solution surface we have

$$\mathbf{v}\cdot\nabla f = 0,$$

or $\mathbf{v}$ is perpendicular to ∇f. Since ∇f is normal to the surface, $\mathbf{v} = (a,b,c)$ is tangent to the surface. Geometrically, $\mathbf{v}$ defines a direction field, called the characteristic field. These are shown in Figure 7.2.

The characteristic field.

7.3.2 Characteristics

We seek the forms of the characteristic curves such as the one shown in Figure 7.2. Recall that one can parametrize space curves,

$$\mathbf{c}(t) = (x(t), y(t), u(t)), \quad t \in [t_1, t_2].$$

The tangent to the curve is then

$$\mathbf{v}(t) = \frac{d\mathbf{c}(t)}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt}\right).$$

However, in the last section we saw that $\mathbf{v}(t) = (a, b, c)$ for the partial differential equation $a(x,y,u)u_x + b(x,y,u)u_y - c(x,y,u) = 0$. This gives the parametric form of the characteristic curves as

$$\frac{dx}{dt} = a, \frac{dy}{dt} = b, \frac{du}{dt} = c. \tag{7.17}$$

Another form of these equations is found by relating the differentials, dx, dy, du, to the coefficients in the differential equation. Since $x = x(t)$ and $y = y(t)$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b}{a}.$$

Similarly, we can show that

$$\frac{du}{dx} = \frac{c}{a}, \quad \frac{du}{dy} = \frac{c}{b}.$$

All of these relations can be summarized in the form

$$dt = \frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{7.18}$$

How do we use these characteristics to solve quasilinear partial differential equations? Consider the next example.

Example 7.2. Find the general solution: $u_x + u_y - u = 0$.

We first identify $a = 1$, $b = 1$, and $c = u$. The relations between the differentials is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u}.$$

We can pair the differentials in three ways:

$$\frac{dy}{dx} = 1, \quad \frac{du}{dx} = u, \quad \frac{du}{dy} = u.$$

Only two of these relations are independent. We focus on the first pair.

The first equation gives the characteristic curves in the xy-plane. This equation is easily solved to give

$$y = x + c_1.$$

The second equation can be solved to give $u = c_2 e^x$.

The goal is to find the general solution to the differential equation. Since $u = u(x, y)$, the integration "constant" is not really a constant, but is constant with respect to x. It is in fact an arbitrary constant function. In fact, we could view it as a function of c_1, the constant of integration in the first equation. Thus, we let $c_2 = G(c_1)$ for G and arbitrary function. Since $c_1 = y - x$, we can write the general solution of the differential equation as

$$u(x, y) = G(y - x)e^x.$$

Example 7.3. Solve the advection equation, $u_t + cu_x = 0$, for c a constant, and $u = u(x, t)$, $|x| < \infty$, $t > 0$.

The characteristic equations are

$$d\tau = \frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} \tag{7.19}$$

and the parametric equations are given by

$$\frac{dx}{d\tau} = c, \quad \frac{du}{d\tau} = 0. \tag{7.20}$$

These equations imply that

- $u =$ const. $= c_1$.
- $x = ct +$ const. $= ct + c_2$.

Traveling waves.

As before, we can write c_1 as an arbitrary function of c_2. However, before doing so, let's replace c_1 with the variable ξ and then we have that

$$\xi = x - ct, \quad u(x, t) = f(\xi) = f(x - ct)$$

where f is an arbitrary function. Furthermore, we see that $u(x, t) = f(x - ct)$ indicates that the solution is a wave moving in one direction in the shape of the initial function, $f(x)$. This is known as a traveling wave. A typical traveling wave is shown in Figure 7.3.

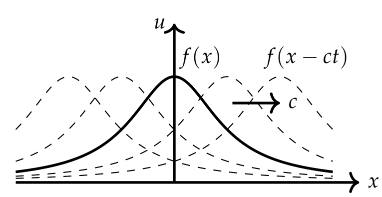

Figure 7.3: Depiction of a traveling wave. $u(x, t) = f(x)$ at $t = 0$ travels without changing shape.

Note that since $u = u(x, t)$, we have

$$\begin{aligned} 0 &= u_t + cu_x \\ &= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \\ &= \frac{du(x(t), t}{dt}. \end{aligned} \tag{7.21}$$

This implies that $u(x, t) =$ constant along the characteristics, $\frac{dx}{dt} = c$.

As with ordinary differential equations, the general solution provides an infinite number of solutions of the differential equation. If we want to pick out a particular solution, we need to specify some side conditions. We investigate this by way of examples.

Side conditions.

Example 7.4. Find solutions of $u_x + u_y - u = 0$ subject to $u(x, 0) = 1$.

We found the general solution to the partial differential equation as $u(x,y) = G(y-x)e^x$. The side condition tells us that $u = 1$ along $y = 0$. This requires

$$1 = u(x,0) = G(-x)e^x.$$

Thus, $G(-x) = e^{-x}$. Replacing x with $-z$, we find

$$G(z) = e^z.$$

Thus, the side condition has allowed for the determination of the arbitrary function $G(y-x)$. Inserting this function, we have

$$u(x,y) = G(y-x)e^x = e^{y-x}e^x = e^y.$$

Side conditions could be placed on other curves. For the general line, $y = mx + d$, we have $u(x, mx+d) = g(x)$ and for $x = d$, $u(d,y) = g(y)$. As we will see, it is possible that a given side condition may not yield a solution. We will see that conditions have to be given on non-characteristic curves in order to be useful.

Example 7.5. Find solutions of $3u_x - 2u_y + u = x$ for a) $u(x,x) = x$ and b) $u(x,y) = 0$ on $3y + 2x = 1$.

Before applying the side condition, we find the general solution of the partial differential equation. Rewriting the differential equation in standard form, we have

$$3u_x - 2u_y = x = u.$$

The characteristic equations are

$$\frac{dx}{3} = \frac{dy}{-2} = \frac{du}{x-u}. \tag{7.22}$$

These equations imply that

- $-2dx = 3dy$

  This implies that the characteristic curves (lines) are $2x + 3y = c_1$.

- $\frac{du}{dx} = \frac{1}{3}(x-u)$.

  This is a linear first order differential equation, $\frac{du}{dx} + \frac{1}{3}u = \frac{1}{3}x$. It can be solved using the integrating factor,

$$\mu(x) = \exp\left(\frac{1}{3}\int^x d\xi\right) = e^{x/3}.$$

$$\begin{aligned}
\frac{d}{dx}\left(ue^{x/3}\right) &= \frac{1}{3}xe^{x/3} \\
ue^{x/3} &= \frac{1}{3}\int^x \xi e^{\xi/3}\,d\xi + c_2 \\
&= (x-3)e^{x/3} + c_2 \\
u(x,y) &= x - 3 + c_2e^{-x/3}.
\end{aligned} \tag{7.23}$$

As before, we write c_2 as an arbitrary function of $c_1 = 2x + 3y$. This gives the general solution

$$u(x, y) = x - 3 + G(2x + 3y)e^{-x/3}.$$

Note that this is the same answer that we had found in Example 7.1

Now we can look at any side conditions and use them to determine particular solutions by picking out specific G's.

a $u(x, x) = x$

This states that $u = x$ along the line $y = x$. Inserting this condition into the general solution, we have

$$x = x - 3 + G(5x)e^{-x/3},$$

or

$$G(5x) = 3e^{x/3}.$$

Letting $z = 5x$,

$$G(z) = 3e^{z/15}.$$

The particular solution satisfying this side condition is

$$\begin{aligned} u(x, y) &= x - 3 + G(2x + 3y)e^{-x/3} \\ &= x - 3 + 3e^{(2x+3y)/15}e^{-x/3} \\ &= x - 3 + 3e^{(y-x)/5}. \end{aligned} \quad (7.24)$$

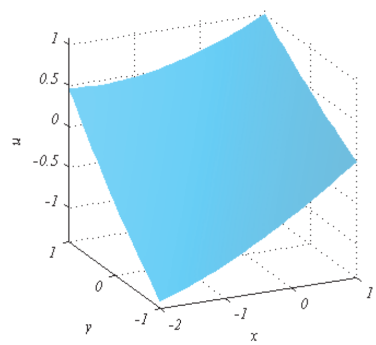

Figure 7.4: Integral surface found in Example 7.5.

This surface is shown in Figure 7.5.

In Figure 7.5 we superimpose the values of $u(x, y)$ along the characteristic curves. The characteristic curves are the red lines and the images of these curves are the black lines. The side condition is indicated with the blue curve drawn along the surface.

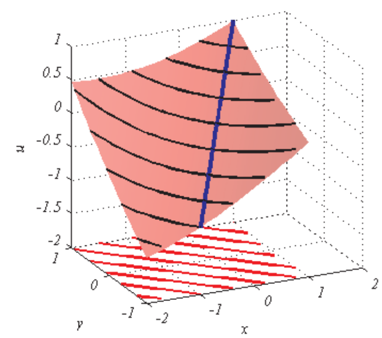

Figure 7.5: Integral surface with side condition and characteristics for Example 7.5.

The values of $u(x, y)$ are found from the side condition as follows. For $x = \xi$ on the blue curve, we know that $y = \xi$ and $u(\xi, \xi) = \xi$. Now, the characteristic lines are given by $2x + 3y = c_1$. The constant c_1 is found on the blue curve from the point of intersection with one of the black characteristic lines. For $x = y = \xi$, we have $c_1 = 5\xi$. Then, the equation of the characteristic line, which is red in Figure 7.5, is given by $y = \frac{1}{3}(5\xi - 2x)$.

Along these lines we need to find $u(x, y) = x - 3 + c_2e^{-x/3}$. First we have to find c_2. We have on the blue curve, that

$$\begin{aligned} \xi &= u(\xi, \xi) \\ &= \xi - 3 + c_2e^{-\xi/3}. \end{aligned} \quad (7.25)$$

Therefore, $c_2 = 3e^{\xi/3}$. Inserting this result into the expression for the solution, we have

$$u(x, y) = x - 3 + e^{(\xi - x)/3}.$$

So, for each ξ, one can draw a family of spacecurves

$$\left(x, \frac{1}{3}(5\xi - 2x), x - 3 + e^{(\xi - x)/3}\right)$$

yielding the integral surface.

b $u(x,y) = 0$ on $3y + 2x = 1$.
For this condition, we have

$$0 = x - 3 + G(1)e^{-x/3}.$$

We note that G is not a function in this expression. We only have one value for G. So, we cannot solve for $G(x)$. Geometrically, this side condition corresponds to one of the black curves in Figure 7.5.

7.4 Applications

7.4.1 Conservation Laws

THERE ARE MANY APPLICATIONS OF QUASILINEAR EQUATIONS, especially in fluid dynamics. The advection equation is one such example and generalizations of this example to nonlinear equations leads to some interesting problems. These equations fall into a category of equations called conservation laws. We will first discuss one-dimensional (in space) conservations laws and then look at simple examples of nonlinear conservation laws.

Conservation laws are useful in modeling several systems. They can be boiled down to determining the rate of change of some stuff, $Q(t)$, in a region, $a \le x \le b$, as depicted in Figure 7.6. The simples model is to think of fluid flowing in one dimension, such as water flowing in a stream. Or, it could be the transport of mass, such as a pollutant. One could think of traffic flow down a straight road.

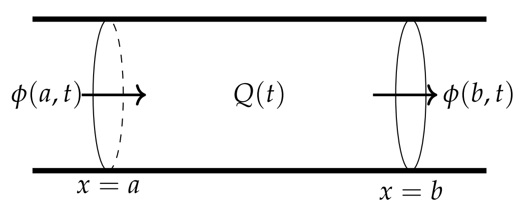

Figure 7.6: The rate of change of Q between $x = a$ and $x = b$ depends on the rates of flow through each end.

This is an example of a typical mixing problem. The rate of change of $Q(t)$ is given as

$$\text{the rate of change of } Q = \text{Rate in} - \text{Rate Out} + \text{ source term.}$$

Here the "Rate in" is how much is flowing into the region in Figure 7.6 from the $x = a$ boundary. Similarly, the "Rate out" is how much is flowing into the region from the $x = b$ boundary. [Of course, this could be the other way, but we can imagine for now that q is flowing from left to right.] We can describe this flow in terms of the flux, $\phi(x,t)$ over the ends of the region. On the left side we have a gain of $\phi(a,t)$ and on the right side of the region there is a loss of $\phi(b,t)$.

The source term would be some other means of adding or removing Q from the region. In terms of fluid flow, there could be a source of fluid inside the region such as a faucet adding more water. Or, there could be a drain letting water escape. We can denote this by the total source over the interval, $\int_a^b f(x,t)\,dx$. Here $f(x,t)$ is the source density.

In summary, the rate of change of $Q(x,t)$ can be written as

$$\frac{dQ}{dt} = \phi(a,t) - \phi(b,t) + \int_a^b f(x,y)\,dx.$$

We can write this in a slightly different form by noting that $\phi(a,t) - \phi(b,t)$ can be viewed as the evaluation of antiderivatives in the Fundamental Theorem of Calculus. Namely, we can recall that

$$\int_a^b \frac{\partial \phi(x,t)}{\partial x}\,dx = \phi(b,t) - \phi(a,t).$$

The difference is not exactly in the order that we desire, but it is easy to see that

Integral form of conservation law.

$$\frac{dQ}{dt} = -\int_a^b \frac{\partial \phi(x,t)}{\partial x}\,dx + \int_a^b f(x,t)\,dx. \tag{7.26}$$

This is the integral form of the conservation law.

We can rewrite the conservation law in differential form. First, we introduce the density function, $u(x,t)$, so that the total amount of stuff at a given time is

$$Q(t) = \int_a^b u(x,t)\,dx.$$

Introducing this form into the integral conservation law, we have

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = -\int_a^b \frac{\partial \phi}{\partial x}\,dx + \int_a^b f(x,t)\,dx. \tag{7.27}$$

Assuming that a and b are fixed in time and that the integrand is continuous, we can bring the time derivative inside the integrand and collect the three terms into one to find

$$\int_a^b \left(u_t(x,t) + \phi_x(x,t) - f(x,t)\right)\,dx = 0, \quad \forall x \in [a,b].$$

We cannot simply set the integrant to zero just because the integral vanishes. However, if this result holds for every region $[a,b]$, then we can conclude the integrand vanishes. So, under that assumption, we have the local conservation law,

Differential form of conservation law.

$$u_t(x,t) + \phi_x(x,t) = f(x,t). \tag{7.28}$$

This partial differential equation is actually an equation in terms of two unknown functions, assuming we know something about the source function. We would like to have a single unknown function. So, we need some additional information. This added information comes from the constitutive relation, a function relating the flux to the density function. Namely, we will assume that we can find the relationship $\phi = \phi(u)$. If so, then we can write

$$\frac{\partial \phi}{\partial x} = \frac{d\phi}{du}\frac{\partial u}{\partial x},$$

or $\phi_x = \phi'(u)u_x$.

Example 7.6. Inviscid Burgers' Equation Find the equation satisfied by $u(x,t)$ for $\phi(u) = \frac{1}{2}u^2$ and $f(x,t) \equiv 0$.

For this flux function we have $\phi_x = \phi'(u)u_x = uu_x$. The resulting equation is then $u_t + uu_x = 0$. This is the inviscid Burgers' equation. We will later discuss Burgers' equation.

Example 7.7. Traffic Flow

This is a simple model of one-dimensional traffic flow. Let $u(x,t)$ be the density of cars. Assume that there is no source term. For example, there is no way for a car to disappear from the flow by turning off the road or falling into a sinkhole. Also, there is no source of additional cars.

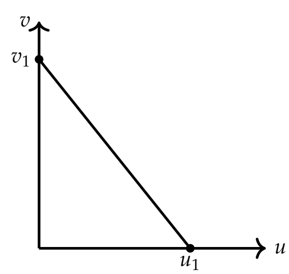

Figure 7.7: Car velocity as a function of car density.

Let $\phi(x,t)$ denote the number of cars per hour passing position x at time t. Note that the units are given by cars/mi times mi/hr. Thus, we can write the flux as $\phi = uv$, where v is the velocity of the carts at position x and time t.

In order to continue we need to assume a relationship between the car velocity and the car density. Let's assume the simplest form, a linear relationship. The more dense the traffic, we expect the speeds to slow down. So, a function similar to that in Figure 7.7 is in order. This is a straight line between the two intercepts $(0, v_1)$ and $(u_1, 0)$. It is easy to determine the equation of this line. Namely the relationship is given as

$$v = v_1 - \frac{v_1}{u_1}u.$$

This gives the flux as

$$\phi = uv = v_1\left(u - \frac{u^2}{u_1}\right).$$

We can now write the equation for the car density,

$$\begin{aligned} 0 &= u_t + \phi' u_x \\ &= u_t + v_1\left(1 - \frac{2u}{u_1}\right)u_x. \end{aligned} \tag{7.29}$$

7.4.2 *Nonlinear Advection Equations*

In this section we consider equations of the form $u_t + c(u)u_x = 0$. When $c(u)$ is a constant function, we have the advection equation. In the last two examples we have seen cases in which $c(u)$ is not a constant function.

We will apply the method of characteristics to these equations. First, we will recall how the method works for the advection equation.

The advection equation is given by $u_t + cu_x = 0$. The characteristic equations are given by

$$\frac{dx}{dt} = c, \quad \frac{du}{dt} = 0.$$

These are easily solved to give the result that

$$u(x,t) = \text{constant along the lines } x = ct + x_0,$$

where x_0 is an arbitrary constant.

The characteristic lines are shown in Figure 7.8. We note that $u(x,t) = u(x_0, 0) = f(x_0)$. So, if we know u initially, we can determine what u is at a later time.

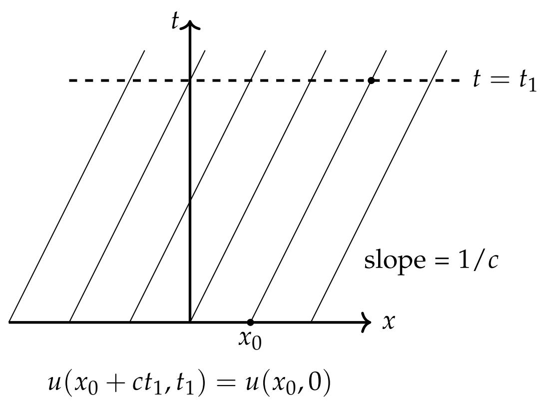

Figure 7.8: The characteristics lines the xt-plane.

In Figure 7.8 we see that the value of $u(x_0,)$ at $t = 0$ and $x = x_0$ propagates along the characteristic to a point at time $t = t_1$. From $x - ct = x_0$, we can solve for x in terms of t_1 and find that $u(x_0 + ct_1, t_1) = u(x_0, 0)$.

Plots of solutions $u(x,t)$ versus x for specific times give traveling waves as shown in Figure 7.3. In Figure 7.9 we show how each wave profile for different times are constructed for a given initial condition.

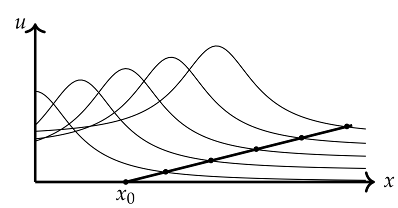

Figure 7.9: For each $x = x_0$ at $t = 0$, $u(x_0 + ct, t) = u(x_0, 0)$.

The nonlinear advection equation is given by $u_t + c(u)u_x = 0$, $|x| < \infty$. Let $u(x,0) = u_0(x)$ be the initial profile. The characteristic equations are given by

$$\frac{dx}{dt} = c(u), \quad \frac{du}{dt} = 0.$$

These are solved to give the result that

$$u(x,t) = \text{constant},$$

along the characteristic curves $x'(t) = c(u)$. The lines passing though $u(x_0,) = u_0(x_0)$ have slope $1/c(u_0(x_0))$.

Example 7.8. Solve $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

For this problem u = constant along

$$\frac{dx}{dt} = u.$$

Since u is constant, this equation can be integrated to yield $x = u(x_0, 0)t + x_0$. Inserting the initial condition, $x = e^{-x_0^2}t + x_0$. Therefore, the solution is

$$u(x,t) = e^{-x_0^2} \text{ along } x = e^{-x_0^2}t + x_0.$$

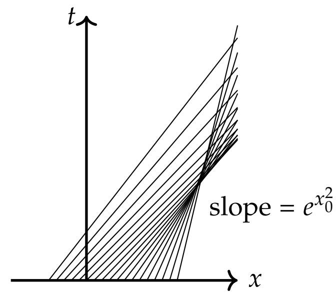

Figure 7.10: The characteristics lines the xt-plane for the nonlinear advection equation.

In Figure 7.10 the characteristics a shown. In this case we see that the characteristics intersect. In Figure charlines3 we look more specifically at the intersection of the characteristic lines for $x_0 = 0$ and $x_0 = 1$. These are approximately the first lines to intersect; i.e., there are (almost) no intersections at earlier times. At the intersection point the function $u(x,t)$ appears to take on more than one value. For the case shown, the solution wants to take the values $u = 0$ and $u = 1$.

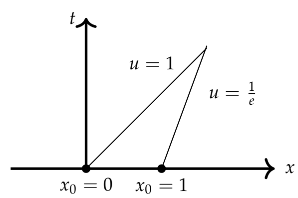

Figure 7.11: The characteristics lines for $x_0 = 0, 1$ in the xt-plane for the nonlinear advection equation.

In Figure 7.12 we see the development of the solution. This is found using a parametric plot of the points $(x_0 + te^{-x_0^2}, e^{-x_0^2})$ for different times. The initial profile propagates to the right with the higher points traveling faster than the lower points since $x'(t) = u > 0$. Around $t = 1.0$ the wave breaks and becomes multivalued. The time at which the function becomes multivalued is called the breaking time.

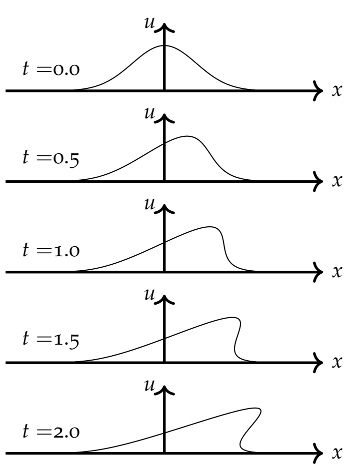

Figure 7.12: The development of a gradient catastrophe in Example 7.8 leading to a multivalued function.

7.4.3 *The Breaking Time*

In the last example we saw that for nonlinear wave speeds a gradient catastrophe might occur. The first time at which a catastrophe occurs is called the breaking time. We will determine the breaking time for the nonlinear advection equation, $u_t + c(u)u_x = 0$. For the characteristic corresponding to $x_0 = \xi$, the wavespeed is given by

$$F(\xi) = c(u_0(\xi))$$

and the characteristic line is given by

$$x = \xi + tF(\xi).$$

The value of the wave function along this characteristic is

$u_0(\xi) = u(\xi, 0)$.

$$\begin{aligned} u(x,t) &= u(\xi + tF(\xi), t) \\ &= . \end{aligned} \tag{7.30}$$

Therefore, the solution is

$$u(x,t) = u_0(\xi) \text{ along } x = \xi + tF(\xi).$$

This means that

$$u_x = u_0'(\xi)\xi_x \quad \textit{and} \quad u_t = u_0'(\xi)\xi_t.$$

We can determine ξ_x and ξ_t using the characteristic line

$$\xi = x - tF(\xi).$$

Then, we have

$$\begin{aligned} \xi_x &= 1 - tF'(\xi)\xi_x \\ &= \frac{1}{1 + tF'(\xi)}. \\ \xi_t &= \frac{\partial}{\partial t}(x - tF(\xi)) \\ &= -F(\xi) - tF'(\xi)\xi_t \\ &= \frac{-F(\xi)}{1 + tF'(\xi)}. \end{aligned} \tag{7.31}$$

Note that ξ_x and ξ_t are undefined if the denominator in both expressions vanishes, $1 + tF'(\xi) = 0$, or at time

$$t = -\frac{1}{F'(\xi)}.$$

The minimum time for this to happen in the breaking time,

The breaking time.

$$t_b = \min\left\{-\frac{1}{F'(\xi)}\right\}. \tag{7.32}$$

Example 7.9. Find the breaking time for $u_t + uu_x = 0$, $u(x,0) = e^{-x^2}$.

Since $c(u) = u$, we have

$$F(\xi) = c(u_0(\xi)) = e^{-\xi^2}$$

and

$$F'(\xi) = -2\xi e^{-\xi^2}.$$

This gives

$$t = \frac{1}{2\xi e^{-\xi^2}}.$$

We need to find the minimum time. Thus, we set the derivative equal to zero and solve for ξ.

$$\begin{aligned} 0 &= \frac{d}{d\xi}\left(\frac{e^{\xi^2}}{2\xi}\right) \\ &= \left(2-\frac{1}{\xi^2}\right)\frac{e^{\xi^2}}{2}. \end{aligned} \tag{7.33}$$

Thus, the minimum occurs for $2-\frac{1}{\xi^2}=0$, or $\xi = 1/\sqrt{2}$. This gives

$$t_b = t\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\frac{2}{\sqrt{2}e^{-1/2}}} = \sqrt{\frac{e}{2}} \approx 1.16. \tag{7.34}$$

7.4.4 *Shock Waves*

Solutions of nonlinear advection equations can become multivalued due to a gradient catastrophe. Namely, the derivatives u_t and u_x become undefined. We would like to extend solutions past the catastrophe. However, this leads to the possibility of discontinuous solutions. Such solutions which may not be differentiable or continuous in the domain are known as weak solutions. In particular, consider the initial value problem

Weak solutions.

$$u_t + \phi_x = 0, \quad x \in R, \quad t > 0, \quad u(x,0) = u_0(x).$$

Then, $u(x,t)$ is a weak solution of this problem if

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + \phi v_x]\, dxdt + \int_{-\infty}^\infty u_0(x)v(x,0)\, dx = 0$$

for all smooth functions $v \in C^\infty(R \times [0,\infty))$ with compact support, i.e., $v \equiv = 0$ outside some compact subset of the domain.

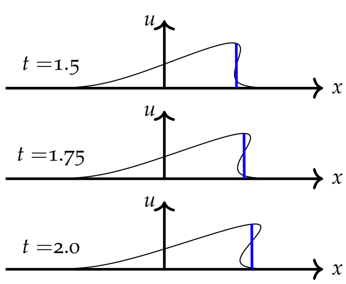

Figure 7.13: The shock solution after the breaking time.

Effectively, the weak solution that evolves will be a piecewise smooth function with a discontinuity, the shock wave, that propagates with shock speed. It can be shown that the form of the shock will be the discontinuity shown in Figure 7.13 such that the areas cut from the solutions will cancel leaving the total area under the solution constant. [See G. B. Whitham's *Linear and Nonlinear Waves*, 1973.] We will consider the discontinuity as shown in Figure 7.14.

We can find the equation for the shock path by using the integral form of the conservation law,

$$\frac{d}{dt}\int_a^b u(x,t)\, dx = \phi(a,t) - \phi(b,t).$$

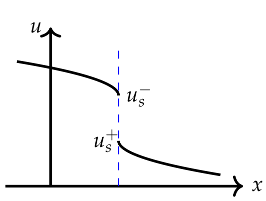

Figure 7.14: Depiction of the jump discontinuity at the shock position.

Recall that one can differentiate under the integral if $u(x,t)$ and $u_t(x,t)$ are continuous in x and t in an appropriate subset of the domain. In particular, we will integrate over the interval $[a,b]$ as shown in Figure 7.15. The domains on either side of shock path are denoted as R^+ and R^- and the limits of $x(t)$ and $u(x,t)$ as one approaches from the left of the shock are denoted by $x_s^-(t)$ and $u^- = u(x_s^-, t)$. Similarly, the limits of $x(t)$ and $u(x, t)$ as one approaches from the right of the shock are denoted by $x_s^+(t)$ and $u^+ = u(x_s^+, t)$.

We need to be careful in differentiating under the integral,

$$\begin{aligned} \frac{d}{dt}\int_a^b u(x,t)\,dx &= \frac{d}{dt}\left[\int_a^{x_s^-(t)} u(x,t)\,dx + \int_{x_s^+(t)}^b u(x,t)\,dx\right] \\ &= \int_a^{x_s^-(t)} u_t(x,t)\,dx + \int_{x_s^+(t)}^b u_t(x,t)\,dx \\ &\quad + u(x_s^-,t)\frac{dx_s^-}{dt} - u(x_s^+,t)\frac{dx_s^+}{dt} \\ &= \phi(a,t) - \phi(b,t). \end{aligned} \tag{7.35}$$

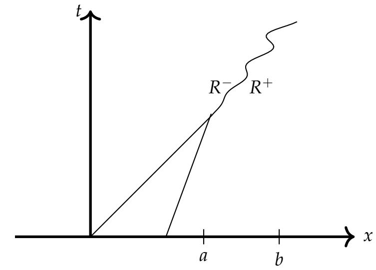

Figure 7.15: Domains on either side of shock path are denoted as R^+ and R^-.

Taking the limits $a \to x_s^-$ and $b \to x_s^+$, we have that

$$\left(u(x_s^-,t) - u(x_s^+,t)\right)\frac{dx_s}{dt} = \phi(x_s^-,t) - \phi(x_s^+,t).$$

Adopting the notation

$$[f] = f(x_s^+) - f(x_s^-),$$

we arrive at the Rankine-Hugonoit jump condition

The Rankine-Hugonoit jump condition.

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}. \tag{7.36}$$

This gives the equation for the shock path as will be shown in the next example.

Example 7.10. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases}$$

The characteristics for this partial differential equation are familiar by now. The initial condition and characteristics are shown in Figure 7.16. From $x'(t) = u$, there are two possibilities. If $u = 0$, then we have a constant. If $u = 1$ along the characteristics, the we have straight lines of slope one. Therefore, the characteristics are given by

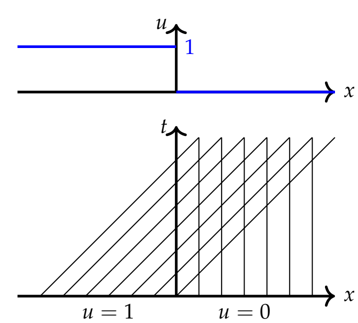

Figure 7.16: Initial condition and characteristics for Example 7.10.

$$x(t) = \begin{cases} x_0, & x > 0, \\ t + x_0, & x < 0. \end{cases}$$

As seen in Figure 7.16 the characteristics intersect immediately at $t = 0$. The shock path is found from the Rankine-Hugonoit jump condition. We first note that $\phi(u) = \frac{1}{2}u^2$, since $\phi_x = uu_x$. Then, we have

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{2}u^{+2} - \frac{1}{2}u^{-2}}{u^+ - u^-} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}\frac{(u^+ + u^-)(u^+ - u^-)}{u^+ - u^-} \\ &= \frac{1}{2}(u^+ + u^-) \\ &= \frac{1}{2}(0+1) = \frac{1}{2}. \end{aligned} \tag{7.37}$$

Now we need only solve the ordinary differential equation $x_s'(t) = \frac{1}{2}$ with initial condition $x_s(0) = 0$. This gives $x_s(t) = \frac{t}{2}$. This line separates the characteristics on the left and right side of the shock solution. The solution is given by

$$u(x,t) = \begin{cases} 1, & x \le t/2, \\ 0, & x > t/2. \end{cases}$$

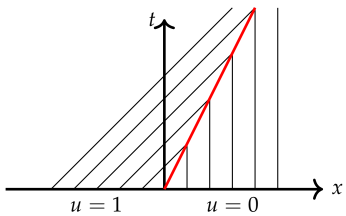

Figure 7.17: The characteristic lines end at the shock path (in red). On the left $u = 1$ and on the right $u = 0$.

In Figure 7.17 we show the characteristic lines ending at the shock path (in red) with $u = 0$ and on the right and $u = 1$ on the left of the shock path. This is consistent with the solution. One just sees the initial step function moving to the right with speed 1/2 without changing shape.

7.4.5 *Rarefaction Waves*

Shocks are not the only type of solutions encountered when the velocity is a function of u. There may sometimes be regions where the characteristic lines do not appear. A simple example is the following.

Example 7.11. Draw the characteristics for the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}$$

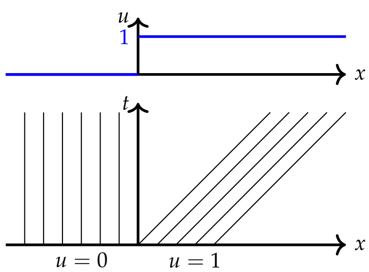

Figure 7.18: Initial condition and characteristics for Example 7.14.

In this case the solution is zero for negative values of x and positive for positive values of x as shown in Figure 7.18. Since the wavespeed is given by u, the $u = 1$ initial values have the waves on the right moving to the right and the values on the left stay fixed. This leads to the characteristics in Figure 7.18 showing a region in the xt-plane that has no characteristics. In this section we will discover how to fill in the missing characteristics and, thus, the details about the solution between the $u = 0$ and $u = 1$ values.

As motivation, we consider a smoothed out version of this problem.

Example 7.12. Draw the characteristics for the initial condition

$$u(x,0) = \begin{cases} 0, & x \le -\epsilon, \\ \frac{x+\epsilon}{2\epsilon}, & |x| \le \epsilon, \\ 1, & x > \epsilon. \end{cases}$$

The function is shown in the top graph in Figure 7.19. The leftmost and rightmost characteristics are the same as the previous example.

The only new part is determining the equations of the characteristics for $|x| \le \epsilon$. These are found using the method of characteristics as

$$x = \xi + u_0(\xi)t, \quad u_0(\xi) = \frac{\xi + \epsilon}{2\epsilon}t.$$

These characteristics are drawn in Figure 7.19 in red. Note that these lines take on slopes varying from infinite slope to slope one, corresponding to speeds going from zero to one.

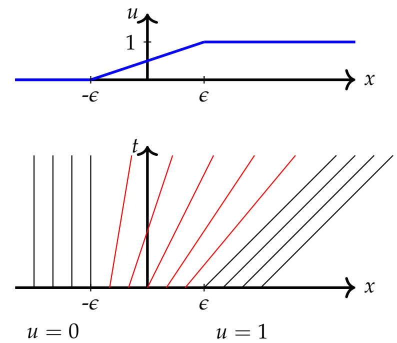

Figure 7.19: The function and characteristics for the smoothed step function.

Characteristics for rarefaction, or expansion, waves are fan-like characteristics.

Comparing the last two examples, we see that as ϵ approaches zero, the last example converges to the previous example. The characteristics in the region where there were none become a "fan". We can see this as follows. Since $|\xi| < \epsilon$ for the fan region, as ϵ gets small, so does this interval. Let's scale ξ as $\xi = \sigma\epsilon$, $\sigma \in [-1, 1]$. Then,

$$x = \sigma\epsilon + u_0(\sigma\epsilon)t, \quad u_0(\sigma\epsilon) = \frac{\sigma\epsilon + \epsilon}{2\epsilon}t = \frac{1}{2}(\sigma + 1)t.$$

For each $\sigma \in [-1, 1]$ there is a characteristic. Letting $\epsilon \to 0$, we have

$$x = ct, \quad c = \frac{1}{2}(\sigma + 1)t.$$

Thus, we have a family of straight characteristic lines in the xt-plane passing through $(0, 0)$ of the form $x = ct$ for c varying from $c = 0$ to $c = 1$. These are shown as the red lines in Figure 7.20.

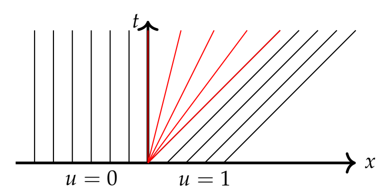

Figure 7.20: The characteristics for Example 7.14 showing the "fan" characteristics.

The fan characteristics can be written as $x/t =$ constant. So, we can seek to determine these characteristics analytically and in a straight forward manner by seeking solutions of the form $u(x, t) = g(\frac{x}{t})$.

Example 7.13. Determine solutions of the form $u(x, t) = g(\frac{x}{t})$ to $u_t + uu_x = 0$.

Inserting this guess into the differential equation, we have

$$\begin{aligned} 0 &= u_t + uu_x \\ &= \frac{1}{t}g'\left(g - \frac{x}{t}\right). \end{aligned} \tag{7.38}$$

Seek rarefaction fan waves using $u(x, t) = g(\frac{x}{t})$.

Thus, either $g' = 0$ or $g = \frac{x}{t}$. The first case will not work since this gives constant solutions. The second solution is exactly what we had obtained before. Recall that solutions along characteristics give $u(x, t) = \frac{x}{t} =$ constant. The characteristics and solutions for $t = 0, 1, 2$ are shown in Figure rarefactionfig4. At a specific time one can draw a line (dashed lines in figure) and follow the characteristics back to the $t = 0$ values, $u(\xi, 0)$ in order to construct $u(x, t)$.

As a last example, let's investigate a nonlinear model which possesses both shock and rarefaction waves.

Example 7.14. Solve the initial value problem $u_t + u^2u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x, 0) = \begin{cases} 0, & x \le 0, \\ 1, & 0 < x < 2, \\ 0, & x \ge 2. \end{cases}$$

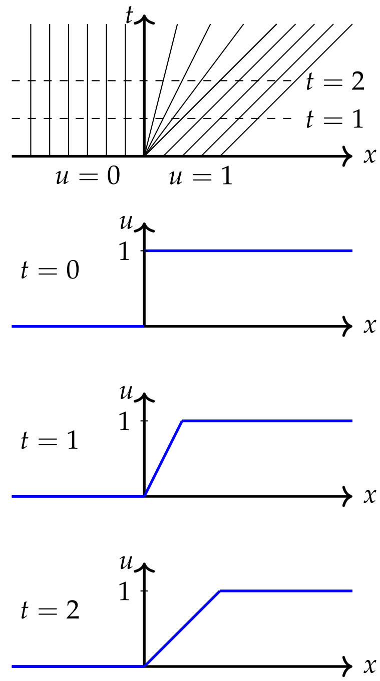

Figure 7.21: The characteristics and solutions for $t = 0, 1, 2$ for Example 7.14

The method of characteristics gives

$$\frac{dx}{dt} = u^2, \quad \frac{du}{dt} = 0.$$

Therefore,

$$u(x,t) = u_0(\xi) = \text{ const. along the lines } x(t) = u_0^2(\xi)t + \xi.$$

There are three values of $u_0(\xi)$,

$$u_0(\xi) = \begin{cases} 0, & \xi \leq 0, \\ 1, & 0 < \xi < 2, \\ 0, & \xi \geq 2. \end{cases}$$

In Figure 7.22 we see that there is a rarefaction and a gradient catastrophe.

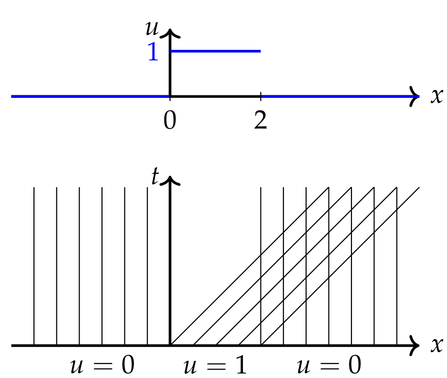

Figure 7.22: In this example there occurs a rarefaction and a gradient catastrophe.

In order to fill in the fan characteristics, we need to find solutions $u(x,t) = g(x/t)$. Inserting this guess into the differential equation, we have

$$\begin{aligned} 0 &= u_t + u^2 u_x \\ &= \frac{1}{t} g' \left(g^2 - \frac{x}{t} \right). \end{aligned} \tag{7.39}$$

Thus, either $g' = 0$ or $g^2 = \frac{x}{t}$. The first case will not work since this gives constant solutions. The second solution gives

$$g\left(\frac{x}{t}\right) = \sqrt{\frac{x}{t}}.$$

. Therefore, along the fan characteristics the solutions are $u(x,t) = \sqrt{\frac{x}{t}} =$ constant. These fan characteristics are added in Figure 7.23.

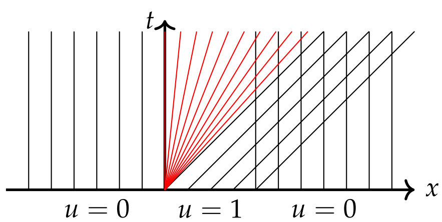

Figure 7.23: The fan characteristics are added to the other characteristic lines.

Next, we turn to the shock path. We see that the first intersection occurs at the point $(x,t) = (2,0)$. The Rankine-Hugonoit condition gives

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-} \\ &= \frac{1}{3}\frac{(u^+ - u^-)(u^{+2} + u^+u^- + u^{-2})}{u^+ - u^-} \\ &= \frac{1}{3}(u^{+2} + u^+u^- + u^{-2}) \\ &= \frac{1}{3}(0 + 0 + 1) = \frac{1}{3}. \end{aligned} \tag{7.40}$$

Thus, the shock path is given by $x_s'(t) = \frac{1}{3}$ with initial condition $x_s(0) = 2$. This gives $x_s(t) = \frac{t}{3} + 2$. In Figure 7.24 the shock path is shown in red with the fan characteristics and vertical lines meeting the path. Note that the fan lines and vertical lines cross the shock path. This leads to a change in the shock path.

The new path is found using the Rankine-Hugonoit condition with $u^+ = 0$ and $u^- = \sqrt{\frac{x}{t}}$. Thus,

$$\frac{dx_s}{dt} = \frac{[\phi]}{[u]}$$

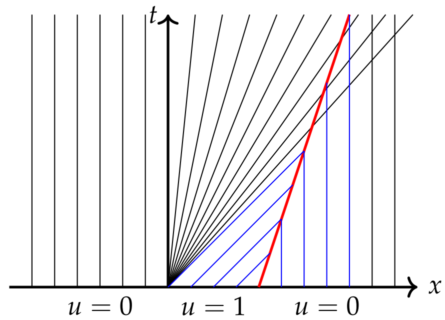

Figure 7.24: The shock path is shown in red with the fan characteristics and vertical lines meeting the path.

$$\begin{aligned}
&= \frac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-} \\
&= \frac{1}{3}\frac{(u^+ - u^-)(u^{+2} + u^+u^- + u^{-2})}{u^+ - u^-} \\
&= \frac{1}{3}(u^{+2} + u^+u^- + u^{-2}) \\
&= \frac{1}{3}(0 + 0 + \sqrt{\frac{x_s}{t}}) = \frac{1}{3}\frac{x_s}{t}.
\end{aligned} \tag{7.41}$$

We need to solve the initial value problem

$$\frac{dx_s}{dt} = \frac{1}{3}\frac{x_s}{t}, \quad x_s(3) = 3.$$

This can be done using separation of variables. Namely,

$$\int \frac{dx_s}{x_s} = \frac{1}{3}\int \frac{dt}{t}.$$

This gives the solution

$$\ln x_s = \frac{1}{3}\ln t + c \quad \Rightarrow \quad x_s = At^{1/3}.$$

Since the second shock solution starts at the point $(3, 3)$, we can determine $A = 3^{2/3}$. This gives the shock path as

$$x_s(t) = 3^{2/3}t^{1/3}.$$

In Figure 7.25 we show this shock path and the other characteristics ending on the path.

It is interesting to construct the solution at different times based on the characteristics. For a given time, t, one draws a horizontal line in the xt-plane and reads off the values of $u(x, t)$ using the values at $t = 0$ and the rarefaction solutions. This is shown in Figure 7.26. The right discontinuity in the initial profile continues as a shock front until $t = 3$. At that time the back rarefaction wave has caught up to the shock. After $t = 3$, the shock propagates forward slightly slower and the height of the shock begins to decrease. Due to the fact that the partial differential equation is a conservation law, the area under the shock remains constant as it stretches and decays in amplitude.

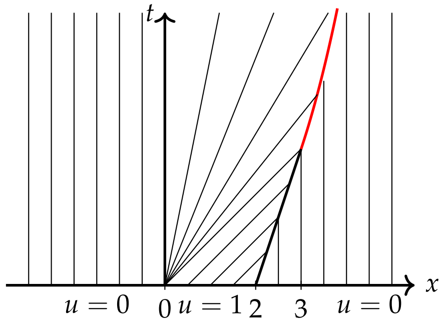

Figure 7.25: The second shock path is shown in red with the characteristics shown in all regions.

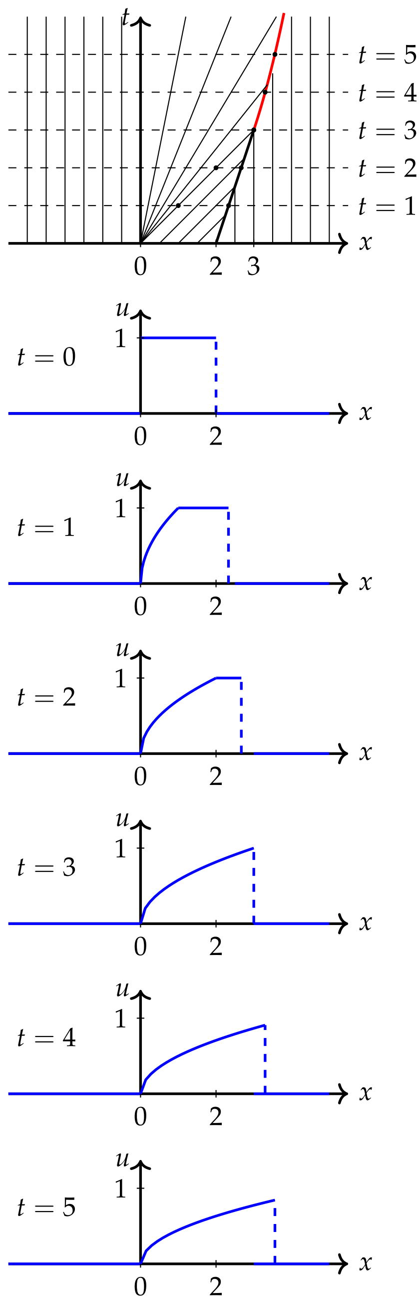

Figure 7.26: Solutions for the shock-rarefaction example.

7.4.6 *Traffic Flow*

An interesting application is that of traffic flow. We had already derived the flux function. Let's investigate examples with varying initial conditions that lead to shock or rarefaction waves. As we had seen earlier in modeling traffic flow, we can consider the flux function

$$\phi = uv = v_1 \left(u - \frac{u^2}{u_1} \right),$$

which leads to the conservation law

$$u_t + v_1(1 - \frac{2u}{u_1})u_x = 0.$$

Here $u(x,t)$ represents the density of the traffic and u_1 is the maximum density and v_1 is the initial velocity.

First, consider the flow of traffic vas it approaches a red light as shown in Figure 7.27. The traffic that is stopped has reached the maximum density u_1. The incoming traffic has a lower density, u_0. For this red light problem, we consider the initial condition

$$u(x,0) = \begin{cases} u_0, & x < 0, \\ u_1, & x \geq 0. \end{cases}$$

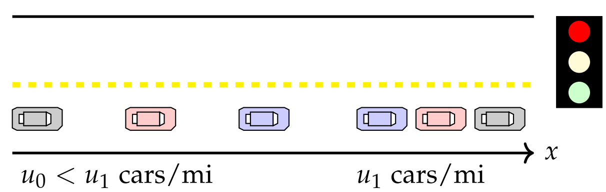

Figure 7.27: Cars approaching a red light.

The characteristics for this problem are given by

$$x = c(u(x_0,t))t + x_0,$$

where

$$c(u(x_0,t)) = v_1(1 - \frac{2u(x_0,0)}{u_1}).$$

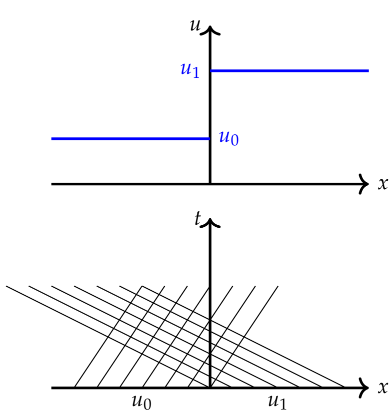

Figure 7.28: Initial condition and characteristics for the red light problem.

Since the initial condition is a piecewise-defined function, we need to consider two cases.

First, for $x \geq 0$, we have

$$c(u(x_0,t)) = c(u_1) = v_1(1 - \frac{2u_1}{u_1}) = -v_1.$$

Therefore, the slopes of the characteristics, $x = -v_1 t + x_0$ are $-1/v_1$.

For $x_0 < 0$, we have

$$c(u(x_0,t)) = c(u_0) = v_1(1 - \frac{2u_0}{u_1}).$$

So, the characteristics are $x = -v_1(1 - \frac{2u_0}{u_1})t + x_0$.

In Figure 7.28 we plot the initial condition and the characteristics for $x < 0$ and $x > 0$. We see that there are crossing characteristics and the begin crossing at $t = 0$. Therefore, the breaking time is $t_b = 0$. We need to find the shock path satisfying $x_s(0) = 0$. The Rankine-Hugonoit conditions give

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{[\phi]}{[u]} \\ &= \frac{\frac{1}{2}u^{+2} - \frac{1}{2}u^{-2}}{u^+ - u^-} \\ &= \frac{1}{2}\frac{0 - v_1\frac{u_0^2}{u_1}}{u_1 - u_0} \\ &= -v_1\frac{u_0}{u_1}. \end{aligned} \tag{7.42}$$

Thus, the shock path is found as $x_s(t) = -v_1\frac{u_0}{u_1}$.

In Figure 7.29 we show the shock path. In the top figure the red line shows the path. In the lower figure the characteristics are stopped on the shock path to give the complete picture of the characteristics. The picture was drawn with $v_1 = 2$ and $u_0/u_1 = 1/3$.

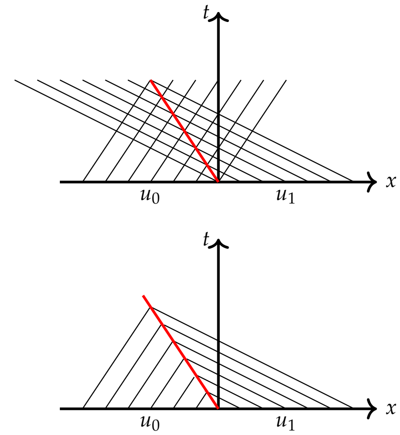

Figure 7.29: The addition of the shock path for the red light problem.

The next problem to consider is stopped traffic as the light turns green. The cars in Figure 7.30 begin to fan out when the traffic light turns green. In this model the initial condition is given by

$$u(x,0) = \begin{cases} u_1, & x \le 0, \\ 0, & x > 0. \end{cases}$$

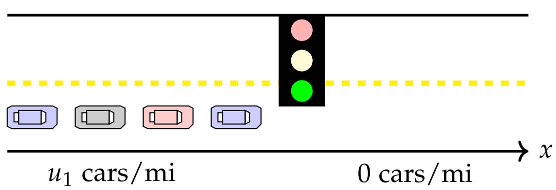

Figure 7.30: Cars begin to fan out when the traffic light turns green.

Again,

$$c(u(x_0,t)) = v_1(1 - \frac{2u(x_0,0)}{u_1}).$$

Inserting the initial values of u into this expression, we obtain constant speeds, $\pm v_1$. The resulting characteristics are given by

$$x(t) = \begin{cases} -v_1 t + x_0, & x \le 0, \\ v_1 t + x_0, & x > 0. \end{cases}$$

This leads to a rarefaction wave with the solution in the rarefaction region given by

$$u(x,t) = g(x/t) = \frac{1}{2}u_1\left(1 - \frac{1}{v_1}\frac{x}{t}\right).$$

The characteristics are shown in Figure 7.30. The full solution is then

$$u(x,t) = \begin{cases} u_1, & x \le -v_1 t, \\ g(x/t), & |x| < v_1 t, \\ 0, & x > v_1 t. \end{cases}$$

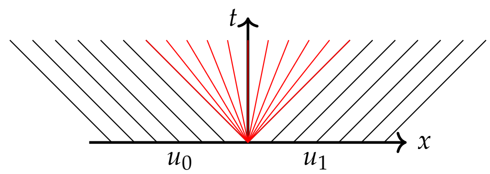

Figure 7.31: The characteristics for the green light problem.

7.5 General First Order PDEs

We have spent time solving quasilinear first order partial differential equations. We now turn to nonlinear first order equations of the form

$$F(x,y,u,u_x,u_y) = 0,$$

for $u = u(x,y)$.

If we introduce new variables, $p = u_x$ and $q = u_y$, then the differential equation takes the form

$$F(x,y,u,p,q) = 0.$$

Note that for $u(x,t)$ a function with continuous derivatives, we have

$$p_y = u_{xy} = u_{yx} = q_x.$$

We can view $F = 0$ as a surface in a five dimensional space. Since the arguments are functions of x and y, we have from the multivariable Chain Rule that

$$\begin{aligned} \frac{dF}{dx} &= F_x + F_u \frac{\partial u}{\partial x} + F_p \frac{\partial p}{\partial x} + F_q \frac{\partial q}{\partial x} \\ 0 &= F_x + pF_u + p_x F_p + p_y F_q. \end{aligned} \tag{7.43}$$

This can be rewritten as a quasilinear equation for $p(x,y)$:

$$F_p p_x + F_q p_x = -F_x - pF_u.$$

The characteristic equations are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = -\frac{dp}{F_x + pF_u}.$$

Similarly, from $\frac{dF}{dy} = 0$ we have that

$$\frac{dx}{F_p} = \frac{dy}{F_q} = -\frac{dq}{F_y + qF_u}.$$

Furthermore, since $u = u(x, y)$,

$$\begin{aligned} du &= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \\ &= pdx + qdy \\ &= pdx + q\frac{F_q}{F_p}dx \\ &= \left(p + q\frac{F_q}{F_p}\right). \end{aligned} \tag{7.44}$$

Therefore,

$$\frac{dx}{F_p} = \frac{du}{pF_p + qF_q}.$$

Combining these results we have the Charpit Equations

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}. \tag{7.45}$$

The Charpit equations. These were named after the French mathematician Paul Charpit Villecourt, who was probably the first to present the method in his thesis the year of his death, 1784. His work was further extended in 1797 by Lagrange and given a geometric explanation by Gaspard Monge (1746-1818) in 1808. This method is often called the Lagrange-Charpit method.

These equations can be used to find solutions of nonlinear first order partial differential equations as seen in the following examples.

Example 7.15. Find the general solutio of $u_x^2 + yu_y - u = 0$.

First, we introduce $u_x = p$ and $u_y = q$. Then,

$$F(x, y, u, p, q) = p^2 + qy - u = 0.$$

Next we identify

$$F_p = 2p, \quad F_q = y, \quad F_u = -1, \quad F_x = 0, \quad , F_y = q.$$

Then,

$$\begin{aligned} pF_p + qF_q &= 2p^2 + qy, \\ F_x + pF_u &= -p, \\ F_y + qF_u &= q - q = 0. \end{aligned}$$

The Charpit equations are then

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{du}{2p^2 + qy} = \frac{dp}{p} = \frac{dq}{0}.$$

The first conclusion is that $q = c_1 =$ constant. So, from the partial differential equation we have $u = p^2 + c_1 y$.

Since $du = pdx + qdy = pdx + c_1 dy$, then

$$du - cdy = \sqrt{u - c_1 y}\,dx.$$

Therefore,

$$\begin{aligned} \int \frac{d(u - c_1 y)}{\sqrt{u - c_1 y}} &= \int dx \\ \int \frac{z}{\sqrt{z}} &= x + c_2 \\ 2\sqrt{u - c_1 y} &= x + c_2. \end{aligned} \tag{7.46}$$

Solving for u, we have

$$u(x,y) = \frac{1}{4}(x+c_2)^2 + c_1 y.$$

This example required a few tricks to implement the solution. Sometimes one needs to find parametric solutions. Also, if an initial condition is given, one needs to find the particular solution. In the next example we show how parametric solutions are found to the initial value problem.

Example 7.16. Solve the initial value problem $u_x^2 + u_y + u = 0$, $u(x,0) = x$.

We consider the parametric form of the Charpit equations,

$$dt = \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}. \tag{7.47}$$

This leads to the system of equations

$$\begin{aligned}
\frac{dx}{dt} &= F_p = 2p. \\
\frac{dy}{dt} &= F_q = 1. \\
\frac{du}{dt} &= pF_p + qF_q = 2p^2 + q. \\
\frac{dp}{dt} &= -(F_x + pF_u) = -p. \\
\frac{dq}{dt} &= -(F_y + qF_u) = -q.
\end{aligned}$$

The second, fourth, and fifth equations can be solved to obtain

$$\begin{aligned}
y &= t + c_1. \\
p &= c_2 e^{-t}. \\
q &= c_3 e^{-t}.
\end{aligned}$$

Inserting these results into the remaining equations, we have

$$\begin{aligned}
\frac{dx}{dt} &= 2c_2 e^{-t}. \\
\frac{du}{dt} &= 2c_2^2 e^{-2t} + c_3 e^{-t}.
\end{aligned}$$

These equations can be integrated to find Inserting these results into the remaining equations, we have

$$\begin{aligned}
x &= -2c_2 e^{-t} + c_4. \\
u &= -c_2^2 e^{-2t} - c_3 e^{-t} + c_5.
\end{aligned}$$

This is a parametric set of equations for $u(x,t)$. Since

$$e^{-t} = \frac{x - c_4}{-2c_2},$$

we have

$$\begin{aligned} u(x,y) &= -c_2^2 e^{-2t} - c_3 e^{-t} + c_5. \\ &= -c_2^2 \left(\frac{x-c_4}{-2c_2}\right)^2 - c_3 \left(\frac{x-c_4}{-2c_2}\right) + c_5 \\ &= \frac{1}{4}(x-c_4)^2 + \frac{c_3}{2c_2}(x-c_4). \end{aligned} \tag{7.48}$$

We can use the initial conditions by first parametrizing the conditions. Let $x(s,0) = s$ and $y(s,0) = 0$, Then, $u(s,0) = s$. Since $u(x,0) = x$, $u_x(x,0) = 1$, or $p(s,0) = 1$.

From the partial differential equation, we have $p^2 + q + u = 0$. Therefore,

$$q(s,0) = -p^2(s,0) - u(s,0) = -(1+s).$$

These relations imply that

$$\begin{aligned} y(s,t)|_{t-0} &= 0 \Rightarrow c_1 = 0. \\ p(s,t)|_{t-0} &= 1 \Rightarrow c_2 = 1. \\ q(s,t)|_{t-0} &= -(1+s) = c_3. \end{aligned}$$

So,

$$\begin{aligned} y(s,t) &= t. \\ p(s,t) &= e^{-t}. \\ q(s,t) &= -(1+s)e^{-t}. \end{aligned}$$

The conditions on x and u give

$$x(s,t) = (s+2) - 2e^{-t},$$

$$u(s,t) = (s+1)e^{-t} - e^{-2t}.$$

7.6 Modern Nonlinear PDEs

The study of nonlinear partial differential equations is a hot research topic. We will (eventually) describe some examples of important evolution equations and discuss their solutions in the last chapter.

Problems

1. Write the following equations in conservation law form, $u_t + \phi_x = 0$ by finding the flux function $\phi(u)$.

a. $u_t + cu_x = 0$.

b. $u_t + uu_x - \mu u_{xx} = 0$.

c. $u_t + 6uu_x + u_{xxx} = 0.$

d. $u_t + u^2u_x + u_{xxx} = 0.$

2. Consider the Klein-Gordon equation, $u_{tt} - au_{xx} = bu$ for a and b constants. Find traveling wave solutions $u(x,t) = f(x - ct)$.

3. Find the general solution $u(x,y)$ to the following problems.

a. $u_x = 0.$

b. $yu_x - xu_y = 0.$

c. $2u_x + 3u_y = 1.$

d. $u_x + u_y = u.$

4. Solve the following problems.

a. $u_x + 2u_y = 0$, $u(x,0) = \sin x$.

b. $u_t + 4u_x = 0$, $u(x,0) = \frac{1}{1+x^2}$.

c. $yu_x - xu_y = 0$, $u(x,0) = x$.

d. $u_t + xtu_x = 0$, $u(x,0) = \sin x$.

e. $yu_x + xu_y = 0$, $u(0,y) = e^{-y^2}$.

f. $xu_t - 2xtu_x = 2tu$, $u(x,0) = x^2$.

g. $(y - u)u_x + (u - x)u_y = x - y$, $u = 0$ on $xy = 1$.

h. $yu_x + xu_y = xy$, $x, y > 0$, for $u(x,0) = e^{-x^2}, x > 0$ and $u(0,y) = e^{-y^2}, y > 0$.

5. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition $u(x,0) = \frac{1}{1+x^2}$.

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Plot solutions $u(x,t)$ at times before and after the breaking time.

6. Consider the problem $u_t + u^2u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition $u(x,0) = \frac{1}{1+x^2}$.

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Plot solutions $u(x,t)$ at times before and after the breaking time.

7. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 2, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

8. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 2, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

9. Consider the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \leq -1, \\ 2, & |x| < 1, \\ 1, & x > 1. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

10. Solve the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 1 - \frac{x}{a}, & 0 < x < a, \\ 0, & x \geq a. \end{cases}$$

11. Solve the problem $u_t + uu_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 0, & x \leq 0, \\ \frac{x}{a}, & 0 < x < a, \\ 1, & x \geq a. \end{cases}$$

12. Consider the problem $u_t + u^2u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 2, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Graphically locate where a gradient catastrophe might occur. Estimate from your plot the breaking time.

c. Analytically determine the breaking time.

d. Find the shock wave solution.

13. Consider the problem $u_t + u^2 u_x = 0$, $|x| < \infty$, $t > 0$ satisfying the initial condition

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 2, & x > 0. \end{cases}$$

a. Find and plot the characteristics.

b. Find and plot the fan characteristics.

c. Write out the rarefaction wave solution for all regions of the xt-plane.

14. Solve the initial-value problem $u_t + uu_x = 0$ $|x| < \infty$, $t > 0$ satisfying

$$u(x,0) = \begin{cases} 1, & x \leq 0, \\ 1-x, & 0 \leq x \leq 1, \\ 0, & x \geq 1. \end{cases}$$

15. Consider the stopped traffic problem in a situation where the maximum car density is 200 cars per mile and the maximum speed is 50 miles per hour. Assume that the cars are arriving at 30 miles per hour. Find the solution of this problem and determine the rate at which the traffic is backing up. How does the answer change if the cars were arriving at 15 miles per hour.

16. Solve the following nonlinear equations where $p = u_x$ and $q = u_y$.

a. $p^2 + q^2 = 1$, $u(x,x) = x$.

b. $pq = u$, $u(0,y) = y^2$.

c. $p + q = pq$, $u(x,0) = x$.

d. $pq = u^2$

e. $p^2 + qy = u$.

17. Find the solution of $xp + qy - p^2 q - u = 0$ in parametric form for the initial conditions at $t = 0$:

$$x(t,s) = s, \quad y(t,s) = 2, \quad u(t,s) = s + 1$$

.

8
Green's Functions and Nonhomogeneous Problems

"The young theoretical physicists of a generation or two earlier subscribed to the belief that: If you haven't done something important by age 30, you never will. Obviously, they were unfamiliar with the history of George Green, the miller of Nottingham." Julian Schwinger (1918-1994)

The wave equation, heat equation, and Laplace's equation are typical homogeneous partial differential equations. They can be written in the form

$$\mathcal{L}u(x) = 0,$$

where $\mathcal{L}$ is a differential operator. For example, these equations can be written as

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u &= 0, \\ \left(\frac{\partial}{\partial t} - k\nabla^2\right) u &= 0, \\ \nabla^2 u &= 0. \end{aligned} \tag{8.1}$$

In this chapter we will explore solutions of nonhomogeneous partial differential equations,

$$\mathcal{L}u(x) = f(x),$$

by seeking out the so-called Green's function. The history of the Green's function dates back to 1828, when George Green published work in which he sought solutions of Poisson's equation $\nabla^2 u = f$ for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function". In this chapter we will derive the initial value Green's function for ordinary differential equations. Later in the chapter we will return to boundary value Green's functions and Green's functions for partial differential equations.

George Green (1793-1841), a British mathematical physicist who had little formal education and worked as a miller and a baker, published *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* in which he not only introduced what is now known as Green's function, but he also introduced potential theory and Green's Theorem in his studies of electricity and magnetism. Recently his paper was posted at arXiv.org, arXiv:0807.0088.

As a simple example, consider Poisson's equation,

$$\nabla^2 u(\mathbf{r}) = f(\mathbf{r}).$$

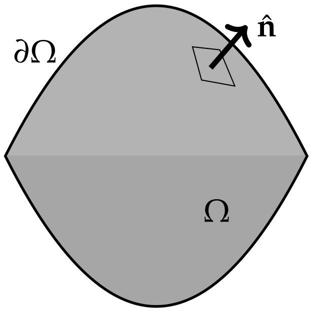

Figure 8.1: Let Poisson's equation hold inside region Ω bounded by surface $\partial\Omega$.

Let Poisson's equation hold inside a region Ω bounded by the surface $\partial\Omega$ as shown in Figure 8.1. This is the nonhomogeneous form of Laplace's equation. The nonhomogeneous term, $f(\mathbf{r})$, could represent a heat source in a steady-state problem or a charge distribution (source) in an electrostatic problem.

Now think of the source as a point source in which we are interested in the response of the system to this point source. If the point source is located at a point $\mathbf{r}'$, then the response to the point source could be felt at points $\mathbf{r}$. We will call this response $G(\mathbf{r},\mathbf{r}')$. The response function would satisfy a point source equation of the form

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}').$$

The Dirac delta function satisfies

$$\delta(\mathbf{r}) = 0, \quad \mathbf{r} \neq \mathbf{0},$$
$$\int_\Omega \delta(\mathbf{r})\, dV = 1.$$

Here $\delta(\mathbf{r}-\mathbf{r}')$ is the Dirac delta function, which we will consider in more detail in Section 10.4. A key property of this generalized function is the sifting property,

$$\int_\Omega \delta(\mathbf{r}-\mathbf{r}') f(\mathbf{r})\, dV = f(\mathbf{r}').$$

The connection between the Green's function and the solution to Poisson's equation can be found from Green's second identity:

$$\int_{\partial\Omega} [\phi\nabla\psi - \psi\nabla\phi]\cdot\mathbf{n}\, dS = \int_\Omega [\phi\nabla^2\psi - \psi\nabla^2\phi]\, dV.$$

Letting $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r},\mathbf{r}')$, we have[1]

$$\begin{aligned} &\int_{\partial\Omega} [u(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla u(\mathbf{r})]\cdot\mathbf{n}\, dS \\ &= \int_\Omega \left[u(\mathbf{r})\nabla^2 G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla^2 u(\mathbf{r})\right] dV \\ &= \int_\Omega \left[u(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') - G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\right] dV \\ &= u(\mathbf{r}') - \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\, dV. \qquad (8.2) \end{aligned}$$

[1] We note that in the following the volume and surface integrals and differentiation using ∇ are performed using the $\mathbf{r}$-coordinates.

Solving for $u(\mathbf{r}')$, we have

$$\begin{aligned} u(\mathbf{r}') &= \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\, dV \\ &\quad + \int_{\partial\Omega} [u(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla u(\mathbf{r})]\cdot\mathbf{n}\, dS. \qquad (8.3) \end{aligned}$$

If both $u(\mathbf{r})$ and $G(\mathbf{r},\mathbf{r}')$ satisfied Dirichlet conditions, $u = 0$ on $\partial\Omega$, then the last integral vanishes and we are left with[2]

$$u(\mathbf{r}') = \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\, dV.$$

[2] In many applications there is a symmetry,

$$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r}).$$

Then, the result can be written as

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r}')\, dV'.$$

So, if we know the Green's function, we can solve the nonhomogeneous differential equation. In fact, we can use the Green's function to solve nonhomogenous boundary value and initial value problems. That is what we will see develop in this chapter as we explore nonhomogeneous problems in more detail. We will begin with the search for Green's functions for ordinary differential equations.

8.1 Initial Value Green's Functions

In this section we will investigate the solution of initial value problems involving nonhomogeneous differential equations using Green's functions. Our goal is to solve the nonhomogeneous differential equation

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \tag{8.4}$$

subject to the initial conditions

$$y(0) = y_0 \quad y'(0) = v_0.$$

Since we are interested in initial value problems, we will denote the independent variable as a time variable, t.

Equation (8.4) can be written compactly as

$$L[y] = f,$$

where L is the differential operator

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t).$$

The solution is formally given by

$$y = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(t) = \int G(t,\tau) f(\tau)\, d\tau.$$

The function $G(t,\tau)$ is referred to as the kernel of the integral operator and is called the Green's function.

$G(t,\tau)$ is called a Green's function.

In the last section we solved nonhomogeneous equations like (8.4) using the Method of Variation of Parameters. Letting,

$$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t), \tag{8.5}$$

we found that we have to solve the system of equations

$$\begin{aligned} c_1'(t)y_1(t) + c_2'(t)y_2(t) &= 0. \\ c_1'(t)y_1'(t) + c_2'(t)y_2'(t) &= \frac{f(t)}{q(t)}. \end{aligned} \tag{8.6}$$

This system is easily solved to give

$$\begin{aligned} c_1'(t) &= -\frac{f(t)y_2(t)}{a(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]} \\ c_2'(t) &= \frac{f(t)y_1(t)}{a(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]}. \end{aligned} \tag{8.7}$$

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem, which is given by the determinant

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

When $y_1(t)$ and $y_2(t)$ are linearly independent, then the Wronskian is not zero and we are guaranteed a solution to the above system.

So, after an integration, we find the parameters as

$$\begin{aligned} c_1(t) &= -\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau \\ c_2(t) &= \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau, \end{aligned} \tag{8.8}$$

where t_0 and t_1 are arbitrary constants to be determined from the initial conditions.

Therefore, the particular solution of (8.4) can be written as

$$y_p(t) = y_2(t) \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t) \int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{8.9}$$

We begin with the particular solution (8.9) of the nonhomogeneous differential equation (8.4). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y_p(t) = c_1 y_1(t) + c_2 y_2(t) + y_2(t) \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t) \int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{8.10}$$

However, an appropriate choice of t_0 and t_1 can be found so that we need not explicitly write out the solution to the homogeneous problem, $c_1 y_1(t) + c_2 y_2(t)$. However, setting up the solution in this form will allow us to use t_0 and t_1 to determine particular solutions which satisfies certain homogeneous conditions. In particular, we will show that Equation (8.10) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \int_0^t G(t, \tau) f(\tau)\, d\tau, \tag{8.11}$$

where the function $G(t, \tau)$ will be identified as the Green's function.

The goal is to develop the Green's function technique to solve the initial value problem

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0. \tag{8.12}$$

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$a(t)y_h''(t) + b(t)y_h'(t) + c(t)y_h(t) = 0, \quad y_h(0) = y_0, \quad y_h'(0) = v_0. \tag{8.13}$$

We then assume that the particular solution satisfies the problem

$$a(t)y_p''(t) + b(t)y_p'(t) + c(t)y_p(t) = f(t), \quad y_p(0) = 0, \quad y_p'(0) = 0. \tag{8.14}$$

Since the differential equation is linear, then we know that

$$y(t) = y_h(t) + y_p(t)$$

is a solution of the nonhomogeneous equation. Also, this solution satisfies the initial conditions:

$$y(0) = y_h(0) + y_p(0) = y_0 + 0 = y_0,$$
$$y'(0) = y_h'(0) + y_p'(0) = v_0 + 0 = v_0.$$

Therefore, we need only focus on finding a particular solution that satisfies homogeneous initial conditions. This will be done by finding values for t_0 and t_1 in Equation (8.9) which satisfy the homogeneous initial conditions, $y_p(0) = 0$ and $y_p'(0) = 0$.

First, we consider $y_p(0) = 0$. We have

$$y_p(0) = y_2(0) \int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(0) \int_{t_0}^{0} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{8.15}$$

Here, $y_1(t)$ and $y_2(t)$ are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2 \neq (0) = 0$. Then, we have

$$y_p(0) = y_2(0) \int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau \tag{8.16}$$

We can force $y_p(0) = 0$ if we set $t_1 = 0$.

Now, we consider $y_p'(0) = 0$. First we differentiate the solution and find that

$$y_p'(t) = y_2'(t) \int_{0}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1'(t) \int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau, \tag{8.17}$$

since the contributions from differentiating the integrals will cancel. Evaluating this result at $t = 0$, we have

$$y_p'(0) = -y_1'(0) \int_{t_0}^{0} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{8.18}$$

Assuming that $y_1'(0) \neq 0$, we can set $t_0 = 0$.

Thus, we have found that

$$\begin{aligned} y_p(x) &= y_2(t) \int_{0}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t) \int_{0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau \\ &= \int_{0}^{t} \left[\frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \right] f(\tau)\, d\tau. \end{aligned} \tag{8.19}$$

This result is in the correct form and we can identify the temporal, or initial value, Green's function. So, the particular solution is given as

$$y_p(t) = \int_{0}^{t} G(t,\tau) f(\tau)\, d\tau, \tag{8.20}$$

where the initial value Green's function is defined as

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}.$$

We summarize

Solution of IVP Using the Green's Function

The solution of the initial value problem,

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0,$$

takes the form

$$y(t) = y_h(t) + \int_0^t G(t,\tau)f(\tau)\,d\tau, \tag{8.21}$$

where

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \tag{8.22}$$

is the Green's function and y_1, y_2, y_h are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0, y_h(0) = y_0, y_h'(0) = v_0.$$

Example 8.1. Solve the forced oscillator problem

$$x'' + x = 2\cos t, \quad x(0) = 4, \quad x'(0) = 0.$$

We first solve the homogeneous problem with nonhomogeneous initial conditions:

$$x_h'' + x_h = 0, \quad x_h(0) = 4, \quad x_h'(0) = 0.$$

The solution is easily seen to be $x_h(t) = 4\cos t$.

Next, we construct the Green's function. We need two linearly independent solutions, $y_1(x)$, $y_2(x)$, to the homogeneous differential equation satisfying different homogeneous conditions, $y_1(0) = 0$ and $y_2'(0) = 0$. The simplest solutions are $y_1(t) = \sin t$ and $y_2(t) = \cos t$. The Wronskian is found as

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\sin^2 t - \cos^2 t = -1.$$

Since $a(t) = 1$ in this problem, we compute the Green's function,

$$\begin{aligned} G(t,\tau) &= \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \\ &= \sin t\cos\tau - \sin\tau\cos t \\ &= \sin(t-\tau). \end{aligned} \tag{8.23}$$

Note that the Green's function depends on $t - \tau$. While this is useful in some contexts, we will use the expanded form when carrying out the integration.

We can now determine the particular solution of the nonhomogeneous differential equation. We have

$$\begin{aligned}
x_p(t) &= \int_0^t G(t,\tau) f(\tau)\, d\tau \\
&= \int_0^t (\sin t \cos\tau - \sin\tau\cos t)\,(2\cos\tau)\ d\tau \\
&= 2\sin t \int_0^t \cos^2\tau d\tau - 2\cos t \int_0^t \sin\tau\cos\tau d\tau \\
&= 2\sin t\left[\frac{\tau}{2} + \frac{1}{2}\sin 2\tau\right]_0^t - 2\cos t\left[\frac{1}{2}\sin^2\tau\right]_0^t \\
&= t\sin t. \qquad (8.24)
\end{aligned}$$

Therefore, the solution of the nonhomogeneous problem is the sum of the solution of the homogeneous problem and this particular solution: $x(t) = 4\cos t + t\sin t$.

8.2 Boundary Value Green's Functions

We solved nonhomogeneous initial value problems in Section 8.1 using a Green's function. In this section we will extend this method to the solution of nonhomogeneous boundary value problems using a boundary value Green's function. Recall that the goal is to solve the nonhomogeneous differential equation

$$L[y] = f, \quad a \le x \le b,$$

where L is a differential operator and $y(x)$ satisfies boundary conditions at $x = a$ and $x = b$.. The solution is formally given by

$$y = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi.$$

The function $G(x,\xi)$ is referred to as the kernel of the integral operator and is called the *Green's function*.

We will consider boundary value problems in Sturm-Liouville form,

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b, \qquad (8.25)$$

with fixed values of $y(x)$ at the boundary, $y(a) = 0$ and $y(b) = 0$. However, the general theory works for other forms of homogeneous boundary conditions.

We seek the Green's function by first solving the nonhomogeneous differential equation using the Method of Variation of Parameters. Recall this method from Section B.3.3. We assume a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

which is formed from two linearly independent solution of the homogeneous problem, $y_i(x)$, $i = 1, 2$. We had found that the coefficient functions satisfy the equations

$$\begin{aligned} c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{p(x)}. \end{aligned} \tag{8.26}$$

Solving this system, we obtain

$$c_1'(x) = -\frac{fy_2}{pW(y_1, y_2)},$$

$$c_1'(x) = \frac{fy_1}{pW(y_1, y_2)},$$

where $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the Wronskian. Integrating these forms and inserting the results back into the particular solution, we find

$$y(x) = y_2(x)\int_{x_1}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi - y_1(x)\int_{x_0}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi,$$

where x_0 and x_1 are to be determined using the boundary values. In particular, we will seek x_0 and x_1 so that the solution to the boundary value problem can be written as a single integral involving a Green's function. Note that we can absorb the solution to the homogeneous problem, $y_h(x)$, into the integrals with an appropriate choice of limits on the integrals.

We now look to satisfy the conditions $y(a) = 0$ and $y(b) = 0$. First we use solutions of the homogeneous differential equation that satisfy $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$. Evaluating $y(x)$ at $x = 0$, we have

$$\begin{aligned} y(a) &= y_2(a)\int_{x_1}^{a} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi - y_1(a)\int_{x_0}^{a} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi \\ &= y_2(a)\int_{x_1}^{a} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi. \end{aligned} \tag{8.27}$$

We can satisfy the condition at $x = a$ if we choose $x_1 = a$.

Similarly, at $x = b$ we find that

$$\begin{aligned} y(b) &= y_2(b)\int_{x_1}^{b} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi - y_1(b)\int_{x_0}^{b} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi \\ &= -y_1(b)\int_{x_0}^{b} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi. \end{aligned} \tag{8.28}$$

This expression vanishes for $x_0 = b$.

The general solution of the boundary value problem.

So, we have found that the solution takes the form

$$y(x) = y_2(x)\int_{a}^{x} \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}\, d\xi - y_1(x)\int_{b}^{x} \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi. \tag{8.29}$$

This solution can be written in a compact form just like we had done for the initial value problem in Section 8.1. We seek a Green's function so that the solution can be written as a single integral. We can move the functions of x under the integral. Also, since $a < x < b$, we can flip the limits in the second integral. This gives

$$y(x) = \int_a^x \frac{f(\xi)y_1(\xi)y_2(x)}{p(\xi)W(\xi)}\, d\xi + \int_x^b \frac{f(\xi)y_1(x)y_2(\xi)}{p(\xi)W(\xi)}\, d\xi. \tag{8.30}$$

This result can now be written in a compact form:

Boundary Value Green's Function

The solution of the boundary value problem

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b,$$
$$y(a) = 0, \quad y(b) = 0. \tag{8.31}$$

takes the form

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi, \tag{8.32}$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \begin{cases} \dfrac{y_1(\xi)y_2(x)}{pW}, & a \le \xi \le x, \\ \dfrac{y_1(x)y_2(\xi)}{pW}, & x \le \xi \le b, \end{cases} \tag{8.33}$$

where $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous problem satisfying $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$.

The Green's function satisfies several properties, which we will explore further in the next section. For example, the Green's function satisfies the boundary conditions at $x = a$ and $x = b$. Thus,

$$G(a,\xi) = \frac{y_1(a)y_2(\xi)}{pW} = 0,$$

$$G(b,\xi) = \frac{y_1(\xi)y_2(b)}{pW} = 0.$$

Also, the Green's function is symmetric in its arguments. Interchanging the arguments gives

$$G(\xi,x) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{pW}, & a \le x \le \xi, \\ \dfrac{y_1(\xi)y_2(x)}{pW}. & \xi \le x \le b, \end{cases} \tag{8.34}$$

But a careful look at the original form shows that

$$G(x,\xi) = G(\xi,x).$$

We will make use of these properties in the next section to quickly determine the Green's functions for other boundary value problems.

Example 8.2. Solve the boundary value problem $y'' = x^2$, $y(0) = 0 = y(1)$ using the boundary value Green's function.

We first solve the homogeneous equation, $y'' = 0$. After two integrations, we have $y(x) = Ax + B$, for A and B constants to be determined.

We need one solution satisfying $y_1(0) = 0$ Thus,

$$0 = y_1(0) = B.$$

So, we can pick $y_1(x) = x$, since A is arbitrary.

The other solution has to satisfy $y_2(1) = 0$. So,

$$0 = y_2(1) = A + B.$$

This can be solved for $B = -A$. Again, A is arbitrary and we will choose $A = -1$. Thus, $y_2(x) = 1 - x$.

For this problem $p(x) = 1$. Thus, for $y_1(x) = x$ and $y_2(x) = 1 - x$,

$$p(x)W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = x(-1) - 1(1-x) = -1.$$

Note that $p(x)W(x)$ is a constant, as it should be.

Now we construct the Green's function. We have

$$G(x,\xi) = \begin{cases} -\xi(1-x), & 0 \le \xi \le x, \\ -x(1-\xi), & x \le \xi \le 1. \end{cases} \tag{8.35}$$

Notice the symmetry between the two branches of the Green's function. Also, the Green's function satisfies homogeneous boundary conditions: $G(0,\xi) = 0$, from the lower branch, and $G(1,\xi) = 0$, from the upper branch.

Finally, we insert the Green's function into the integral form of the solution and evaluate the integral.

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi)f(\xi)\,d\xi \\ &= \int_0^1 G(x,\xi)\xi^2\,d\xi \\ &= -\int_0^x \xi(1-x)\xi^2\,d\xi - \int_x^1 x(1-\xi)\xi^2\,d\xi \\ &= -(1-x)\int_0^x \xi^3\,d\xi - x\int_x^1 (\xi^2 - \xi^3)\,d\xi \\ &= -(1-x)\left[\frac{\xi^4}{4}\right]_0^x - x\left[\frac{\xi^3}{3} - \frac{\xi^4}{4}\right]_x^1 \\ &= -\frac{1}{4}(1-x)x^4 - \frac{1}{12}x(4-3) + \frac{1}{12}x(4x^3 - 3x^4) \\ &= \frac{1}{12}(x^4 - x). \end{aligned} \tag{8.36}$$

Checking the answer, we can easily verify that $y'' = x^2$, $y(0) = 0$, and $y(1) = 0$.

8.2.1 Properties of Green's Functions

We have noted some properties of Green's functions in the last section. In this section we will elaborate on some of these properties as a tool for quickly constructing Green's functions for boundary value problems. We list five basic properties:

1. **Differential Equation:**

   The boundary value Green's function satisfies the differential equation $\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, x \neq \xi$.

   This is easily established. For $x < \xi$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, since $y_1(x)$ is a solution of the homogeneous equation, then so is $G(x,\xi)$. For $x > \xi$ we are on the first branch and $G(x,\xi)$ is proportional to $y_2(x)$. So, once again $G(x,\xi)$ is a solution of the homogeneous problem.

2. **Boundary Conditions:**

   In the example in the last section we had seen that $G(a,\xi) = 0$ and $G(b,\xi) = 0$. For example, for $x = a$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, whatever condition $y_1(x)$ satisfies, $G(x,\xi)$ will satisfy. A similar statement can be made for $x = b$.

3. **Symmetry or Reciprocity:** $G(x,\xi) = G(\xi, x)$

   We had shown this reciprocity property in the last section.

4. **Continuity of G at** $x = \xi$**:** $G(\xi^+,\xi) = G(\xi^-,\xi)$

   Here we define $\xi^{\pm}$ through the limits of a function as x approaches ξ from above or below. In particular,

   $$G(\xi^+, x) = \lim_{x\downarrow\xi} G(x,\xi), \quad x > \xi,$$

   $$G(\xi^-, x) = \lim_{x\uparrow\xi} G(x,\xi), \quad x < \xi.$$

   Setting $x = \xi$ in both branches, we have

   $$\frac{y_1(\xi)y_2(\xi)}{pW} = \frac{y_1(\xi)y_2(\xi)}{pW}.$$

   Therefore, we have established the continuity of $G(x,\xi)$ between the two branches at $x = \xi$.

5. **Jump Discontinuity of** $\frac{\partial G}{\partial x}$ **at** $x = \xi$**:**

   $$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}$$

   This case is not as obvious. We first compute the derivatives by noting which branch is involved and then evaluate the derivatives and subtract them. Thus, we have

$$\begin{aligned}\frac{\partial G(\xi^+,\xi)}{\partial x}-\frac{\partial G(\xi^-,\xi)}{\partial x} &= -\frac{1}{pW}y_1(\xi)y_2'(\xi)+\frac{1}{pW}y_1'(\xi)y_2(\xi)\\ &= -\frac{y_1'(\xi)y_2(\xi)-y_1(\xi)y_2'(\xi)}{p(\xi)(y_1(\xi)y_2'(\xi)-y_1'(\xi)y_2(\xi))}\\ &= \frac{1}{p(\xi)}. \qquad (8.37)\end{aligned}$$

Here is a summary of the properties of the boundary value Green's function based upon the previous solution.

Properties of the Green's Function

1. **Differential Equation:**
   $\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right)+q(x)G(x,\xi)=0, x\neq\xi$
2. **Boundary Conditions:** Whatever conditions $y_1(x)$ and $y_2(x)$ satisfy, $G(x,\xi)$ will satisfy.
3. **Symmetry or Reciprocity:** $G(x,\xi)=G(\xi,x)$
4. **Continuity of G at** $x=\xi$**:** $G(\xi^+,\xi)=G(\xi^-,\xi)$
5. **Jump Discontinuity of** $\frac{\partial G}{\partial x}$ **at** $x=\xi$**:**

$$\frac{\partial G(\xi^+,\xi)}{\partial x}-\frac{\partial G(\xi^-,\xi)}{\partial x}=\frac{1}{p(\xi)}$$

We now show how a knowledge of these properties allows one to quickly construct a Green's function with an example.

Example 8.3. Construct the Green's function for the problem

$$y''+\omega^2y=f(x),\quad 0<x<1,$$

$$y(0)=0=y(1),$$

with $\omega\neq 0$.

I. **Find solutions to the homogeneous equation.**

A general solution to the homogeneous equation is given as

$$y_h(x)=c_1\sin\omega x+c_2\cos\omega x.$$

Thus, for $x\neq\xi$,

$$G(x,\xi)=c_1(\xi)\sin\omega x+c_2(\xi)\cos\omega x.$$

II. **Boundary Conditions.**

First, we have $G(0,\xi)=0$ for $0\leq x\leq\xi$. So,

$$G(0,\xi)=c_2(\xi)\cos\omega x=0.$$

So,

$$G(x, \xi) = c_1(\xi) \sin \omega x, \quad 0 \leq x \leq \xi.$$

Second, we have $G(1, \xi) = 0$ for $\xi \leq x \leq 1$. So,

$$G(1, \xi) = c_1(\xi) \sin \omega + c_2(\xi) \cos \omega. = 0$$

A solution can be chosen with

$$c_2(\xi) = -c_1(\xi) \tan \omega.$$

This gives

$$G(x, \xi) = c_1(\xi) \sin \omega x - c_1(\xi) \tan \omega \cos \omega x.$$

This can be simplified by factoring out the $c_1(\xi)$ and placing the remaining terms over a common denominator. The result is

$$\begin{aligned} G(x, \xi) &= \frac{c_1(\xi)}{\cos \omega} [\sin \omega x \cos \omega - \sin \omega \cos \omega x] \\ &= -\frac{c_1(\xi)}{\cos \omega} \sin \omega (1 - x). \end{aligned} \tag{8.38}$$

Since the coefficient is arbitrary at this point, as can write the result as

$$G(x, \xi) = d_1(\xi) \sin \omega (1 - x), \quad \xi \leq x \leq 1.$$

We note that we could have started with $y_2(x) = \sin \omega (1 - x)$ as one of the linearly independent solutions of the homogeneous problem in anticipation that $y_2(x)$ satisfies the second boundary condition.

III. **Symmetry or Reciprocity**

We now impose that $G(x, \xi) = G(\xi, x)$. To this point we have that

$$G(x, \xi) = \begin{cases} c_1(\xi) \sin \omega x, & 0 \leq x \leq \xi, \\ d_1(\xi) \sin \omega (1 - x), & \xi \leq x \leq 1. \end{cases}$$

We can make the branches symmetric by picking the right forms for $c_1(\xi)$ and $d_1(\xi)$. We choose $c_1(\xi) = C \sin \omega (1 - \xi)$ and $d_1(\xi) = C \sin \omega \xi$. Then,

$$G(x, \xi) = \begin{cases} C \sin \omega (1 - \xi) \sin \omega x, & 0 \leq x \leq \xi, \\ C \sin \omega (1 - x) \sin \omega \xi, & \xi \leq x \leq 1. \end{cases}$$

Now the Green's function is symmetric and we still have to determine the constant C. We note that we could have gotten to this point using the Method of Variation of Parameters result where $C = \frac{1}{pW}$.

IV. **Continuity of** $G(x, \xi)$

We already have continuity by virtue of the symmetry imposed in the last step.

V. **Jump Discontinuity in $\frac{\partial}{\partial x}G(x,\xi)$.**

We still need to determine C. We can do this using the jump discontinuity in the derivative:

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}.$$

For this problem $p(x) = 1$. Inserting the Green's function, we have

$$\begin{aligned} 1 &= \frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} \\ &= \frac{\partial}{\partial x}\left[C \sin\omega(1-x)\sin\omega\xi\right]_{x=\xi} - \frac{\partial}{\partial x}\left[C\sin\omega(1-\xi)\sin\omega x\right]_{x=\xi} \\ &= -\omega C\cos\omega(1-\xi)\sin\omega\xi - \omega C\sin\omega(1-\xi)\cos\omega\xi \\ &= -\omega C\sin\omega(\xi+1-\xi) \\ &= -\omega C\sin\omega. \end{aligned} \tag{8.39}$$

Therefore,

$$C = -\frac{1}{\omega\sin\omega}.$$

Finally, we have the Green's function:

$$G(x,\xi) = \begin{cases} -\dfrac{\sin\omega(1-\xi)\sin\omega x}{\omega\sin\omega}, & 0 \le x \le \xi, \\ -\dfrac{\sin\omega(1-x)\sin\omega\xi}{\omega\sin\omega}, & \xi \le x \le 1. \end{cases} \tag{8.40}$$

It is instructive to compare this result to the Variation of Parameters result.

Example 8.4. Use the Method of Variation of Parameters to solve

$$y'' + \omega^2 y = f(x), \quad 0 < x < 1,$$

$$y(0) = 0 = y(1), \quad \omega \neq 0.$$

We have the functions $y_1(x) = \sin\omega x$ and $y_2(x) = \sin\omega(1-x)$ as the solutions of the homogeneous equation satisfying $y_1(0) = 0$ and $y_2(1) = 0$. We need to compute pW:

$$\begin{aligned} p(x)W(x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= -\omega\sin\omega x\cos\omega(1-x) - \omega\cos\omega x\sin\omega(1-x) \\ &= -\omega\sin\omega \end{aligned} \tag{8.41}$$

Inserting this result into the Variation of Parameters result for the Green's function leads to the same Green's function as above.

8.2.2 *The Differential Equation for the Green's Function*

As we progress in the book we will develop a more general theory of Green's functions for ordinary and partial differential equations. Much of this theory relies on understanding that the Green's function really is the system response function to a point source. This begins with recalling that the boundary value Green's function satisfies a homogeneous differential equation for $x \neq \xi$,

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, \quad x \neq \xi. \tag{8.42}$$

For $x = \xi$, we have seen that the derivative has a jump in its value. This is similar to the step, or Heaviside, function,

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

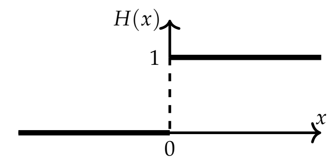

Figure 8.2: The Heaviside step function, $H(x)$.

This function is shown in Figure 8.2 and we see that the derivative of the step function is zero everywhere except at the jump, or discontinuity . At the jump, there is an infinite slope, though technically, we have learned that there is no derivative at this point. We will try to remedy this situation by introducing the Dirac delta function,

$$\delta(x) = \frac{d}{dx}H(x).$$

We will show that the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{8.43}$$

However, we will first indicate why this knowledge is useful for the general theory of solving differential equations using Green's functions.

The Dirac delta function is described in more detail in Section 10.4. The key property we will need here is the sifting property,

$$\int_a^b f(x)\delta(x-\xi)\,dx = f(\xi)$$

for $a < \xi < b$.

As noted, the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi) \tag{8.44}$$

and satisfies homogeneous conditions. We will use the Green's function to solve the nonhomogeneous equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x). \tag{8.45}$$

These equations can be written in the more compact forms

$$\begin{aligned} \mathcal{L}[y] &= f(x) \\ \mathcal{L}[G] &= \delta(x-\xi). \end{aligned} \tag{8.46}$$

Using these equations, we can determine the solution, $y(x)$, in terms of the Green's function. Multiplying the first equation by $G(x,\xi)$, the second equation by $y(x)$, and then subtracting, we have

$$G\mathcal{L}[y] - y\mathcal{L}[G] = f(x)G(x,\xi) - \delta(x-\xi)y(x).$$

Now, integrate both sides from $x = a$ to $x = b$. The left hand side becomes

$$\int_a^b \left[f(x)G(x,\xi) - \delta(x-\xi)y(x)\right]\,dx = \int_a^b f(x)G(x,\xi)\,dx - y(\xi).$$

Using Green's Identity from Section 4.2.2, the right side is

$$\int_a^b \left(G\mathcal{L}[y] - y\mathcal{L}[G]\right)\, dx = \left[p(x)\left(G(x,\xi)y'(x) - y(x)\frac{\partial G}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}.$$

Recall that Green's identity is given by

$$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\, dx = [p(uv' - vu')]_a^b.$$

The general solution in terms of the boundary value Green's function with corresponding surface terms.

Combining these results and rearranging, we obtain

$$\begin{aligned} y(\xi) &= \int_a^b f(x)G(x,\xi)\, dx \\ &\quad - \left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi) - G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b}. \end{aligned} \tag{8.47}$$

We will refer to the extra terms in the solution,

$$S(b,\xi) - S(a,\xi) = \left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi) - G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b},$$

as the boundary, or surface, terms. Thus,

$$y(\xi) = \int_a^b f(x)G(x,\xi)\, dx - [S(b,\xi) - S(a,\xi)].$$

The result in Equation (8.47) is the key equation in determining the solution of a nonhomogeneous boundary value problem. The particular set of boundary conditions in the problem will dictate what conditions $G(x,\xi)$ has to satisfy. For example, if we have the boundary conditions $y(a) = 0$ and $y(b) = 0$, then the boundary terms yield

$$\begin{aligned} y(\xi) &= \int_a^b f(x)G(x,\xi)\, dx - \left[p(b)\left(y(b)\frac{\partial G}{\partial x}(b,\xi) - G(b,\xi)y'(b)\right)\right] \\ &\quad + \left[p(a)\left(y(a)\frac{\partial G}{\partial x}(a,\xi) - G(a,\xi)y'(a)\right)\right] \\ &= \int_a^b f(x)G(x,\xi)\, dx + p(b)G(b,\xi)y'(b) - p(a)G(a,\xi)y'(a). \end{aligned} \tag{8.48}$$

The right hand side will only vanish if $G(x,\xi)$ also satisfies these homogeneous boundary conditions. This then leaves us with the solution

$$y(\xi) = \int_a^b f(x)G(x,\xi)\, dx.$$

We should rewrite this as a function of x. So, we replace ξ with x and x with ξ. This gives

$$y(x) = \int_a^b f(\xi)G(\xi,x)\, d\xi.$$

However, this is not yet in the desirable form. The arguments of the Green's function are reversed. But, in this case $G(x,\xi)$ is symmetric in its arguments. So, we can simply switch the arguments getting the desired result.

We can now see that the theory works for other boundary conditions. If we had $y'(a) = 0$, then the $y(a)\frac{\partial G}{\partial x}(a,\xi)$ term in the boundary terms could be made to vanish if we set $\frac{\partial G}{\partial x}(a, \xi) = 0$. So, this confirms that other boundary value problems can be posed besides the one elaborated upon in the chapter so far.

We can even adapt this theory to nonhomogeneous boundary conditions. We first rewrite Equation (8.47) as

$$y(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi - \left[p(\xi) \left(y(\xi) \frac{\partial G}{\partial \xi}(x, \xi) - G(x, \xi) y'(\xi) \right) \right]_{\xi=a}^{\xi=b}. \tag{8.49}$$

Let's consider the boundary conditions $y(a) = \alpha$ and $y'(b) = \beta$. We also assume that $G(x, \xi)$ satisfies homogeneous boundary conditions,

$$G(a, \xi) = 0, \quad \frac{\partial G}{\partial \xi}(b, \xi) = 0.$$

in both x and ξ since the Green's function is symmetric in its variables. Then, we need only focus on the boundary terms to examine the effect on the solution. We have

$$\begin{aligned} S(b, x) - S(a, x) &= \left[p(b) \left(y(b) \frac{\partial G}{\partial \xi}(x, b) - G(x, b) y'(b) \right) \right] \\ &\quad - \left[p(a) \left(y(a) \frac{\partial G}{\partial \xi}(x, a) - G(x, a) y'(a) \right) \right] \\ &= -\beta p(b) G(x, b) - \alpha p(a) \frac{\partial G}{\partial \xi}(x, a). \end{aligned} \tag{8.50}$$

Therefore, we have the solution

General solution satisfying the nonhomogeneous boundary conditions $y(a) = \alpha$ and $y'(b) = \beta$. Here the Green's function satisfies homogeneous boundary conditions, $G(a, \xi) = 0$, $\frac{\partial G}{\partial \xi}(b, \xi) = 0$.

$$y(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi + \beta p(b) G(x, b) + \alpha p(a) \frac{\partial G}{\partial \xi}(x, a). \tag{8.51}$$

This solution satisfies the nonhomogeneous boundary conditions.

Example 8.5. Solve $y'' = x^2, \quad y(0) = 1, y(1) = 2$ using the boundary value Green's function.

This is a modification of Example 8.2. We can use the boundary value Green's function that we found in that problem,

$$G(x, \xi) = \begin{cases} -\xi(1 - x), & 0 \le \xi \le x, \\ -x(1 - \xi), & x \le \xi \le 1. \end{cases} \tag{8.52}$$

We insert the Green's function into the general solution (8.51) and use the given boundary conditions to obtain

$$\begin{aligned} y(x) &= \int_0^1 G(x, \xi) \xi^2\, d\xi - \left[y(\xi) \frac{\partial G}{\partial \xi}(x, \xi) - G(x, \xi) y'(\xi) \right]_{\xi=0}^{\xi=1} \\ &= \int_0^x (x - 1) \xi^3\, d\xi + \int_x^1 x(\xi - 1) \xi^2\, d\xi + y(0) \frac{\partial G}{\partial \xi}(x, 0) - y(1) \frac{\partial G}{\partial \xi}(x, 1) \\ &= \frac{(x - 1) x^4}{4} + \frac{x(1 - x^4)}{4} - \frac{x(1 - x^3)}{3} + (x - 1) - 2x \\ &= \frac{x^4}{12} + \frac{35}{12} x - 1. \end{aligned} \tag{8.53}$$

Of course, this problem can be solved by direct integration. The general solution is

$$y(x) = \frac{x^4}{12} + c_1 x + c_2.$$

Inserting this solution into each boundary condition yields the same result.

The Green's function satisfies a delta function forced differential equation.

We have seen how the introduction of the Dirac delta function in the differential equation satisfied by the Green's function, Equation (8.44), can lead to the solution of boundary value problems. The Dirac delta function also aids in the interpretation of the Green's function. We note that the Green's function is a solution of an equation in which the nonhomogeneous function is $\delta(x-\xi)$. Note that if we multiply the delta function by $f(\xi)$ and integrate, we obtain

$$\int_{-\infty}^{\infty} \delta(x-\xi) f(\xi)\, d\xi = f(x).$$

We can view the delta function as a unit impulse at $x = \xi$ which can be used to build $f(x)$ as a sum of impulses of different strengths, $f(\xi)$. Thus, the Green's function is the response to the impulse as governed by the differential equation and given boundary conditions.

Derivation of the jump condition for the Green's function.

In particular, the delta function forced equation can be used to derive the jump condition. We begin with the equation in the form

$$\frac{\partial}{\partial x}\left(p(x) \frac{\partial G(x,\xi)}{\partial x} \right) + q(x) G(x,\xi) = \delta(x-\xi). \tag{8.54}$$

Now, integrate both sides from $\xi - \epsilon$ to $\xi + \epsilon$ and take the limit as $\epsilon \to 0$. Then,

$$\begin{aligned} \lim_{\epsilon\to 0} \int_{\xi-\epsilon}^{\xi+\epsilon} \left[\frac{\partial}{\partial x}\left(p(x) \frac{\partial G(x,\xi)}{\partial x} \right) + q(x) G(x,\xi) \right] dx &= \lim_{\epsilon\to 0} \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\, dx \\ &= 1. \end{aligned} \tag{8.55}$$

Since the $q(x)$ term is continuous, the limit as $\epsilon \to 0$ of that term vanishes. Using the Fundamental Theorem of Calculus, we then have

$$\lim_{\epsilon\to 0} \left[p(x) \frac{\partial G(x,\xi)}{\partial x} \right]_{\xi-\epsilon}^{\xi+\epsilon} = 1. \tag{8.56}$$

This is the jump condition that we have been using!

8.2.3 *Series Representations of Green's Functions*

There are times that it might not be so simple to find the Green's function in the simple closed form that we have seen so far. However, there is a method for determining the Green's functions of Sturm-Liouville boundary value problems in the form of an eigenfunction expansion. We will finish our discussion of Green's functions for ordinary differential equations by showing how one obtains such series representations. (Note that we are really just repeating the steps towards developing eigenfunction expansion which we had seen in Section 4.3.)

We will make use of the complete set of eigenfunctions of the differential operator, $\mathcal{L}$, satisfying the homogeneous boundary conditions:

$$\mathcal{L}[\phi_n] = -\lambda_n \sigma \phi_n, \quad n = 1, 2, \ldots$$

We want to find the particular solution y satisfying $\mathcal{L}[y] = f$ and homogeneous boundary conditions. We assume that

$$y(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Inserting this into the differential equation, we obtain

$$\mathcal{L}[y] = \sum_{n=1}^{\infty} a_n \mathcal{L}[\phi_n] = -\sum_{n=1}^{\infty} \lambda_n a_n \sigma \phi_n = f.$$

This has resulted in the generalized Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \sigma \phi_n(x)$$

with coefficients

$$c_n = -\lambda_n a_n.$$

We have seen how to compute these coefficients earlier in section 4.3. We multiply both sides by $\phi_k(x)$ and integrate. Using the orthogonality of the eigenfunctions,

$$\int_a^b \phi_n(x)\phi_k(x)\sigma(x)\, dx = N_k \delta_{nk},$$

one obtains the expansion coefficients (if $\lambda_k \neq 0$)

$$a_k = -\frac{(f, \phi_k)}{N_k \lambda_k},$$

where $(f, \phi_k) \equiv \int_a^b f(x)\phi_k(x)\, dx$.

As before, we can rearrange the solution to obtain the Green's function. Namely, we have

$$y(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{-N_n \lambda_n} \phi_n(x) = \int_a^b \underbrace{\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-N_n \lambda_n}}_{G(x,\xi)} f(\xi)\, d\xi$$

Therefore, we have found the Green's function as an expansion in the eigenfunctions:

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n}. \tag{8.57}$$

Green's function as an expansion in the eigenfunctions.

We will conclude this discussion with an example. We will solve this problem three different ways in order to summarize the methods we have used in the text.

Example using the Green's function eigenfunction expansion.

Example 8.6. Solve

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0$$

using the Green's function eigenfunction expansion.

The Green's function for this problem can be constructed fairly quickly for this problem once the eigenvalue problem is solved. The eigenvalue problem is

$$\phi''(x) + 4\phi(x) = -\lambda\phi(x),$$

where $\phi(0) = 0$ and $\phi(1) = 0$. The general solution is obtained by rewriting the equation as

$$\phi''(x) + k^2\phi(x) = 0,$$

where

$$k^2 = 4 + \lambda.$$

Solutions satisfying the boundary condition at $x = 0$ are of the form

$$\phi(x) = A \sin kx.$$

Forcing $\phi(1) = 0$ gives

$$0 = A \sin k \Rightarrow k = n\pi, \quad k = 1, 2, 3 \ldots.$$

So, the eigenvalues are

$$\lambda_n = n^2\pi^2 - 4, \quad n = 1, 2, \ldots$$

and the eigenfunctions are

$$\phi_n = \sin n\pi x, \quad n = 1, 2, \ldots.$$

We also need the normalization constant, N_n. We have that

$$N_n = \|\phi_n\|^2 = \int_0^1 \sin^2 n\pi x = \frac{1}{2}.$$

We can now construct the Green's function for this problem using Equation (8.57).

$$G(x,\xi) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4 - n^2\pi^2)}. \tag{8.58}$$

Using this Green's function, the solution of the boundary value problem becomes

$$\begin{aligned}
y(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\
&= \int_0^1 \left(2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4 - n^2\pi^2)}\right) \xi^2\, d\xi \\
&= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4 - n^2\pi^2)} \int_0^1 \xi^2 \sin n\pi\xi\, d\xi \\
&= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4 - n^2\pi^2)} \left[\frac{(2 - n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right]
\end{aligned} \tag{8.59}$$

We can compare this solution to the one we would obtain if we did not employ Green's functions directly. The eigenfunction expansion method for solving boundary value problems, which we saw earlier is demonstrated in the next example.

Example 8.7. Solve

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0$$

using the eigenfunction expansion method.

Example using the eigenfunction expansion method.

We assume that the solution of this problem is in the form

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Inserting this solution into the differential equation $\mathcal{L}[y] = x^2$, gives

$$\begin{aligned} x^2 &= \mathcal{L}\left[\sum_{n=1}^{\infty} c_n \sin n\pi x\right] \\ &= \sum_{n=1}^{\infty} c_n \left[\frac{d^2}{dx^2}\sin n\pi x + 4\sin n\pi x\right] \\ &= \sum_{n=1}^{\infty} c_n [4 - n^2\pi^2] \sin n\pi x \end{aligned} \tag{8.60}$$

This is a Fourier sine series expansion of $f(x) = x^2$ on $[0,1]$. Namely,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

In order to determine the c_n's in Equation (8.60), we will need the Fourier sine series expansion of x^2 on $[0,1]$. Thus, we need to compute

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 x^2 \sin n\pi x \\ &= 2\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right], \quad n = 1,2,\ldots. \end{aligned} \tag{8.61}$$

The resulting Fourier sine series is

$$x^2 = 2\sum_{n=1}^{\infty}\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right] \sin n\pi x.$$

Inserting this expansion in Equation (8.60), we find

$$2\sum_{n=1}^{\infty}\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right] \sin n\pi x = \sum_{n=1}^{\infty} c_n [4 - n^2\pi^2] \sin n\pi x.$$

Due to the linear independence of the eigenfunctions, we can solve for the unknown coefficients to obtain

$$c_n = 2\frac{(2-n^2\pi^2)(-1)^n - 2}{(4-n^2\pi^2)n^3\pi^3}.$$

Therefore, the solution using the eigenfunction expansion method is

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\ &= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4-n^2\pi^2)} \left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right]. \end{aligned} \tag{8.62}$$

We note that the solution in this example is the same solution as we had obtained using the Green's function obtained in series form in the previous example.

One remaining question is the following: Is there a closed form for the Green's function and the solution to this problem? The answer is yes!

Example 8.8. Find the closed form Green's function for the problem

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0$$

and use it to obtain a closed form solution to this boundary value problem.

Using the closed form Green's function.

We note that the differential operator is a special case of the example done in section 8.2. Namely, we pick $\omega = 2$. The Green's function was already found in that section. For this special case, we have

$$G(x,\xi) = \begin{cases} -\dfrac{\sin 2(1-\xi)\sin 2x}{2\sin 2}, & 0 \le x \le \xi, \\ -\dfrac{\sin 2(1-x)\sin 2\xi}{2\sin 2}, & \xi \le x \le 1. \end{cases} \tag{8.63}$$

Using this Green's function, the solution to the boundary value problem is readily computed

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\ &= -\int_0^x \frac{\sin 2(1-x)\sin 2\xi}{2\sin 2}\xi^2\, d\xi + \int_x^1 \frac{\sin 2(\xi-1)\sin 2x}{2\sin 2}\xi^2\, d\xi \\ &= -\frac{1}{4\sin 2}\left[-x^2\sin 2 + (1-\cos^2 x)\sin 2 + \sin x\cos x(1+\cos 2)\right]. \\ &= -\frac{1}{4\sin 2}\left[-x^2\sin 2 + 2\sin^2 x\sin 1\cos 1 + 2\sin x\cos x\cos^2 1)\right]. \\ &= -\frac{1}{8\sin 1\cos 1}\left[-x^2\sin 2 + 2\sin x\cos 1(\sin x\sin 1 + \cos x\cos 1)\right]. \\ &= \frac{x^2}{4} - \frac{\sin x\cos(1-x)}{4\sin 1}. \end{aligned} \tag{8.64}$$

In Figure 8.3 we show a plot of this solution along with the first five terms of the series solution. The series solution converges quickly to the closed form solution.

As one last check, we solve the boundary value problem directly, as we had done in the last chapter.

Example 8.9. Solve directly:

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0.$$

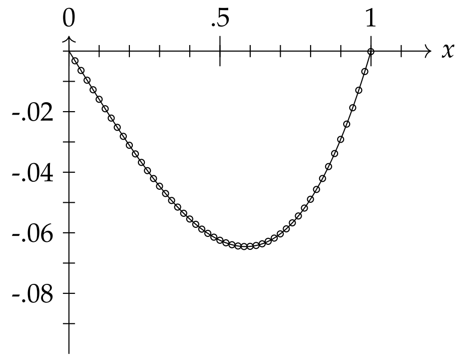

Figure 8.3: Plots of the exact solution to Example 8.6 with the first five terms of the series solution.

The problem has the general solution

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + y_p(x),$$

where y_p is a particular solution of the nonhomogeneous differential equation. Using the Method of Undetermined Coefficients, we assume a solution of the form

$$y_p(x) = Ax^2 + Bx + C.$$

Direct solution of the boundary value problem.

Inserting this guess into the nonhomogeneous equation, we have

$$2A + 4(Ax^2 + Bx + C) = x^2,$$

Thus, $B = 0$, $4A = 1$ and $2A + 4C = 0$. The solution of this system is

$$A = \frac{1}{4}, \quad B = 0, \quad C = -\frac{1}{8}.$$

So, the general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2}{4} - \frac{1}{8}.$$

We next determine the arbitrary constants using the boundary conditions. We have

$$\begin{aligned} 0 &= y(0) \\ &= c_1 - \frac{1}{8} \\ 0 &= y(1) \\ &= c_1 \cos 2 + c_2 \sin 2 + \frac{1}{8} \end{aligned} \tag{8.65}$$

Thus, $c_1 = \frac{1}{8}$ and

$$c_2 = -\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}.$$

Inserting these constants into the solution we find the same solution as before.

$$y(x) = \frac{1}{8}\cos 2x - \left[\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}\right] \sin 2x + \frac{x^2}{4} - \frac{1}{8}$$

$$\begin{aligned}
&= \frac{(\cos 2x - 1)\sin 2 - \sin 2x(1+\cos 2)}{8\sin 2} + \frac{x^2}{4} \\
&= \frac{(-2\sin^2 x)2\sin 1\cos 1 - \sin 2x(2\cos^2 1)}{16\sin 1\cos 1} + \frac{x^2}{4} \\
&= -\frac{(\sin^2 x)\sin 1 + \sin x\cos x(\cos 1)}{4\sin 1} + \frac{x^2}{4} \\
&= \frac{x^2}{4} - \frac{\sin x\cos(1-x)}{4\sin 1}. \qquad (8.66)
\end{aligned}$$

8.2.4 *The Generalized Green's Function*

WHEN SOLVING $Lu = f$ USING EIGENFUCTION EXPANSIONS, there can be a problem when there are zero eigenvalues. Recall from Section 4.3 the solution of this problem is given by

$$\begin{aligned}
y(x) &= \sum_{n=1}^{\infty} c_n\phi_n(x), \\
c_n &= -\frac{\int_a^b f(x)\phi_n(x)\,dx}{\lambda_m\int_a^b \phi_n^2(x)\sigma(x)\,dx}. \qquad (8.67)
\end{aligned}$$

Here the eigenfunctions, $\phi_n(x)$, satisfy the eigenvalue problem

$$\mathcal{L}\phi_n(x) = -\lambda_n\sigma(x)\phi_n(x), \quad x \in [a,b]$$

subject to given homogeneous boundary conditions.

Note that if $\lambda_m = 0$ for some value of $n = m$, then c_m is undefined. However, if we require

$$(f,\phi_m) = \int_a^b f(x)\phi_n(x)\,dx = 0,$$

The Fredholm Alternative.

then there is no problem. This is a form of the Fredholm Alternative. Namely, if $\lambda_n = 0$ for some n, then there is no solution unless $f,\phi_m) = 0$; i.e., f is orthogonal to ϕ_n. In this case, a_n will be arbitrary and there are an infinite number of solutions.

Example 8.10. $u'' = f(x)$, $u'(0) = 0$, $u'(L) = 0$.

The eigenfunctions satisfy $\phi_n''(x) = -\lambda_n\phi_n(x)$, $\phi_n'(0) = 0$, $\phi_n'(L) = 0$. There are the usual solutions,

$$\phi_n(x) = \cos\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,\ldots.$$

However, when $\lambda_n = 0$, $\phi_0''(x) = 0$. So, $\phi_0(x) = Ax + B$. The boundary conditions are satisfied if $A = 0$. So, we can take $\phi_0(x) = 1$. Therefore, there exists an eigenfunction corresponding to a zero eigenvalue. Thus, in order to have a solution, we have to require

$$\int_0^L f(x)\,dx = 0.$$

Example 8.11. $u'' + \pi^2 u = \beta + 2x$, $u(0) = 0$, $u(1) = 0$.

In this problem we check to see if there is an eigenfunctions with a zero eigenvalue. The eigenvalue problem is

$$\phi'' + \pi^2\phi = 0, \quad \phi(0) = 0, \quad \phi(1) = 0.$$

A solution satisfying this problem is easily founds as

$$\phi(x) = \sin \pi x.$$

Therefore, there is a zero eigenvalue. For a solution to exist, we need to require

$$\begin{aligned} 0 &= \int_0^1 (\beta + 2x)\sin \pi x \, dx \\ &= -\frac{\beta}{\pi}\cos \pi x\Big|_0^1 + 2\left[\frac{1}{\pi}x\cos \pi x - \frac{1}{\pi^2}\sin \pi x\right]_0^1 \\ &= -\frac{2}{\pi}(\beta + 1). \end{aligned} \tag{8.68}$$

Thus, either $\beta = -1$ or there are no solutions.

Recall the series representation of the Green's function for a Sturm-Liouville problem in Equation (8.57),

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n}. \tag{8.69}$$

We see that if there is a zero eigenvalue, then we also can run into trouble as one of the terms in the series is undefined.

Recall that the Green's function satisfies the differential equation $LG(x,\xi) = \delta(x-\xi)$, $x, \xi \in [a,b]$ and satisfies some appropriate set of boundary conditions. Using the above analysis, if there is a zero eigenvalue, then $L\phi_h(x) = 0$. In order for a solution to exist to the Green's function differential equation, then $f(x) = \delta(x-\xi)$ and we have to require

$$0 = (f, \phi_h) = \int_a^b \phi_h(x)\delta(x-\xi)\, dx = \phi_h(\xi),$$

for and $\xi \in [a,b]$. Therefore, the Green's function does not exist.

We can fix this problem by introducing a modified Green's function. Let's consider a modified differential equation,

$$LG_M(x,\xi) = \delta(x-\xi) + c\phi_h(x)$$

for some constant c. Now, the orthogonality condition becomes

$$\begin{aligned} 0 = (f, \phi_h) &= \int_a^b \phi_h(x)[\delta(x-\xi) + c\phi_h(x)]\, dx \\ &= \phi_h(\xi) + c\int_a^b \phi_h^2(x)\, dx. \end{aligned} \tag{8.70}$$

Thus, we can choose

$$c = -\frac{\phi_h(\xi)}{\int_a^b \phi_h^2(x)\, dx}$$

Using the modified Green's function, we can obtain solutions to $Lu = f$. We begin with Green's identity from Section 4.2.2, given by

$$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx = [p(uv' - vu')]_a^b.$$

Letting $v = G_M$, we have

$$\int_a^b (G_M\mathcal{L}[u] - u\mathcal{L}[G_M])\;dx = \left[p(x)\left(G_M(x,\xi)u'(x) - u(x)\frac{\partial G_M}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}.$$

Applying homogeneous boundary conditions, the right hand side vanishes. Then we have

$$\begin{aligned} 0 &= \int_a^b (G_M(x,\xi)\mathcal{L}[u(x)] - u(x)\mathcal{L}[G_M(x,\xi)])\;dx \\ &= \int_a^b (G_M(x,\xi)f(x) - u(x)[\delta(x-\xi) + c\phi_h(x)])\;dx \\ u(\xi) &= \int_a^b G_M(x,\xi)f(x)\,dx - c\int_a^b u(x)\phi_h(x)\,dx. \end{aligned} \tag{8.71}$$

Noting that $u(x,t) = c_1\phi_h(x) + u_p(x)$,, the last integral gives

$$-c\int_a^b u(x)\phi_h(x)\,dx = \frac{\phi_h(\xi)}{\int_a^b \phi_h^2(x)\,dx}\int_a^b \phi_h^2(x)\,dx = c_1\phi_h(\xi).$$

Therefore, the solution can be written as

$$u(x) = \int_a^b f(\xi)G_M(x,\xi)\,d\xi + c_1\phi_h(x).$$

Here we see that there are an infinite number of solutions when solutions exist.

Example 8.12. Use the modified Green's function to solve $u'' + \pi^2 u = 2x - 1$, $u(0) = 0$, $u(1) = 0$.

We have already seen that a solution exists for this problem, where we have set $\beta = -1$ in Example 8.11.

We construct the modified Green's function from the solutions of

$$\phi_n'' + \pi^2\phi_n = -\lambda_n\phi_n, \quad \phi(0) = 0, \quad \phi(1) = 0.$$

The general solutions of this equation are

$$\phi_n(x) = c_1\cos\sqrt{\pi^2 + \lambda_n}x + c_2\sin\sqrt{\pi^2 + \lambda_n}x.$$

Applying the boundary conditions, we have $c_1 = 0$ and $\sqrt{\pi^2 + \lambda_n} = n\pi$. Thus, the eigenfunctions and eigenvalues are

$$\phi_n(x) = \sin n\pi x, \quad \lambda_n = (n^2 - 1)\pi^2, \quad n = 1, 2, 3, \ldots.$$

Note that $\lambda_1 = 0$.

The modified Green's function satisfies

$$\frac{d^2}{dx^2}G_M(x,\xi) + \pi^2 G_M(x,\xi) = \delta(x-\xi) + c\phi_h(x),$$

where

$$\begin{aligned} c &= -\frac{\phi_1(\xi)}{\int_0^1 \phi_1^2(x)\,dx} \\ &= -\frac{\sin \pi\xi}{\int_0^1 \sin^2 \pi\xi, dx} \\ &= -2\sin \pi\xi. \end{aligned} \tag{8.72}$$

We need to solve for $G_M(x,\xi)$. The modified Green's function satisfies

$$\frac{d^2}{dx^2}G_M(x,\xi) + \pi^2 G_M(x,\xi) = \delta(x-\xi) - 2\sin\pi\xi\sin\pi x,$$

and the boundary conditions $G_M(0,\xi) = 0$ and $G_M(1,\xi) = 0$. We assume an eigenfunction expansion,

$$G_M(x,\xi) = \sum_{n=1}^{\infty} c_n(\xi)\sin n\pi x.$$

Then,

$$\begin{aligned} \delta(x-\xi) - 2\sin\pi\xi\sin\pi x &= \frac{d^2}{dx^2}G_M(x,\xi) + \pi^2 G_M(x,\xi) \\ &= -\sum_{n=1}^{\infty} \lambda_n c_n(\xi)\sin n\pi x \end{aligned} \tag{8.73}$$

The coefficients are found as

$$\begin{aligned} -\lambda_n c_n &= 2\int_0^1 [\delta(x-\xi) - 2\sin\pi\xi\sin\pi x]\sin n\pi x\,dx \\ &= 2\sin n\pi\xi - 2\sin\pi\xi\delta_{n1}. \end{aligned} \tag{8.74}$$

Therefore, $c_1 = 0$ and $c_n = 2\sin n\pi\xi$, for $n > 1$.

We have found the modified Green's function as

$$G_M(x,\xi) = -2\sum_{n=2}^{\infty}\frac{\sin n\pi x\sin n\pi\xi}{\lambda_n}.$$

We can use this to find the solution. Namely, we have (for $c_1 = 0$)

$$\begin{aligned} u(x) &= \int_0^1 (2\xi-1)G_M(x,\xi)\,d\xi \\ &= -2\sum_{n=2}^{\infty}\frac{\sin n\pi x}{\lambda_n}\int_0^1 (2\xi-1)\sin n\pi\xi\,dx \\ &= -2\sum_{n=2}^{\infty}\frac{\sin n\pi x}{(n^2-1)\pi^2}\left[-\frac{1}{n\pi}(2\xi-1)\cos n\pi\xi + \frac{1}{n^2\pi^2}\sin n\pi\xi\right]_0^1 \\ &= 2\sum_{n=2}^{\infty}\frac{1+\cos n\pi}{n(n^2-1)\pi^3}\sin n\pi x. \end{aligned} \tag{8.75}$$

We can also solve this problem exactly. The general solution is given by

$$u(x) = c_1\sin\pi x + c_2\cos\pi x + \frac{2x-1}{\pi^2}.$$

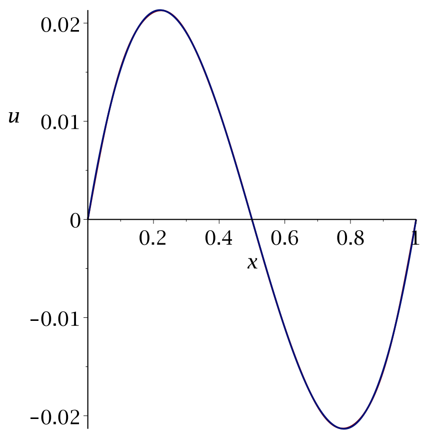

Figure 8.4: The solution for Example 8.12.

Imposing the boundary conditions, we obtain

$$u(x) = c_1 \sin \pi x + \frac{1}{\pi^2} \cos \pi x + \frac{2x-1}{\pi^2}.$$

Notice that there are an infinite number of solutions. Choosing $c_1 = 0$, we have the particular solution

$$u(x) = \frac{1}{\pi^2} \cos \pi x + \frac{2x-1}{\pi^2}.$$

In Figure 8.4 we plot this solution and that obtained using the modified Green's function. The result is that they are in complete agreement.

8.3 *The Nonhomogeneous Heat Equation*

Boundary value Green's functions do not only arise in the solution of nonhomogeneous ordinary differential equations. They are also important in arriving at the solution of nonhomogeneous partial differential equations. In this section we will show that this is the case by turning to the nonhomogeneous heat equation.

8.3.1 *Nonhomogeneous Time Independent Boundary Conditions*

Consider the nonhomogeneous heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} u_t - k u_{xx} &= h(x), \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a, \quad u(L,t) = b, \\ u(x,0) &= f(x). \end{aligned} \tag{8.76}$$

We are interested in finding a particular solution to this initial-boundary value problem. In fact, we can represent the solution to the general nonhomogeneous heat equation as the sum of two solutions that solve different problems.

First, we let $v(x,t)$ satisfy the homogeneous problem

$$\begin{aligned} v_t - k v_{xx} &= 0, \qquad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \\ v(x,0) &= g(x), \end{aligned} \tag{8.77}$$

which has homogeneous boundary conditions.

The steady state solution, $w(t)$, satisfies a nonhomogeneous differential equation with nonhomogeneous boundary conditions. The transient solution, $v(t)$, satisfies the homogeneous heat equation with homogeneous boundary conditions and satisfies a modified initial condition.

We will also need a steady state solution to the original problem. A steady state solution is one that satisfies $u_t = 0$. Let $w(x)$ be the steady state solution. It satisfies the problem

$$\begin{aligned} -k w_{xx} &= h(x), \qquad 0 \le x \le L. \\ w(0,t) &= a, \quad w(L,t) = b. \end{aligned} \tag{8.78}$$

Now consider $u(x,t) = w(x) + v(x,t)$, the sum of the steady state solution, $w(x)$, and the transient solution, $v(x,t)$. We first note that $u(x,t)$ satisfies the nonhomogeneous heat equation,

$$\begin{aligned} u_t - ku_{xx} &= (w+v)_t - (w+v)_{xx} \\ &= v_t - kv_{xx} - kw_{xx} \equiv h(x). \end{aligned} \tag{8.79}$$

The boundary conditions are also satisfied. Evaluating, $u(x,t)$ at $x = 0$ and $x = L$, we have

$$\begin{aligned} u(0,t) &= w(0) + v(0,t) = a, \\ u(L,t) &= w(L) + v(L,t) = b. \end{aligned} \tag{8.80}$$

Finally, the initial condition gives

The transient solution satisfies

$$v(x,0) = f(x) - w(x).$$

$$u(x,0) = w(x) + v(x,0) = w(x) + g(x).$$

Thus, if we set $g(x) = f(x) - w(x)$, then $u(x,t) = w(x) + v(x,t)$ will be the solution of the nonhomogeneous boundary value problem. We all ready know how to solve the homogeneous problem to obtain $v(x,t)$. So, we only need to find the steady state solution, $w(x)$.

There are several methods we could use to solve Equation (8.78) for the steady state solution. One is the Method of Variation of Parameters, which is closely related to the Green's function method for boundary value problems which we described in the last several sections. However, we will just integrate the differential equation for the steady state solution directly to find the solution. From this solution we will be able to read off the Green's function.

Integrating the steady state equation (8.78) once, yields

$$\frac{dw}{dx} = -\frac{1}{k}\int_0^x h(z)\,dz + A,$$

where we have been careful to include the integration constant, $A = w'(0)$. Integrating again, we obtain

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)\,dz\right) dy + Ax + B,$$

where a second integration constant has been introduced. This gives the general solution for Equation (8.78).

The boundary conditions can now be used to determine the constants. It is clear that $B = a$ for the condition at $x = 0$ to be satisfied. The second condition gives

$$b = w(L) = -\frac{1}{k}\int_0^L \left(\int_0^y h(z)\,dz\right) dy + AL + a.$$

Solving for A, we have

$$A = \frac{1}{kL}\int_0^L \left(\int_0^y h(z)\,dz\right) dy + \frac{b-a}{L}.$$

Inserting the integration constants, the solution of the boundary value problem for the steady state solution is then

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)\,dz\right)\,dy + \frac{x}{kL}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + \frac{b-a}{L}x + a.$$

The steady state solution.

This is sufficient for an answer, but it can be written in a more compact form. In fact, we will show that the solution can be written in a way that a Green's function can be identified.

First, we rewrite the double integrals as single integrals. We can do this using integration by parts. Consider integral in the first term of the solution,

$$I = \int_0^x \left(\int_0^y h(z)\,dz\right)\,dy.$$

Setting $u = \int_0^y h(z)\,dz$ and $dv = dy$ in the standard integration by parts formula, we obtain

$$\begin{aligned} I &= \int_0^x \left(\int_0^y h(z)\,dz\right)\,dy \\ &= y\int_0^y h(z)\,dz\Big|_0^x - \int_0^x yh(y)\,dy \\ &= \int_0^x (x-y)h(y)\,dy. \end{aligned} \tag{8.81}$$

Thus, the double integral has now collapsed to a single integral. Replacing the integral in the solution, the steady state solution becomes

$$w(x) = -\frac{1}{k}\int_0^x (x-y)h(y)\,dy + \frac{x}{kL}\int_0^L (L-y)h(y)\,dy + \frac{b-a}{L}x + a.$$

We can make a further simplification by combining these integrals. This can be done if the integration range, $[0, L]$, in the second integral is split into two pieces, $[0, x]$ and $[x, L]$. Writing the second integral as two integrals over these subintervals, we obtain

$$\begin{aligned} w(x) &= -\frac{1}{k}\int_0^x (x-y)h(y)\,dy + \frac{x}{kL}\int_0^x (L-y)h(y)\,dy \\ &\quad + \frac{x}{kL}\int_x^L (L-y)h(y)\,dy + \frac{b-a}{L}x + a. \end{aligned} \tag{8.82}$$

Next, we rewrite the integrands,

$$\begin{aligned} w(x) &= -\frac{1}{k}\int_0^x \frac{L(x-y)}{L}h(y)\,dy + \frac{1}{k}\int_0^x \frac{x(L-y)}{L}h(y)\,dy \\ &\quad + \frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)\,dy + \frac{b-a}{L}x + a. \end{aligned} \tag{8.83}$$

It can now be seen how we can combine the first two integrals:

$$w(x) = -\frac{1}{k}\int_0^x \frac{y(L-x)}{L}h(y)\,dy + \frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)\,dy + \frac{b-a}{L}x + a.$$

The resulting integrals now take on a similar form and this solution can be written compactly as

$$w(x) = -\int_0^L G(x,y)[-\frac{1}{k}h(y)]\,dy + \frac{b-a}{L}x + a,$$

where

$$G(x,y) = \begin{cases} \dfrac{x(L-y)}{L}, & 0 \le x \le y, \\ \dfrac{y(L-x)}{L}, & y \le x \le L, \end{cases}$$

is the Green's function for this problem.

The Green's function for the steady state problem.

The full solution to the original problem can be found by adding to this steady state solution a solution of the homogeneous problem,

$$\begin{aligned} u_t - ku_{xx} &= 0, \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \\ u(x,0) &= f(x) - w(x). \end{aligned} \tag{8.84}$$

Example 8.13. Solve the nonhomogeneous problem,

$$\begin{aligned} u_t - u_{xx} &= 10, \quad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 20, \quad u(1,t) = 0, \\ u(x,0) &= 2x(1-x). \end{aligned} \tag{8.85}$$

In this problem we have a rod initially at a temperature of $u(x,0) = 2x(1-x)$. The ends of the rod are maintained at fixed temperatures and the bar is continually heated at a constant temperature, represented by the source term, 10.

First, we find the steady state temperature, $w(x)$, satisfying

$$\begin{aligned} -w_{xx} &= 10, \quad 0 \le x \le 1. \\ w(0,t) &= 20, \quad w(1,t) = 0. \end{aligned} \tag{8.86}$$

Using the general solution, we have

$$w(x) = \int_0^1 10G(x,y)\,dy - 20x + 20,$$

where

$$G(x,y) = \begin{cases} x(1-y), & 0 \le x \le y, \\ y(1-x), & y \le x \le 1, \end{cases}$$

we compute the solution

$$\begin{aligned} w(x) &= \int_0^x 10y(1-x)\,dy + \int_x^1 10x(1-y)\,dy - 20x + 20 \\ &= 5(x - x^2) - 20x + 20, \\ &= 20 - 15x - 5x^2. \end{aligned} \tag{8.87}$$

Checking this solution, it satisfies both the steady state equation and boundary conditions.

The transient solution satisfies

$$\begin{aligned} v_t - v_{xx} &= 0, \quad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(1,t) = 0, \\ v(x,0) &= x(1-x) - 10. \end{aligned} \tag{8.88}$$

Recall, that we have determined the solution of this problem as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where the Fourier sine coefficients are given in terms of the initial temperature distribution,

$$b_n = 2\int_0^1 [x(1-x) - 10] \sin n\pi x \, dx, \quad n = 1, 2, \ldots.$$

Therefore, the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x + 20 - 15x - 5x^2.$$

Note that for large t, the transient solution tends to zero and we are left with the steady state solution as expected.

8.3.2 Time Dependent Boundary Conditions

In the last section we solved problems with time independent boundary conditions using equilibrium solutions satisfying the steady state heat equation sand nonhomogeneous boundary conditions. When the boundary conditions are time dependent, we can also convert the problem to an auxiliary problem with homogeneous boundary conditions.

Consider the problem

$$\begin{aligned} u_t - ku_{xx} &= h(x), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a(t), \quad u(L,t) = b(t), \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{8.89}$$

We define $u(x,t) = v(x,t) + w(x,t)$, where $w(x,t)$ is a modified form of the steady state solution from the last section,

$$w(x,t) = a(t) + \frac{b(t) - a(t)}{L} x.$$

Noting that

$$\begin{aligned} u_t &= v_t + \dot{a} + \frac{\dot{b} - \dot{a}}{L} x, \\ u_{xx} &= v_{xx}, \end{aligned} \tag{8.90}$$

we find that $v(x,t)$ is a solution of the problem

$$\begin{aligned} v_t - kv_{xx} &= h(x) - \left[\dot{a}(t) + \frac{\dot{b}(t) - \dot{a}(t)}{L} x\right], \quad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= f(x) - \left[a(0) + \frac{b(0) - a(0)}{L} x\right], \quad 0 \le x \le L. \end{aligned} \tag{8.91}$$

Thus, we have converted the original problem into a nonhomogeneous heat equation with homogeneous boundary conditions and a new source term and new initial condition.

Example 8.14. Solve the problem

$$\begin{aligned} u_t - u_{xx} &= x, \qquad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 2, \quad u(L,t) = t, \quad t > 0 \\ u(x,0) &= 3\sin 2\pi x + 2(1-x), \quad 0 \le x \le 1. \end{aligned} \tag{8.92}$$

We first define

$$u(x,t) = v(x,t) + 2 + (t-2)x.$$

Then, $v(x,t)$ satisfies the problem

$$\begin{aligned} v_t - v_{xx} &= 0, \qquad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= 3\sin 2\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{8.93}$$

This problem is easily solved. The general solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}.$$

We can see that the Fourier coefficients all vanish except for b_2. This gives $v(x,t) = 3\sin 2\pi x e^{-4\pi^2 t}$ and, therefore, we have found the solution

$$u(x,t) = 3\sin 2\pi x e^{-4\pi^2 t} + 2 + (t-2)x.$$

8.4 *Green's Functions for 1D Partial Differential Equations*

In Section 8.1 we encountered the initial value Green's function for initial value problems for ordinary differential equations. In that case we were able to express the solution of the differential equation $L[y] = f$ in the form

$$y(t) = \int G(t,\tau) f(\tau)\, d\tau,$$

where the Green's function $G(t,\tau)$ was used to handle the nonhomogeneous term in the differential equation. In a similar spirit, we can introduce Green's functions of different types to handle nonhomogeneous terms, nonhomogeneous boundary conditions, or nonhomogeneous initial conditions. Occasionally, we will stop and rearrange the solutions of different problems and recast the solution and identify the Green's function for the problem.

In this section we will rewrite the solutions of the heat equation and wave equation on a finite interval to obtain an initial value Green;s function. Assuming homogeneous boundary conditions and a homogeneous differential operator, we can write the solution of the heat equation in the form

$$u(x,t) = \int_0^L G(x,\xi;t,t_0) f(\xi)\, d\xi.$$

where $u(x,t_0) = f(x)$, and the solution of the wave equation as

$$u(x,t) = \int_0^L G_c(x,\xi,t,t_0) f(\xi)\, d\xi + \int_0^L G_s(x,\xi,t,t_0) g(\xi)\, d\xi.$$

where $u(x,t_0) = f(x)$ and $u_t(x,t_0) = g(x)$. The functions $G(x,\xi;t,t_0)$, $G(x,\xi;t,t_0)$, and $G(x,\xi;t,t_0)$ are initial value Green's functions and we will need to explore some more methods before we can discuss the properties of these functions. [For example, see Section.]

We will now turn to showing that for the solutions of the one dimensional heat and wave equations with fixed, homogeneous boundary conditions, we can construct the particular Green's functions.

8.4.1 Heat Equation

In Section 2.5 we obtained the solution to the one dimensional heat equation on a finite interval satisfying homogeneous Dirichlet conditions,

$$\begin{aligned} u_t &= k u_{xx}, \quad 0 < t, \quad 0 \leq x \leq L, \\ u(x,0) &= f(x), \quad 0 < x < L, \\ u(0,t) &= 0, \quad t > 0, \\ u(L,t) &= 0, \quad t > 0. \end{aligned} \tag{8.94}$$

The solution we found was the Fourier sine series

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n k t} \sin \frac{n\pi x}{L},$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

and the Fourier sine coefficients are given in terms of the initial temperature distribution,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots.$$

Inserting the coefficients b_n into the solution, we have

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(\xi) \sin \frac{n\pi\xi}{L}\, d\xi \right) e^{\lambda_n k t} \sin \frac{n\pi x}{L}.$$

Interchanging the sum and integration, we obtain

$$u(x,t) = \int_0^L \left(\frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi\xi}{L} e^{\lambda_n k t} \right) f(\xi)\, d\xi.$$

This solution is of the form

$$u(x,t) = \int_0^L G(x,\xi;t,0) f(\xi)\, d\xi.$$

Here the function $G(x,\xi;t,0)$ is the initial value Green's function for the heat equation in the form

$$G(x,\xi;t,0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} e^{\lambda_n kt}.$$

which involves a sum over eigenfunctions of the spatial eigenvalue problem, $X_n(x) = \sin\frac{n\pi x}{L}$.

8.4.2 Wave Equation

The solution of the one dimensional wave equation (1.2),

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < t, \quad 0 \le x \le L, \\ u(0,t) &= 0, \quad u(L,0) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L, \end{aligned} \tag{8.95}$$

was found as

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right] \sin\frac{n\pi x}{L}.$$

The Fourier coefficients were determined from the initial conditions,

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}, \\ g(x) &= \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\frac{n\pi x}{L}, \end{aligned} \tag{8.96}$$

as

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(\xi) \sin\frac{n\pi\xi}{L}\, d\xi, \\ B_n &= \frac{L}{n\pi c}\frac{2}{L}\int_0^L f(\xi) \sin\frac{n\pi\xi}{L}\, d\xi. \end{aligned} \tag{8.97}$$

Inserting these coefficients into the solution and interchanging integration with summation, we have

$$\begin{aligned} u(x,t) &= \int_0^{\infty}\left[\frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} \cos\frac{n\pi ct}{L}\right] f(\xi)\, d\xi \\ &\quad + \int_0^{\infty}\left[\frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} \frac{\sin\frac{n\pi ct}{L}}{n\pi c/L}\right] g(\xi)\, d\xi \\ &= \int_0^L G_c(x,\xi,t,0) f(\xi)\, d\xi + \int_0^L G_s(x,\xi,t,0) g(\xi)\, d\xi. \end{aligned} \tag{8.98}$$

In this case, we have defined two Green's functions,

$$\begin{aligned} G_c(x,\xi,t,0) &= \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} \cos\frac{n\pi ct}{L}, \\ G_s(x,\xi,t,0) &= \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} \frac{\sin\frac{n\pi ct}{L}}{n\pi c/L}. \end{aligned} \tag{8.99}$$

The first, G_c, provides the response to the initial profile and the second, G_s, to the initial velocity.

8.5 Green's Functions for the 2D Poisson Equation

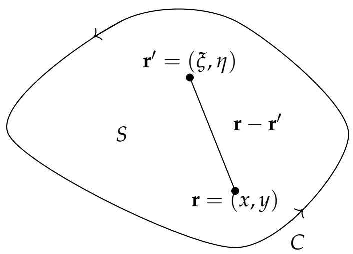

Figure 8.5: Domain for solving Poisson's equation.

In this section we consider the two dimensional Poisson equation with Dirichlet boundary conditions. We consider the problem

$$\begin{aligned} \nabla^2 u &= f, \quad \text{in } D, \\ u &= g, \quad \text{on } C, \end{aligned} \tag{8.100}$$

for the domain in Figure 8.5

We seek to solve this problem using a Green's function. As in earlier discussions, the Green's function satisfies the differential equation and homogeneous boundary conditions. The associated problem is given by

$$\begin{aligned} \nabla^2 G &= \delta(\xi - x, \eta - y), \quad \text{in } D, \\ G &\equiv 0, \quad \text{on } C. \end{aligned} \tag{8.101}$$

However, we need to be careful as to which variables appear in the differentiation. Many times we just make the adjustment after the derivation of the solution, assuming that the Green's function is symmetric in its arguments. However, this is not always the case and depends on things such as the self-adjointedness of the problem. Thus, we will assume that the Green's function satisfies

$$\nabla^2_{r'} G = \delta(\xi - x, \eta - y),$$

where the notation $\nabla_{r'}$ means differentiation with respect to the variables ξ and η. Thus,

$$\nabla^2_{r'} G = \frac{\partial^2 G}{\partial \xi^2} + \frac{\partial^2 G}{\partial \eta^2}.$$

With this notation in mind, we now apply Green's second identity for two dimensions from Problem 8 in Chapter 9. We have

$$\int_D (u \nabla^2_{r'} G - G \nabla^2_{r'} u)\, dA' = \int_C (u \nabla_{r'} G - G \nabla_{r'} u) \cdot d\mathbf{s}'. \tag{8.102}$$

Inserting the differential equations, the left hand side of the equation becomes

$$\begin{aligned} &\int_D \left[u \nabla^2_{r'} G - G \nabla^2_{r'} u \right] \, dA' \\ = \quad & \int_D \left[u(\xi, \eta) \delta(\xi - x, \eta - y) - G(x, y; \xi, \eta) f(\xi, \eta) \right] \, d\xi d\eta \\ = \quad & u(x, y) - \int_D G(x, y; \xi, \eta) f(\xi, \eta)\, d\xi d\eta. \end{aligned} \tag{8.103}$$

Using the boundary conditions, $u(\xi, \eta) = g(\xi, \eta)$ on C and $G(x, y; \xi, \eta) = 0$ on C, the right hand side of the equation becomes

$$\int_C (u \nabla_{r'} G - G \nabla_{r'} u) \cdot d\mathbf{s}' = \int_C g(\xi, \eta) \nabla_{r'} G \cdot d\mathbf{s}'. \tag{8.104}$$

Solving for $u(x,y)$, we have the solution written in terms of the Green's function,

$$u(x,y) = \int_D G(x,y;\xi,\eta) f(\xi,\eta)\, d\xi d\eta + \int_C g(\xi,\eta) \nabla_{r'} G \cdot d\mathbf{s}'.$$

Now we need to find the Green's function. We find the Green's functions for several examples.

Example 8.15. Find the two dimensional Green's function for the antisymmetric Poisson equation; that is, we seek solutions that are θ-independent.

The problem we need to solve in order to find the Green's function involves writing the Laplacian in polar coordinates,

$$v_{rr} + \frac{1}{r} v_r = \delta(r).$$

For $r \neq 0$, this is a Cauchy-Euler type of differential equation. The general solution is $v(r) = A \ln r + B$.

Due to the singularity at $r = 0$, we integrate over a domain in which a small circle of radius ϵ is cut form the plane and apply the two dimensional Divergence Theorem. In particular, we have

$$\begin{aligned} 1 &= \int_{D_\epsilon} \delta(r)\, dA \\ &= \int_{D_\epsilon} \nabla^2 v\, dA \\ &= \int_{C_\epsilon} \nabla v \cdot \cdot d\mathbf{s} \\ &= \int_{C_\epsilon} \frac{\partial v}{\partial r}\, dS = 2\pi A. \end{aligned} \tag{8.105}$$

Therefore, $A = 1/2\pi$. We note that B is arbitrary, so we will take $B = 0$ in the remaining discussion.

Using this solution for a source of the form $\delta(\mathbf{r} - \mathbf{r}')$, we obtain the Green's function for Poisson's equation as

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}'|.$$

Green's function for the infinite plane.

Example 8.16. Find the Green's function for the infinite plane.

From Figure 8.5 we have $|\mathbf{r} - \mathbf{r}'| = \sqrt{(x-\xi)^2 + (y-\eta)^2}$. Therefore, the Green's function from the last example gives

$$G(x,y,\xi,\eta) = \frac{1}{4\pi} \ln((\xi - x)^2 + (\eta - y)^2).$$

Green's function for the half plane using the Method of Images.

Example 8.17. Find the Green's function for the half plane, $\{(x,y) | y > 0\}$, using the Method of Images

This problem can be solved using the result for the Green's function for the infinite plane. We use the Method of Images to construct a function such that $G = 0$ on the boundary, $y = 0$. Namely, we use the image of the point (x,y) with respect to the x-axis, $(x, -y)$.

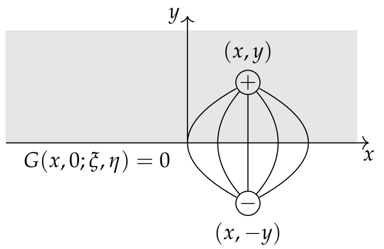

Figure 8.6: The Method of Images: The source and image source for the Green's function for the half plane. Imagine two opposite charges forming a dipole. The electric field lines are depicted indicating that the electric potential, or Green's function, is constant along $y=0$.

Imagine that the Green's function $G(x,y,\xi,\eta)$ represents a point charge at (x,y) and $G(x,y,\xi,\eta)$ provides the electric potential, or response, at (ξ,η). This single charge cannot yield a zero potential along the x-axis (y=0). One needs an additional charge to yield a zero equipotential line. This is shown in Figure 8.6.

The positive charge has a source of $\delta(\mathbf{r}-\mathbf{r}')$ at $\mathbf{r}=(x,y)$ and the negative charge is represented by the source $-\delta(\mathbf{r}^*-\mathbf{r}')$ at $\mathbf{r}^*=(x,-y)$. We construct the Green's functions at these two points and introduce a negative sign for the negative image source. Thus, we have

$$G(x,y,\xi,\eta)=\frac{1}{4\pi}\ln((\xi-x)^2+(\eta-y)^2)-\frac{1}{4\pi}\ln((\xi-x)^2+(\eta+y)^2).$$

These functions satisfy the differential equation and the boundary condition

$$G(x,0,\xi,\eta)=\frac{1}{4\pi}\ln((\xi-x)^2+(\eta)^2)-\frac{1}{4\pi}\ln((\xi-x)^2+(\eta)^2)=0.$$

Example 8.18. Solve the homogeneous version of the problem; i.e., solve Laplace's equation on the half plane with a specified value on the boundary.

We want to solve the problem

$$\begin{aligned}\nabla^2 u&=0, \quad \text{in } D,\\ u&=f, \quad \text{on } C,\end{aligned} \tag{8.106}$$

This is displayed in Figure 8.7.

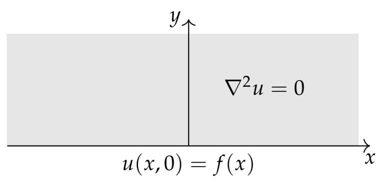

Figure 8.7: This is the domain for a semi-infinite slab with boundary value $u(x,0)=f(x)$ and governed by Laplace's equation.

From the previous analysis, the solution takes the form

$$u(x,y)=\int_C f\nabla G\cdot\mathbf{n}\,ds=\int_C f\frac{\partial G}{\partial n}\,ds.$$

Since

$$G(x,y,\xi,\eta)=\frac{1}{4\pi}\ln((\xi-x)^2+(\eta-y)^2)-\frac{1}{4\pi}\ln((\xi-x)^2+(\eta+y)^2),$$

$$\frac{\partial G}{\partial n}=\frac{\partial G(x,y,\xi,\eta)}{\partial\eta}\Big|_{\eta=0}=\frac{1}{\pi}\frac{y}{(\xi-x)^2+y^2}.$$

We have arrived at the same surface Green's function as we had found in Example 10.11.2 and the solution is

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\xi)^2+y^2}f(\xi)\,d\xi.$$

8.6 Method of Eigenfunction Expansions

We have seen that the use of eigenfunction expansions is another technique for finding solutions of differential equations. In this section we will show how we can use eigenfunction expansions to find the solutions to nonhomogeneous partial differential equations. In particular, we will apply this technique to solving nonhomogeneous versions of the heat and wave equations.

8.6.1 The Nonhomogeneous Heat Equation

In this section we solve the one dimensional heat equation with a source using an eigenfunction expansion. Consider the problem

$$\begin{aligned} u_t &= ku_{xx} + Q(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < L. \end{aligned} \tag{8.107}$$

The homogeneous version of this problem is given by

$$\begin{aligned} v_t &= kv_{xx}, \quad 0 < x < L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0. \end{aligned} \tag{8.108}$$

We know that a separation of variables leads to the eigenvalue problem

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0.$$

The eigenfunctions and eigenvalues are given by

$$\phi_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

We can use these eigenfunctions to obtain a solution of the nonhomogeneous problem (8.107). We begin by assuming the solution is given by the eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x). \tag{8.109}$$

In general, we assume that $v(x,t)$ and $\phi_n(x)$ satisfy the same boundary conditions and that $v(x,t)$ and $v_x(x,t)$ are continuous functions.Note that the difference between this eigenfunction expansion and that in Section 4.3 is that the expansion coefficients are functions of time.

In order to carry out the full process, we will also need to expand the initial profile, $f(x)$, and the source term, $Q(x,t)$, in the basis of eigenfunctions. Thus, we assume the forms

$$\begin{aligned} f(x) &= u(x,0) \\ &= \sum_{n=1}^{\infty} a_n(0)\phi_n(x), \quad (8.110) \\ Q(x,t) &= \sum_{n=1}^{\infty} q_n(t)\phi_n(x). \quad (8.111) \end{aligned}$$

Recalling from Chapter 4, the generalized Fourier coefficients are given by

$$a_n(0) = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_0^L f(x)\phi_n(x)\,dx, \tag{8.112}$$

$$q_n(t) = \frac{\langle Q, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_0^L Q(x,t)\phi_n(x)\,dx. \tag{8.113}$$

The next step is to insert the expansions (8.109) and (8.111) into the nonhomogeneous heat equation (8.107). We first note that

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} \dot{a}_n(t)\phi_n(x), \\ u_{xx}(x,t) &= -\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x). \end{aligned} \tag{8.114}$$

Inserting these expansions into the heat equation (8.107), we have

$$\begin{aligned} u_t &= ku_{xx} + Q(x,t), \\ \sum_{n=1}^{\infty} \dot{a}_n(t)\phi_n(x) &= -k\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x). \end{aligned} \tag{8.115}$$

Collecting like terms, we have

$$\sum_{n=1}^{\infty} [\dot{a}_n(t) + k\lambda_n a_n(t) - q_n(t)]\phi_n(x) = 0, \quad \forall x \in [0,L].$$

Due to the linear independence of the eigenfunctions, we can conclude that

$$\dot{a}_n(t) + k\lambda_n a_n(t) = q_n(t), \quad n = 1,2,3,\ldots.$$

This is a linear first order ordinary differential equation for the unknown expansion coefficients.

We further note that the initial condition can be used to specify the initial condition for this first order ODE. In particular,

$$f(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x).$$

The coefficients can be found as generalized Fourier coefficients in an expansion of $f(x)$ in the basis $\phi_n(x)$. These are given by Equation (8.112).

Recall from Appendix B that the solution of a first order ordinary differential equation of the form

$$y'(t) + a(t)y(t) = p(t)$$

is found using the integrating factor

$$\mu(t) = \exp\int^t a(\tau)\,d\tau.$$

Multiplying the ODE by the integrating factor, one has

$$\frac{d}{dt}\left[y(t)\exp\int^t a(\tau)\,d\tau\right] = p(t)\exp\int^t a(\tau)\,d\tau.$$

After integrating, the solution can be found providing the integral is doable.

For the current problem, we have

$$\dot{a}_n(t) + k\lambda_n a_n(t) = q_n(t), \quad n = 1,2,3,\ldots.$$

Then, the integrating factor is

$$\mu(t) = \exp \int^t k\lambda_n \, d\tau = e^{k\lambda_n t}.$$

Multiplying the differential equation by the integrating factor, we find

$$\begin{aligned} [\dot{a}_n(t) + k\lambda_n a_n(t)]e^{k\lambda_n t} &= q_n(t)e^{k\lambda_n t} \\ \frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) &= q_n(t)e^{k\lambda_n t}. \end{aligned} \tag{8.116}$$

Integrating, we have

$$a_n(t)e^{k\lambda_n t} - a_n(0) = \int_0^t q_n(\tau)e^{k\lambda_n \tau} \, d\tau,$$

or

$$a_n(t) = a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)} \, d\tau.$$

Using these coefficients, we can write out the general solution.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)} \, d\tau \right] \phi_n(x). \end{aligned} \tag{8.117}$$

We will apply this theory to a more specific problem which not only has a heat source but also has nonhomogeneous boundary conditions.

Example 8.19. Solve the following nonhomogeneous heat problem using eigenfunction expansions:

$$\begin{aligned} u_t - u_{xx} &= x + t\sin 3\pi x, \qquad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 2, \quad u(L,t) = t, \quad t > 0 \\ u(x,0) &= 3\sin 2\pi x + 2(1-x), \quad 0 \le x \le 1. \end{aligned} \tag{8.118}$$

This problem has the same nonhomogeneous boundary conditions as those in Example 8.14. Recall that we can define

$$u(x,t) = v(x,t) + 2 + (t-2)x$$

to obtain a new problem for $v(x,t)$. The new problem is

$$\begin{aligned} v_t - v_{xx} &= t\sin 3\pi x, \qquad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= 3\sin 2\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{8.119}$$

We can now apply the method of eigenfunction expansions to find $v(x,t)$. The eigenfunctions satisfy the homogeneous problem

$$\phi_n'' + \lambda_n\phi_n = 0, \quad \phi_n(0) = 0, \quad \phi_n(1) = 0.$$

The solutions are

$$\phi_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Now, let

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x.$$

Inserting $v(x,t)$ into the PDE, we have

$$\sum_{n=1}^{\infty} [\dot{a}_n(t) + n^2\pi^2 a_n(t)] \sin n\pi x = t \sin 3\pi x.$$

Due to the linear independence of the eigenfunctions, we can equate the coefficients of the $\sin n\pi x$ terms. This gives

$$\begin{aligned} \dot{a}_n(t) + n^2\pi^2 a_n(t) &= 0, \quad n \neq 3, \\ \dot{a}_3(t) + 9\pi^2 a_3(t) &= t, \quad n = 3. \end{aligned} \tag{8.120}$$

This is a system of first order ordinary differential equations. The first set of equations are separable and are easily solved. For $n \neq 3$, we seek solutions of

$$\frac{d}{dt} a_n = -n^2\pi^2 a_n(t).$$

These are given by

$$a_n(t) = a_n(0) e^{-n^2\pi^2 t}, \quad n \neq 3.$$

In the case $n = 3$, we seek solutions of

$$\frac{d}{dt} a_3 + 9\pi^2 a_3(t) = t.$$

The integrating factor for this first order equation is given by

$$\mu(t) = e^{9\pi^2 t}.$$

Multiplying the differential equation by the integrating factor, we have

$$\frac{d}{dt}\left(a_3(t) e^{9\pi^2 t}\right) = t e^{9\pi^2 t}.$$

Integrating, we obtain the solution

$$\begin{aligned} a_3(t) &= a_3(0) e^{-9\pi^2 t} + e^{-9\pi^2 t} \int_0^t \tau e^{9\pi^2 \tau}\, d\tau, \\ &= a_3(0) e^{-9\pi^2 t} + e^{-9\pi^2 t} \left[\frac{1}{9\pi^2} \tau e^{9\pi^2 \tau} - \frac{1}{(9\pi^2)^2} e^{9\pi^2 \tau} \right]_0^t, \\ &= a_3(0) e^{-9\pi^2 t} + \frac{1}{9\pi^2} t - \frac{1}{(9\pi^2)^2} \left[1 - e^{-9\pi^2 \tau} \right]. \end{aligned} \tag{8.121}$$

Up to this point, we have the solution

$$\begin{aligned} u(x,t) &= v(x,t) + w(x,t) \\ &= \sum_{n=1}^{\infty} a_n(t) \sin n\pi x + 2 + (t-2)x, \end{aligned} \tag{8.122}$$

where

$$\begin{aligned} a_n(t) &= a_n(0)e^{-n^2\pi^2 t}, \quad n \neq 3 \\ a_3(t) &= a_3(0)e^{-9\pi^2 t} + \frac{1}{9\pi^2}t - \frac{1}{(9\pi^2)^2}\left[1 - e^{-9\pi^2\tau}\right]. \end{aligned} \tag{8.123}$$

We still need to find $a_n(0)$, $n = 1, 2, 3, \ldots$.

The initial values of the expansion coefficients are found using the initial condition

$$v(x, 0) = 3\sin 2\pi x = \sum_{n=1}^{\infty} a_n(0)\sin n\pi x.$$

It is clear that we have $a_n(0) = 0$ for $n \neq 2$ and $a_2(0) = 3$. Thus, the series for $v(x, t)$ has two nonvanishing coefficients,

$$\begin{aligned} a_2(t) &= 3e^{-4\pi^2 t}, \\ a_3(t) &= \frac{1}{9\pi^2}t - \frac{1}{(9\pi^2)^2}\left[1 - e^{-9\pi^2\tau}\right]. \end{aligned} \tag{8.124}$$

Therefore, the final solution is given by

$$u(x, t) = 2 + (t - 2)x + 3e^{-4\pi^2 t}\sin 2\pi x + \frac{9\pi^2 t - (1 - e^{-9\pi^2\tau})}{81\pi^4}\sin 3\pi x.$$

8.6.2 The Forced Vibrating Membrane

We now consider the forced vibrating membrane. A two-dimensional membrane is stretched over some domain D. We assume Dirichlet conditions on the boundary, $u = 0$ on ∂D. The forced membrane can be modeled as

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u + Q(\mathbf{r}, t), \quad \mathbf{r} \in D, \quad t > 0, \\ u(\mathbf{r}, t) &= 0, \quad \mathbf{r} \in \partial D, \quad t > 0, \\ u(\mathbf{r}, 0) &= f(\mathbf{r}), \quad u_t(\mathbf{r}, 0) = g(\mathbf{r}), \quad \mathbf{r} \in D. \end{aligned} \tag{8.125}$$

The method of eigenfunction expansions relies on the use of eigenfunctions, $\phi_\alpha(\mathbf{r})$, for $\alpha \in J \subset Z^2$ a set of indices typically of the form (i, j) in some lattice grid of integers. The eigenfunctions satisfy the eigenvalue equation

$$\nabla^2\phi_\alpha(\mathbf{r}) = -\lambda_\alpha\phi_\alpha(\mathbf{r}), \quad \phi_\alpha(\mathbf{r}) = 0, \text{ on } \partial D.$$

We assume that the solution and forcing function can be expanded in the basis of eigenfunctions,

$$\begin{aligned} u(\mathbf{r}, t) &= \sum_{\alpha \in J} a_\alpha(t)\phi_\alpha(\mathbf{r}), \\ Q(\mathbf{r}, t) &= \sum_{\alpha \in J} q_\alpha(t)\phi_\alpha(\mathbf{r}). \end{aligned} \tag{8.126}$$

Inserting this form into the forced wave equation (8.125), we have

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u + Q(\mathbf{r},t) \\ \sum_{\alpha\in J} \ddot{a}_\alpha(t)\phi_\alpha(\mathbf{r}) &= -c^2\sum_{\alpha\in J}\lambda_\alpha a_\alpha(t)\phi_\alpha(\mathbf{r}) + \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}) \\ 0 &= \sum_{\alpha\in J}[\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) - q_\alpha(t)]\phi_\alpha(\mathbf{r}). \end{aligned} \tag{8.127}$$

The linear independence of the eigenfunctions then gives the ordinary differential equation

$$\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) = q_\alpha(t).$$

We can solve this equation with initial conditions $a_\alpha(0)$ and $\dot{a}_\alpha(0)$ found from

$$\begin{aligned} f(\mathbf{r}) &= u(\mathbf{r},0) = \sum_{\alpha\in J} a_\alpha(0)\phi_\alpha(\mathbf{r}), \\ g(\mathbf{r}) &= u_t(\mathbf{r},0) = \sum_{\alpha\in J} \dot{a}_\alpha(0)\phi_\alpha(\mathbf{r}). \end{aligned} \tag{8.128}$$

Example 8.20. Periodic Forcing, $Q(\mathbf{r},t) = G(\mathbf{r})\cos\omega t$.

It is enough to specify $Q(\mathbf{r},t)$ in order to solve for the time dependence of the expansion coefficients. A simple example is the case of periodic forcing, $Q(\mathbf{r},t) = h(\mathbf{r})\cos\omega t$. In this case, we expand Q in the basis of eigenfunctions,

$$\begin{aligned} Q(\mathbf{r},t) &= \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}), \\ G(\mathbf{r})\cos\omega t &= \sum_{\alpha\in J} \gamma_\alpha \cos\omega t\phi_\alpha(\mathbf{r}). \end{aligned} \tag{8.129}$$

Inserting these expressions into the forced wave equation (8.125), we obtain a system of differential equations for the expansion coefficients,

$$\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) = \gamma_\alpha\cos\omega t.$$

In order to solve this equation we borrow the methods from a course on ordinary differential equations for solving nonhomogeneous equations. In particular we can use the Method of Undetermined Coefficients as reviewed in Section B.3.1. The solution of these equations are of the form

$$a_\alpha(t) = a_{\alpha h}(t) + a_{\alpha p}(t),$$

where $a_{\alpha h}(t)$ satisfies the homogeneous equation,

$$\ddot{a}_{\alpha h}(t) + c^2\lambda_\alpha a_{\alpha h}(t) = 0, \tag{8.130}$$

and $a_{\alpha p}(t)$ is a particular solution of the nonhomogeneous equation,

$$\ddot{a}_{\alpha p}(t) + c^2\lambda_\alpha a_{\alpha p}(t) = \gamma_\alpha\cos\omega t. \tag{8.131}$$

The solution of the homogeneous problem (8.130) is easily founds as

$$a_{\alpha h}(t) = c_{1\alpha}\cos(\omega_{0\alpha}t) + c_{2\alpha}\sin(\omega_{0\alpha}t),$$

where $\omega_{0\alpha} = c\sqrt{\lambda_\alpha}$.

The particular solution is found by making the guess $a_{\alpha p}(t) = A_\alpha \cos \omega t$. Inserting this guess into Equation (ceqn2), we have

$$[-\omega^2 + c^2\lambda_\alpha]A_\alpha \cos \omega t = \gamma_\alpha \cos \omega t.$$

Solving for A_α, we obtain

$$A_\alpha = \frac{\gamma_\alpha}{-\omega^2 + c^2\lambda_\alpha}, \quad \omega^2 \neq c^2\lambda_\alpha.$$

Then, the general solution is given by

$$a_\alpha(t) = c_{1\alpha}\cos(\omega_{0\alpha}t) + c_{2\alpha}\sin(\omega_{0\alpha}t) + \frac{\gamma_\alpha}{-\omega^2 + c^2\lambda_\alpha}\cos \omega t,$$

where $\omega_{0\alpha} = c\sqrt{\lambda_\alpha}$ and $\omega^2 \neq c^2\lambda_\alpha$.

In the case where $\omega^2 = c^2\lambda_\alpha$, we have a resonant solution. This is discussed in Section FO on forced oscillations. In this case the Method of Undetermined Coefficients fails and we need the Modified Method of Undetermined Coefficients. This is because the driving term, $\gamma_\alpha \cos \omega t$, is a solution of the homogeneous problem. So, we make a different guess for the particular solution. We let

$$a_{\alpha p}(t) = t(A_\alpha \cos \omega t + B_\alpha \sin \omega t).$$

Then, the needed derivatives are

$$\begin{aligned} a_{\alpha p}(t) &= \omega t(-A_\alpha \sin \omega t + B_\alpha \cos \omega t) + A_\alpha \cos \omega t + B_\alpha \sin \omega t, \\ a_{\alpha p}(t) &= -\omega^2 t(A_\alpha \cos \omega t + B_\alpha \sin \omega t) - 2\omega A_\alpha \sin \omega t + 2\omega B_\alpha \cos \omega t, \\ &= -\omega^2 a_{\alpha p}(t) - 2\omega A_\alpha \sin \omega t + 2\omega B_\alpha \cos \omega t. \end{aligned} \tag{8.132}$$

Inserting this guess into Equation (ceqn2) and noting that $\omega^2 = c^2\lambda_\alpha$, we have

$$-2\omega A_\alpha \sin \omega t + 2\omega B_\alpha \cos \omega t = \gamma_\alpha \cos \omega t.$$

Therefore, $A_\alpha = 0$ and

$$B_\alpha = \frac{\gamma_\alpha}{2\omega}.$$

So, the particular solution becomes

$$a_{\alpha p}(t) = \frac{\gamma_\alpha}{2\omega} t \sin \omega t.$$

The full general solution is then

$$a_\alpha(t) = c_{1\alpha}\cos(\omega t) + c_{2\alpha}\sin(\omega t) + \frac{\gamma_\alpha}{2\omega} t \sin \omega t,$$

where $\omega = c\sqrt{\lambda_\alpha}$.

We see from this result that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency for one of the frequencies in the system. A typical plot of such a solution is given in Figure 8.8.

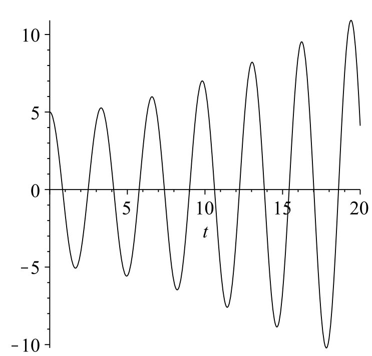

Figure 8.8: Plot of a solution showing resonance.

8.7 Green's Function Solution of Nonhomogeneous Heat Equation

We solved the one dimensional heat equation with a source using an eigenfunction expansion. In this section we rewrite the solution and identify the Green's function form of the solution. Recall that the solution of the nonhomogeneous problem,

$$\begin{aligned} u_t &= k u_{xx} + Q(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < L, \end{aligned} \tag{8.133}$$

is given by Equation (8.117)

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty} \left[a_n(0) e^{-k\lambda_n t} + \int_0^t q_n(\tau) e^{-k\lambda_n(t-\tau)}\, d\tau \right] \phi_n(x). \end{aligned} \tag{8.134}$$

The generalized Fourier coefficients for $a_n(0)$ and $q_n(t)$ are given by

$$a_n(0) = \frac{1}{\|\phi_n\|^2} \int_0^L f(x)\phi_n(x)\, dx, \tag{8.135}$$

$$q_n(t) = \frac{1}{\|\phi_n\|^2} \int_0^L Q(x,t)\phi_n(x)\, dx. \tag{8.136}$$

The solution in Equation (8.134) can be rewritten using the Fourier coefficients in Equations (8.135) and (8.136).

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left[a_n(0) e^{-k\lambda_n t} + \int_0^t q_n(\tau) e^{-k\lambda_n(t-\tau)}\, d\tau \right] \phi_n(x) \\ &= \sum_{n=1}^{\infty} a_n(0) e^{-k\lambda_n t}\phi_n(x) + \int_0^t \sum_{n=1}^{\infty} \left(q_n(\tau) e^{-k\lambda_n(t-\tau)}\phi_n(x) \right) d\tau \\ &= \sum_{n=1}^{\infty} \frac{1}{\|\phi_n\|^2} \left(\int_0^L f(\xi)\phi_n(\xi)\, d\xi \right) e^{-k\lambda_n t}\phi_n(x) \\ &\quad + \int_0^t \sum_{n=1}^{\infty} \frac{1}{\|\phi_n\|^2} \left(\int_0^L Q(\xi,\tau)\phi_n(\xi)\, d\xi \right) e^{-k\lambda_n(t-\tau)}\phi_n(x)\, d\tau \\ &= \int_0^L \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi) e^{-k\lambda_n t}}{\|\phi_n\|^2} \right) f(\xi)\, d\xi \\ &\quad + \int_0^t \int_0^L \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi) e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2} \right) Q(\xi,\tau)\, d\xi\, d\tau. \end{aligned} \tag{8.137}$$

Defining

$$G(x,t;\xi,\tau) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi) e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2},$$

we see that the solution can be written in the form

$$u(x,t) = \int_0^L G(x,t;\xi,0) f(\xi)\, d\xi + \int_0^t \int_0^L G(x,t;\xi,\tau) Q(\xi,\tau)\, d\xi\, d\tau.$$

The solution can be written in terms of the initial value Green's function, $G(x,t;\xi,0)$, and the general Green's function, $G(x,t;\xi,\tau)$.

Thus, we see that $G(x,t;\xi,0)$ is the initial value Green's function and $G(x,t;\xi,\tau)$ is the general Green's function for this problem.

The only thing left is to introduce nonhomogeneous boundary conditions into this solution. So, we modify the original problem to the fully nonhomogeneous heat equation:

$$\begin{aligned} u_t &= ku_{xx} + Q(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= \alpha(t), \quad u(L,t) = \beta(t), \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < L, \end{aligned} \tag{8.138}$$

As before, we begin with the expansion of the solution in the basis of eigenfunctions,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

However, due to potential convergence problems, we cannot expect that u_{xx} can be obtained by simply differentiating the series twice and expecting the resulting series to converge to u_{xx}. So, we need to be a little more careful.

We first note that

$$u_t = \sum_{n=1}^{\infty} \dot{a}_n(t)\phi_n(x) = ku_{xx} + Q(x,t).$$

Solving for the expansion coefficients, we have

$$\dot{a}(t) = \frac{\int_0^L (ku_{xx} + Q(x,t))\phi_n(x)\,dx}{\|\phi_n\|^2}.$$

In order to proceed, we need an expression for $\int_a^b u_{xx}\phi_n(x)\,dx$. We can find this using Green's identity from Section 4.2.2.

We start with

$$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx = [p(uv' - vu')]_a^b$$

and let $v = \phi_n$. Then,

$$\begin{aligned} \int_0^L (u(x,t)\phi_n''(x) - \phi_n(x)u_{xx}(x,t))\,dx &= [u(x,t)\phi_n'(x) - \phi_n(x)u_x(x,t))]_0^L \\ \int_0^L (-\lambda_n u(x,t) + u_{xx}(x,t))\phi_n(x)\,dx &= [u(L,t)\phi_n'(L) - \phi_n(L)u_x(L,t))] \\ &\quad -[u(0,t)\phi_n'(0) - \phi_n(0)u_x(0,t))] \\ -\lambda_n a_n\|\phi_n\|^2 - \int_0^L u_{xx}(x,t)\phi_n(x)\,dx &= \beta(t)\phi_n'(L) - \alpha(t)\phi_n'(0). \end{aligned} \tag{8.139}$$

Thus,

$$\int_0^L u_{xx}(x,t)\phi_n(x)\,dx = -\lambda_n a_n\|\phi_n\|^2 + \alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L).$$

Inserting this result into the equation for $\dot{a}_n(t)$, we have

$$\dot{a}(t) = -k\lambda_n a_n(t) + q_n(t) + k\frac{\alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L)}{\|\phi_n\|^2}.$$

As we had seen before, this first order equation can be solved using the integrating factor

$$\mu(t) = \exp \int^t k\lambda_n \, d\tau = e^{k\lambda_n t}.$$

Multiplying the differential equation by the integrating factor, we find

$$\begin{aligned}
[\dot{a}_n(t) + k\lambda_n a_n(t)]e^{k\lambda_n t} &= \left[q_n(t) + k\frac{\alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L)}{\|\phi_n\|^2} \right] e^{k\lambda_n t} \\
\frac{d}{dt}\left(a_n(t)e^{k\lambda_n t} \right) &= \left[q_n(t) + k\frac{\alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L)}{\|\phi_n\|^2} \right] e^{k\lambda_n t}.
\end{aligned} \tag{8.140}$$

Integrating, we have

$$a_n(t)e^{k\lambda_n t} - a_n(0) = \int_0^t \left[q_n(\tau) + k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2} \right] e^{k\lambda_n \tau} \, d\tau,$$

or

$$a_n(t) = a_n(0)e^{-k\lambda_n t} + \int_0^t \left[q_n(\tau) + k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2} \right] e^{-k\lambda_n (t-\tau)} \, d\tau.$$

We can now insert these coefficients into the solution and see how to extract the Green's function contributions. Inserting the coefficients, we have

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \\
&= \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n (t-\tau)} \, d\tau \right] \phi_n(x) \\
&\quad + \sum_{n=1}^{\infty} \left(\int_0^t \left[k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2} \right] e^{-k\lambda_n (t-\tau)} \, d\tau \right) \phi_n(x).
\end{aligned} \tag{8.141}$$

Recall that the generalized Fourier coefficients for $a_n(0)$ and $q_n(t)$ are given by

$$a_n(0) = \frac{1}{\|\phi_n\|^2} \int_0^L f(x)\phi_n(x) \, dx, \tag{8.142}$$

$$q_n(t) = \frac{1}{\|\phi_n\|^2} \int_0^L Q(x,t)\phi_n(x) \, dx. \tag{8.143}$$

The solution in Equation (8.141) can be rewritten using the Fourier coefficients in Equations (8.142) and (8.143).

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n (t-\tau)} \, d\tau \right] \phi_n(x) \\
&\quad + \sum_{n=1}^{\infty} \left(\int_0^t \left[k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2} \right] e^{-k\lambda_n (t-\tau)} \, d\tau \right) \phi_n(x) \\
&= \sum_{n=1}^{\infty} a_n(0)e^{-k\lambda_n t}\phi_n(x) + \int_0^t \sum_{n=1}^{\infty} \left(q_n(\tau)e^{-k\lambda_n (t-\tau)}\phi_n(x) \right) \, d\tau
\end{aligned}$$

$$\begin{aligned}
&+\int_0^t \sum_{n=1}^{\infty}\left(\left[k\frac{\alpha(\tau)\phi_n'(0)-\beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\right)\phi_n(x)d\tau \\
=&\sum_{n=1}^{\infty}\frac{1}{\|\phi_n\|^2}\left(\int_0^L f(\xi)\phi_n(\xi)\,d\xi\right)e^{-k\lambda_n t}\phi_n(x) \\
&+\int_0^t \sum_{n=1}^{\infty}\frac{1}{\|\phi_n\|^2}\left(\int_0^L Q(\xi,\tau)\phi_n(\xi)\,d\xi\right)e^{-k\lambda_n(t-\tau)}\phi_n(x)\,d\tau \\
&+\int_0^t \sum_{n=1}^{\infty}\left(\left[k\frac{\alpha(\tau)\phi_n'(0)-\beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\right)\phi_n(x)d\tau \\
=&\int_0^L\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n t}}{\|\phi_n\|^2}\right)f(\xi)\,d\xi \\
&+\int_0^t\int_0^L\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)Q(\xi,\tau)\,d\xi\,d\tau. \\
&+k\int_0^t\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n'(0)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)\alpha(\tau)\,d\tau \\
&-k\int_0^t\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n'(L)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)\beta(\tau)\,d\tau. \qquad (8.144)
\end{aligned}$$

As before, we can define the general Green's function as

$$G(x,t;\xi,\tau)=\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}.$$

Then, we can write the solution to the fully homogeneous problem as

$$\begin{aligned}
u(x,t) =& \int_0^t\int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\,d\xi\,d\tau+\int_0^L G(x,t;\xi,0)f(\xi)\,d\xi \\
&+k\int_0^t\left[\alpha(\tau)\frac{\partial G}{\partial\xi}(x,0;t,\tau)-\beta(\tau)\frac{\partial G}{\partial\xi}(x,L;t,\tau)\right]d\tau. \qquad (8.145)
\end{aligned}$$

The first integral handles the source term, the second integral handles the initial condition, and the third term handles the fixed boundary conditions.

This general form can be deduced from the differential equation for the Green's function and original differential equation by using a more general form of Green's identity. Let the heat equation operator be defined as $\mathcal{L}=\frac{\partial}{\partial t}-k\frac{\partial^2}{\partial x^2}$. The differential equations for $u(x,t)$ and $G(x,t;\xi,\tau)$ for $0\leq x,\xi\leq L$ and $t,\tau\geq 0$, are taken to be

$$\begin{aligned}
\mathcal{L}u(x,t)&=Q(x,t), \\
\mathcal{L}G(x,t;\xi,\tau)&=\delta(x-\xi)\delta(t-\tau). \qquad (8.146)
\end{aligned}$$

Multiplying the first equation by $G(x,t;\xi,\tau)$ and the second by $u(x,t)$, we obtain

$$\begin{aligned}
G(x,t;\xi,\tau)\mathcal{L}u(x,t)&=G(x,t;\xi,\tau)Q(x,t), \\
u(x,t)\mathcal{L}G(x,t;\xi,\tau)&=\delta(x-\xi)\delta(t-\tau)u(x,t). \qquad (8.147)
\end{aligned}$$

Now, we subtract the equations and integrate with respect to x and t. This gives

$$\int_0^\infty\int_0^L\left[G(x,t;\xi,\tau)\mathcal{L}u(x,t)-u(x,t)\mathcal{L}G(x,t;\xi,\tau)\right]\,dxdt$$

$$\begin{aligned}
&= \int_0^\infty \int_0^L \left[G(x,t;\xi,\tau)Q(x,t) - \delta(x-\xi)\delta(t-\tau)u(x,t)\right]\, dxdt \\
&= \int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(x,t)\, dxdt - u(\xi,\tau). \qquad (8.148)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty \int_0^L \left[G(x,t;\xi,\tau)\mathcal{L}u(x,t) - u(x,t)\mathcal{L}G(x,t;\xi,\tau)\right]\, dxdt \\
&= \int_0^L \int_0^\infty \left[G(x,t;\xi,\tau)u_t - u(x,t)G_t(x,t;\xi,\tau)\right]\, dtdx \\
&\quad -k\int_0^\infty \int_0^L \left[G(x,t;\xi,\tau)u_{xx}(x,t) - u(x,t)G_{xx}(x,t;\xi,\tau)\right]\, dxdt \\
&= \int_0^L \left[G(x,t;\xi,\tau)u_t\Big|_0^\infty - 2\int_0^\infty u(x,t)G_t(x,t;\xi,\tau)\, dt\right]\, dx \\
&\quad -k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial x}(x,t) - u(x,t)\frac{\partial G}{\partial x}(x,t;\xi,\tau)\right]_0^L\, dxdt
\end{aligned} \qquad (8.149)$$

Equating these two results and solving for $u(\xi,\tau)$, we have

$$\begin{aligned}
u(\xi,\tau) &= \int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(x,t)\, dxdt \\
&\quad +k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial x}(x,t) - u(x,t)\frac{\partial G}{\partial x}(x,t;\xi,\tau)\right]_0^L\, dxdt \\
&\quad +\int_0^L \left[G(x,0;\xi,\tau)u(x,0) + 2\int_0^\infty u(x,t)G_t(x,t;\xi,\tau)\, dt\right]\, dx.
\end{aligned} \qquad (8.150)$$

Exchanging (ξ,τ) with (x,t) and assuming that the Green's function is symmetric in these arguments, we have

$$\begin{aligned}
u(x,t) &= \int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\, d\xi d\tau \\
&\quad +k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial \xi}(\xi,\tau) - u(\xi,\tau)\frac{\partial G}{\partial \xi}(x,t;\xi,\tau)\right]_0^L\, dxdt \\
&\quad +\int_0^L G(x,t;\xi,0)u(\xi,0)\, d\xi + 2\int_0^L \int_0^\infty u(\xi,\tau)G_\tau(x,t;\xi,\tau)\, d\tau d\xi.
\end{aligned} \qquad (8.151)$$

This result is almost in the desired form except for the last integral. Thus, if

$$\int_0^L \int_0^\infty u(\xi,\tau)G_\tau(x,t;\xi,\tau)\, d\tau d\xi = 0,$$

then we have

$$\begin{aligned}
u(x,t) &= \int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\, d\xi d\tau + \int_0^L G(x,t;\xi,0)u(\xi,0)\, d\xi \\
&\quad +k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial \xi}(\xi,\tau) - u(\xi,\tau)\frac{\partial G}{\partial \xi}(x,t;\xi,\tau)\right]_0^L\, dxdt.
\end{aligned} \qquad (8.152)$$

8.8 Summary

We have seen throughout the chapter that Green's functions are the solutions of a differential equation representing the effect of a point impulse on either source terms, or initial and boundary conditions. The Green's function is obtained from transform methods or as an eigenfunction expansion. In the text we have occasionally rewritten solutions of differential equations in term's of Green's functions. We will first provide a few of these examples and then present a compilation of Green's Functions for generic partial differential equations.

For example, in section 8.4 we wrote the solution of the one dimensional heat equation as

$$u(x,t) = \int_0^L G(x,\xi;t,0) f(\xi)\, d\xi,$$

where

$$G(x,\xi;t,0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} e^{\lambda_n kt},$$

and the solution of the wave equation as

$$u(x,t) = \int_0^L G_c(x,\xi,t,0) f(\xi)\, d\xi + \int_0^L G_s(x,\xi,t,0) g(\xi)\, d\xi,$$

where

$$\begin{aligned} G_c(x,\xi,t,0) &= \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} \cos\frac{n\pi ct}{L}, \\ G_s(x,\xi,t,0) &= \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} \frac{\sin\frac{n\pi ct}{L}}{n\pi c/L}. \end{aligned}$$

We note that setting $t = 0$ in $G_c(x,\xi;t,0)$, we obtain

$$G_c(x,\xi,0,0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L}.$$

This is the Fourier sine series representation of the Dirac delta function, $\delta(x-\xi)$. Similarly, if we differentiate $G_s(x,\xi,t,0)$ with repsect to t and set $t = 0$, we once again obtain the Fourier sine series representation of the Dirac delta function.

It is also possible to find closed form expression for Green's functions, which we had done for the heat equation on the infinite interval,

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t;\xi,0) f(\xi)\, d\xi,$$

where

$$G(x,t;\xi,0) = \frac{e^{-(x-\xi)^2/4t}}{\sqrt{4\pi t}},$$

and for Poisson's equation,

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}') f(\mathbf{r}')\, d^3r',$$

where the three dimensional Green's function is given by

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

We can construct Green's functions for other problems which we have seen in the book. For example, the solution of the two dimensional wave equation on a rectangular membrane was found in Equation (6.37) as

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \quad (8.153)$$

where

$$a_{nm} = \frac{4}{LH}\int_0^H\int_0^L f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dxdy, \quad (8.154)$$

$$b_{nm} = \frac{4}{\omega_{nm}LH}\int_0^H\int_0^L g(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dxdy, \quad (8.155)$$

where the angular frequencies are given by

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \quad (8.156)$$

Rearranging the solution, we have

$$u(x,y,t) = \int_0^H\int_0^L [G_c(x,y;\xi,\eta;t,0)f(\xi,\eta) + G_s(x,y;\xi,\eta;t,0)g(\xi,\eta)]\,d\xi d\eta,$$

where

$$G_c(x,y;\xi,\eta;t,0) = \frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\sin\frac{m\pi y}{H}\sin\frac{m\pi\eta}{H}\cos\omega_{nm}t$$

and

$$G_s(x,y;\xi,\eta;t,0) = \frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\sin\frac{m\pi y}{H}\sin\frac{m\pi\eta}{H}\frac{\sin\omega_{nm}t}{\omega_{nm}}.$$

Once again, we note that setting $t = 0$ in $G_c(x,\xi;t,0)$ and setting $t = 0$ in $\frac{\partial G_c(x,\xi;t,0)}{\partial t}$, we obtain a Fourier series representation of the Dirac delta function in two dimensions,

$$\delta(x-\xi)\delta(y-\eta) = \frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\sin\frac{m\pi y}{H}\sin\frac{m\pi\eta}{H}.$$

Another example was the solution of the two dimensional Laplace equation on a disk given by Equation 6.87. We found that

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)\,r^n. \quad (8.157)$$

$$a_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta, \quad n = 0,1,\ldots, \quad (8.158)$$

$$b_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta \quad n = 1,2\ldots. \quad (8.159)$$

We saw that this solution can be written as

$$u(r,\theta) = \int_{-\pi}^{\pi} G(\theta,\phi;r,a) f(\phi)\, d\phi,$$

where the Green's function could be summed giving the Poisson kernel

$$G(\theta,\phi;r,a) = \frac{1}{2\pi}\frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta-\phi)}.$$

We had also investigated the nonhomogeneous heat equation in section 10.11.4,

$$\begin{aligned} u_t - ku_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0. \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le . \end{aligned} \tag{8.160}$$

We found that the solution of the heat equation is given by

$$u(x,t) = \int_0^L f(\xi) G(x,\xi;t,0) d\xi + \int_0^t \int_0^L h(\xi,\tau) G(x,\xi;t,\tau)\, d\xi d\tau,$$

where

$$G(x,\xi;t,\tau) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} e^{-\omega_n^2(t-\tau)}.$$

Note that setting $t=\tau$, we again get a Fourier sine series representation of the Dirac delta function.

In general, Green's functions based on eigenfunction expansions over eigenfunctions of Sturm-Liouville eigenvalue problems are a common way to construct Green's functions. For example, surface and initial value Green's functions are constructed in terms of a modification of delta function representations modified by factors which make the Green's function a solution of the given differential equations and a factor taking into account the boundary or initial condition plus a restoration of the delta function when applied to the condition. Examples with an indication of these factors are shown below.

1. Surface Green's Function: Cube $[0,a] \times [0,b] \times [0,c]$

$$g(x,y,z;x',y',c) = \sum_{\ell,n} \underbrace{\frac{2}{a}\sin\frac{\ell\pi x}{a}\sin\frac{\ell\pi x'}{a}\frac{2}{b}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}}_{\delta-\text{function}} \left[\underbrace{\sinh\gamma_{\ell n} z}_{\text{D.E.}} / \underbrace{\sinh\gamma_{\ell n} c}_{\text{restore } \delta}\right].$$

2. Surface Green's Function: Sphere $[0,a] \times [0,\pi] \times [0,2\pi]$

$$g(r,\phi,\theta;a,\phi',\theta') = \sum_{\ell,m} \underbrace{Y_\ell^{m*}(\psi'\,\theta') Y_\ell^{m*}(\psi\,\theta)}_{\delta-\text{function}} \left[\underbrace{r^\ell}_{\text{D.E.}} / \underbrace{a^\ell}_{\text{restore } \delta}\right].$$

3. Initial Value Green's Function: 1D Heat Equation on $[0,L]$, $k_n = \frac{n\pi}{L}$

$$g(x,t;x',t_0) = \sum_n \underbrace{\frac{2}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi x'}{L}}_{\delta-\text{function}} \left[\underbrace{e^{-a^2k_n^2 t}}_{\text{D.E.}} / \underbrace{e^{-a^2k_n^2 t_0}}_{\text{restore } \delta}\right].$$

4. Initial Value Green's Function: 1D Heat Equation on infinite domain

$$g(x,t;x',0) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x-x')}}_{\delta-\text{function}} \underbrace{e^{-a^2k^2t}}_{\text{D.E.}} = \frac{e^{-(x-x')^2/4a^2t}}{\sqrt{4\pi a^2 t}}.$$

We can extend this analysis to a more general theory of Green's functions. This theory is based upon Green's Theorems, or identities.

1. Green's First Theorem

$$\oint_S \varphi \nabla\chi \cdot \hat{\mathbf{n}}\, dS = \int_V (\nabla\varphi \cdot \nabla\chi + \varphi\nabla^2\chi)\, dV.$$

   This is easily proven starting with the identity

$$\nabla \cdot (\varphi\nabla\chi) = \nabla\varphi \cdot \nabla\chi + \varphi\nabla^2\chi,$$

   integrating over a volume of space and using Gauss' Integral Theorem.

2. Green's Second Theorem

$$\int_V (\varphi\nabla^2\chi - \chi\nabla^2\varphi)\, dV = \oint_S (\varphi\nabla\chi - \chi\nabla\varphi) \cdot \hat{\mathbf{n}}\, dS.$$

   This is proven by interchanging φ and χ in the first theorem and subtracting the two versions of the theorem.

The next step is to let $\varphi = u$ and $\chi = G$. Then,

$$\int_V (u\nabla^2 G - G\nabla^2 u)\, dV = \oint_S (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, dS.$$

As we had seen earlier for Poisson's equation, inserting the differential equation yields

$$u(x,y) = \int_V Gf\, dV + \oint_S (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, dS.$$

If we have the Green's function, we only need to know the source term and boundary conditions in order to obtain the solution to a given problem.

In the next sections we provide a summary of these ideas as applied to some generic partial differential equations.[3]

[3] This is an adaptation of notes from J. Franklin's course on mathematical physics.

8.8.1 Laplace's Equation: $\nabla^2\psi = 0$.

1. **Boundary Conditions**
   (a) *Dirichlet* - ψ is given on the surface.
   (b) *Neumann* - $\hat{\mathbf{n}} \cdot \nabla\psi = \frac{\partial\psi}{\partial n}$ is given on the surface.

   Note: Boundary conditions can be Dirichlet on part of the surface and Neumann on part. If they are Neumann on the whole surface, then the Divergence Theorem requires the constraint

$$\int \frac{\partial\psi}{\partial n}\, dS = 0.$$

2. **Solution by Surface Green's Function,** $g(\vec{\mathbf{r}}, \vec{\mathbf{r}}')$.

   (a) Dirichlet conditions

$$\nabla^2 g_D(\vec{\mathbf{r}}, \vec{\mathbf{r}}') = 0,$$

$$g_D(\vec{\mathbf{r}}_s, \vec{\mathbf{r}}'_s) = \delta^{(2)}(\vec{\mathbf{r}}_s - \vec{\mathbf{r}}'_s),$$

$$\psi(\vec{\mathbf{r}}) = \int g_D(\vec{\mathbf{r}}, \vec{\mathbf{r}}'_{\mathbf{s}})\psi(\vec{\mathbf{r}}'_s)\, dS'.$$

   (b) Neumann conditions

$$\nabla^2 g_N(\vec{\mathbf{r}}, \vec{\mathbf{r}}') = 0,$$

$$\frac{\partial g_N}{\partial n}(\vec{\mathbf{r}}_s, \vec{\mathbf{r}}'_s) = \delta^{(2)}(\vec{\mathbf{r}}_s - \vec{\mathbf{r}}'_s),$$

$$\psi(\vec{\mathbf{r}}) = \int g_N(\vec{\mathbf{r}}, \vec{\mathbf{r}}'_s)\frac{\partial \psi}{\partial n}(\vec{\mathbf{r}}'_s)\, dS'.$$

Note: Use of g is readily generalized to any number of dimensions.

8.8.2 *Homogeneous Time Dependent Equations*

1. **Typical Equations**

   (a) Diffusion/Heat Equation $\nabla^2\Psi = \frac{1}{a^2}\frac{\partial}{\partial t}\Psi$.

   (b) Schrödinger Equation $-\nabla^2\Psi + U\Psi = i\frac{\partial}{\partial t}\Psi$.

   (c) Wave Equation $\nabla^2\Psi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi$.

   (d) General form: $\mathcal{D}\Psi = \mathcal{T}\Psi$.

2. **Initial Value Green's Function,** $g(\vec{\mathbf{r}}, \vec{\mathbf{r}}'; t, t')$.

   (a) **Homogeneous Boundary Conditions**

      i. Diffusion, or Schrödinger Equation (1st order in time), $\mathcal{D}g = \mathcal{T}g$.

$$\Psi(\vec{\mathbf{r}}, t) = \int g(\vec{\mathbf{r}}, \vec{\mathbf{r}}'; t, t_0)\Psi(\mathbf{r}', t_0)\, d^3\mathbf{r}',$$

      where

$$g(\mathbf{r}, \mathbf{r}'; t_0, t_0) = \delta(\mathbf{r} - \mathbf{r}'),$$

      $g(\mathbf{r}_s)$ satisfies homogeneous boundary conditions.

      ii. Wave Equation

$$\Psi(\mathbf{r}, t) = \int [g_c(\mathbf{r}, \mathbf{r}'; t, t_0)\Psi(\mathbf{r}', t_0) + g_s(\mathbf{r}, \mathbf{r}'; t, t_0)\dot{\Psi}(\mathbf{r}', t_0)]\, d^3\mathbf{r}'.$$

      The first two properties in (a) above hold, but

$$g_c(\mathbf{r}, \mathbf{r}'; t_0, t_0) = \delta(\mathbf{r} - \mathbf{r}')$$

$$\dot{g}_s(\mathbf{r}, \mathbf{r}'; t_0, t_0) = \delta(\mathbf{r} - \mathbf{r}')$$

      Note: For the diffusion and Schrödinger equations the initial condition is Dirichlet in time. For the wave equation the initial condition is Cauchy, where Ψ and $\dot{\Psi}$ are given.

(b) **Inhomogeneous, Time Independent (steady) Boundary Conditions**

i. Solve Laplace's equation, $\nabla^2\psi_s = 0$, for inhomogeneous B.C.'s

ii. Solve homogeneous, time-dependent equation for

$$\Psi_t(\mathbf{r},t) \text{ satisfying } \Psi_t(\mathbf{r},t_0) = \Psi(\mathbf{r},t_0) - \psi_s(\mathbf{r}).$$

iii. Then $\Psi(\mathbf{r},t) = \Psi_t(\mathbf{r},t) + \psi_s(\mathbf{r})$.
Note: Ψ_t is the *transient part* and ψ_s is the *steady state part*.

3. **Time Dependent Boundary Conditions with Homogeneous Initial Conditions**

(a) Use the Boundary Value Green's Function, $h(\mathbf{r},\mathbf{r}'_s;t,t')$, which is similar to the surface Green's function in an earlier section.

$$\Psi(\mathbf{r},t) = \int_{t_0}^{\infty} h_D(\mathbf{r},\mathbf{r}'_s;t,t')\Psi(\mathbf{r}'_s,t')\,dt',$$

or

$$\Psi(\mathbf{r},t) = \int_{t_0}^{\infty} \frac{\partial h_N}{\partial n}(\mathbf{r},\mathbf{r}'_s;t,t')\Psi(\mathbf{r}'_s,t')\,dt'.$$

(b) Properties of $h(\mathbf{r},\mathbf{r}'_s;t,t')$:

$$\mathcal{D}h = \mathcal{T}h$$

$$h_D(\mathbf{r}_s,\mathbf{r}'_s;t,t') = \delta(t-t'), \text{ or} \frac{\partial h_N}{\partial n}(\mathbf{r}_s,\mathbf{r}'_s;t,t') = \delta(t-t'),$$

$$h(\mathbf{r},\mathbf{r}'_s;t,t') = 0, \quad t' > t, \text{ (causality)}.$$

(c) **Note:** For inhomogeneous I.C.,

$$\Psi = \int g\Psi(\mathbf{r}',t_0) + \int dt' h_D \Psi(\mathbf{r}'_s,t')\,d^3\mathbf{r}'.$$

8.8.3 Inhomogeneous Steady State Equation

1. **Poisson's Equation**

$$\nabla^2\psi(\mathbf{r},t) = f(\mathbf{r}), \qquad \psi(\mathbf{r}_s) \quad \text{or} \quad \frac{\partial\psi}{\partial n}(\mathbf{r}_s) \quad \text{given.}$$

(a) Green's Theorem:

$$\int [\psi(\mathbf{r}')\nabla'^2 G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla'^2\psi(\mathbf{r}')]\,d^3\mathbf{r}'$$
$$= \int [\psi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla'\psi(\mathbf{r}')] \cdot \vec{dS}',$$

where ∇' denotes differentiation with respect to r'.

(b) Properties of $G(\mathbf{r},\mathbf{r}')$:

i. $\nabla'^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$.

ii. $G|_s = 0$ or $\frac{\partial G}{\partial n'}|_s = 0$.

iii. Solution

$$\psi(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\,d^3\mathbf{r}' + \int [\psi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla'\psi(\mathbf{r}')]\cdot \vec{dS}'. \tag{8.161}$$

(c) For the case of pure Neumann B.C.'s, the Divergence Theorem leads to the constraint

$$\int \nabla\psi \cdot \vec{dS} = \int f\,d^3\mathbf{r}.$$

If there are pure Neumann conditions and S is finite and $\int f\,d^3\mathbf{r} \neq 0$ by symmetry, then $\vec{n}' \cdot \nabla' G|_s \neq 0$ and the Green's function method is much more complicated to solve.

(d) From the above result:

$$\vec{n}' \cdot \nabla' G(\mathbf{r},\mathbf{r}_s') = g_D(\mathbf{r},\mathbf{r}_s')$$

or

$$G_N(\mathbf{r},\mathbf{r}_s') = -g_N(\mathbf{r},\mathbf{r}_s').$$

It is often simpler to use G for $\int d^3\mathbf{r}'$ and g for $\int \vec{dS}'$, separately.

(e) G satisfies a reciprocity property,$G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$ for either Dirichlet or Neumann boundary conditions.

(f) $G(\mathbf{r},\mathbf{r}')$ can be considered as a potential at $\mathbf{r}$ due to a point charge $q = -1/4\pi$ at $\mathbf{r}'$, with all surfaces being grounded conductors.

8.8.4 *Inhomogeneous, Time Dependent Equations*

1. **Diffusion/Heat Flow** $\nabla^2\Psi - \frac{1}{a^2}\dot{\Psi} = f(\mathbf{r},t)$.

(a)

$$[\nabla^2 - \frac{1}{a^2}\frac{\partial}{\partial t}]G(\mathbf{r},\mathbf{r}';t,t') = [\nabla'^2 + \frac{1}{a^2}\frac{\partial}{\partial t'}]G(\mathbf{r},\mathbf{r}';t,t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{8.162}$$

(b) Green's Theorem in 4 dimensions $(\mathbf{r},t)$ yields

$$\Psi(\mathbf{r},t) = \int\int_{t_0}^{\infty} G(\mathbf{r},\mathbf{r}';t,t')f(\mathbf{r}',t')\,dt'd^3\mathbf{r}' - \frac{1}{a^2}\int G(\mathbf{r},\mathbf{r}';t,t_0)\Psi(\mathbf{r}',t_0)\,d^3\mathbf{r}' + \int_{t_0}^{\infty}\int [\Psi(\mathbf{r}_s',t)\nabla' G_D(\mathbf{r},\mathbf{r}_s';t,t') - G_N(\mathbf{r},\mathbf{r}_s';t,t')\nabla'\Psi(\mathbf{r}_s',t')]\cdot\vec{dS}'dt'.$$

(c) Either $G_D(\mathbf{r}_s') = 0$ or $G_N(\mathbf{r}_s') = 0$ on S at any point $\mathbf{r}_s'$.

(d) $\hat{\mathbf{n}}' \cdot \nabla' G_D(\mathbf{r}_s') = h_D(\mathbf{r}_s')$, $G_N(\mathbf{r}_s') = -h_N(\mathbf{r}_s')$, and $-\frac{1}{a^2}G(\mathbf{r},\mathbf{r}';t,t_0) = g(\mathbf{r},\mathbf{r}';t,t_0)$.

2. **Wave Equation** $\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial^2 t} = f(\mathbf{r},t)$.

(a)

$$
\begin{aligned}
[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]G(\mathbf{r},\mathbf{r}';t,t') &= [\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]G(\mathbf{r},\mathbf{r}';t,t') \\
&= \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \qquad (8.163)
\end{aligned}
$$

(b) Green's Theorem in 4 dimensions $(\mathbf{r},t)$ yields

$$
\begin{aligned}
\Psi(\mathbf{r},t) &= \int\int_{t_0}^{\infty} G(\mathbf{r},\mathbf{r}';t,t')f(\mathbf{r}',t')\,dt'd^3\mathbf{r}' \\
&- \frac{1}{c^2}\int [G(\mathbf{r},\mathbf{r}';t,t_0)\frac{\partial}{\partial t'}\Psi(\mathbf{r}',t_0) - \Psi(\mathbf{r}',t_0)\frac{\partial}{\partial t'}G(\mathbf{r},\mathbf{r}';t,t_0)]\,d^3\mathbf{r}' \\
&+ \int_{t_0}^{\infty}\int [\Psi(\mathbf{r}'_s,t)\nabla' G_D(\mathbf{r},\mathbf{r}'_s;t,t') - G_N(\mathbf{r},\mathbf{r}'_s;t,t')\nabla'\psi(\mathbf{r}'_s,t')]\cdot \vec{dS}'\,dt'.
\end{aligned}
$$

(c) Cauchy initial conditions are given: $\Psi(t_0)$ and $\dot{\Psi}(t_0)$.

(d) The wave and diffusion equations satisfy a causality condition $G(t,t') = 0, \quad t' > t$.

Problems

1. Find the solution of each initial value problem using the appropriate initial value Green's function.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

b. $y'' + y = 2\sin 3x, \quad y(0) = 5, \quad y'(0) = 0.$

c. $y'' + y = 1 + 2\cos x, \quad y(0) = 2, \quad y'(0) = 0.$

d. $x^2y'' - 2xy' + 2y = 3x^2 - x, \quad y(1) = \pi, \quad y'(1) = 0.$

2. Use the initial value Green's function for $x'' + x = f(t)$, $x(0) = 4$, $x'(0) = 0$, to solve the following problems.

a. $x'' + x = 5t^2.$

b. $x'' + x = 2\tan t.$

3. For the problem $y'' - k^2y = f(x)$, $y(0) = 0$, $y'(0) = 1$,

a. Find the initial value Green's function.

b. Use the Green's function to solve $y'' - y = e^{-x}$.

c. Use the Green's function to solve $y'' - 4y = e^{2x}$.

4. Find and use the initial value Green's function to solve

$$x^2y'' + 3xy' - 15y = x^4e^x, \quad y(1) = 1, y'(1) = 0.$$

5. Consider the problem $y'' = \sin x$, $y'(0) = 0$, $y(\pi) = 0$.

a. Solve by direct integration.

b. Determine the Green's function.

c. Solve the boundary value problem using the Green's function.

d. Change the boundary conditions to $y'(0) = 5$, $y(\pi) = -3$.

   i. Solve by direct integration.

   ii. Solve using the Green's function.

6. Let C be a closed curve and D the enclosed region. Prove the identity

$$\int_C \phi \nabla \phi \cdot \mathbf{n}\, ds = \int_D (\phi \nabla^2 \phi + \nabla \phi \cdot \nabla \phi)\, dA.$$

7. Let S be a closed surface and V the enclosed volume. Prove Green's first and second identities, respectively.

a. $\int_S \phi \nabla \psi \cdot \mathbf{n}\, dS = \int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi)\, dV$.

b. $\int_S [\phi \nabla \psi - \psi \nabla \phi] \cdot \mathbf{n}\, dS = \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV$.

8. Let C be a closed curve and D the enclosed region. Prove Green's identities in two dimensions.

a. First prove

$$\int_D (v \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla v)\, dA = \int_C (v\mathbf{F}) \cdot d\mathbf{s}.$$

b. Let $\mathbf{F} = \nabla u$ and obtain Green's first identity,

$$\int_D (v \nabla^2 u + \nabla u \cdot \nabla v)\, dA = \int_C (v \nabla u) \cdot d\mathbf{s}.$$

c. Use Green's first identity to prove Green's second identity,

$$\int_D (u \nabla^2 v - v \nabla^2 u)\, dA = \int_C (u \nabla v - v \nabla u) \cdot d\mathbf{s}.$$

9. Consider the problem:

$$\frac{\partial^2 G}{\partial x^2} = \delta(x - x_0), \quad \frac{\partial G}{\partial x}(0, x_0) = 0, \quad G(\pi, x_0) = 0.$$

a. Solve by direct integration.

b. Compare this result to the Green's function in part b of the last problem.

c. Verify that G is symmetric in its arguments.

10. Consider the boundary value problem: $y'' - y = x$, $x \in (0, 1)$, with boundary conditions $y(0) = y(1) = 0$.

a. Find a closed form solution without using Green's functions.

b. Determine the closed form Green's function using the properties of Green's functions. Use this Green's function to obtain a solution of the boundary value problem.

c. Determine a series representation of the Green's function. Use this Green's function to obtain a solution of the boundary value problem.

d. Confirm that all of the solutions obtained give the same results.

11. Rewrite the solution to Problem 15 and identify the initial value Green's function.

12. Rewrite the solution to Problem 16 and identify the initial value Green's functions.

13. Find the Green's function for the homogeneous fixed values on the boundary of the quarter plane $x > 0$, $y > 0$, for Poisson's equation using the infinite plane Green's function for Poisson's equation. Use the method of images.

14. Find the Green's function for the one dimensional heat equation with boundary conditions $u(0,t) = 0$ $u_x(L,t)$, $t > 0$.

15. Consider Laplace's equation on the rectangular plate in Figure 6.8. Construct the Green's function for this problem.

16. Construct the Green's function for Laplace's equation in the spherical domain in Figure 6.18.

17. Find the solution to the heat equation:

PDE: $u_t = 2u_{xx}$, $0 \le x \le 1$, $t > 0$.
BC: $u(0,t) = -1$, $u_x(1,t) = 1$.
IC: $u(x,0) = x + \sin\frac{3\pi x}{2} - 1$.

18. Find the solution to the heat equation:

PDE: $u_t = 5u_{xx}$, $0 \le x \le 10$, $t > 0$.
BC: $u_x(0,t) = 2$, $u_x(10,t) = 3$.
IC: $u(x,0) = \frac{x^2}{20} + 2x + \cos\pi x$.

19. Find the solution to the nonhomogeneous heat equation:

PDE: $u_t - u_{xx} = t\sin x$, $0 \le x \le \pi$, $t > 0$.
BC: $u(0,t) = 0$, $u(\pi,t) = 0$.
IC: $u(x,0) = 0$.

20. Find the solution to the nonhomogeneous heat equation:

PDE: $u_t - u_{xx} = t(\sin 2\pi x + 2x)$, $0 \le x \le 1$, $t > 0$.
BC: $u(0,t) = 1$, $u(1,t) = t^2$.
IC: $u(x,0) = 1 + \sin 3\pi x - x$.

21. Find the general solution to the heat equation, $u_t - u_{xx} = 0$, on $[0,\pi]$ satisfying the boundary conditions $u_x(0,t) = 0$ and $u(\pi,t) = 0$. Determine the solution satisfying the initial condition,

$$u(x,0) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \le x \le \pi, \end{cases}$$

Rewrite the solution, identify the initial value Green's function, and write the solution in terms of the Green's function.

22. Find the general solution to the wave equation $u_{tt} = 2u_{xx}$, on $[0, 2\pi]$ satisfying the boundary conditions $u(0, t) = 0$ and $u_x(2\pi, t) = 0$. Determine the solution satisfying the initial conditions, $u(x, 0) = x(4\pi - x)$, and $u_t(x, 0) = 0$. Rewrite the solution, identify the initial value Green's functions, and write the solution in terms of the Green's functions.

23. Find the solution to Laplace's equation, $u_{xx} + u_{yy} = 0$, on the unit square, $[0, 1] \times [0, 1]$ satisfying the boundary conditions $u(0, y) = 0$, $u(1, y) = y(1 - y)$, $u(x, 0) = 0$, and $u(x, 1) = 0$. Rewrite the solution, identify the boundary value Green's function, and write the solution in terms of the Green's function.

24. Find the Green's function for homogeneous fixed values on the boundary of the quarter plane $x > 0$, $y > 0$, for Poisson's Equation using the infinite plane Green's function for Poisson's equation. Use the method of images.

25. Consider the forced heat equation with homogeneous boundary conditions:

PDE: $u_t - 5u_{xx} = e^{-t}$, $0 \leq x \leq 1$, $t > 0$.

BC: $u(0, t) = 0$, $u(1, t) = 0$.

IC: $u(x, 0) = x(1 - x)$.

Solve this problem by assuming a solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin n\pi x$$

and solving for $b_n(t)$, $n = 1, 2, \ldots$.

26. Consider the forced wave equation with homogeneous initial conditions:

PDE: $u_{tt} - 4u_{xx} = x(t + 1)$, $0 \leq x \leq \pi$, $t > 0$.

BC: $u(0, t) = 0$, $u(\pi, t) = \sin t$.

IC: $u(x, 0) = 0$, $u_t(x, 0) = 0$.

Assuming that

$$u(x, t) = v(x, t) + \frac{x}{\pi} \sin t,$$

where

$$v(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin nx,$$

find the coefficients $b_n(t)$, $n = 1, 2, \ldots$, and write out the full solution.

9
Complex Representations of Functions

"He is not a true man of science who does not bring some sympathy to his studies, and expect to learn something by behavior as well as by application. It is childish to rest in the discovery of mere coincidences, or of partial and extraneous laws. The study of geometry is a petty and idle exercise of the mind, if it is applied to no larger system than the starry one. Mathematics should be mixed not only with physics but with ethics; that is mixed mathematics. The fact which interests us most is the life of the naturalist. The purest science is still biographical." Henry David Thoreau (1817-1862)

9.1 Complex Representations of Waves

We have seen that we can determine the frequency content of a function $f(t)$ defined on an interval $[0, T]$ by looking for the Fourier coefficients in the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T}.$$

The coefficients take forms like

$$a_n = \frac{2}{T} \int_0^T f(t) \cos \frac{2\pi nt}{T} \, dt.$$

However, trigonometric functions can be written in a complex exponential form. Using Euler's formula, which was obtained using the Maclaurin expansion of e^x in Example A.36,

$$e^{i\theta} = \cos\theta + i \sin\theta,$$

the complex conjugate is found by replacing i with $-i$ to obtain

$$e^{-i\theta} = \cos\theta - i \sin\theta.$$

Adding these expressions, we have

$$2\cos\theta = e^{i\theta} + e^{-i\theta}.$$

Subtracting the exponentials leads to an expression for the sine function. Thus, we have the important result that sines and cosines can be written as complex exponentials:

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}, \\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i}. \end{aligned} \tag{9.1}$$

So, we can write

$$\cos\frac{2\pi nt}{T} = \frac{1}{2}(e^{\frac{2\pi int}{T}} + e^{-\frac{2\pi int}{T}}).$$

Later we will see that we can use this information to rewrite the series as a sum over complex exponentials in the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi int}{T}},$$

where the Fourier coefficients now take the form

$$c_n = \int_0^T f(t) e^{-\frac{2\pi int}{T}}\, dt.$$

In fact, when one considers the representation of analogue signals defined over an infinite interval and containing a continuum of frequencies, we will see that Fourier series sums become integrals of complex functions and so do the Fourier coefficients. Thus, we will naturally find ourselves needing to work with functions of complex variables and perform complex integrals.

We can also develop a complex representation for waves. Recall from the discussion in Section 2.6 on finite length strings that a solution to the wave equation was given by

$$u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} A_n \sin k_n(x+ct) + \sum_{n=1}^{\infty} A_n \sin k_n(x-ct)\right]. \tag{9.2}$$

We can replace the sines with their complex forms as

$$\begin{aligned} u(x,t) &= \frac{1}{4i}\left[\sum_{n=1}^{\infty} A_n \left(e^{ik_n(x+ct)} - e^{-ik_n(x+ct)}\right)\right. \\ &\quad \left. + \sum_{n=1}^{\infty} A_n \left(e^{ik_n(x-ct)} - e^{-ik_n(x-ct)}\right)\right]. \end{aligned} \tag{9.3}$$

Defining $k_{-n} = -k_n$, $n = 1, 2, \ldots$, we can rewrite this solution in the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} \left[c_n e^{ik_n(x+ct)} + d_n e^{ik_n(x-ct)}\right]. \tag{9.4}$$

Such representations are also possible for waves propagating over the entire real line. In such cases we are not restricted to discrete frequencies and wave numbers. The sum of the harmonics will then be a sum over a continuous range, which means that the sums become integrals. So, we are lead to the complex representation

$$u(x,t) = \int_{-\infty}^{\infty} \left[c(k)e^{ik(x+ct)} + d(k)e^{ik(x-ct)}\right] dk. \tag{9.5}$$

The forms $e^{ik(x+ct)}$ and $e^{ik(x-ct)}$ are complex representations of what are called plane waves in one dimension. The integral represents a general wave form consisting of a sum over plane waves. The Fourier coefficients in the representation can be complex valued functions and the evaluation of the integral may be done using methods from complex analysis. We would like to be able to compute such integrals.

With the above ideas in mind, we will now take a tour of complex analysis. We will first review some facts about complex numbers and then introduce complex functions. This will lead us to the calculus of functions of a complex variable, including the differentiation and integration complex functions. This will set up the methods needed to explore Fourier transforms in the next chapter.

9.2 *Complex Numbers*

COMPLEX NUMBERS WERE FIRST INTRODUCED in order to solve some simple problems. The history of complex numbers only extends about five hundred years. In essence, it was found that we need to find the roots of equations such as $x^2 + 1 = 0$. The solution is $x = \pm\sqrt{-1}$. Due to the usefulness of this concept, which was not realized at first, a special symbol was introduced - the imaginary unit, $i = \sqrt{-1}$. In particular, Girolamo Cardano $(1501 - 1576)$ was one of the first to use square roots of negative numbers when providing solutions of cubic equations. However, complex numbers did not become an important part of mathematics or science until the late seventh and eighteenth centuries after people like Abraham de Moivre (1667-1754), the Bernoulli[1] family and Euler took them seriously.

[1] The Bernoulli's were a family of Swiss mathematicians spanning three generations. It all started with Jacob Bernoulli (1654-1705) and his brother Johann Bernoulli (1667-1748). Jacob had a son, Nicolaus Bernoulli (1687-1759) and Johann (1667-1748) had three sons, Nicolaus Bernoulli II (1695-1726), Daniel Bernoulli (1700-1872), and Johann Bernoulli II (1710-1790). The last generation consisted of Johann II's sons, Johann Bernoulli III (1747-1807) and Jacob Bernoulli II (1759-1789). Johann, Jacob and Daniel Bernoulli were the most famous of the Bernoulli's. Jacob studied with Leibniz, Johann studied under his older brother and later taught Leonhard Euler and Daniel Bernoulli, who is known for his work in hydrodynamics.

A complex number is a number of the form $z = x + iy$, where x and y are real numbers. x is called the real part of z and y is the imaginary part of z. Examples of such numbers are $3 + 3i$, $-1i = -i$, $4i$ and 5. Note that $5 = 5 + 0i$ and $4i = 0 + 4i$.

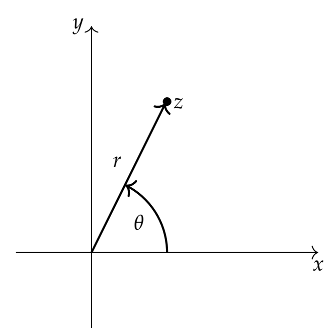

Figure 9.1: The Argand diagram for plotting complex numbers in the complex z-plane.

There is a geometric representation of complex numbers in a two dimensional plane, known as the complex plane C. This is given by the Argand diagram as shown in Figure 9.1. Here we can think of the complex number $z = x + iy$ as a point (x, y) in the z-complex plane or as a vector. The magnitude, or length, of this vector is called the complex modulus of z, denoted by $|z| = \sqrt{x^2 + y^2}$. We can also use the geometric picture to develop a polar representation of complex numbers. From Figure 9.1 we can see that in terms of r and θ we have that

The complex modulus, $|z| = \sqrt{x^2 + y^2}$.

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned} \tag{9.6}$$

Thus,

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{9.7}$$

Complex numbers can be represented in rectangular (Cartesian), $z = x + iy$, or polar form, $z = re^{i\theta}$. Here we define the argument, θ, and modulus, $|z| = r$ of complex numbers.

So, given r and θ we have $z = re^{i\theta}$. However, given the Cartesian form,

$z = x + iy$, we can also determine the polar form, since

$$r = \sqrt{x^2 + y^2},$$
$$\tan\theta = \frac{y}{x}. \quad (9.8)$$

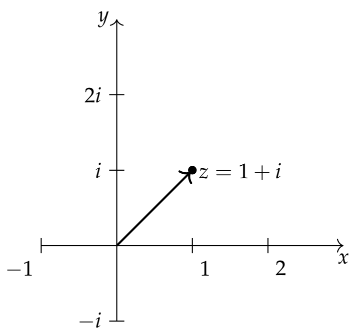

Figure 9.2: Locating $1 + i$ in the complex z-plane.

Note that $r = |z|$.

Locating $1 + i$ in the complex plane, it is possible to immediately determine the polar form from the angle and length of the "complex vector." This is shown in Figure 9.2. It is obvious that $\theta = \frac{\pi}{4}$ and $r = \sqrt{2}$.

Example 9.1. Write $z = 1 + i$ in polar form.

If one did not see the polar form from the plot in the z-plane, then one could systematically determine the results. First, write $z = 1 + i$ in polar form, $z = re^{i\theta}$, for some r and θ.

Using the above relations between polar and Cartesian representations, we have $r = \sqrt{x^2 + y^2} = \sqrt{2}$ and $\tan\theta = \frac{y}{x} = 1$. This gives $\theta = \frac{\pi}{4}$. So, we have found that

$$1 + i = \sqrt{2}e^{i\pi/4}.$$

We can also define binary operations of addition, subtraction, multiplication and division of complex numbers to produce a new complex number.

We can easily add, subtract, multiply and divide complex numbers.

The addition of two complex numbers is simply done by adding the real and imaginary parts of each number. So,

$$(3 + 2i) + (1 - i) = 4 + i.$$

Subtraction is just as easy,

$$(3 + 2i) - (1 - i) = 2 + 3i.$$

We can multiply two complex numbers just like we multiply any binomials, though we now can use the fact that $i^2 = -1$. For example, we have

$$(3 + 2i)(1 - i) = 3 + 2i - 3i + 2i(-i) = 5 - i.$$

We can even divide one complex number into another one and get a complex number as the quotient. Before we do this, we need to introduce the complex conjugate, $\bar{z}$, of a complex number. The complex conjugate of $z = x + iy$, where x and y are real numbers, is given as

The complex conjugate of $z = x + iy$, is given as $\bar{z} = x - iy$.

$$\bar{z} = x - iy.$$

Complex conjugates satisfy the following relations for complex numbers z and w and real number x.

$$\begin{aligned} \overline{z + w} &= \bar{z} + \bar{w} \\ \overline{zw} &= \bar{z}\bar{w} \\ \bar{\bar{z}} &= z \\ \bar{x} &= x. \end{aligned} \quad (9.9)$$

One consequence is that the complex conjugate of $re^{i\theta}$ is

$$\overline{re^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = re^{-i\theta}.$$

Another consequence is that

$$z\bar{z} = re^{i\theta}re^{-i\theta} = r^2.$$

Thus, the product of a complex number with its complex conjugate is a real number. We can also prove this result using the Cartesian form

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2.$$

Now we are in a position to write the quotient of two complex numbers in the standard form of a real plus an imaginary number.

Example 9.2. Simplify the expression $z = \frac{3+2i}{1-i}$.

This simplification is accomplished by multiplying the numerator and denominator of this expression by the complex conjugate of the denominator:

$$z = \frac{3+2i}{1-i} = \frac{3+2i}{1-i}\frac{1+i}{1+i} = \frac{1+5i}{2}.$$

Therefore, the quotient is a complex number and in standard form it is given by $z = \frac{1}{2} + \frac{5}{2}i$.

We can also consider powers of complex numbers. For example,

$$(1+i)^2 = 2i,$$

$$(1+i)^3 = (1+i)(2i) = 2i - 2.$$

But, what is $(1+i)^{1/2} = \sqrt{1+i}$?

In general, we want to find the nth root of a complex number. Let $t = z^{1/n}$. To find t in this case is the same as asking for the solution of

$$z = t^n$$

given z. But, this is the root of an nth degree equation, for which we expect n roots. If we write z in polar form, $z = re^{i\theta}$, then we would naively compute

$$\begin{aligned} z^{1/n} &= \left(re^{i\theta}\right)^{1/n} \\ &= r^{1/n}e^{i\theta/n} \\ &= r^{1/n}\left[\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right]. \end{aligned} \tag{9.10}$$

For example,

$$(1+i)^{1/2} = \left(\sqrt{2}e^{i\pi/4}\right)^{1/2} = 2^{1/4}e^{i\pi/8}.$$

But this is only one solution. We expected two solutions for $n = 2$..

The function $f(z) = z^{1/n}$ is multivalued. $z^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}$, $k = 0, 1, \ldots, n-1$.

The reason we only found one solution is that the polar representation for z is not unique. We note that

$$e^{2k\pi i} = 1, \quad k = 0, \pm 1, \pm 2, \ldots .$$

So, we can rewrite z as $z = re^{i\theta}e^{2k\pi i} = re^{i(\theta+2k\pi)}$. Now, we have that

$$z^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1.$$

Note that these are the only distinct values for the roots. We can see this by considering the case $k = n$. Then, we find that

$$e^{i(\theta+2\pi i n)/n} = e^{i\theta/n}e^{2\pi i} = e^{i\theta/n}.$$

So, we have recovered the $n = 0$ value. Similar results can be shown for the other k values larger than n.

Now, we can finish the example we had started.

Example 9.3. Determine the square roots of $1+i$, or $\sqrt{1+i}$.

As we have seen, we first write $1+i$ in polar form, $1+i = \sqrt{2}e^{i\pi/4}$. Then, introduce $e^{2k\pi i} = 1$ and find the roots:

$$\begin{aligned}
(1+i)^{1/2} &= \left(\sqrt{2}e^{i\pi/4}e^{2k\pi i}\right)^{1/2}, \quad k = 0, 1, \\
&= 2^{1/4}e^{i(\pi/8+k\pi)}, \quad k = 0, 1, \\
&= 2^{1/4}e^{i\pi/8}, 2^{1/4}e^{9\pi i/8}.
\end{aligned} \tag{9.11}$$

The nth roots of unity, $\sqrt[n]{1}$.

Finally, what is $\sqrt[n]{1}$? Our first guess would be $\sqrt[n]{1} = 1$. But, we now know that there should be n roots. These roots are called the nth roots of unity. Using the above result with $r = 1$ and $\theta = 0$, we have that

$$\sqrt[n]{1} = \left[\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right], \quad k = 0, \ldots, n-1.$$

For example, we have

$$\sqrt[3]{1} = \left[\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right], \quad k = 0, 1, 2.$$

These three roots can be written out as

$$\sqrt[3]{1} = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

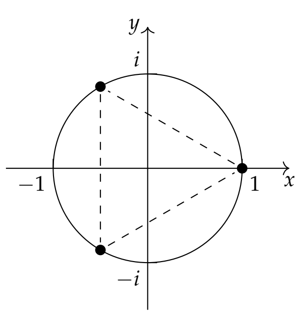

Figure 9.3: Locating the cube roots of unity in the complex z-plane.

We can locate these cube roots of unity in the complex plane. In Figure 9.3 we see that these points lie on the unit circle and are at the vertices of an equilateral triangle. In fact, all nth roots of unity lie on the unit circle and are the vertices of a regular n-gon with one vertex at $z = 1$.

9.3 *Complex Valued Functions*

We would like to next explore complex functions and the calculus of complex functions. We begin by defining a function that takes complex numbers into complex numbers, $f : C \to C$. It is difficult to visualize such functions. For real functions of one variable, $f : R \to R$, we graph these functions by first drawing two intersecting copies of R and then proceed to map the domain into the range of f.

It would be more difficult to do this for complex functions. Imagine placing together two orthogonal copies of the complex plane, C. One would need a four dimensional space in order to complete the visualization. Instead, typically uses two copies of the complex plane side by side in order to indicate how such functions behave. Over the years there have been several ways to visualize complex functions. We will describe a few of these in this chapter.

We will assume that the domain lies in the z-plane and the image lies in the w-plane. We will then write the complex function as $w = f(z)$. We show these planes in Figure 9.4 and the mapping between the planes.

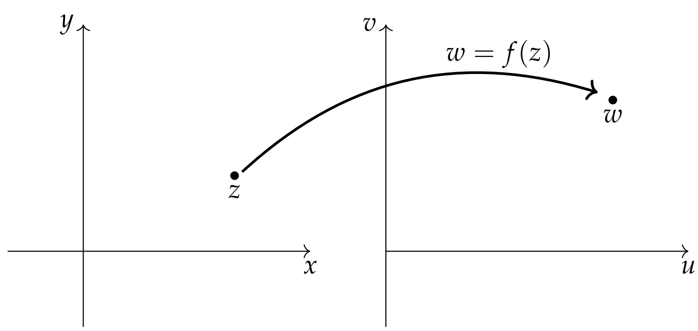

Figure 9.4: Defining a complex valued function, $w = f(z)$, on C for $z = x + iy$ and $w = u + iv$.

Letting $z = x + iy$ and $w = u + iv$, we can write the real and imaginary parts of $f(z)$:

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

We see that one can view this function as a function of z or a function of x and y. Often, we have an interest in writing out the real and imaginary parts of the function, $u(x, y)$ and $v(x, y)$, which are functions of two real variables, x and y. We will look at several functions to determine the real and imaginary parts.

Example 9.4. Find the real and imaginary parts of $f(z) = z^2$.

For example, we can look at the simple function $f(z) = z^2$. It is a simple matter to determine the real and imaginary parts of this function. Namely, we have

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Therefore, we have that

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

In Figure 9.5 we show how a grid in the z-plane is mapped by $f(z) = z^2$ into the w-plane. For example, the horizontal line $x =$

1 is mapped to $u(1, y) = 1 - y^2$ and $v(1, y) = 2y$. Eliminating the "parameter" y between these two equations, we have $u = 1 - v^2/4$. This is a parabolic curve. Similarly, the horizontal line $y = 1$ results in the curve $u = v^2/4 - 1$.

If we look at several curves, x =const and y =const, then we get a family of intersecting parabolae, as shown in Figure 9.5.

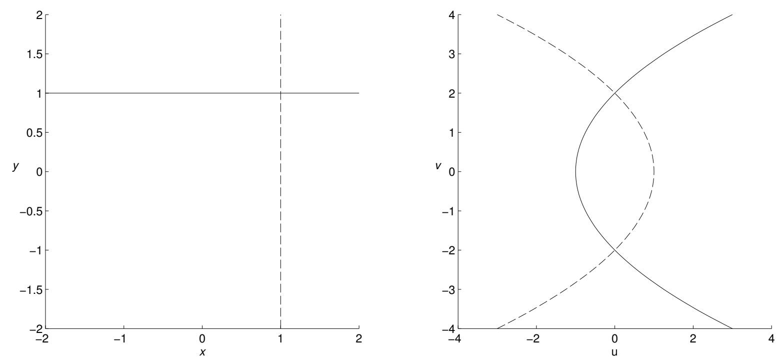

Figure 9.5: 2D plot showing how the function $f(z) = z^2$ maps the lines $x = 1$ and $y = 1$ in the z-plane into parabolae in the w-plane.

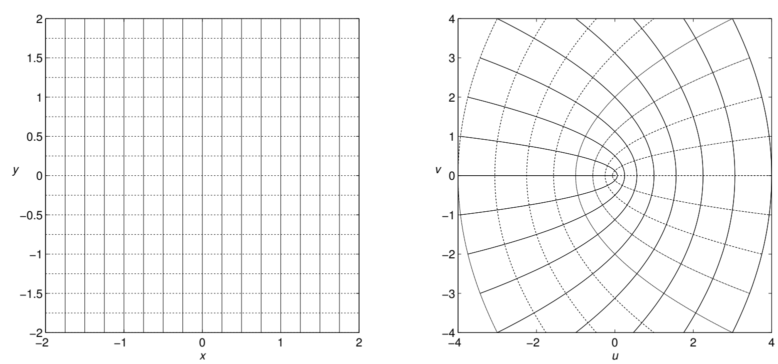

Figure 9.6: 2D plot showing how the function $f(z) = z^2$ maps a grid in the z-plane into the w-plane.

Example 9.5. Find the real and imaginary parts of $f(z) = e^z$.

For this case, we make use of Euler's Formula.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y). \end{aligned} \tag{9.12}$$

Thus, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. In Figure 9.7 we show how a grid in the z-plane is mapped by $f(z) = e^z$ into the w-plane.

Example 9.6. Find the real and imaginary parts of $f(z) = z^{1/2}$.

We have that

$$z^{1/2} = \sqrt{x^2 + y^2} \left(\cos(\theta + k\pi) + i \sin(\theta + k\pi)\right), \quad k = 0, 1. \tag{9.13}$$

Thus,

$$u = |z| \cos(\theta + k\pi), u = |z| \cos(\theta + k\pi),$$

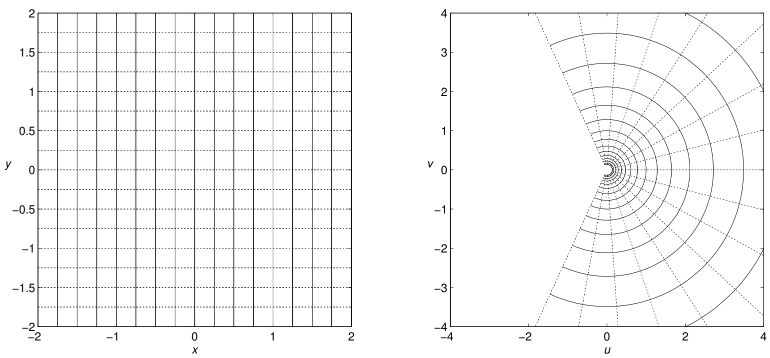

Figure 9.7: 2D plot showing how the function $f(z) = e^z$ maps a grid in the z-plane into the w-plane.

for $|z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$. For each k-value one has a different surface and curves of constant θ give $u/v = c_1$, and curves of constant nonzero complex modulus give concentric circles, $u^2 + v^2 = c_2$, for c_1 and c_2 constants.

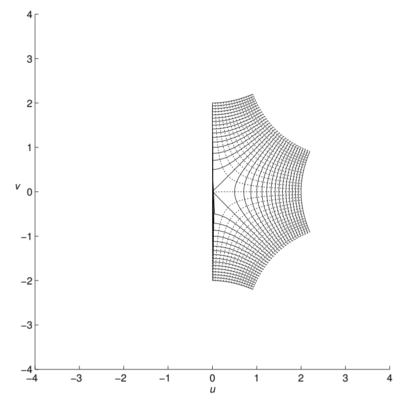

Figure 9.8: 2D plot showing how the function $f(z) = \sqrt{z}$ maps a grid in the z-plane into the w-plane.

Example 9.7. Find the real and imaginary parts of $f(z) = \ln z$.

In this case we make use of the polar form of a complex number, $z = re^{i\theta}$. Our first thought would be to simply compute

$$\ln z = \ln r + i\theta.$$

However, the natural logarithm is multivalued, just like the square root function. Recalling that $e^{2\pi ik} = 1$ for k an integer, we have $z = re^{i(\theta+2\pi k)}$. Therefore,

$$\ln z = \ln r + i(\theta + 2\pi k), \qquad k = \text{ integer}.$$

The natural logarithm is a multivalued function. In fact there are an infinite number of values for a given z. Of course, this contradicts the definition of a function that you were first taught.

Thus, one typically will only report the principal value, $\operatorname{Log} z = \ln r + i\theta$, for θ restricted to some interval of length 2π, such as $[0, 2\pi)$. In order to account for the multivaluedness, one introduces a way to extend the complex plane so as to include all of the branches. This is done by assigning a plane to each branch, using (branch) cuts along lines, and then gluing the planes together at the branch cuts to form what is called a Riemann surface. We will not elaborate upon this any further here and refer the interested reader to more advanced texts. Comparing the multivalued logarithm to the principal value logarithm, we have

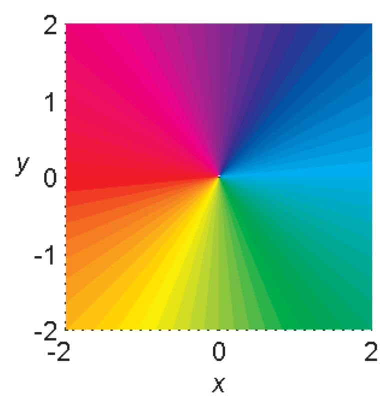

Figure 9.9: Domain coloring of the complex z-plane assigning colors to $\arg(z)$.

$$\ln z = \operatorname{Log} z + 2n\pi i.$$

We should not that some books use $\log z$ instead of $\ln z$. It should not be confused with the common logarithm.

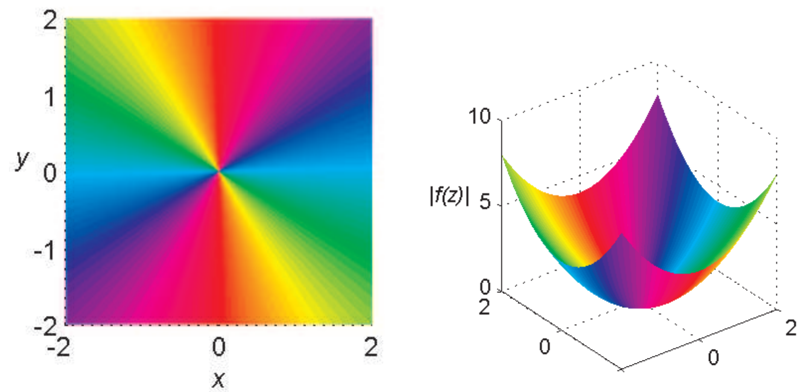

Figure 9.10: Domain coloring for $f(z) = z^2$. The left figure shows the phase coloring. The right figure show the colored surface with height $|f(z)|$.

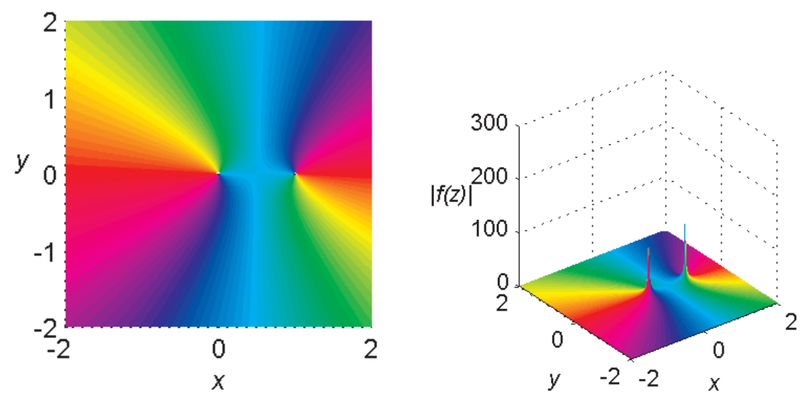

Figure 9.11: Domain coloring for $f(z) = 1/z(1-z)$. The left figure shows the phase coloring. The right figure show the colored surface with height $|f(z)|$.

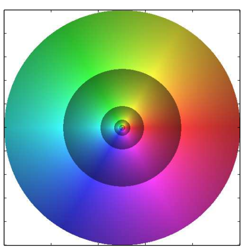

Figure 9.12: Domain coloring for the function $f(z) = z$ showing a coloring for $\arg(z)$ and brightness based on $|f(z)|$.

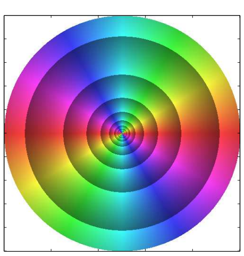

Figure 9.13: Domain coloring for the function $f(z) = z^2$.

9.3.1 Complex Domain Coloring

ANOTHER METHOD FOR VISUALIZING COMPLEX FUNCTIONS is domain coloring. The idea was described by Frank A. Farris. There are a few approaches to this method. The main idea is that one colors each point of the z-plane (the domain) according to $\arg(z)$ as shown in Figure 9.9. The modulus, $|f(z)|$ is then plotted as a surface. Examples are shown for $f(z) = z^2$ in Figure 9.10 and $f(z) = 1/z(1-z)$ in Figure 9.11.

We would like to put all of this information in one plot. We can do this by adjusting the brightness of the colored domain by using the modulus of the function. In the plots that follow we use the fractional part of $\ln|z|$. In Figure 9.12 we show the effect for the z-plane using $f(z) = z$. In the figures that follow we look at several other functions. In these plots we have chosen to view the functions in a circular window.

One can see the rich behavior hidden in these figures. As you progress in your reading, especially after the next chapter, you should return to these figures and locate the zeros, poles, branch points and branch cuts. A search online will lead you to other colorings and superposition of the uv grid on these figures.

As a final picture, we look at iteration in the complex plane. Consider the function $f(z) = z^2 - 0.75 - 0.2i$. Interesting figures result when studying the iteration in the complex plane. In Figure 9.15 we show $f(z)$ and $f^{20}(z)$, which is the iteration of f twenty times. It leads to an interesting coloring. What happens when one keeps iterating? Such iterations lead to the study of Julia and Mandelbrot sets . In Figure 9.16 we show six iterations of $f(z) = (1 - i/2)\sin x$.

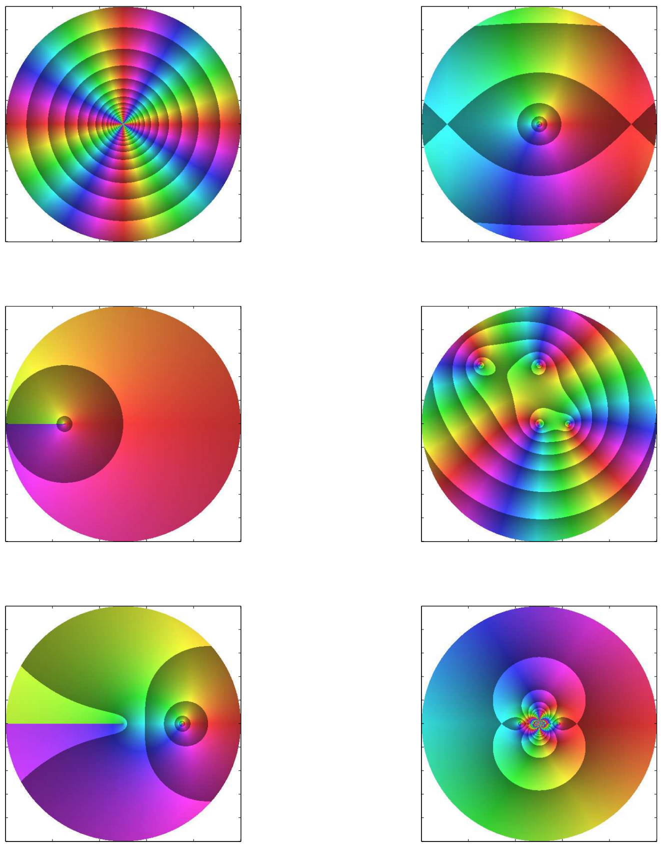

Figure 9.14: Domain coloring for several functions. On the top row the domain coloring is shown for $f(z) = z^4$ and $f(z) = \sin z$. On the second row plots for $f(z) = \sqrt{1+z}$ and $f(z) = \frac{1}{z(1/2-z)(z-i)(z-i+1)}$ are shown. In the last row domain colorings for $f(z) = \ln z$ and $f(z) = \sin(1/z)$ are shown.

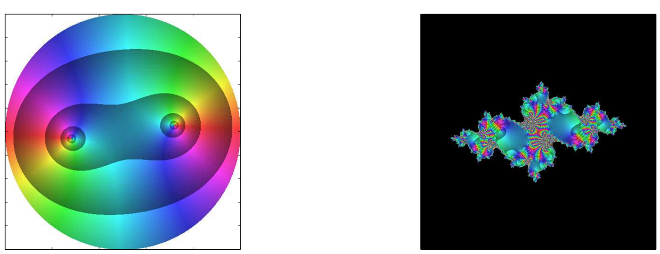

Figure 9.15: Domain coloring for $f(z) = z^2 - 0.75 - 0.2i$. The left figure shows the phase coloring. On the right is the plot for $f^{20}(z)$.

The following code was used in MATLAB to produce these figures.

```
fn = @(x) (1-i/2)*sin(x);
xmin=-2; xmax=2; ymin=-2; ymax=2;
Nx=500;
Ny=500;
x=linspace(xmin,xmax,Nx);
```

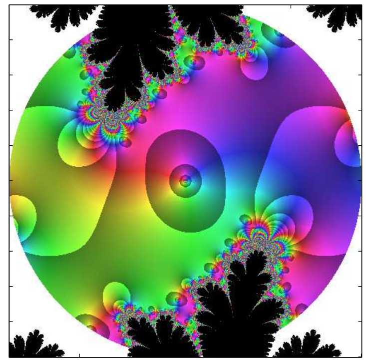

Figure 9.16: Domain coloring for six iterations of $f(z) = (1 - i/2)\sin x$.

```
y=linspace(ymin,ymax,Ny);
[X,Y] = meshgrid(x,y); z = complex(X,Y);
tmp=z; for n=1:6
    tmp = fn(tmp);
end Z=tmp;
XX=real(Z);
YY=imag(Z);
R2=max(max(X.^2));
R=max(max(XX.^2+YY.^2));

circle(:,:,1) = X.^2+Y.^2 < R2;
circle(:,:,2)=circle(:,:,1);
circle(:,:,3)=circle(:,:,1);

addcirc(:,:,1)=circle(:,:,1)==0;
addcirc(:,:,2)=circle(:,:,1)==0;
addcirc(:,:,3)=circle(:,:,1)==0;

warning off MATLAB:divideByZero;
hsvCircle=ones(Nx,Ny,3);
hsvCircle(:,:,1)=atan2(YY,XX)*180/pi+(atan2(YY,XX)*180/pi<0)*360;
hsvCircle(:,:,1)=hsvCircle(:,:,1)/360; lgz=log(XX.^2+YY.^2)/2;
hsvCircle(:,:,2)=0.75; hsvCircle(:,:,3)=1-(lgz-floor(lgz))/2;
hsvCircle(:,:,1) = flipud((hsvCircle(:,:,1)));

hsvCircle(:,:,2) = flipud((hsvCircle(:,:,2)));

hsvCircle(:,:,3) =flipud((hsvCircle(:,:,3)));

rgbCircle=hsv2rgb(hsvCircle);
rgbCircle=rgbCircle.*circle+addcirc;
```

```
image(rgbCircle)
axis square
set(gca,'XTickLabel',{})
set(gca,'YTickLabel',{})
```

9.4 Complex Differentiation

NEXT WE WANT TO DIFFERENTIATE COMPLEX FUNCTIONS. We generalize the definition from single variable calculus,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}, \tag{9.14}$$

provided this limit exists.

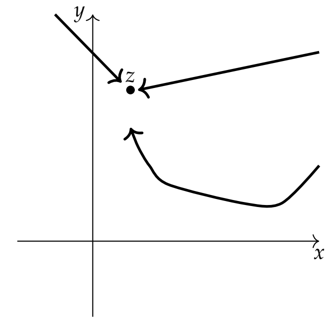

Figure 9.17: There are many paths that approach z as $\Delta z \to 0$.

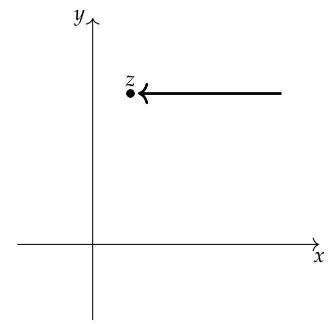

Figure 9.18: A path that approaches z with $y =$ constant.

The computation of this limit is similar to what one sees in multivariable calculus for limits of real functions of two variables. Letting $z = x + iy$ and $\delta z = \delta x + i\delta y$, then

$$z + \delta x = (x + \delta x) + i(y + \delta y).$$

Letting $\Delta z \to 0$ means that we get closer to z. There are many paths that one can take that will approach z. [See Figure 9.17.]

It is sufficient to look at two paths in particular. We first consider the path $y =$ constant. This horizontal path is shown in Figure 9.18. For this path, $\Delta z = \Delta x + i\Delta y = \Delta x$, since y does not change along the path. The derivative, if it exists, is then computed as

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - (u(x,y) + iv(x,y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}. \end{aligned} \tag{9.15}$$

The last two limits are easily identified as partial derivatives of real valued functions of two variables. Thus, we have shown that when $f'(z)$ exists,

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.} \tag{9.16}$$

A similar computation can be made if instead we take the vertical path, $x =$ constant, in Figure 9.17). In this case $\Delta z = i\Delta y$ and

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - (u(x,y) + iv(x,y))}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}. \end{aligned} \tag{9.17}$$

Therefore,

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{9.18}$$

We have found two different expressions for $f'(z)$ by following two different paths to z. If the derivative exists, then these two expressions must be the same. Equating the real and imaginary parts of these expressions, we have

The Cauchy-Riemann Equations.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned} \tag{9.19}$$

These are known as the Cauchy-Riemann equations[2].

[2] Augustin-Louis Cauchy (1789-1857) was a French mathematician well known for his work in analysis. Georg Friedrich Bernhard Riemann (1826-1866) was a German mathematician who made major contributions to geometry and analysis.

Theorem 9.1. *$f(z)$ is holomorphic (differentiable) if and only if the Cauchy-Riemann equations are satisfied.*

Example 9.8. $f(z) = z^2$.

In this case we have already seen that $z^2 = x^2 - y^2 + 2ixy$. Therefore, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We first check the Cauchy-Riemann equations.

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= 2y = -\frac{\partial u}{\partial y}. \end{aligned} \tag{9.20}$$

Therefore, $f(z) = z^2$ is differentiable.

We can further compute the derivative using either Equation (9.16) or Equation (9.18). Thus,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i(2y) = 2z.$$

This result is not surprising.

Example 9.9. $f(z) = \bar{z}$.

In this case we have $f(z) = x - iy$. Therefore, $u(x,y) = x$ and $v(x,y) = -y$. But, $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial y} = -1$. Thus, the Cauchy-Riemann equations are not satisfied and we conclude the $f(z) = \bar{z}$ is not differentiable.

Harmonic functions satisfy Laplace's equation.

Another consequence of the Cauchy-Riemann equations is that both $u(x,y)$ and $v(x,y)$ are harmonic functions. A real-valued function $u(x,y)$ is harmonic if it satisfies Laplace's equation in 2D, $\nabla^2 u = 0$, or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Theorem 9.2. *$f(z) = u(x,y) + iv(x,y)$ is differentiable if and only if u and v are harmonic functions.*

This is easily proven using the Cauchy-Riemann equations.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \\
&= \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\
&= -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\
&= -\frac{\partial^2 u}{\partial y^2}. \qquad (9.21)
\end{aligned}$$

Example 9.10. Is $u(x,y) = x^2 + y^2$ harmonic?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 \neq 0.$$

No, it is not.

Example 9.11. Is $u(x,y) = x^2 - y^2$ harmonic?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

Yes, it is.

Given a harmonic function $u(x,y)$, can one find a function, $v(x,y)$, such $f(z) = u(x,y) + iv(x,y)$ is differentiable? In this case, v are called the harmonic conjugate of u.

The harmonic conjugate function.

Example 9.12. Find the harmonic conjugate of $u(x,y) = x^2 - y^2$ and determine $f(z) = u + iv$ such that $u + iv$ is differentiable.

The Cauchy-Riemann equations tell us the following about the unknown function, $v(x,y)$:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y,$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x.$$

We can integrate the first of these equations to obtain

$$v(x,y) = \int 2y\, dx = 2xy + c(y).$$

Here $c(y)$ is an arbitrary function of y. One can check to see that this works by simply differentiating the result with respect to x.

However, the second equation must also hold. So, we differentiate the result with respect to y to find that

$$\frac{\partial v}{\partial y} = 2x + c'(y).$$

Since we were supposed to get $2x$, we have that $c'(y) = 0$. Thus, $c(y) = k$ is a constant.

We have just shown that we get an infinite number of functions,

$$v(x,y) = 2xy + k,$$

such that

$$f(z) = x^2 - y^2 + i(2xy + k)$$

is differentiable. In fact, for $k = 0$ this is nothing other than $f(z) = z^2$.

9.5 Complex Integration

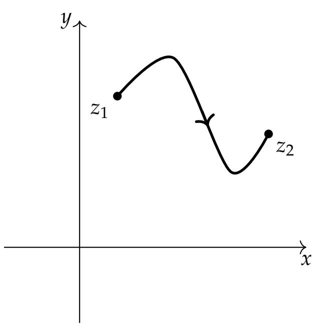

Figure 9.19: We would like to integrate a complex function $f(z)$ over the path Γ in the complex plane.

We have introduced functions of a complex variable. We also established when functions are differentiable as complex functions, or holomorphic. In this chapter we will turn to integration in the complex plane. We will learn how to compute complex path integrals, or contour integrals. We will see that contour integral methods are also useful in the computation of some of the real integrals that we will face when exploring Fourier transforms in the next chapter.

9.5.1 Complex Path Integrals

In this section we will investigate the computation of complex path integrals. Given two points in the complex plane, connected by a path Γ as shown in Figure 9.19, we would like to define the integral of $f(z)$ along Γ,

$$\int_\Gamma f(z)\, dz.$$

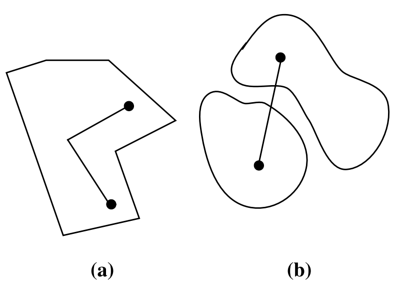

Figure 9.20: Examples of (a) a connected set and (b) a disconnected set.

A natural procedure would be to work in real variables, by writing

$$\int_\Gamma f(z)\, dz = \int_\Gamma [u(x,y) + iv(x,y)]\,(dx + idy),$$

since $z = x + iy$ and $dz = dx + idy$.

In order to carry out the integration, we then have to find a parametrization of the path and use methods from a multivariate calculus class. Namely, let u and v be continuous in domain D, and Γ a piecewise smooth curve in D. Let $(x(t), y(t))$ be a parametrization of Γ for $t_0 \le t \le t_1$ and $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$. Then

$$\int_\Gamma f(z)\, dz = \int_{t_0}^{t_1} [u(x(t), y(t)) + iv(x(t), y(t))]\,(\frac{dx}{dt} + i\frac{dy}{dt})dt. \tag{9.22}$$

Here we have used

$$dz = dx + idy = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt.$$

Furthermore, a set D is called a domain if it is both open and connected.

Before continuing, we first define open and connected. A set D is connected if and only if for all z_1, and z_2 in D there exists a piecewise smooth curve connecting z_1 to z_2 and lying in D. Otherwise it is called disconnected. Examples are shown in Figure 9.20

A set D is open if and only if for all z_0 in D there exists an open disk $|z - z_0| < \rho$ in D. In Figure 9.21 we show a region with two disks.

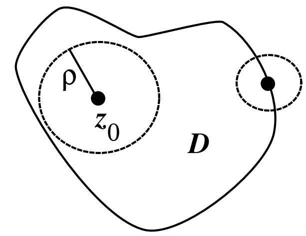

Figure 9.21: Locations of open disks inside and on the boundary of a region.

For all points on the interior of the region one can find at least one disk contained entirely in the region. The closer one is to the boundary, the smaller the radii of such disks. However, for a point on the boundary, every such disk would contain points inside and outside the disk. Thus, an open set in the complex plane would not contain any of its boundary points.

We now have a prescription for computing path integrals. Let's see how this works with a couple of examples.

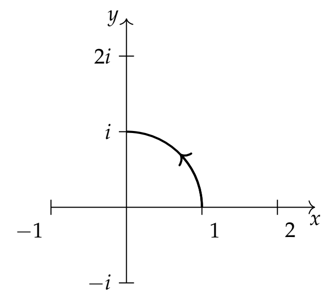

Figure 9.22: Contour for Example 9.13.

Example 9.13. Evaluate $\int_C z^2\,dz$, where C = the arc of the unit circle in the first quadrant as shown in Figure 9.22.

There are two ways we could carry out the parametrization. First, we note that the standard parametrization of the unit circle is

$$(x(\theta), y(\theta)) = (\cos\theta, \sin\theta), \quad 0 \le \theta \le 2\pi.$$

For a quarter circle in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$, we let $z = \cos\theta + i\sin\theta$. Therefore, $dz = (-\sin\theta + i\cos\theta)\,d\theta$ and the path integral becomes

$$\int_C z^2\,dz = \int_0^{\frac{\pi}{2}} (\cos\theta + i\sin\theta)^2(-\sin\theta + i\cos\theta)\,d\theta.$$

We can expand the integrand and integrate, having to perform some trigonometric integrations.

$$\int_0^{\frac{\pi}{2}} [\sin^3\theta - 3\cos^2\theta\sin\theta + i(\cos^3\theta - 3\cos\theta\sin^2\theta)]\,d\theta.$$

The reader should work out these trigonometric integrations and confirm the result. For example, you can use

$$\sin^3\theta = \sin\theta(1 - \cos^2\theta))$$

to write the real part of the integrand as

$$\sin\theta - 4\cos^2\theta\sin\theta.$$

The resulting antiderivative becomes

$$-\cos\theta + \frac{4}{3}\cos^3\theta.$$

The imaginary integrand can be integrated in a similar fashion.

While this integral is doable, there is a simpler procedure. We first note that $z = e^{i\theta}$ on C. So, $dz = ie^{i\theta}d\theta$. The integration then becomes

$$\int_C z^2\,dz = \int_0^{\frac{\pi}{2}} (e^{i\theta})^2 ie^{i\theta}\,d\theta$$

$$= i\int_0^{\frac{\pi}{2}} e^{3i\theta}\, d\theta$$
$$= \left.\frac{ie^{3i\theta}}{3i}\right|_0^{\pi/2}$$
$$= -\frac{1+i}{3}. \qquad (9.23)$$

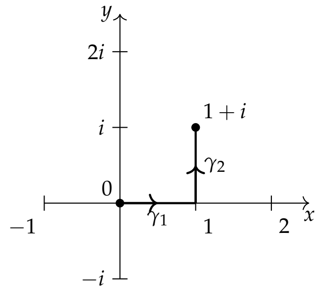

Figure 9.23: Contour for Example 9.14 with $\Gamma = \gamma_1 \cup \gamma_2$.

Example 9.14. Evaluate $\int_\Gamma z\,dz$, for the path $\Gamma = \gamma_1 \cup \gamma_2$ shown in Figure 9.23.

In this problem we have a path that is a piecewise smooth curve. We can compute the path integral by computing the values along the two segments of the path and adding the results. Let the two segments be called γ_1 and γ_2 as shown in Figure 9.23 and parametrize each path separately.

Over γ_1 we note that $y = 0$. Thus, $z = x$ for $x \in [0,1]$. It is natural to take x as the parameter. So, we let $dz = dx$ to find

$$\int_{\gamma_1} z\,dz = \int_0^1 x\,dx = \frac{1}{2}.$$

For path γ_2 we have that $z = 1 + iy$ for $y \in [0,1]$ and $dz = i\,dy$. Inserting this parametrization into the integral, the integral becomes

$$\int_{\gamma_2} z\,dz = \int_0^1 (1+iy)\, idy = i - \frac{1}{2}.$$

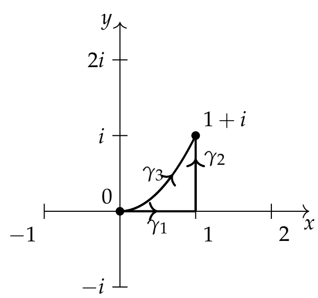

Figure 9.24: Contour for Example 9.15.

Combining the results for the paths γ_1 and γ_2, we have $\int_\Gamma z\,dz = \frac{1}{2} + (i - \frac{1}{2}) = i$.

Example 9.15. Evaluate $\int_{\gamma_3} z\,dz$, where γ_3, is the path shown in Figure 9.24.

In this case we take a path from $z = 0$ to $z = 1 + i$ along a different path than in the last example. Let $\gamma_3 = \{(x,y)|y = x^2, x \in [0,1]\} = \{z|z = x + ix^2, x \in [0,1]\}$. Then, $dz = (1 + 2ix)\, dx$.

The integral becomes

$$\begin{aligned}\int_{\gamma_3} z\,dz &= \int_0^1 (x+ix^2)(1+2ix)\,dx \\ &= \int_0^1 (x + 3ix^2 - 2x^3)\,dx = \\ &= \left[\frac{1}{2}x^2 + ix^3 - \frac{1}{2}x^4\right]_0^1 = i. \qquad (9.24)\end{aligned}$$

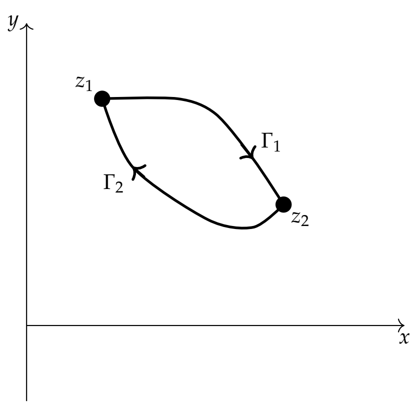

Figure 9.25: $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ for all paths from z_1 to z_2 when the integral of $f(z)$ is path independent.

In the last case we found the same answer as we had obtained in Example 9.14. But we should not take this as a general rule for all complex path integrals. In fact, it is not true that integrating over different paths always yields the same results. However, when this is true, then we refer to this property as path independence. In particular, the integral $\int f(z)\,dz$ is path independent if

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

for all paths from z_1 to z_2 as shown in Figure 9.25.

We can show that if $\int f(z)\,dz$ is path independent, then the integral of $f(z)$ over all closed loops is zero,

$$\int_{\text{closed loops}} f(z)\,dz = 0.$$

A common notation for integrating over closed loops is $\oint_C f(z)\,dz$. But first we have to define what we mean by a closed loop. A simple closed contour is a path satisfying

A simple closed contour.

a The end point is the same as the beginning point. (This makes the loop closed.)

b The are no self-intersections. (This makes the loop simple.)

A loop in the shape of a figure eight is closed, but it is not simple.

Now, consider an integral over the closed loop C shown in Figure 9.26. We pick two points on the loop breaking it into two contours, C_1 and C_2. Then we make use of the path independence by defining C_2^- to be the path along C_2 but in the opposite direction. Then,

$\oint_C f(z)\,dz = 0$ if the integral is path independent.

$$\begin{aligned}\oint_C f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{C_2^-} f(z)\,dz. \qquad (9.25)\end{aligned}$$

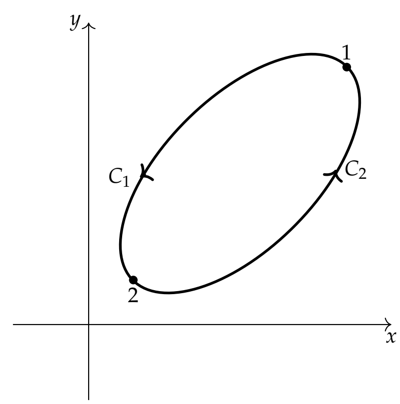

Figure 9.26: The integral $\oint_C f(z)\,dz$ around C is zero if the integral $\int_\Gamma f(z)\,dz$ is path independent.

Assuming that the integrals from point 1 to point 2 are path independent, then the integrals over C_1 and C_2^- are equal. Therefore, we have $\oint_C f(z)\,dz = 0$.

Example 9.16. Consider the integral $\oint_C z\,dz$ for C the closed contour shown in Figure 9.24 starting at $z = 0$ following path γ_1, then γ_2 and returning to $z = 0$. Based on the earlier examples and the fact that going backwards on γ_3 introduces a negative sign, we have

$$\oint_C z\,dz = \int_{\gamma_1} z\,dz + \int_{\gamma_2} z\,dz - \int_{\gamma_3} z\,dz = \frac{1}{2} + \left(i - \frac{1}{2}\right) - i = 0.$$

9.5.2 Cauchy's Theorem

NEXT WE WANT TO INVESTIGATE if we can determine that integrals over simple closed contours vanish without doing all the work of parametrizing the contour. First, we need to establish the direction about which we traverse the contour. We can define the orientation of a curve by referring to the normal of the curve.

A curve with parametrization $(x(t), y(t))$ has a normal $(n_x, n_y) = (-\frac{dx}{dt}, \frac{dy}{dt})$.

Recall that the normal is a perpendicular to the curve. There are two such perpendiculars. The above normal points outward and the other normal points towards the interior of a closed curve. We will define a positively oriented contour as one that is traversed with the outward normal pointing to the right. As one follows loops, the interior would then be on the left.

We now consider $\oint_C (u + iv)\, dz$ over a simple closed contour. This can be written in terms of two real integrals in the xy-plane.

$$\begin{aligned}\oint_C (u + iv)\, dz &= \int_C (u + iv)(dx + i\, dy) \\ &= \int_C u\, dx - v\, dy + i \int_C v\, dx + u\, dy. \qquad (9.26)\end{aligned}$$

These integrals in the plane can be evaluated using Green's Theorem in the Plane. Recall this theorem from your last semester of calculus:

Green's Theorem in the Plane.

Theorem 9.3. *Let $P(x, y)$ and $Q(x, y)$ be continuously differentiable functions on and inside the simple closed curve C as shown in Figure 9.27. Denoting the enclosed region S, we have*

$$\int_C P\, dx + Q\, dy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy. \qquad (9.27)$$

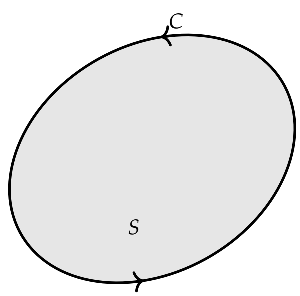

Figure 9.27: Region used in Green's Theorem.

Green's Theorem in the Plane is one of the major integral theorems of vector calculus. It was discovered by George Green (1793-1841) and published in 1828, about four years before he entered Cambridge as an undergraduate.

Using Green's Theorem to rewrite the first integral in (9.26), we have

$$\int_C u\, dx - v\, dy = \int\int_S \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy.$$

If u and v satisfy the Cauchy-Riemann equations (9.19), then the integrand in the double integral vanishes. Therefore,

$$\int_C u\, dx - v\, dy = 0.$$

In a similar fashion, one can show that

$$\int_C v\, dx + u\, dy = 0.$$

We have thus proven the following theorem:

Cauchy's Theorem

Theorem 9.4. *If u and v satisfy the Cauchy-Riemann equations (9.19) inside and on the simple closed contour C, then*

$$\oint_C (u + iv)\, dz = 0. \qquad (9.28)$$

Corollary $\oint_C f(z)\, dz = 0$ *when f is differentiable in domain D with $C \subset D$.*

Either one of these is referred to as **Cauchy's Theorem**.

Example 9.17. Evaluate $\oint_{|z-1|=3} z^4\, dz$.

Since $f(z) = z^4$ is differentiable inside the circle $|z - 1| = 3$, this integral vanishes.

We can use Cauchy's Theorem to show that we can deform one contour into another, perhaps simpler, contour.

Theorem 9.5. *If $f(z)$ is holomorphic between two simple closed contours, C and C', then $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$.*

One can deform contours into simpler ones.

Proof. We consider the two curves C and C' as shown in Figure 9.28. Connecting the two contours with contours Γ_1 and Γ_2 (as shown in the figure), C is seen to split into contours C_1 and C_2 and C' into contours C_1' and C_2'. Note that $f(z)$ is differentiable inside the newly formed regions between the curves. Also, the boundaries of these regions are now simple closed curves. Therefore, Cauchy's Theorem tells us that the integrals of $f(z)$ over these regions are zero.

Noting that integrations over contours opposite to the positive orientation are the negative of integrals that are positively oriented, we have from Cauchy's Theorem that

$$\int_{C_1} f(z)\,dz + \int_{\Gamma_1} f(z)\,dz - \int_{C_1'} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0$$

and

$$\int_{C_2} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz - \int_{C_2'} f(z)\,dz - \int_{\Gamma_1} f(z)\,dz = 0.$$

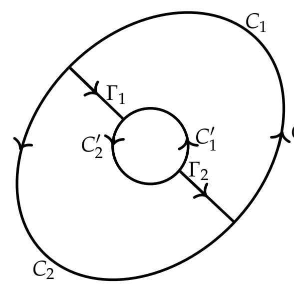

Figure 9.28: The contours needed to prove that $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$ when $f(z)$ is holomorphic between the contours C and C'.

In the first integral we have traversed the contours in the following order: C_1, Γ_1, C_1' backwards, and Γ_2. The second integral denotes the integration over the lower region, but going backwards over all contours except for C_2.

Combining these results by adding the two equations above, we have

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz - \int_{C_1'} f(z)\,dz - \int_{C_2'} f(z)\,dz = 0.$$

Noting that $C = C_1 + C_2$ and $C' = C_1' + C_2'$, we have

$$\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz,$$

as was to be proven. □

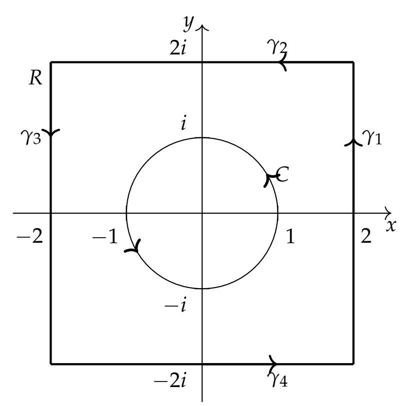

Figure 9.29: The contours used to compute $\oint_R \frac{dz}{z}$. Note that to compute the integral around R we can deform the contour to the circle C since $f(z)$ is differentiable in the region between the contours.

Example 9.18. Compute $\oint_R \frac{dz}{z}$ for R the rectangle $[-2,2] \times [-2i, 2i]$.

We can compute this integral by looking at four separate integrals over the sides of the rectangle in the complex plane. One simply parametrizes each line segment, perform the integration and sum the four separate results. From the last theorem, we can instead integrate over a simpler contour by deforming the rectangle into a circle as long as $f(z) = \frac{1}{z}$ is differentiable in the region bounded by the rectangle and the circle. So, using the unit circle, as shown in Figure 9.29, the integration might be easier to perform.

More specifically, the last theorem tells us that

$$\oint_R \frac{dz}{z} = \oint_{|z|=1} \frac{dz}{z}$$

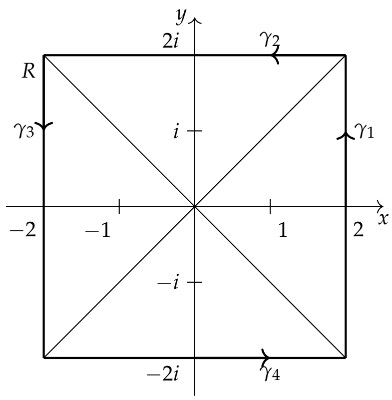

Figure 9.30: The contours used to compute $\oint_R \frac{dz}{z}$. The added diagonals are for the reader to easily see the arguments used in the evaluation of the limits when integrating over the segments of the square R.

The latter integral can be computed using the parametrization $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Thus,

$$\begin{aligned}\oint_{|z|=1} \frac{dz}{z} &= \int_0^{2\pi} \frac{ie^{i\theta}\, d\theta}{e^{i\theta}} \\ &= i\int_0^{2\pi} d\theta = 2\pi i. \end{aligned} \tag{9.29}$$

Therefore, we have found that $\oint_R \frac{dz}{z} = 2\pi i$ by deforming the original simple closed contour.

For fun, let's do this the long way to see how much effort was saved. We will label the contour as shown in Figure 9.30. The lower segment, γ_4 of the square can be simple parametrized by noting that along this segment $z = x - 2i$ for $x \in [-2, 2]$. Then, we have

$$\begin{aligned}\oint_{\gamma_4} \frac{dz}{z} &= \int_{-2}^{2} \frac{dx}{x-2i} \\ &= \ln\left|x - 2i\right|_{-2}^{2} \\ &= \left(\ln(2\sqrt{2}) - \frac{\pi i}{4}\right) - \left(\ln(2\sqrt{2}) - \frac{3\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{9.30}$$

We note that the arguments of the logarithms are determined from the angles made by the diagonals provided in Figure 9.30.

Similarly, the integral along the top segment, $z = x + 2i$, $x \in [-2, 2]$, is computed as

$$\begin{aligned}\oint_{\gamma_2} \frac{dz}{z} &= \int_{2}^{-2} \frac{dx}{x+2i} \\ &= \ln\left|x + 2i\right|_{2}^{-2} \\ &= \left(\ln(2\sqrt{2}) + \frac{3\pi i}{4}\right) - \left(\ln(2\sqrt{2}) + \frac{\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{9.31}$$

The integral over the right side, $z = 2 + iy$, $y \in [-2, 2]$, is

$$\begin{aligned}\oint_{\gamma_1} \frac{dz}{z} &= \int_{-2}^{2} \frac{i\,dy}{2+iy} \\ &= \ln\left|2 + iy\right|_{-2}^{2} \\ &= \left(\ln(2\sqrt{2}) + \frac{\pi i}{4}\right) - \left(\ln(2\sqrt{2}) - \frac{\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{9.32}$$

Finally, the integral over the left side, $z = -2 + iy$, $y \in [-2, 2]$, is

$$\oint_{\gamma_3} \frac{dz}{z} = \int_{2}^{-2} \frac{i\,dy}{-2+iy}$$

$$\begin{aligned} &= \ln\Big| -2+iy\Big|_{-2}^{2} \\ &= \left(\ln(2\sqrt{2})+\frac{5\pi i}{4}\right)-\left(\ln(2\sqrt{2})+\frac{3\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{9.33}$$

Therefore, we have that

$$\begin{aligned} \oint_R \frac{dz}{z} &= \int_{\gamma_1}\frac{dz}{z}+\int_{\gamma_2}\frac{dz}{z}+\int_{\gamma_3}\frac{dz}{z}+\int_{\gamma_4}\frac{dz}{z} \\ &= \frac{\pi i}{2}+\frac{\pi i}{2}+\frac{\pi i}{2}+\frac{\pi i}{2} \\ &= 4\left(\frac{\pi i}{2}\right)=2\pi i. \end{aligned} \tag{9.34}$$

This gives the same answer we had found using a simple contour deformation.

The converse of Cauchy's Theorem is not true, namely $\oint_C f(z)\,dz = 0$ does not always imply that $f(z)$ is differentiable. What we do have is Morera's Theorem(Giacinto Morera, 1856-1909):

Morera's Theorem.

Theorem 9.6. *Let f be continuous in a domain D. Suppose that for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$. Then f is differentiable in D.*

The proof is a bit more detailed than we need to go into here. However, this theorem is useful in the next section.

9.5.3 *Analytic Functions and Cauchy's Integral Formula*

In the previous section we saw that Cauchy's Theorem was useful for computing particular integrals without having to parametrize the contours or for deforming contours into simpler contours. The integrand needs to possess certain differentiability properties. In this section, we will generalize the functions that we can integrate slightly so that we can integrate a larger family of complex functions. This will lead us to the Cauchy's Integral Formula, which extends Cauchy's Theorem to functions analytic in an annulus. However, first we need to explore the concept of analytic functions.

There are various types of complex-valued functions.

A holomorphic function is (complex) differentiable in a neighborhood of every point in its domain.

An analytic function has a convergent Taylor series expansion in a neighborhood of each point in its domain. We see here that analytic functions are holomorphic and vice versa.

If a function is holomorphic throughout the complex plane, then it is called an entire function.

Finally, a function which is holomorphic on all of its domain except at a set of isolated poles (to be defined later), then it is called a meromorphic function.

A function $f(z)$ is analytic in domain D if for every open disk $|z - z_0| < \rho$ lying in D, $f(z)$ can be represented as a power series in z_0. Namely,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

This series converges uniformly and absolutely inside the circle of convergence, $|z - z_0| < R$, with radius of convergence R. [See the Appendix for a review of convergence.]

Since $f(z)$ can be written as a uniformly convergent power series, we can integrate it term by term over any simple closed contour in D containing z_0. In particular, we have to compute integrals like $\oint_C (z - z_0)^n\,dz$. As we will see in the homework exercises, these integrals evaluate to zero for most n. Thus, we can show that for $f(z)$ analytic in D and on any closed contour C lying in D, $\oint_C f(z)\,dz = 0$. Also, f is a uniformly convergent sum of continuous functions, so $f(z)$ is also continuous. Thus, by Morera's Theorem, we have that $f(z)$ is differentiable if it is analytic. Often terms like analytic, differentiable and holomorphic are used interchangeably, though there is a subtle distinction due to their definitions.

As examples of series expansions about a given point, we will consider series expansions and regions of convergence for $f(z) = \frac{1}{1+z}$.

Example 9.19. Find the series expansion of $f(z) = \frac{1}{1+z}$ about $z_0 = 0$.

This case is simple. From Chapter 1 we recall that $f(z)$ is the sum of a geometric series for $|z| < 1$. We have

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n.$$

Thus, this series expansion converges inside the unit circle ($|z| < 1$) in the complex plane.

Example 9.20. Find the series expansion of $f(z) = \frac{1}{1+z}$ about $z_0 = \frac{1}{2}$.

We now look into an expansion about a different point. We could compute the expansion coefficients using Taylor's formula for the coefficients. However, we can also make use of the formula for geometric series after rearranging the function. We seek an expansion in powers of $z - \frac{1}{2}$. So, we rewrite the function in a form that has is a function of $z - \frac{1}{2}$. Thus,

$$f(z) = \frac{1}{1+z} = \frac{1}{1 + (z - \frac{1}{2} + \frac{1}{2})} = \frac{1}{\frac{3}{2} + (z - \frac{1}{2})}.$$

This is not quite in the form we need. It would be nice if the denominator were of the form of one plus something. [Note: This is similar to what we had seen in Example A.35.] We can get the denominator into such a form by factoring out the $\frac{3}{2}$. Then we would have

$$f(z) = \frac{2}{3} \frac{1}{1 + \frac{2}{3}(z - \frac{1}{2})}.$$

The second factor now has the form $\frac{1}{1-r}$, which would be the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{2}{3}(z - \frac{1}{2})$ provided that |r|<1. Therefore, we have found that

$$f(z) = \frac{2}{3} \sum_{n=0}^{\infty} \left[-\frac{2}{3}(z - \frac{1}{2}) \right]^n$$

for

$$\left| -\frac{2}{3}(z - \frac{1}{2}) \right| < 1.$$

This convergence interval can be rewritten as

$$\left| z - \frac{1}{2} \right| < \frac{3}{2},$$

which is a circle centered at $z = \frac{1}{2}$ with radius $\frac{3}{2}$.

In Figure 9.31 we show the regions of convergence for the power series expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. We note that the first expansion gives that $f(z)$ is at least analytic inside the region $|z| < 1$. The second expansion shows that $f(z)$ is analytic in a larger region, $|z - \frac{1}{2}| < \frac{3}{2}$. We will see later that there are expansions which converge outside of these regions and that some yield expansions involving negative powers of $z - z_0$.

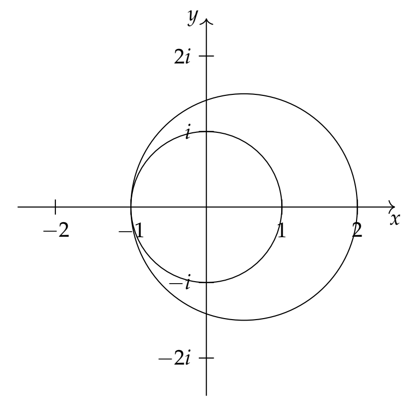

Figure 9.31: Regions of convergence for expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$.

We now present the main theorem of this section:

Cauchy Integral Formula

Theorem 9.7. *Let $f(z)$ be analytic in $|z - z_0| < \rho$ and let C be the boundary (circle) of this disk. Then,*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{9.35}$$

Proof. In order to prove this, we first make use of the analyticity of $f(z)$. We insert the power series expansion of $f(z)$ about z_0 into the integrand. Then we have

$$\begin{aligned}
\frac{f(z)}{z - z_0} &= \frac{1}{z - z_0}\left[\sum_{n=0}^{\infty} c_n (z - z_0)^n\right] \\
&= \frac{1}{z - z_0}\left[c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots\right] \\
&= \frac{c_0}{z - z_0} + \underbrace{c_1 + c_2(z - z_0) + \ldots}_{\text{analytic function}}.
\end{aligned} \tag{9.36}$$

As noted the integrand can be written as

$$\frac{f(z)}{z - z_0} = \frac{c_0}{z - z_0} + h(z),$$

where $h(z)$ is an analytic function, since $h(z)$ is representable as a series expansion about z_0. We have already shown that analytic functions are differentiable, so by Cauchy's Theorem $\oint_C h(z)\, dz = 0$.

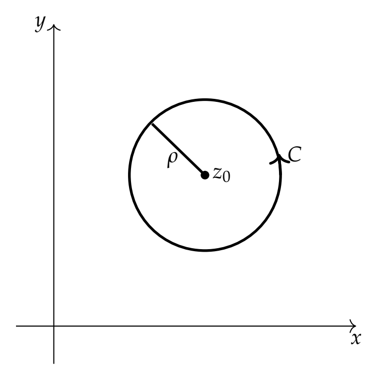

Figure 9.32: Circular contour used in proving the Cauchy Integral Formula.

Noting also that $c_0 = f(z_0)$ is the first term of a Taylor series expansion about $z = z_0$, we have

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_C \left[\frac{c_0}{z - z_0} + h(z)\right] dz = f(z_0) \oint_C \frac{1}{z - z_0}\, dz.$$

We need only compute the integral $\oint_C \frac{1}{z - z_0}\, dz$ to finish the proof of Cauchy's Integral Formula. This is done by parametrizing the circle, $|z - z_0| = \rho$, as shown in Figure 9.32. This is simply done by letting

$$z - z_0 = \rho e^{i\theta}.$$

(Note that this has the right complex modulus since $|e^{i\theta}| = 1$. Then $dz = i\rho e^{i\theta} d\theta$. Using this parametrization, we have

$$\oint_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\, d\theta}{\rho e^{i\theta}} = i \int_0^{2\pi} d\theta = 2\pi i.$$

Therefore,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{1}{z-z_0}\,dz = 2\pi i f(z_0),$$

as was to be shown. □

Example 9.21. Compute $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\,dz$.

In order to apply the Cauchy Integral Formula, we need to factor the denominator, $z^2-6z+5=(z-1)(z-5)$. We next locate the zeros of the denominator. In Figure 9.33 we show the contour and the points $z=1$ and $z=5$. The only point inside the region bounded by the contour is $z=1$. Therefore, we can apply the Cauchy Integral Formula for $f(z)=\frac{\cos z}{z-5}$ to the integral

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\,dz = \int_{|z|=4} \frac{f(z)}{(z-1)}\,dz = 2\pi i f(1).$$

Therefore, we have

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\,dz = -\frac{\pi i\cos(1)}{2}.$$

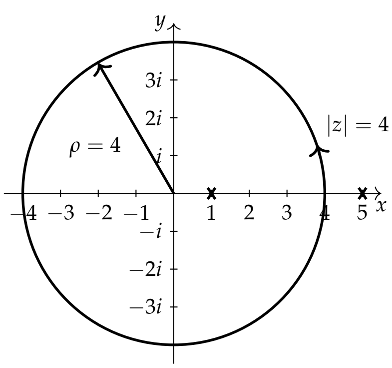

Figure 9.33: Circular contour used in computing $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\,dz$.

We have shown that $f(z_0)$ has an integral representation for $f(z)$ analytic in $|z-z_0|<\rho$. In fact, all derivatives of an analytic function have an integral representation. This is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz. \tag{9.37}$$

This can be proven following a derivation similar to that for the Cauchy Integral Formula. Inserting the Taylor series expansion for $f(z)$ into the integral on the right hand side, we have

$$\begin{aligned}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz &= \sum_{m=0}^{\infty} c_m \oint_C \frac{(z-z_0)^m}{(z-z_0)^{n+1}}\,dz \\ &= \sum_{m=0}^{\infty} c_m \oint_C \frac{dz}{(z-z_0)^{n-m+1}}.\end{aligned} \tag{9.38}$$

Picking $k=n-m$, the integrals in the sum can be computed by using the following result:

$$\oint_C \frac{dz}{(z-z_0)^{k+1}} = \begin{cases} 0, & k\neq 0 \\ 2\pi i, & k=0. \end{cases} \tag{9.39}$$

The proof is left for the exercises.

The only nonvanishing integrals, $\oint_C \frac{dz}{(z-z_0)^{n-m+1}}$, occur when $k=n-m=0$, or $m=n$. Therefore, the series of integrals collapses to one term and we have

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i c_n.$$

We finish the proof by recalling that the coefficients of the Taylor series expansion for $f(z)$ are given by

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Then,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

and the result follows.

9.5.4 *Laurent Series*

UNTIL THIS POINT WE HAVE ONLY TALKED about series whose terms have nonnegative powers of $z - z_0$. It is possible to have series representations in which there are negative powers. In the last section we investigated expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. The regions of convergence for each series was shown in Figure 9.31. Let us reconsider each of these expansions, but for values of z outside the region of convergence previously found.

Example 9.22. $f(z) = \frac{1}{1+z}$ for $|z| > 1$.

As before, we make use of the geometric series . Since $|z| > 1$, we instead rewrite the function as

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\frac{1}{1+\frac{1}{z}}.$$

We now have the function in a form of the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{1}{z}$. We note that $|z| > 1$ implies that $|r| < 1$. Thus, we have the geometric series

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n.$$

This can be re-indexed[3] as

$$f(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \sum_{j=1}^{\infty}(-1)^{j-1}z^{-j}.$$

Note that this series, which converges outside the unit circle, $|z| > 1$, has negative powers of z.

Example 9.23. $f(z) = \frac{1}{1+z}$ for $|z - \frac{1}{2}| > \frac{3}{2}$.

As before, we express this in a form in which we can use a geometric series expansion. We seek powers of $z - \frac{1}{2}$. So, we add and subtract $\frac{1}{2}$ to the z to obtain:

$$f(z) = \frac{1}{1+z} = \frac{1}{1+(z-\frac{1}{2}+\frac{1}{2})} = \frac{1}{\frac{3}{2}+(z-\frac{1}{2})}.$$

[3] Re-indexing a series is often useful in series manipulations. In this case, we have the series

$$\sum_{n=0}^{\infty}(-1)^n z^{-n-1} = z^{-1} - z^{-2} + z^{-3} + \ldots.$$

The index is n. You can see that the index does not appear when the sum is expanded showing the terms. The summation index is sometimes referred to as a dummy index for this reason. Reindexing allows one to rewrite the shorthand summation notation while capturing the same terms. In this example, the exponents are $-n-1$. We can simplify the notation by letting $-n-1 = -j$, or $j = n+1$. Noting that $j = 1$ when $n = 0$, we get the sum $\sum_{j=1}^{\infty}(-1)^{j-1}z^{-j}$.

Instead of factoring out the $\frac{3}{2}$ as we had done in Example 9.20, we factor out the $(z-\frac{1}{2})$ term. Then, we obtain

$$f(z)=\frac{1}{1+z}=\frac{1}{(z-\frac{1}{2})}\frac{1}{\left[1+\frac{3}{2}(z-\frac{1}{2})^{-1}\right]}.$$

Now we identify $a=1$ and $r=-\frac{3}{2}(z-\frac{1}{2})^{-1}$. This leads to the series

$$\begin{aligned} f(z) &= \frac{1}{z-\frac{1}{2}}\sum_{n=0}^{\infty}\left(-\frac{3}{2}(z-\frac{1}{2})^{-1}\right)^n \\ &= \sum_{n=0}^{\infty}\left(-\frac{3}{2}\right)^n\left(z-\frac{1}{2}\right)^{-n-1}. \end{aligned} \tag{9.40}$$

This converges for $|z-\frac{1}{2}|>\frac{3}{2}$ and can also be re-indexed to verify that this series involves negative powers of $z-\frac{1}{2}$.

This leads to the following theorem:

Theorem 9.8. *Let $f(z)$ be analytic in an annulus, $R_1<|z-z_0|<R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus as shown in Figure 9.34. Then,*

$$f(z)=\sum_{j=0}^{\infty}a_j(z-z_0)^j+\sum_{j=1}^{\infty}b_j(z-z_0)^{-j},$$

with

$$a_j=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^{j+1}}\,dz,$$

and

$$b_j=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^{-j+1}}\,dz.$$

The above series can be written in the more compact form

$$f(z)=\sum_{j=-\infty}^{\infty}c_j(z-z_0)^j.$$

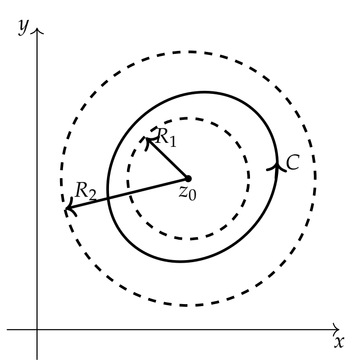

Figure 9.34: This figure shows an annulus, $R_1<|z-z_0|<R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus.

Such a series expansion is called a Laurent series expansion named after its discoverer Pierre Alphonse Laurent (1813-1854).

Example 9.24. Expand $f(z)=\frac{1}{(1-z)(2+z)}$ in the annulus $1<|z|<2$.

Using partial fractions , we can write this as

$$f(z)=\frac{1}{3}\left[\frac{1}{1-z}+\frac{1}{2+z}\right].$$

We can expand the first fraction, $\frac{1}{1-z}$, as an analytic function in the region $|z|>1$ and the second fraction, $\frac{1}{2+z}$, as an analytic function in $|z|<2$. This is done as follows. First, we write

$$\frac{1}{2+z}=\frac{1}{2[1-(-\frac{z}{2})]}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n.$$

Then, we write

$$\frac{1}{1-z} = -\frac{1}{z[1-\frac{1}{z}]} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}.$$

Therefore, in the common region, $1 < |z| < 2$, we have that

$$\begin{aligned} \frac{1}{(1-z)(2+z)} &= \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right] \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{6(2^n)}z^n + \sum_{n=1}^{\infty}\frac{(-1)}{3}z^{-n}. \end{aligned} \quad (9.41)$$

We note that this is not a Taylor series expansion due to the existence of terms with negative powers in the second sum.

Example 9.25. Find series representations of $f(z) = \frac{1}{(1-z)(2+z)}$ throughout the complex plane.

In the last example we found series representations of $f(z) = \frac{1}{(1-z)(2+z)}$ in the annulus $1 < |z| < 2$. However, we can also find expansions which converge for other regions. We first write

$$f(z) = \frac{1}{3}\left[\frac{1}{1-z} + \frac{1}{2+z}\right].$$

We then expand each term separately.

The first fraction, $\frac{1}{1-z}$, can be written as the sum of the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

This series converges inside the unit circle. We indicate this by region 1 in Figure 9.35.

In the last example, we showed that the second fraction, $\frac{1}{2+z}$, has the series expansion

$$\frac{1}{2+z} = \frac{1}{2[1-(-\frac{z}{2})]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n.$$

which converges in the circle $|z| < 2$. This is labeled as region 2 in Figure 9.35.

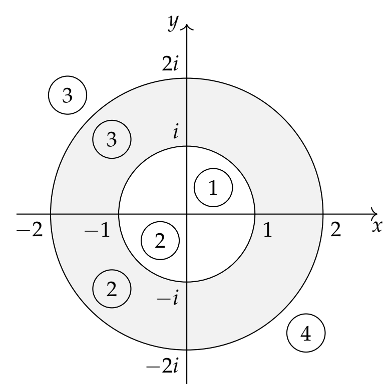

Figure 9.35: Regions of convergence for Laurent expansions of $f(z) = \frac{1}{1+z}$.

Regions 1 and 2 intersect for $|z| < 1$, so, we can combine these two series representations to obtain

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n\right], \quad |z| < 1.$$

In the annulus, $1 < |z| < 2$, we had already seen in the last example that we needed a different expansion for the fraction $\frac{1}{1-z}$. We looked for an expansion in powers of $1/z$ which would converge for large values of z. We had found that

$$\frac{1}{1-z} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}, \quad |z| > 1.$$

This series converges in region 3 in Figure 9.35. Combining this series with the one for the second fraction, we obtain a series representation that converges in the overlap of regions 2 and 3. Thus, in the annulus $1 < |z| < 2$ we have

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right].$$

So far, we have series representations for $|z| < 2$. The only region not covered yet is outside this disk, $|z| > 2$. In in Figure 9.35 we see that series 3, which converges in region 3, will converge in the last section of the complex plane. We just need one more series expansion for $1/(2+z)$ for large z. Factoring out a z in the denominator, we can write this as a geometric series with $r = 2/z$,

$$\frac{1}{2+z} = \frac{1}{z[\frac{2}{z}+1]} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n.$$

This series converges for $|z| > 2$. Therefore, it converges in region 4 and the final series representation is

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right].$$

9.5.5 Singularities and The Residue Theorem

In the last section we found that we could integrate functions satisfying some analyticity properties along contours without using detailed parametrizations around the contours. We can deform contours if the function is analytic in the region between the original and new contour. In this section we will extend our tools for performing contour integrals.

Singularities of complex functions.

The integrand in the Cauchy Integral Formula was of the form $g(z) = \frac{f(z)}{z-z_0}$, where $f(z)$ is well behaved at z_0. The point $z = z_0$ is called a singularity of $g(z)$, as $g(z)$ is not defined there. More specifically, a singularity of $f(z)$ is a point at which $f(z)$ fails to be analytic.

We can also classify these singularities. Typically these are isolated singularities. As we saw from the proof of the Cauchy Integral Formula, $g(z) = \frac{f(z)}{z-z_0}$ has a Laurent series expansion about $z = z_0$, given by

$$g(z) = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{1}{2}f''(z_0)(z-z_0) + \ldots.$$

It is the nature of the first term that gives information about the type of singularity that $g(z)$ has. Namely, in order to classify the singularities of $f(z)$, we look at the principal part of the Laurent series of $f(z)$ about $z = z_0$, $\sum_{j-1}^{\infty} b_j(z-z_0)^{-j}$, which consists of the negative powers of $z - z_0$.

Classification of singularities.

There are three types of singularities, removable, poles, and essential singularities. They are defined as follows:

1. If $f(z)$ is bounded near z_0, then z_0 is a removable singularity.
2. If there are a finite number of terms in the principal part of the Laurent series of $f(z)$ about $z = z_0$, then z_0 is called a pole.
3. If there are an infinite number of terms in the principal part of the Laurent series of $f(z)$ about $z = z_0$, then z_0 is called an essential singularity.

Example 9.26. $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$.

At first it looks like there is a possible singularity at $z = 0$, since the denominator is zero at $z = 0$. However, we know from the first semester of calculus that $\lim_{z\to 0} \frac{\sin z}{z} = 1$. Furthermore, we can expand $\sin z$ about $z = 0$ and see that

$$\frac{\sin z}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \ldots) = 1 - \frac{z^2}{3!} + \ldots .$$

Thus, there are only nonnegative powers in the series expansion. So, $z = 0$ is a removable singularity.

Example 9.27. $f(z) = \frac{e^z}{(z-1)^n}$ has poles at $z = 1$ for n a positive integer.

For $n = 1$ we have $f(z) = \frac{e^z}{z-1}$. This function has a singularity at $z = 1$. The series expansion is found by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{z-1} e^{z-1} = \frac{e}{z-1} + e + \frac{e}{2!}(z-1) + \ldots .$$

Note that the principal part of the Laurent series expansion about $z = 1$ only has one term, $\frac{e}{z-1}$. Therefore, $z = 1$ is a pole. Since the leading term has an exponent of -1, $z = 1$ is called a pole of order one, or a simple pole.

Simple pole.

For $n = 2$ we have $f(z) = \frac{e^z}{(z-1)^2}$. The series expansion is found again by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{(z-1)^2} e^{z-1} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2!} + \frac{e}{3!}(z-1) + \ldots .$$

Note that the principal part of the Laurent series has two terms involving $(z-1)^{-2}$ and $(z-1)^{-1}$. Since the leading term has an exponent of -2, $z = 1$ is called a pole of order 2, or a double pole.

Double pole.

Example 9.28. $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$.

In this case we have the series expansion about $z = 0$ given by

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

We see that there are an infinite number of terms in the principal part of the Laurent series. So, this function has an essential singularity at $z = 0$.

In the above examples we have seen poles of order one (a simple pole) and two (a double pole). In general, we can say that $f(z)$ has a pole of order k at z_0 if and only if $(z - z_0)^k f(z)$ has a removable singularity at z_0, but $(z - z_0)^{k-1} f(z)$ for $k > 0$ does not.

Poles of order k.

Example 9.29. Determine the order of the pole of $f(z) = \cot z \csc z$ at $z = 0$.

First we rewrite $f(z)$ in terms of sines and cosines.

$$f(z) = \cot z \csc z = \frac{\cos z}{\sin^2 z}.$$

We note that the denominator vanishes at $z = 0$.

How do we know that the pole is not a simple pole? Well, we check to see if $(z-0)f(z)$ has a removable singularity at $z = 0$:

$$\begin{aligned} \lim_{z\to 0}(z-0)f(z) &= \lim_{z\to 0}\frac{z\cos z}{\sin^2 z} \\ &= \left(\lim_{z\to 0}\frac{z}{\sin z}\right)\left(\lim_{z\to 0}\frac{\cos z}{\sin z}\right) \\ &= \lim_{z\to 0}\frac{\cos z}{\sin z}. \end{aligned} \tag{9.42}$$

We see that this limit is undefined. So, now we check to see if $(z-0)^2 f(z)$ has a removable singularity at $z = 0$:

$$\begin{aligned} \lim_{z\to 0}(z-0)^2 f(z) &= \lim_{z\to 0}\frac{z^2\cos z}{\sin^2 z} \\ &= \left(\lim_{z\to 0}\frac{z}{\sin z}\right)\left(\lim_{z\to 0}\frac{z\cos z}{\sin z}\right) \\ &= \lim_{z\to 0}\frac{z}{\sin z}\cos(0) = 1. \end{aligned} \tag{9.43}$$

In this case, we have obtained a finite, nonzero, result. So, $z = 0$ is a pole of order 2.

We could have also relied on series expansions. Expanding both the sine and cosine functions in a Taylor series expansion, we have

$$f(z) = \frac{\cos z}{\sin^2 z} = \frac{1 - \frac{1}{2!}z^2 + \ldots}{(z - \frac{1}{3!}z^3 + \ldots)^2}.$$

Factoring a z from the expansion in the denominator,

$$f(z) = \frac{1}{z^2}\frac{1 - \frac{1}{2!}z^2 + \ldots}{(1 - \frac{1}{3!}z + \ldots)^2} = \frac{1}{z^2}\left(1 + O(z^2)\right),$$

we can see that the leading term will be a $1/z^2$, indicating a pole of order 2.

We will see how knowledge of the poles of a function can aid in the computation of contour integrals. We now show that if a function, $f(z)$, has a pole of order k, then

Integral of a function with a simple pole inside C.

Residues of a function with poles of order k.

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}[f(z); z_0],$$

where we have defined $\operatorname{Res}[f(z); z_0]$ as the residue of $f(z)$ at $z = z_0$. In particular, for a pole of order k the residue is given by

Residues - Poles of order k

$$\text{Res}[f(z);z_0] = \lim_{z\to z_0} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]. \tag{9.44}$$

Proof. Let $\phi(z) = (z-z_0)^k f(z)$ be an analytic function. Then $\phi(z)$ has a Taylor series expansion about z_0. As we had seen in the last section, we can write the integral representation of any derivative of ϕ as

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i}\oint_C \frac{\phi(z)}{(z-z_0)^k}\,dz.$$

Inserting the definition of $\phi(z)$, we then have

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i}\oint_C f(z)\,dz.$$

Solving for the integral, we have the result

$$\begin{aligned}\oint_C f(z)\,dz &= \frac{2\pi i}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]_{z=z_0} \\ &\equiv 2\pi i \,\text{Res}[f(z);z_0] \end{aligned} \tag{9.45}$$

□

Note: If z_0 is a simple pole, the residue is easily computed as

$$Res[f(z);z_0] = \lim_{z\to z_0}(z-z_0)f(z).$$

The residue for a simple pole.

In fact, one can show (Problem 18) that for g and h analytic functions at z_0, with $g(z_0)\neq 0$, $h(z_0)=0$, and $h'(z_0)\neq 0$,

$$\text{Res}\left[\frac{g(z)}{h(z)};z_0\right] = \frac{g(z_0)}{h'(z_0)}.$$

Example 9.30. Find the residues of $f(z) = \frac{z-1}{(z+1)^2(z^2+4)}$.

$f(z)$ has poles at $z=-1$, $z=2i$, and $z=-2i$. The pole at $z=-1$ is a double pole (pole of order 2). The other poles are simple poles. We compute those residues first:

$$\begin{aligned}\text{Res}[f(z);2i] &= \lim_{z\to 2i}(z-2i)\frac{z-1}{(z+1)^2(z+2i)(z-2i)} \\ &= \lim_{z\to 2i}\frac{z-1}{(z+1)^2(z+2i)} \\ &= \frac{2i-1}{(2i+1)^2(4i)} = -\frac{1}{50}-\frac{11}{100}i. \end{aligned} \tag{9.46}$$

$$\begin{aligned}\text{Res}[f(z);-2i] &= \lim_{z\to -2i}(z+2i)\frac{z-1}{(z+1)^2(z+2i)(z-2i)} \\ &= \lim_{z\to -2i}\frac{z-1}{(z+1)^2(z-2i)} \\ &= \frac{-2i-1}{(-2i+1)^2(-4i)} = -\frac{1}{50}+\frac{11}{100}i. \end{aligned} \tag{9.47}$$

For the double pole, we have to do a little more work.

$$\begin{aligned} \text{Res}[f(z); -1] &= \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \frac{z-1}{(z+1)^2(z^2+4)}\right] \\ &= \lim_{z\to -1} \frac{d}{dz}\left[\frac{z-1}{z^2+4}\right] \\ &= \lim_{z\to -1} \frac{d}{dz}\left[\frac{z^2+4-2z(z-1)}{(z^2+4)^2}\right] \\ &= \lim_{z\to -1} \frac{d}{dz}\left[\frac{-z^2+2z+4}{(z^2+4)^2}\right] \\ &= \frac{1}{25}. \end{aligned} \tag{9.48}$$

Example 9.31. Find the residue of $f(z) = \cot z$ at $z = 0$.

We write $f(z) = \cot z = \frac{\cos z}{\sin z}$ and note that $z = 0$ is a simple pole. Thus,

$$\text{Res}[\cot z; z = 0] = \lim_{z\to 0} \frac{z \cos z}{\sin z} = \cos(0) = 1.$$

The residue of $f(z)$ at z_0 is the coefficient of the $(z - z_0)^{-1}$ term, $c_{-1} = b_1$, of the Laurent series expansion about z_0.

Another way to find the residue of a function $f(z)$ at a singularity z_0 is to look at the Laurent series expansion about the singularity. This is because the residue of $f(z)$ at z_0 is the coefficient of the $(z - z_0)^{-1}$ term, or $c_{-1} = b_1$.

Example 9.32. Find the residue of $f(z) = \frac{1}{z(3-z)}$ at $z = 0$ using a Laurent series expansion.

First, we need the Laurent series expansion about $z = 0$ of the form $\sum_{-\infty}^{\infty} c_n z^n$. A partial fraction expansion gives

$$f(z) = \frac{1}{z(3-z)} = \frac{1}{3}\left(\frac{1}{z} + \frac{1}{3-z}\right).$$

The first term is a power of z. The second term needs to be written as a convergent series for small z. This is given by

$$\begin{aligned} \frac{1}{3-z} &= \frac{1}{3(1-z/3)} \\ &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n. \end{aligned} \tag{9.49}$$

Thus, we have found

$$f(z) = \frac{1}{3}\left(\frac{1}{z} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right).$$

The coefficient of z^{-1} can be read off to give $\text{Res}[f(z); z = 0] = \frac{1}{3}$.

Example 9.33. Find the residue of $f(z) = z \cos \frac{1}{z}$ at $z = 0$ using a Laurent series expansion.

Finding the residue at an essential singularity.

In this case $z = 0$ is an essential singularity. The only way to find residues at essential singularities is to use Laurent series. Since

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \ldots,$$

then we have

$$\begin{aligned} f(z) &= z\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \dots\right) \\ &= z - \frac{1}{2!z} + \frac{1}{4!z^3} - \frac{1}{6!z^5} + \dots. \end{aligned} \tag{9.50}$$

From the second term we have that $\text{Res}[f(z); z = 0] = -\frac{1}{2}$.

We are now ready to use residues in order to evaluate integrals.

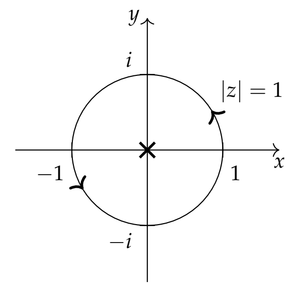

Figure 9.36: Contour for computing $\oint_{|z|=1} \frac{dz}{\sin z}$.

Example 9.34. Evaluate $\oint_{|z|=1} \frac{dz}{\sin z}$.

We begin by looking for the singularities of the integrand. These are located at values of z for which $\sin z = 0$. Thus, $z = 0, \pm\pi, \pm 2\pi, \dots,$ are the singularities. However, only $z = 0$ lies inside the contour, as shown in Figure 9.36. We note further that $z = 0$ is a simple pole, since

$$\lim_{z \to 0}(z - 0)\frac{1}{\sin z} = 1.$$

Therefore, the residue is one and we have

$$\oint_{|z|=1} \frac{dz}{\sin z} = 2\pi i.$$

In general, we could have several poles of different orders. For example, we will be computing

$$\oint_{|z|=2} \frac{dz}{z^2 - 1}.$$

The integrand has singularities at $z^2 - 1 = 0$, or $z = \pm 1$. Both poles are inside the contour, as seen in Figure 9.38. One could do a partial fraction decomposition and have two integrals with one pole each integral. Then, the result could be found by adding the residues from each pole.

In general, when there are several poles, we can use the Residue Theorem.

The Residue Theorem.

The Residue Theorem

Theorem 9.9. *Let $f(z)$ be a function which has poles z_j, $j = 1, \dots, N$ inside a simple closed contour C and no other singularities in this region. Then,*

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} \text{Res}[f(z); z_j], \tag{9.51}$$

where the residues are computed using Equation (9.44),

$$\text{Res}[f(z); z_0] = \lim_{z \to z_0} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z - z_0)^k f(z)\right].$$

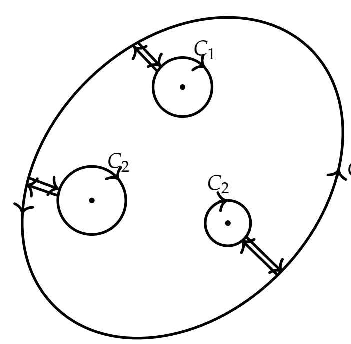

Figure 9.37: A depiction of how one cuts out poles to prove that the integral around C is the sum of the integrals around circles with the poles at the center of each.

The proof of this theorem is based upon the contours shown in Figure 9.37. One constructs a new contour C' by encircling each pole, as show in the figure. Then one connects a path from C to each circle. In the figure two separated paths along the cut are shown only to indicate the direction followed on the cut. The new contour is then obtained by following C and crossing each cut as it is encountered. Then one goes around a circle in the negative sense and returns along the cut to proceed around C. The sum of the contributions to the contour integration involve two integrals for each cut, which will cancel due to the opposing directions. Thus, we are left with

$$\oint_{C'} f(z)\,dz = \oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \oint_{C_3} f(z)\,dz = 0.$$

Of course, the sum is zero because $f(z)$ is analytic in the enclosed region, since all singularities have been cut out. Solving for $\oint_C f(z)\,dz$, one has that this integral is the sum of the integrals around the separate poles, which can be evaluated with single residue computations. Thus, the result is that $\oint_C f(z)\,dz$ is $2\pi i$ times the sum of the residues.

Example 9.35. Evaluate $\oint_{|z|=2} \frac{dz}{z^2-1}$.

We first note that there are two poles in this integral since

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}.$$

In Figure 9.38 we plot the contour and the two poles, denoted by an "x." Since both poles are inside the contour, we need to compute the residues for each one. They are each simple poles, so we have

$$\begin{aligned}\operatorname{Res}\left[\frac{1}{z^2-1}; z=1\right] &= \lim_{z\to 1}(z-1)\frac{1}{z^2-1} \\ &= \lim_{z\to 1}\frac{1}{z+1} = \frac{1}{2},\end{aligned} \tag{9.52}$$

and

$$\begin{aligned}\operatorname{Res}\left[\frac{1}{z^2-1}; z=-1\right] &= \lim_{z\to -1}(z+1)\frac{1}{z^2-1} \\ &= \lim_{z\to -1}\frac{1}{z-1} = -\frac{1}{2}.\end{aligned} \tag{9.53}$$

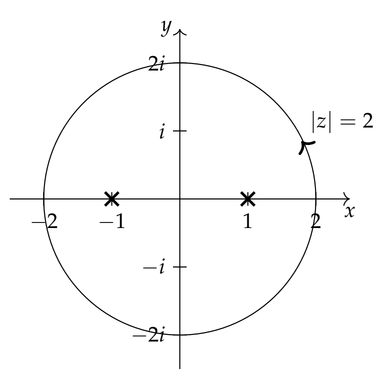

Figure 9.38: Contour for computing $\oint_{|z|=2} \frac{dz}{z^2-1}$.

Then,

$$\oint_{|z|=2} \frac{dz}{z^2-1} = 2\pi i(\frac{1}{2} - \frac{1}{2}) = 0.$$

Example 9.36. Evaluate $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\,dz$.

In this example there are two poles $z = 1, -2$ inside the contour. [See Figure 9.39.] $z = 1$ is a second order pole and $z = -2$ is a simple pole. Therefore, we need to compute the residues at each pole of $f(z) = \frac{z^2+1}{(z-1)^2(z+2)}$:

$$\begin{aligned}\operatorname{Res}[f(z); z=1] &= \lim_{z\to 1}\frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\frac{z^2+1}{(z-1)^2(z+2)}\right] \\ &= \lim_{z\to 1}\left(\frac{z^2+4z-1}{(z+2)^2}\right) \\ &= \frac{4}{9}.\end{aligned} \tag{9.54}$$

$$\begin{aligned}\text{Res}[f(z); z=-2] &= \lim_{z\to -2}(z+2)\frac{z^2+1}{(z-1)^2(z+2)} \\ &= \lim_{z\to -2}\frac{z^2+1}{(z-1)^2} \\ &= \frac{5}{9}. \end{aligned} \tag{9.55}$$

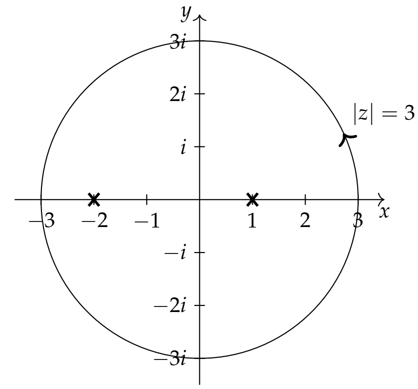

Figure 9.39: Contour for computing $\oint_{|z|=3}\frac{z^2+1}{(z-1)^2(z+2)}\,dz$.

The evaluation of the integral is found by computing $2\pi i$ times the sum of the residues:

$$\oint_{|z|=3}\frac{z^2+1}{(z-1)^2(z+2)}\,dz = 2\pi i\left(\frac{4}{9}+\frac{5}{9}\right) = 2\pi i.$$

Example 9.37. Compute $\oint_{|z|=2} z^3 e^{2/z}\,dz$.

In this case, $z=0$ is an essential singularity and is inside the contour. A Laurent series expansion about $z=0$ gives

$$\begin{aligned} z^3 e^{2/z} &= z^3\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2}{z}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{2^n}{n!}z^{3-n} \\ &= z^3+\frac{2}{2!}z^2+\frac{4}{3!}z+\frac{8}{4!}+\frac{16}{5!z}+\ldots. \end{aligned} \tag{9.56}$$

The residue is the coefficient of z^{-1}, or $\text{Res}[z^3e^{2/z}; z=0] = -\frac{2}{15}$. Therefore,

$$\oint_{|z|=2} z^3 e^{2/z}\,dz = \frac{4}{15}\pi i.$$

Example 9.38. Evaluate $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$.

Here we have a real integral in which there are no signs of complex functions. In fact, we could apply simpler methods from a calculus course to do this integral, attempting to write $1+\cos\theta = 2\cos^2\frac{\theta}{2}$. However, we do not get very far.

Computation of integrals of functions of sines and cosines, $f(\cos\theta,\sin\theta)$.

One trick, useful in computing integrals whose integrand is in the form $f(\cos\theta,\sin\theta)$, is to transform the integration to the complex plane through the transformation $z=e^{i\theta}$. Then,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = -\frac{i}{2}\left(z-\frac{1}{z}\right).$$

Under this transformation, $z=e^{i\theta}$, the integration now takes place around the unit circle in the complex plane. Noting that $dz = ie^{i\theta}\,d\theta = iz\,d\theta$, we have

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \oint_{|z|=1}\frac{\frac{dz}{iz}}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

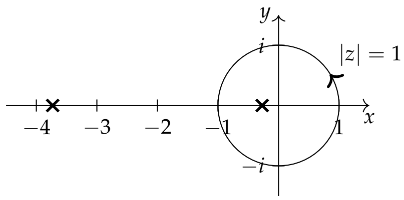

Figure 9.40: Contour for computing $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

$$
\begin{aligned}
&= -i \oint_{|z|=1} \frac{dz}{2z + \frac{1}{2}(z^2+1)} \\
&= -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1}. \qquad (9.57)
\end{aligned}
$$

We can apply the Residue Theorem to the resulting integral. The singularities occur at the roots of $z^2+4z+1=0$. Using the quadratic formula, we have the roots $z=-2\pm\sqrt{3}$.

The location of these poles are shown in Figure 9.40. Only $z = -2+\sqrt{3}$ lies inside the integration contour. We will therefore need the residue of $f(z) = \frac{-2i}{z^2+4z+1}$ at this simple pole:

$$
\begin{aligned}
\operatorname{Res}[f(z); z=-2+\sqrt{3}] &= \lim_{z\to-2+\sqrt{3}} (z-(-2+\sqrt{3}))\frac{-2i}{z^2+4z+1} \\
&= -2i \lim_{z\to-2+\sqrt{3}} \frac{z-(-2+\sqrt{3})}{(z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))} \\
&= -2i \lim_{z\to-2+\sqrt{3}} \frac{1}{z-(-2-\sqrt{3})} \\
&= \frac{-2i}{-2+\sqrt{3}-(-2-\sqrt{3})} \\
&= \frac{-i}{\sqrt{3}} \\
&= \frac{-i\sqrt{3}}{3}. \qquad (9.58)
\end{aligned}
$$

Therefore, we have

$$
\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i\oint_{|z|=1} \frac{dz}{z^2+4z+1} = 2\pi i\left(\frac{-i\sqrt{3}}{3}\right) = \frac{2\pi\sqrt{3}}{3}. \qquad (9.59)
$$

Before moving on to further applications, we note that there is another way to compute the integral in the last example. Karl Theodor Wilhelm Weierstraß (1815-1897) introduced a substitution method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t = \tan\frac{\theta}{2}$ and converts the integrand into a rational function of t. One can show that this substitution implies that

The Weierstraß substitution method.

$$
\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2},
$$

and

$$
d\theta = \frac{2dt}{1+t^2}.
$$

The details are left for Problem 8 and apply the method. In order to see how it works, we will redo the last problem.

Example 9.39. Apply the Weierstraß substitution method to compute $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

$$
\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{-\infty}^{\infty} \frac{1}{2+\frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2}
$$

$$= 2\int_{-\infty}^{\infty} \frac{dt}{t^2+3}$$
$$= \frac{2}{3}\sqrt{3}\left[\tan^{-1}\left(\frac{\sqrt{3}}{3}t\right)\right]_{-\infty}^{\infty} = \frac{2\pi\sqrt{3}}{3}. \qquad (9.60)$$

9.5.6 Infinite Integrals

Infinite integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ occur often in physics. They can represent wave packets, wave diffraction, Fourier transforms, and arise in other applications. In this section we will see that such integrals may be computed by extending the integration to a contour in the complex plane.

Recall from your calculus experience that these integrals are improper integrals and the way that one determines if improper integrals exist, or converge, is to carefully compute these integrals using limits such as

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$$

For example, we evaluate the integral of $f(x) = x$ as

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left(\frac{R^2}{2} - \frac{(-R)^2}{2}\right) = 0.$$

One might also be tempted to carry out this integration by splitting the integration interval, $(-\infty, 0] \cup [0, \infty)$. However, the integrals $\int_0^{\infty} x\,dx$ and $\int_{-\infty}^{0} x\,dx$ do not exist. A simple computation confirms this.

$$\int_0^{\infty} x\,dx = \lim_{R\to\infty}\int_0^{R} x\,dx = \lim_{R\to\infty}\left(\frac{R^2}{2}\right) = \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

does not exist while $\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$ does exist. We will be interested in computing the latter type of integral. Such an integral is called the Cauchy Principal Value Integral and is denoted with either a P, PV, or a bar through the integral:

The Cauchy principal value integral.

$$P\int_{-\infty}^{\infty} f(x)\,dx = PV\int_{-\infty}^{\infty} f(x)\,dx = ⨍_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \qquad (9.61)$$

If there is a discontinuity in the integral, one can further modify this definition of principal value integral to bypass the singularity. For example, if $f(x)$ is continuous on $a \le x \le b$ and not defined at $x = x_0 \in [a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0}\left(\int_a^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^{b} f(x)\,dx\right).$$

In our discussions we will be computing integrals over the real line in the Cauchy principal value sense.

Example 9.40. Compute $\int_{-1}^{1} \frac{dx}{x^3}$ in the Cauchy Principal Value sense.

In this case, $f(x) = \frac{1}{x^3}$ is not defined at $x = 0$. So, we have

$$\begin{aligned} \int_{-1}^{1} \frac{dx}{x^3} &= \lim_{\epsilon \to 0} \left(\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3} \right) \\ &= \lim_{\epsilon \to 0} \left(-\frac{1}{2x^2}\Big|_{-1}^{-\epsilon} - \frac{1}{2x^2}\Big|_{\epsilon}^{1} \right) = 0. \end{aligned} \quad (9.62)$$

Computation of real integrals by embedding the problem in the complex plane.

We now proceed to the evaluation of principal value integrals using complex integration methods. We want to evaluate the integral $\int_{-\infty}^{\infty} f(x)\,dx$. We will extend this into an integration in the complex plane. We extend $f(x)$ to $f(z)$ and assume that $f(z)$ is analytic in the upper half plane ($\text{Im}(z) > 0$) except at isolated poles. We then consider the integral $\int_{-R}^{R} f(x)\,dx$ as an integral over the interval $(-R, R)$. We view this interval as a piece of a larger contour C_R obtained by completing the contour with a semicircle Γ_R of radius R extending into the upper half plane as shown in Figure 9.41. Note, a similar construction is sometimes needed extending the integration into the lower half plane ($\text{Im}(z) < 0$) as we will later see.

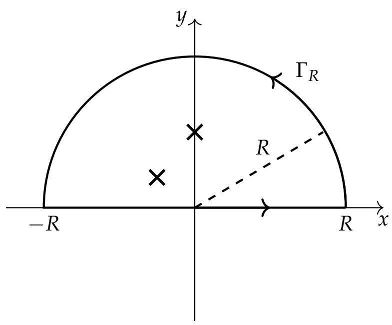

Figure 9.41: Contours for computing $P\int_{-\infty}^{\infty} f(x)\,dx$.

The integral around the entire contour C_R can be computed using the Residue Theorem and is related to integrations over the pieces of the contour by

$$\oint_{C_R} f(z)\,dz = \int_{\Gamma_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz. \quad (9.63)$$

Taking the limit $R \to \infty$ and noting that the integral over $(-R, R)$ is the desired integral, we have

$$P\int_{-\infty}^{\infty} f(x)\,dx = \oint_{C} f(z)\,dz - \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz, \quad (9.64)$$

where we have identified C as the limiting contour as R gets large.

Now the key to carrying out the integration is that the second integral vanishes in the limit. This is true if $R|f(z)| \to 0$ along Γ_R as $R \to \infty$. This can be seen by the following argument. We parametrize the contour Γ_R using $z = Re^{i\theta}$. Then, when $|f(z)| < M(R)$,

$$\begin{aligned} \left| \int_{\Gamma_R} f(z)\,dz \right| &= \left| \int_0^{2\pi} f(Re^{i\theta}) Re^{i\theta}\,d\theta \right| \\ &\le R \int_0^{2\pi} \left| f(Re^{i\theta}) \right| d\theta \\ &< RM(R) \int_0^{2\pi} d\theta \\ &= 2\pi RM(R). \end{aligned} \quad (9.65)$$

So, if $\lim_{R\to\infty} RM(R) = 0$, then $\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0$.

We now demonstrate how to use complex integration methods in evaluating integrals over real valued functions.

Example 9.41. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

We already know how to do this integral using calculus without complex analysis. We have that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \left(2\tan^{-1} R \right) = 2\left(\frac{\pi}{2}\right) = \pi.$$

We will apply the methods of this section and confirm this result. The needed contours are shown in Figure 9.42 and the poles of the integrand are at $z = \pm i$. We first write the integral over the bounded contour C_R as the sum of an integral from $-R$ to R along the real axis plus the integral over the semicircular arc in the upper half complex plane,

$$\int_{C_R} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\Gamma_R} \frac{dz}{1+z^2}.$$

Next, we let R get large.

We first note that $f(z) = \frac{1}{1+z^2}$ goes to zero fast enough on Γ_R as R gets large.

$$R|f(z)| = \frac{R}{|1+R^2 e^{2i\theta}|} = \frac{R}{\sqrt{1+2R^2\cos\theta + R^4}}.$$

Thus, as $R \to \infty$, $R|f(z)| \to 0$ and $C_R \to C$. So,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint_C \frac{dz}{1+z^2}.$$

We need only compute the residue at the enclosed pole, $z = i$.

$$\text{Res}[f(z); z = i] = \lim_{z\to i}(z-i)\frac{1}{1+z^2} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}.$$

Then, using the Residue Theorem, we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \left(\frac{1}{2i}\right) = \pi.$$

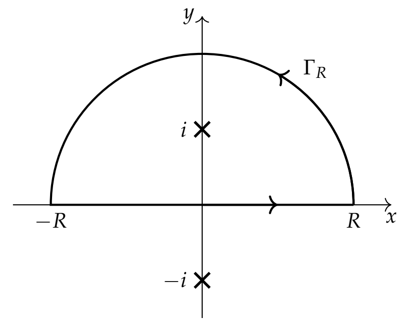

Figure 9.42: Contour for computing $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Example 9.42. Evaluate $P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$.

For this example the integral is unbounded at $z = 0$. Constructing the contours as before we are faced for the first time with a pole lying on the contour. We cannot ignore this fact. We can proceed with the computation by carefully going around the pole with a small semicircle of radius ϵ, as shown in Figure 9.43. Then the principal value integral computation becomes

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{\epsilon\to 0, R\to\infty} \left(\int_{-R}^{-\epsilon} \frac{\sin x}{x}\, dx + \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx \right). \qquad (9.66)$$

We will also need to rewrite the sine function in term of exponentials in this integral. There are two approaches that we could take. First, we could employ the definition of the sine function in terms of complex exponentials. This gives two integrals to compute:

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2i}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx \right). \qquad (9.67)$$

The other approach would be to realize that the sine function is the imaginary part of an exponential, $\text{Im}\, e^{ix} = sinx$. Then, we would have

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \text{Im}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx \right). \qquad (9.68)$$

We first consider $P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx$, which is common to both approaches. We use the contour in Figure 9.43. Then we have

$$\oint_{C_R}\frac{e^{iz}}{z}\,dz=\int_{\Gamma_R}\frac{e^{iz}}{z}\,dz+\int_{-R}^{-\epsilon}\frac{e^{iz}}{z}\,dz+\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz+\int_{\epsilon}^{R}\frac{e^{iz}}{z}\,dz.$$

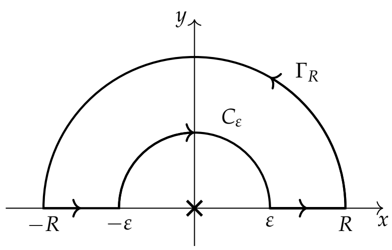

Figure 9.43: Contour for computing $P\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$.

The integral $\oint_{C_R}\frac{e^{iz}}{z}\,dz$ vanishes since there are no poles enclosed in the contour! The sum of the second and fourth integrals gives the integral we seek as $\epsilon\to 0$ and $R\to\infty$. The integral over Γ_R will vanish as R gets large according to Jordan's Lemma.

Jordan's Lemma give conditions as when integrals over Γ_R will vanish as R gets large. We state a version of Jordan's Lemma here for reference and give a proof is at the end of this chapter.

Jordan's Lemma

If $f(z)$ converges uniformly to zero as $z\to\infty$, then

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{ikz}\,dz=0$$

where $k>0$ and C_R is the upper half of the circle $|z|=R$.

A similar result applies for $k<0$, but one closes the contour in the lower half plane. [See Section 9.5.8 for the proof of Jordan's Lemma.]

The remaining integral around the small semicircular arc has to be done separately. We have

$$\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz=\int_{\pi}^{0}\frac{\exp(i\epsilon e^{i\theta})}{\epsilon e^{i\theta}}i\epsilon e^{i\theta}\,d\theta=-\int_0^{\pi} i\exp(i\epsilon e^{i\theta})\,d\theta.$$

Taking the limit as ϵ goes to zero, the integrand goes to i and we have

$$\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz=-\pi i.$$

Note that we have not previously done integrals in which a singularity lies on the contour. One can show, as in this example, that points on the contour can be accounted for by using half of a residue (times $2\pi i$). For the semicircle C_ϵ you can verify this. The negative sign comes from going clockwise around the semicircle.

So far, we have that

$$P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx=-\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz=\pi i.$$

At this point we can get the answer using the second approach in Equation (9.68). Namely,

$$P\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx=\mathrm{Im}\left(P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx\right)=\mathrm{Im}(\pi i)=\pi. \tag{9.69}$$

It is instructive to carry out the first approach in Equation (9.67). We will need to compute $P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx$. This is done in a similar to the above computation, being careful with the sign changes due to the orientations of the contours as shown in Figure 9.44.

We note that the contour is closed in the lower half plane. This is because $k < 0$ in the application of Jordan's Lemma. One can understand why this is the case from the following observation. Consider the exponential in Jordan's Lemma. Let $z = z_R + iz_I$. Then,

$$e^{ikz} = e^{ik(z_R+iz_I)} = e^{-kz_I}e^{ikz_R}.$$

As $|z|$ gets large, the second factor just oscillates. The first factor would go to zero if $kz_I > 0$. So, if $k > 0$, we would close the contour in the upper half plane. If $k < 0$, then we would close the contour in the lower half plane. In the current computation, $k = -1$, so we use the lower half plane.

Working out the details, we find the same value for

$$P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\,dx = \pi i.$$

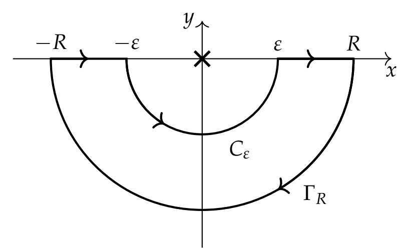

Figure 9.44: Contour in the lower half plane for computing $P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx$.

Finally, we can compute the original integral as

$$\begin{aligned} P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx &= \frac{1}{2i}\left(P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx - P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx\right) \\ &= \frac{1}{2i}\left(\pi i + \pi i\right) \\ &= \pi. \end{aligned} \tag{9.70}$$

This is the same result as we obtained using Equation(9.68).

Example 9.43. Evaluate $\oint_{|z|=1}\frac{dz}{z^2+1}$.

In this example there are two simple poles, $z = \pm i$ lying on the contour, as seen in Figure 9.45. This problem is similar to Problem 1c, except we will do it using contour integration instead of a parametrization. We bypass the two poles by drawing small semicircles around them. Since the poles are not included in the closed contour, then the Residue Theorem tells us that the integral over the path vanishes. We can write the full integration as a sum over three paths, $C_\pm$ for the semicircles and C for the original contour with the poles cut out. Then we take the limit as the semicircle radii go to zero. So,

$$0 = \int_C \frac{dz}{z^2+1} + \int_{C_+}\frac{dz}{z^2+1} + \int_{C_-}\frac{dz}{z^2+1}.$$

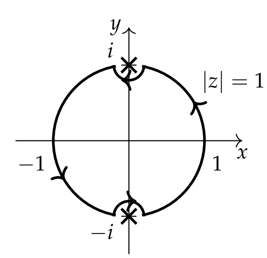

Figure 9.45: Example with poles on contour.

The integral over the semicircle around i can be done using the parametrization $z = i + \epsilon e^{i\theta}$. Then $z^2 + 1 = 2i\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}$. This gives

$$\int_{C_+}\frac{dz}{z^2+1} = \lim_{\epsilon\to 0}\int_0^{-\pi}\frac{i\epsilon e^{i\theta}}{2i\epsilon e^{i\theta}+\epsilon^2 e^{2i\theta}}\,d\theta = \frac{1}{2}\int_0^{-\pi} d\theta = -\frac{\pi}{2}.$$

As in the last example, we note that this is just πi times the residue, $Res\left[\frac{1}{z^2+1}; z = i\right] = \frac{1}{2i}$. Since the path is traced clockwise, we find the contribution is $-\pi i Res = -\frac{\pi}{2}$, which is what we obtained above. A Similar computation will give the contribution from $z = -i$ as $\frac{\pi}{2}$.

Adding these values gives the total contribution from $C_\pm$ as zero. So, the final result is that

$$\oint_{|z|=1} \frac{dz}{z^2+1} = 0.$$

Example 9.44. Evaluate $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$, for $0 < a < 1$.

In dealing with integrals involving exponentials or hyperbolic functions it is sometimes useful to use different types of contours. This example is one such case. We will replace x with z and integrate over the contour in Figure 9.46. Letting $R \to \infty$, the integral along the real axis is the integral that we desire. The integral along the path for $y = 2\pi$ leads to a multiple of this integral since $z = x + 2\pi i$ along this path. Integration along the vertical paths vanish as $R \to \infty$. This is captured in the following integrals:

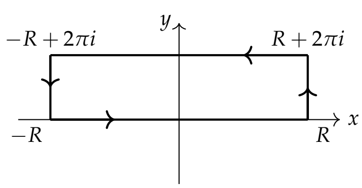

Figure 9.46: Example using a rectangular contour.

$$\begin{aligned}\oint_{C_R} \frac{e^{az}}{1+e^z}dz &= \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx + \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}}\,dy \\ &\quad + \int_{R}^{-R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx + \int_{2\pi}^{0} \frac{e^{a(-R+iy)}}{1+e^{-R+iy}}\,dy \qquad (9.71)\end{aligned}$$

We can now let $R \to \infty$. For large R the second integral decays as $e^{(a-1)R}$ and the fourth integral decays as e^{-aR}. Thus, we are left with

$$\begin{aligned}\oint_{C} \frac{e^{az}}{1+e^z}dz &= \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{e^{ax}}{1+e^x}dx - e^{2\pi i a}\int_{-R}^{R} \frac{e^{ax}}{1+e^x}dx\right) \\ &= (1-e^{2\pi i a})\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx. \qquad (9.72)\end{aligned}$$

We need only evaluate the left contour integral using the Residue Theorem. The poles are found from

$$1 + e^z = 0.$$

Within the contour, this is satisfied by $z = i\pi$. So,

$$Res\left[\frac{e^{az}}{1+e^z}; z = i\pi\right] = \lim_{z\to i\pi}(z - i\pi)\frac{e^{az}}{1+e^z} = -e^{i\pi a}.$$

Applying the Residue Theorem, we have

$$(1-e^{2\pi i a})\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = -2\pi i e^{i\pi a}.$$

Therefore, we have found that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = \frac{-2\pi i e^{i\pi a}}{1-e^{2\pi i a}} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1.$$

9.5.7 *Integration Over Multivalued Functions*

We have seen that some complex functions inherently possess multivaluedness; i.e., such "functions" do not evaluate to a single value, but have many values. The key examples were $f(z) = z^{1/n}$ and $f(z) = \ln z$. The nth roots have n distinct values and logarithms have an infinite number of values as determined by the range of the resulting arguments. We mentioned that the way to handle multivaluedness is to assign different branches to these functions, introduce a branch cut and glue them together at the branch cuts to form Riemann surfaces. In this way we can draw continuous paths along the Riemann surfaces as we move from one Riemann sheet to another.

Before we do examples of contour integration involving multivalued functions, lets first try to get a handle on multivaluedness in a simple case. We will consider the square root function,

$$w = z^{1/2} = r^{1/2} e^{i(\frac{\theta}{2}+k\pi)}, \quad k = 0, 1.$$

There are two branches, corresponding to each k value. If we follow a path not containing the origin, then we stay in the same branch, so the final argument (θ) will be equal to the initial argument. However, if we follow a path that encloses the origin, this will not be true. In particular, for an initial point on the unit circle, $z_0 = e^{i\theta_0}$, we have its image as $w_0 = e^{i\theta_0/2}$. However, if we go around a full revolution, $\theta = \theta_0 + 2\pi$, then

$$z_1 = e^{i\theta_0 + 2\pi i} = e^{i\theta_0},$$

but

$$w_1 = e^{(i\theta_0+2\pi i)/2} = e^{i\theta_0/2} e^{\pi i} \neq w_0.$$

Here we obtain a final argument (θ) that is not equal to the initial argument! Somewhere, we have crossed from one branch to another. Points, such as the origin in this example, are called branch points. Actually, there are two branch points, because we can view the closed path around the origin as a closed path around complex infinity in the compactified complex plane. However, we will not go into that at this time.

We can demonstrate this in the following figures. In Figure 9.47 we show how the points A-E are mapped from the z-plane into the w-plane under the square root function for the principal branch, $k = 0$. As we trace out the unit circle in the z-plane, we only trace out a semicircle in the w-plane. If we consider the branch $k = 1$, we then trace out a semicircle in the lower half plane, as shown in Figure 9.48 following the points from F to J.

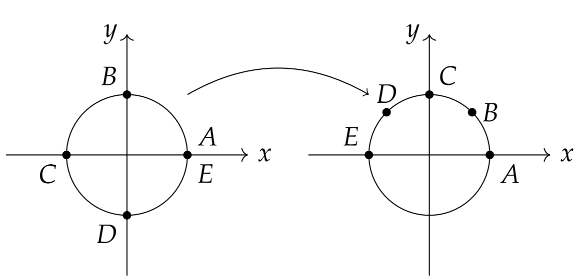

Figure 9.47: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the principal square root function.

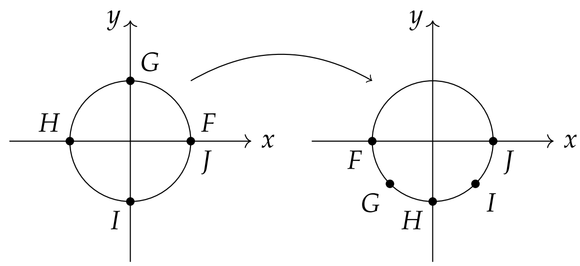

Figure 9.48: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the square root function for the second branch, $k = 1$.

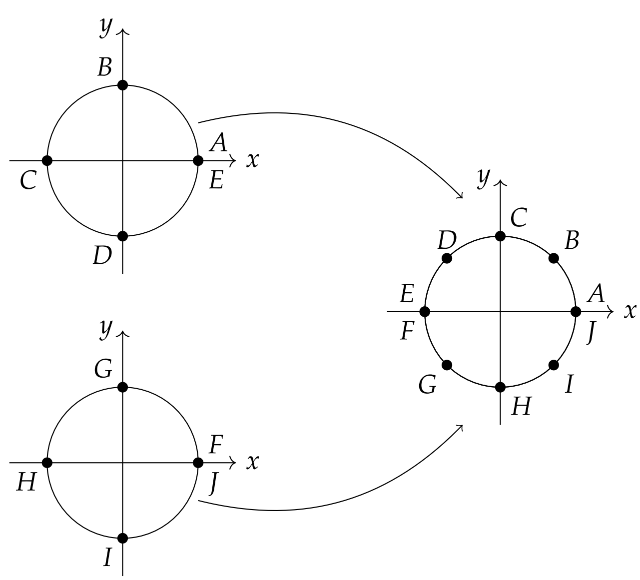

Figure 9.49: In this figure we show the combined mapping using two branches of the square root function.

We can combine these into one mapping depicting how the two complex planes corresponding to each branch provide a mapping to the w-plane. This is shown in Figure 9.49.

A common way to draw this domain, which looks like two separate complex planes, would be to glue them together. Imagine cutting each plane along the positive x-axis, extending between the two branch points, $z = 0$ and $z = \infty$. As one approaches the cut on the principal branch, then one can move onto the glued second branch. Then one continues around the origin on this branch until one once again reaches the cut. This cut is glued to the principal branch in such a way that the path returns to its starting point. The resulting surface we obtain is the Riemann surface shown in Figure 9.50. Note that there is nothing that forces us to place the branch cut at a particular place. For example, the branch cut could be along the positive real axis, the negative real axis, or any path connecting the origin and complex infinity.

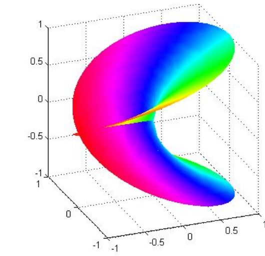

Figure 9.50: Riemann surface for $f(z) = z^{1/2}$.

We now look at examples involving integrals of multivalued functions.

Example 9.45. Evaluate $\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$.

We consider the contour integral $\oint_C \frac{\sqrt{z}}{1+z^2}dz$.

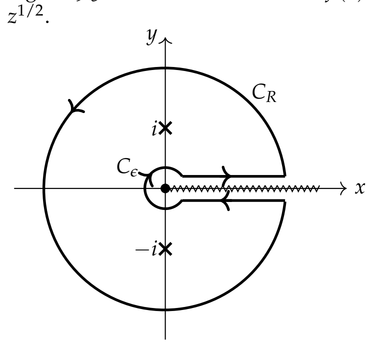

Figure 9.51: An example of a contour which accounts for a branch cut.

The first thing we can see in this problem is the square root function in the integrand. Being there is a multivalued function, we locate the branch point and determine where to draw the branch cut. In Figure 9.51 we show the contour that we will use in this problem. Note that we picked the branch cut along the positive x-axis.

We take the contour C to be positively oriented, being careful to enclose the two poles and to hug the branch cut. It consists of two circles. The outer circle C_R is a circle of radius R and the inner circle C_ϵ will have a radius of ϵ. The sought answer will be obtained by letting $R \to \infty$ and $\epsilon \to 0$. On the large circle we have that the integrand goes to zero fast enough as $R \to \infty$. The integral around the small circle vanishes as $\epsilon \to 0$. We can see this by parametrizing the circle as $z = \epsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$:

$$\begin{aligned}
\oint_{C_\epsilon} \frac{\sqrt{z}}{1+z^2}dz &= \int_0^{2\pi} \frac{\sqrt{\epsilon e^{i\theta}}}{1+(\epsilon e^{i\theta})^2} i\epsilon e^{i\theta}d\theta \\
&= i\epsilon^{3/2}\int_0^{2\pi} \frac{e^{3i\theta/2}}{1+(\epsilon^2 e^{2i\theta})}d\theta. \qquad (9.73)
\end{aligned}$$

It should now be easy to see that as $\epsilon \to 0$ this integral vanishes.

The integral above the branch cut is the one we are seeking, $\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx$. The integral under the branch cut, where $z = re^{2\pi i}$, is

$$\begin{aligned}
\int \frac{\sqrt{z}}{1+z^2}dz &= \int_\infty^0 \frac{\sqrt{re^{2\pi i}}}{1+r^2e^{4\pi i}}dr \\
&= \int_0^\infty \frac{\sqrt{r}}{1+r^2}dr. \qquad (9.74)
\end{aligned}$$

We note that this is the same as that above the cut.

Up to this point, we have that the contour integral, as $R \to \infty$ and $\epsilon \to 0$ is

$$\oint_C \frac{\sqrt{z}}{1+z^2}dz = 2\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx.$$

In order to finish this problem, we need the residues at the two simple poles.

$$Res\left[\frac{\sqrt{z}}{1+z^2}; z=i\right] = \frac{\sqrt{i}}{2i} = \frac{\sqrt{2}}{4}(1+i),$$
$$Res\left[\frac{\sqrt{z}}{1+z^2}; z=-i\right] = \frac{\sqrt{-i}}{-2i} = \frac{\sqrt{2}}{4}(1-i).$$

So,

$$2\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = 2\pi i\left(\frac{\sqrt{2}}{4}(1+i) + \frac{\sqrt{2}}{4}(1-i)\right) = \pi\sqrt{2}.$$

Finally, we have the value of the integral that we were seeking,

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \frac{\pi\sqrt{2}}{2}.$$

Example 9.46. Compute $\int_a^\infty f(x)\,dx$ using contour integration involving logarithms.[4]

In this example we will apply contour integration to the integral

$$\oint_C f(z)\ln(a-z)\,dz$$

for the contour shown in Figure 9.52.

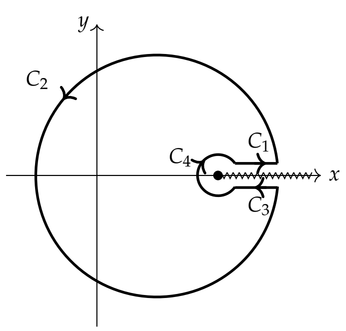

Figure 9.52: Contour needed to compute $\oint_C f(z)\ln(a-z)\,dz$.

We will assume that $f(z)$ is single valued and vanishes as $|z| \to \infty$. We will choose the branch cut to span from the origin along the positive real axis. Employing the Residue Theorem and breaking up the integrals over the pieces of the contour in Figure 9.52, we have schematically that

$$2\pi i \sum Res[f(z)\ln(a-z)] = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) f(z)\ln(a-z)\,dz.$$

First of all, we assume that $f(z)$ is well behaved at $z=a$ and vanishes fast enough as $|z| = R \to \infty$. Then, the integrals over C_2 and C_4 will vanish. For example, for the path C_4, we let $z = a + \epsilon e^{i\theta}$, $0 < \theta < 2\pi$. Then,

$$\int_{C_4} f(z)\ln(a-z)\,dz. = \lim_{\epsilon\to 0}\int_{2\pi}^0 f(a+\epsilon e^{i\theta})\ln(\epsilon e^{i\theta}) i\epsilon e^{i\theta}\,d\theta.$$

If $f(a)$ is well behaved, then we only need to show that $\lim_{\epsilon\to 0}\epsilon\ln\epsilon = 0$. This is left to the reader.

Similarly, we consider the integral over C_2 as R gets large,

$$\int_{C_2} f(z)\ln(a-z)\,dz = \lim_{R\to\infty}\int_0^{2\pi} f(Re^{i\theta})\ln(Re^{i\theta}) iRe^{i\theta}\,d\theta.$$

[4] This approach was originally published in Neville, E. H., 1945, Indefinite integration by means of residues. *The Mathematical Student*, **13**, 16-35, and discussed in Duffy, D. G., *Transform Methods for Solving Partial Differential Equations*, 1994.

Thus, we need only require that

$$\lim_{R\to\infty} R\ln R|f(Re^{i\theta})| = 0.$$

Next, we consider the two straight line pieces. For C_1, the integration along the real axis occurs for $z = x$, so

$$\int_{C_1} f(z)\ln(a-z)\,dz = \int_a^\infty f(x)\ln(a-x)\,dz.$$

However, integration over C_3 requires noting that we need the branch for the logarithm such that $\ln z = \ln(a-x) + 2\pi i$. Then,

$$\int_{C_3} f(z)\ln(a-z)\,dz = \int_\infty^a f(x)[\ln(a-x)+2\pi i]\,dz.$$

Combining these results, we have

$$\begin{aligned} 2\pi i\sum Res[f(z)\ln(a-z)] &= \int_a^\infty f(x)\ln(a-x)\,dz \\ &\quad + \int_\infty^a f(x)[\ln(a-x)+2\pi i]\,dz. \\ &= -2\pi i\int_a^\infty f(x)\,dz. \end{aligned} \tag{9.75}$$

Therefore,

$$\int_a^\infty f(x)\,dx = -\sum Res[f(z)\ln(a-z)].$$

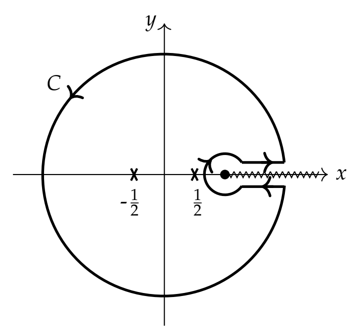

Figure 9.53: Contour needed to compute $\int_1^\infty \frac{dx}{4x^2-1}$.

Example 9.47. Compute $\int_1^\infty \frac{dx}{4x^2-1}$.

We can apply the last example to this case. We see from Figure 9.53 that the two poles at $z = \pm\frac{1}{2}$ are inside contour C. So, we compute the residues of $\frac{\ln(1-z)}{4z^2-1}$ at these poles and find that

$$\begin{aligned} \int_1^\infty \frac{dx}{4x^2-1} &= -Res\left[\frac{\ln(1-z)}{4z^2-1};\frac{1}{2}\right] - Res\left[\frac{\ln(1-z)}{4z^2-1};-\frac{1}{2}\right] \\ &= -\frac{\ln\frac{1}{2}}{4} + \frac{\ln\frac{3}{2}}{4} = \frac{\ln 3}{4}. \end{aligned} \tag{9.76}$$

9.5.8 Appendix: Jordan's Lemma

For completeness, we prove Jordan's Lemma.

Theorem 9.10. *If $f(z)$ converges uniformly to zero as $z\to\infty$, then*

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{ikz}\,dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$.

Proof. We consider the integral

$$I_R = \int_{C_R} f(z)e^{ikz}\,dz,$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$ in the complex plane. Let $z = Re^{i\theta}$ be a parametrization of C_R. Then,

$$I_R = \int_0^{\pi} f(Re^{i\theta}) e^{ikR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta.$$

Since

$$\lim_{|z|\to\infty} f(z) = 0, \quad 0 \le \arg z \le \pi,$$

then for large $|R|$, $|f(z)| < \epsilon$ for some $\epsilon > 0$. Then,

$$\begin{aligned} |I_R| &= \left| \int_0^{\pi} f(Re^{i\theta}) e^{ikR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta \right| \\ &\le \int_0^{\pi} \left| f(Re^{i\theta}) \right| \left| e^{ikR\cos\theta} \right| \left| e^{-aR\sin\theta} \right| \left| iRe^{i\theta} \right| d\theta \\ &\le \epsilon R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta \\ &= 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta. \end{aligned} \tag{9.77}$$

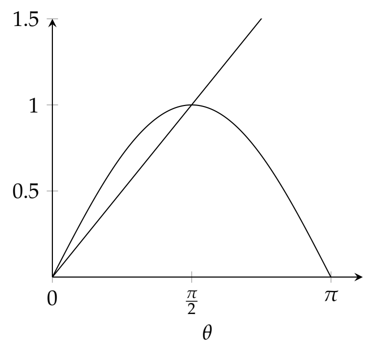

Figure 9.54: Plots of $y = \sin\theta$ and $y = \frac{2}{\pi}\theta$ to show where $\sin\theta \ge \frac{2}{\pi}\theta$.

The last integral still cannot be computed, but we can get a bound on it over the range $\theta \in [0, \pi/2]$. Note from Figure 9.54 that

$$\sin\theta \ge \frac{2}{\pi}\theta, \quad \theta \in [0, \pi/2].$$

Therefore, we have

$$|I_R| \le 2\epsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\, d\theta = \frac{2\epsilon R}{2aR/\pi}(1 - e^{-aR}).$$

For large R we have

$$\lim_{R\to\infty} |I_R| \le \frac{\pi\epsilon}{a}.$$

So, as $\epsilon \to 0$, the integral vanishes.

□

9.6 Laplace's Equation in 2D, Revisited

Harmonic functions are solutions of Laplace's equation. We have seen that the real and imaginary parts of a holomorphic function are harmonic. So, there must be a connection between complex functions and solutions of the two-dimensional Laplace equation. In this section we will describe how conformal mapping can be used to find solutions of Laplace's equation in two dimensional regions.

In Section 1.8 we had first seen applications in two-dimensional steady-state heat flow (or, diffusion), electrostatics, and fluid flow. For example, letting $\phi(\mathbf{r})$ be the electric potential, one has for a static charge distribution, $\rho(\mathbf{r})$, that the electric field, $\mathbf{E} = \nabla\phi$, satisfies

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0.$$

In regions devoid of charge, these equations yield the Laplace equation, $\nabla^2\phi = 0$.

Similarly, we can derive Laplace's equation for an incompressible, $\nabla \cdot \mathbf{v} = 0$, irrotational, , $\nabla \times \mathbf{v} = 0$, fluid flow. From well-known vector identities, we know that $\nabla \times \nabla\phi = 0$ for a scalar function, ϕ. Therefore, we can introduce a velocity potential, ϕ, such that $\mathbf{v} = \nabla\phi$. Thus, $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, the velocity potential satisfies Laplace's equation.

Fluid flow is probably the simplest and most interesting application of complex variable techniques for solving Laplace's equation. So, we will spend some time discussing how conformal mappings have been used to study two-dimensional ideal fluid flow, leading to the study of airfoil design.

9.6.1 Fluid Flow

The study of fluid flow and conformal mappings dates back to Euler, Riemann, and others.[5] The method was further elaborated upon by physicists like Lord Rayleigh (1877) and applications to airfoil theory we presented in papers by Kutta (1902) and Joukowski (1906) on later to be improved upon by others.

[5] "On the Use of Conformal Mapping in Shaping Wing Profiles," MAA lecture by R. S. Burington, 1939, published (1940) in ... 362-373

The physics behind flight and the dynamics of wing theory relies on the ideas of drag and lift. Namely, as the the cross section of a wing, the airfoil, goes through the air, it will experience several forces. The air speed above and belong the wing will differ due to the distance the air has to travel across the top and bottom of the wing. According to Bernoulli's Principle, steady fluid flow satisfies the conservation of energy in the form

$$P + \frac{1}{2}\rho U^2 + \rho g h = \text{ constant}$$

at points on either side of the wing profile. Here P is the pressure, ρ is the air density, U is the fluid speed, h is a reference height, and g is the acceleration due to gravity. The gravitational potential energy, $\rho g h$, is roughly constant on either side of the wing. So, this reduces to

$$P + \frac{1}{2}\rho U^2 = \text{ constant.}$$

Therefore, if the speed of the air below the wing is lower that than above the wing, the pressure below the wing will be higher, resulting in a net upward pressure. Since the pressure is the force per area, this will result in an upward force, a lift force, acting on the wing. This is the simplified version for the lift force. There is also a drag force acting in the direction of the flow. In general, we want to use complex variable methods to model the streamlines of the airflow as the air flows around an airfoil.

We begin by considering the fluid flow across a curve, C as shown in Figure 9.55. We assume that it is an ideal fluid with zero viscosity (i.e., does not flow like molasses) and is incompressible. It is a continuous, homogeneous flow with a constant thickness and represented by a velocity

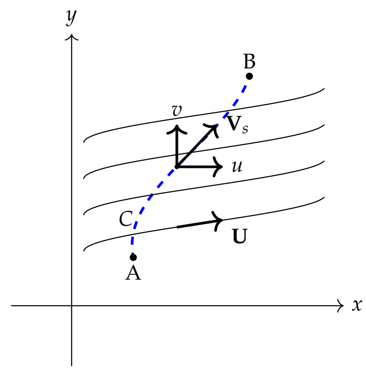

Figure 9.55: Fluid flow U across curve C between the points A and B.

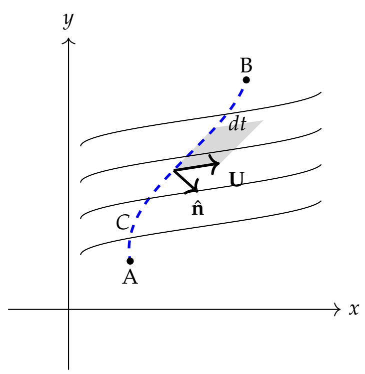

Figure 9.56: An amount of fluid crossing curve c in unit time.

$\mathbf{U} = (u(x,y), v(x,y))$, where u and v are the horizontal components of the flow as shown in Figure 9.55.

We are interested in the flow of fluid across a given curve which crosses several streamlines. The mass that flows over C per unit thickness in time dt can be given by

$$dm = \rho \mathbf{U} \cdot \hat{\mathbf{n}}\, dA dt.$$

Here $\hat{\mathbf{n}}\, dA$ is the normal area to the flow and for unit thickness can be written as $\hat{\mathbf{n}}\, dA = \mathbf{i}\, dy - \mathbf{i}\, dx$. Therefore, for a unit thickness the mass flow rate is given by

$$\frac{dm}{dt} = \rho(u\, dy - v\, dx).$$

Since the total mass flowing across ds in time dt is given by $dm = \rho dV$, for constant density, this also gives the volume flow rate,

$$\frac{dV}{dt} = u\, dy - v\, dx,$$

over a section of the curve. The total volume flow over C is therefore

$$\frac{dV}{dt}\Big|_{\text{total}} = \int_C u\, dy - v\, dx.$$

If this flow is independent of the curve, i.e., the path, then we have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}.$$

[This is just a consequence of Green's Theorem in the Plane. See Equation (9.3).] Another way to say this is that there exists a function, $\psi(x,t)$, such that $d\psi = u\, dy - v\, dx$. Then,

$$\int_C u\, dy - v\, dx = \int_A^B d\psi = \psi_B - \psi_A.$$

However, from basic calculus of several variables, we know that

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = u\, dy - v\, dx.$$

Therefore,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

Streamline functions.

It follows that if $\psi(x,y)$ has continuous second derivatives, then $u_x = -v_y$. This function is called the streamline function.

Furthermore, for constant density, we have

$$\begin{aligned} \nabla \cdot (\rho \mathbf{U}) &= \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &= \rho \left(\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial^2 \psi}{\partial x \partial y} \right) = 0. \end{aligned} \qquad (9.78)$$

This is the conservation of mass formula for constant density fluid flow.

We can also assume that the flow is irrotational. This means that the vorticity of the flow vanishes; i.e., $\nabla \times \mathbf{U} = \mathbf{0}$. Since the curl of the velocity field is zero, we can assume that the velocity is the gradient of a scalar function, $\mathbf{U} = \nabla\phi$. Then, a standard vector identity automatically gives

$$\nabla \times \mathbf{U} = \nabla \times \nabla\phi = 0.$$

For the two-dimensional flow with $\mathbf{U} = (u, v)$, we have

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}.$$

This is the velocity potential function for the flow.

Velocity potential curves.

Let's place the two-dimensional flow in the complex plane. Let an arbitrary point be $z = (x, y)$. Then, we have found two real-valued functions, $\psi(x, y)$ and $\psi(x, y)$, satisfying the relations

$$\begin{aligned} u &= \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \\ v &= \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \end{aligned} \qquad (9.79)$$

These are the Cauchy-Riemann relations for the real and imaginary parts of a complex differentiable function,

$$F(z(x, y) = \phi(x, y) + i\psi(x, y).$$

From its form, $\frac{dF}{dz}$ is called the complex velocity and $\sqrt{\left|\frac{dF}{dz}\right|} = \sqrt{u^2 + v^2}$ is the flow speed.

Furthermore, we have

$$\frac{dF}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv.$$

Integrating, we have

$$\begin{aligned} F &= \int_C (u - iv)\, dz \\ \phi(x, y) + i\psi(x, y) &= \int_{(x_0,y_0)}^{(x,y)} [u(x, y)\, dx + v(x, y)\, dy] \\ &\quad + i\int_{(x_0,y_0)}^{(x,y)} [-v(x, y)\, dx + u(x, y)\, dy]. \end{aligned} \qquad (9.80)$$

Therefore, the streamline and potential functions are given by the integral forms

$$\begin{aligned} \phi(x, y) &= \int_{(x_0,y_0)}^{(x,y)} [u(x, y)\, dx + v(x, y)\, dy], \\ \psi(x, y) &= \int_{(x_0,y_0)}^{(x,y)} [-v(x, y)\, dx + u(x, y)\, dy]. \end{aligned} \qquad (9.81)$$

These integrals give the circulation $\int_C V_s\, ds = \int_C u\, dx + v\, dy$ and the fluid flow per time, $\int_C -v\, dx + u\, dy$.

Streamline and potential curves are orthogonal families of curves.

The streamlines for the flow are given by the level curves $\psi(x, y) = c_1$ and the potential lines are given by the level curves $\phi(x, y) = c_2$. These are two orthogonal families of curves; i.e., these families of curves intersect each other orthogonally at each point as we will see in the examples. Note that these families of curves also provide the field lines and equipotential curves for electrostatic problems.

Example 9.48. Show that $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ are an orthogonal family of curves when $F(z) = \phi(x,y) + i\psi(x,y)$ is holomorphic.

In order to show that these curves are orthogonal, we need to find the slopes of the curves at an arbitrary point, (x,y). For $\phi(x,y) = c_1$, we recall from multivaribale calculus that

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = 0.$$

So, the slope is found as

$$\frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}.$$

Similarly, we have

$$\frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}}.$$

Since $F(z)$ is differentiable, we can use the Cauchy-Riemann equations to find the product of the slopes satisfy

$$\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = -\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial x}} \frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = -1.$$

Therefore, $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ form an orthogonal family of curves.

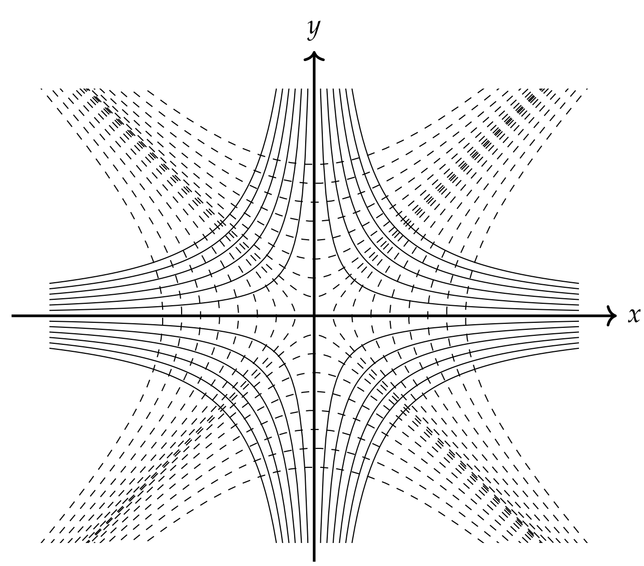

Figure 9.57: Plot of the orthogonal families $\phi = x^2 - y^2 = c_1$ (dashed) and $\phi(x,y) = 2xy = c_2$.

As an example, consider $F(z) = z^2 = x^2 - y^2 + 2ixy$. Then, $\phi(x, y) = x^2 - y^2$ and $\psi(x, y) = 2xy$. The slopes of the families of curves are given by

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}} \\ &= -\frac{2x}{-2y} = \frac{x}{y}. \\ \frac{dy}{dx} &= -\frac{\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}} \\ &= -\frac{2y}{2x} = -\frac{y}{x}. \end{aligned} \tag{9.82}$$

The products of these slopes is -1. The orthogonal families are depicted in Figure 9.57.

We will now turn to some typical examples by writing down some differentiable functions, $F(z)$, and determining the types of flows that result from these examples. We will then turn in the next section to using these basic forms to solve problems in slightly different domains through the use of conformal mappings.

Example 9.49. Describe the fluid flow associated with $F(z) = U_0 e^{-i\alpha} z$, where U_0 and α are real.

For this example, we have

$$\frac{dF}{dz} = U_0 e^{-i\alpha} = u - iv.$$

Thus, the velocity is constant,

$$\mathbf{U} = (U_0 \cos\alpha, U_0 \sin\alpha).$$

Thus, the velocity is a uniform flow at an angle of α.

Since

$$F(z) = U_0 e^{-i\alpha} z = U_0(x\cos\alpha + y\sin\alpha) + iU_0(y\cos\alpha - x\sin\alpha).$$

Thus, we have

$$\begin{aligned} \phi(x, y) &= U_0(x\cos\alpha + y\sin\alpha), \\ \psi(x, y) &= U_0(y\cos\alpha - x\sin\alpha). \end{aligned} \tag{9.83}$$

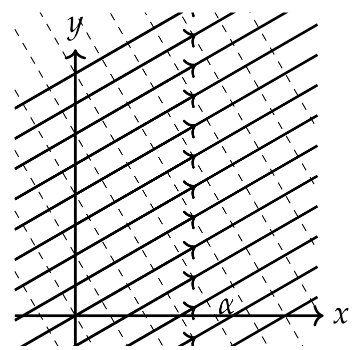

Figure 9.58: Stream lines (solid) and potential lines (dashed) for uniform flow at an angle of α, given by $F(z) = U_0 e^{-i\alpha} z$.

An example of this family of curves is shown in Figure 9.58.

Example 9.50. Describe the flow given by $F(z) = \frac{U_0 e^{-i\alpha}}{z - z_0}$.

We write

$$\begin{aligned} F(z) &= \frac{U_0 e^{-i\alpha}}{z - z_0} \\ &= \frac{U_0(\cos\alpha + i\sin\alpha)}{(x - x_0)^2 + (y - y_0)^2}[(x - x_0) - i(y - y_0)] \end{aligned}$$

$$= \frac{U_0}{(x-x_0)^2+(y-y_0)^2}[(x-x_0)\cos\alpha+(y-y_0)\sin\alpha]$$
$$+i\frac{U_0}{(x-x_0)^2+(y-y_0)^2}[-(y-y_0)\cos\alpha+(x-x_0)\sin\alpha]. \quad (9.84)$$

The level curves become

$$\phi(x,y) = \frac{U_0}{(x-x_0)^2+(y-y_0)^2}[(x-x_0)\cos\alpha+(y-y_0)\sin\alpha] = c_1,$$
$$\psi(x,y) = \frac{U_0}{(x-x_0)^2+(y-y_0)^2}[-(y-y_0)\cos\alpha+(x-x_0)\sin\alpha] = c_2. \quad (9.85)$$

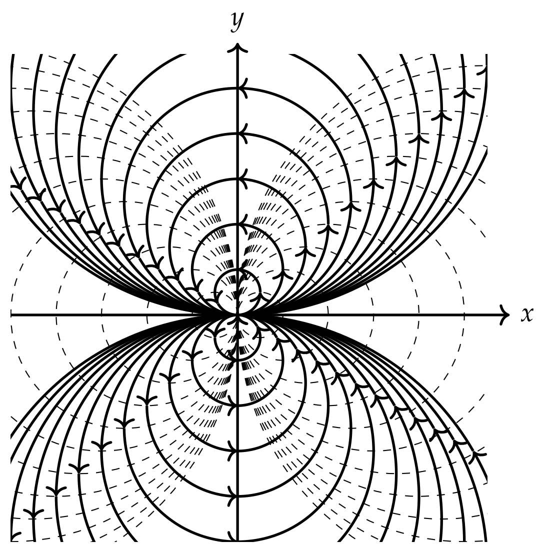

Figure 9.59: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{U_0 e^{-i\alpha}}{z}$ for $\alpha = 0$.

The level curves for the stream and potential functions satisfy equations of the form

$$\beta_i(\Delta x^2+\Delta y^2) - \cos(\alpha+\delta_i)\Delta x - \sin(\alpha+\delta_i)\Delta y = 0,$$

where $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\beta_i = \frac{c_i}{U_0}$, $\delta_1 = 0$, and $\delta_2 = \pi/2.$, These can be written in the more suggestive form

$$(\Delta x - \gamma_i\cos(\alpha-\delta_i))^2 + (\Delta y - \gamma_i\sin(\alpha-\delta_i))^2 = \gamma_i^2$$

for $\gamma_i = \frac{c_i}{2U_0}$, $i = 1,2$. Thus, the stream and potential curves are circles with varying radii (γ_i) and centers $((x_0+\gamma_i\cos(\alpha-\delta_i), y_0+\gamma_i\sin(\alpha-\delta_i)))$. Examples of this family of curves is shown for $\alpha = 0$ in in Figure 9.59 and for $\alpha = \pi/6$ in in Figure 9.60.

The components of the velocity field for $\alpha = 0$ are found from

$$\frac{dF}{dz} = \frac{d}{dz}\left(\frac{U_0}{z-z_0}\right)$$
$$= -\frac{U_0}{(z-z_0)^2}$$

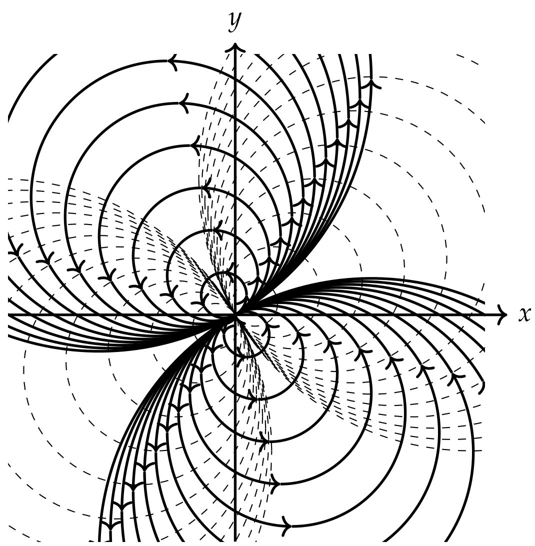

Figure 9.60: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{U_0 e^{-i\alpha}}{z}$ for $\alpha = \pi/6$.

$$
\begin{aligned}
&= -\frac{U_0[(x-x_0) - i(y-y_0)]^2}{[(x-x_0)^2 + (y-y_0)^2]^2} \\
&= -\frac{U_0[(x-x_0)^2 + (y-y_0)^2 - 2i(x-x_0)(y-y_0)]}{[(x-x_0)^2 + (y-y_0)^2]^2} \\
&= -\frac{U_0[(x-x_0)^2 + (y-y_0)^2]}{[(x-x_0)^2 + (y-y_0)^2]^2} + i\frac{U_0[2(x-x_0)(y-y_0)]}{[(x-x_0)^2 + (y-y_0)^2]^2} \\
&= -\frac{U_0}{[(x-x_0)^2 + (y-y_0)^2]} + i\frac{U_0[2(x-x_0)(y-y_0)]}{[(x-x_0)^2 + (y-y_0)^2]^2}. \quad (9.86)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
u &= -\frac{U_0}{[(x-x_0)^2 + (y-y_0)^2]}, \\
v &= \frac{U_0[2(x-x_0)(y-y_0)]}{[(x-x_0)^2 + (y-y_0)^2]^2}. \quad (9.87)
\end{aligned}
$$

Example 9.51. Describe the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$.

We write $F(z)$ in terms of its real and imaginary parts:

$$
\begin{aligned}
F(z) &= \frac{m}{2\pi}\ln(z - z_0) \\
&= \frac{m}{2\pi}\left[\ln\sqrt{(x-x_0)^2 + (y-y_0)^2} + i\tan^{-1}\frac{y-y_0}{x-x_0}\right]. \quad (9.88)
\end{aligned}
$$

The level curves become

$$
\begin{aligned}
\phi(x,y) &= \frac{m}{2\pi}\ln\sqrt{(x-x_0)^2 + (y-y_0)^2} = c_1, \\
\psi(x,y) &= \frac{m}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} = c_2.
\end{aligned}
\quad (9.89)
$$

Rewriting these equations, we have

$$
(x-x_0)^2 + (y-y_0)^2 = e^{4\pi c_1/m},
$$

$$y - y_0 = (x - x_0)\tan\frac{2\pi c_2}{m}. \tag{9.90}$$

In Figure 9.61 we see that the stream lines are those for a source or sink depending if $m > 0$ or $m < 0$, respectively.

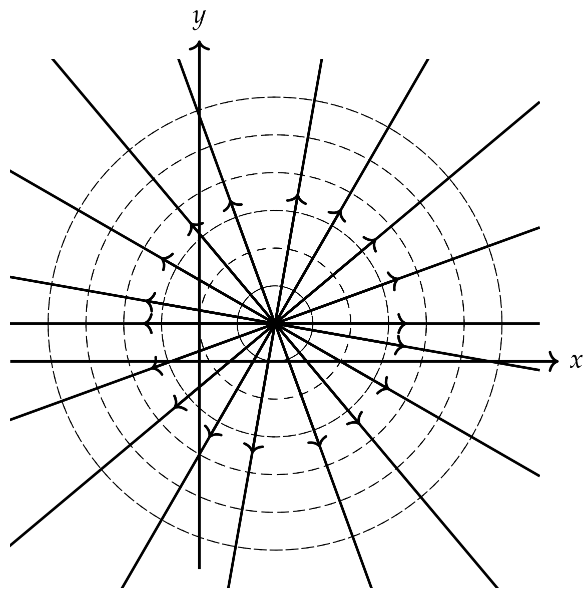

Figure 9.61: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$ for $(x_0, y_0) = (2, 1)$.

Example 9.52. Describe the flow given by $F(z) = -\frac{i\Gamma}{2\pi}\ln\frac{z-z_0}{a}$.

We write $F(z)$ in terms of its real and imaginary parts:

$$\begin{aligned} F(z) &= -\frac{i\Gamma}{2\pi}\ln\frac{z-z_0}{a} \\ &= -i\frac{\Gamma}{2\pi}\ln\sqrt{\left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{a}\right)^2} + \frac{\Gamma}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} \end{aligned} \tag{9.91}$$

The level curves become

$$\begin{aligned} \phi(x,y) &= \frac{\Gamma}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} = c_1, \\ \psi(x,y) &= -\frac{\Gamma}{2\pi}\ln\sqrt{\left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{a}\right)^2} = c_2. \end{aligned} \tag{9.92}$$

Rewriting these equations, we have

$$\begin{aligned} y - y_0 &= (x - x_0)\tan\frac{2\pi c_1}{\Gamma}, \\ \left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{a}\right)^2 &= e^{-2\pi c_2/\Gamma}. \end{aligned} \tag{9.93}$$

In Figure 9.62 we see that the stream lines circles, indicating rotational motion. Therefore, we have a vortex of counterclockwise, or clockwise flow, depending if $\Gamma > 0$ or $\Gamma < 0$, respectively.

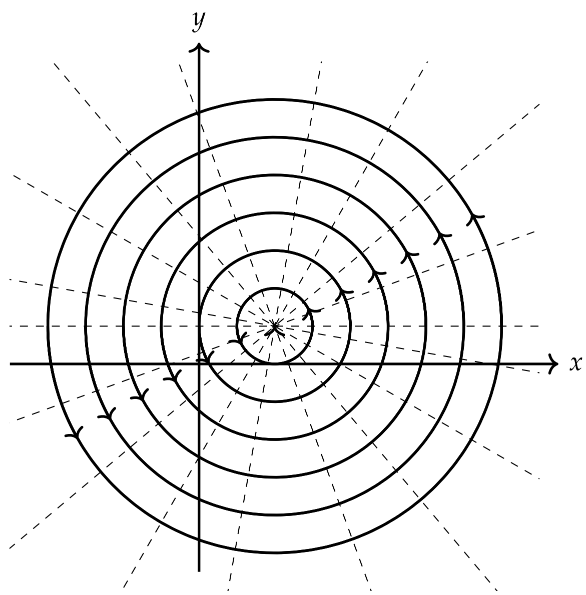

Figure 9.62: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$ for $(x_0, y_0) = (2, 1)$.

Example 9.53. Flow around a cylinder, $F(z) = U_0\left(z + \frac{a^2}{z}\right)$, $a, U_0 \in R$.

For this example, we have

$$\begin{aligned} F(z) &= U_0\left(z + \frac{a^2}{z}\right) \\ &= U_0\left(x + iy + \frac{a^2}{x+iy}\right) \\ &= U_0\left(x + iy + \frac{a^2}{x^2+y^2}(x - iy)\right) \\ &= U_0 x\left(1 + \frac{a^2}{x^2+y^2}\right) + iU_0 y\left(1 - \frac{a^2}{x^2+y^2}\right). \end{aligned} \tag{9.94}$$

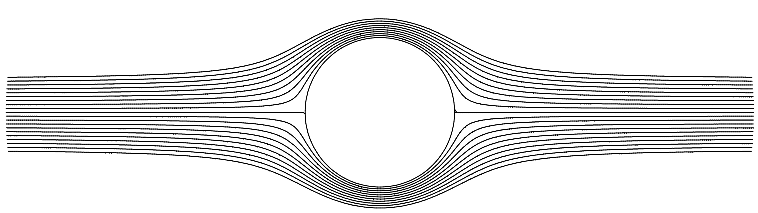

Figure 9.63: Stream lines for the flow given by $F(z) = U_0\left(z + \frac{a^2}{z}\right)$.

The level curves become

$$\begin{aligned} \phi(x,y) &= U_0 x\left(1 + \frac{a^2}{x^2+y^2}\right) = c_1, \\ \psi(x,y) &= U_0 y\left(1 - \frac{a^2}{x^2+y^2}\right) = c_2. \end{aligned} \tag{9.95}$$

Note that for the streamlines when $|z|$ is large, then $\psi \sim Vy$, or horizontal lines. For $x^2 + y^2 = a^2$, we have $\psi = 0$. This behavior is shown in Figure 9.63 where we have graphed the solution for $r \geq a$.

The level curves in Figure 9.63 can be obtained using the implicitplot feature of Maple. An example is shown below:

```
restart: with(plots):
k0:=20:
for k from 0 to k0 do
  P[k]:=implicitplot(sin(t)*(r-1/r)*1=(k0/2-k)/20, r=1..5,
  t=0..2*Pi, coords=polar,view=[-2..2, -1..1], axes=none,
  grid=[150,150],color=black):
  od:
display({seq(P[k],k=1..k0)},scaling=constrained);
```

A slight modification of the last example is if a circulation term is added:

$$F(z) = U_0\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln\frac{r}{a}.$$

The combination of the two terms gives the streamlines,

$$\psi(x, y) = U_0 y\left(1 - \frac{a^2}{x^2 + y^2}\right) - \frac{\Gamma}{2\pi}\ln\frac{r}{a},$$

which are seen in Figure 9.64. We can see interesting features in this flow including what is called a stagnation point. A stagnation point is a point where the flow speed, $\left|\frac{dF}{dz}\right| = 0$.

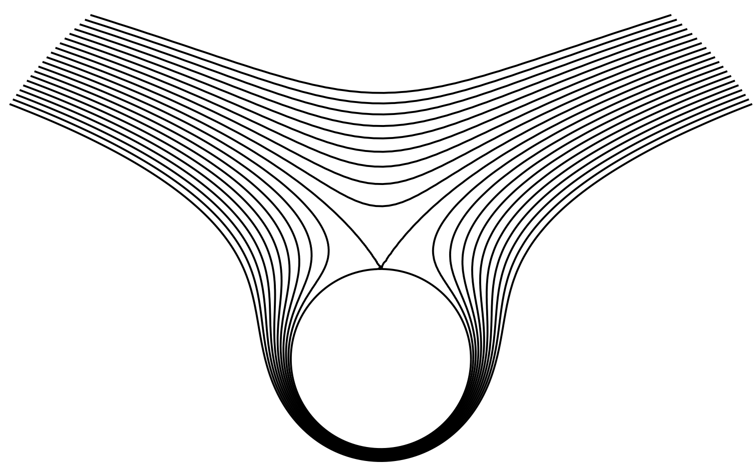

Figure 9.64: Stream lines for the flow given by $F(z) = U_0\left(z + \frac{a^2}{z}\right) - \frac{\Gamma}{2\pi}\ln\frac{z}{a}$.

Example 9.54. Find the stagnation point for the flow $F(z) = \left(z + \frac{1}{z}\right) - i\ln z$.

Since the flow speed vanishes at the stagnation points, we consider

$$\frac{dF}{dz} = 1 - \frac{1}{z^2} - \frac{i}{z} = 0.$$

This can be rewritten as

$$z^2 - iz - 1 = 0.$$

The solutions are $z = \frac{1}{2}(i \pm \sqrt{3})$. Thus, there are two stagnation points on the cylinder about which the flow is circulating. These are shown in Figure 9.65.

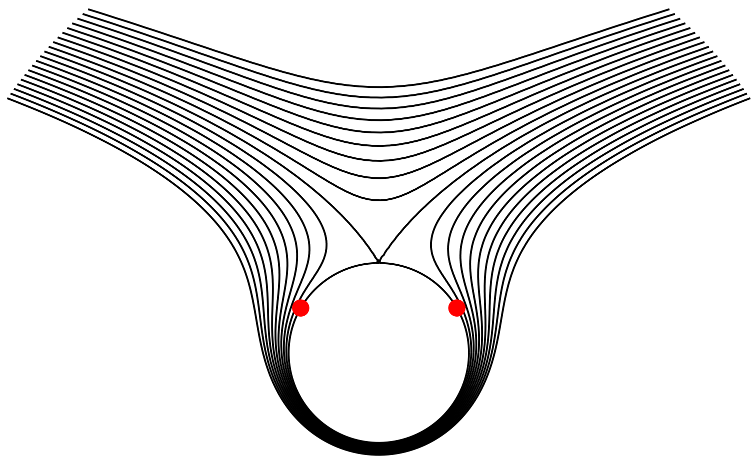

Figure 9.65: Stagnation points (red) on the cylinder are shown for the flow given by $F(z) = \left(z + \frac{1}{z}\right) - i \ln z$.

Example 9.55. Consider the complex potentials $F(z) = \frac{k}{2\pi} \ln \frac{z-a}{z-b}$, where $k = q$ and $k = -iq$ for q real.

We first note that for $z = x + iy$,

$$\begin{aligned} \ln \frac{z-a}{z-b} &= \ln \sqrt{(x-a)^2 + y^2} - \ln \sqrt{(x-a)^2 + y^2}, \\ &\quad + i \tan^{-1} \frac{y}{x-a} - i \tan^{-1} \frac{y}{x-b}. \end{aligned} \tag{9.96}$$

For $k = q$, we have

$$\begin{aligned} \psi(x,y) &= \frac{q}{2\pi} \left[\ln \sqrt{(x-a)^2 + y^2} - \ln \sqrt{(x-a)^2 + y^2} \right] = c_1, \\ \phi(x,y) &= \frac{q}{2\pi} \left[\tan^{-1} \frac{y}{x-a} - \tan^{-1} \frac{y}{x-b} \right] = c_2. \end{aligned} \tag{9.97}$$

The potential lines are circles and the streamlines are circular arcs as shown in Figure 9.66. These correspond to a source at $z = a$ and a sink at $z = b$. One can also view these as the electric field lines and equipotentials for an electric dipole consisting of two point charges of opposite sign at the points $z = a$ and $z = b$.

The equations for the curves are found from[6]

$$\begin{aligned} (x-a)^2 + y^2 &= C_1[(x-b)^2 + y^2], \\ (x-a)(x-b) + y^2 &= C_2 y(a-b), \end{aligned} \tag{9.98}$$

where these can be rewritten, respectively, in the more suggestive forms

$$\left(x - \frac{a - bC_1}{1 - C_1}\right)^2 + y^2 = \frac{C_1(a-b)^2}{(1-C_1)^2},$$

[6] The streamlines are found using the identity

$$\tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\frac{\alpha - \beta}{1 + \alpha\beta}.$$

$$\left(x-\frac{a+b}{2}\right)^2+\left(y-\frac{C_2(a-b)}{2}\right)^2 = (1+C_2^2)\left(\frac{a-b}{2}\right)^2. \quad (9.99)$$

Note that the first family of curves are the potential curves and the second give the streamlines.

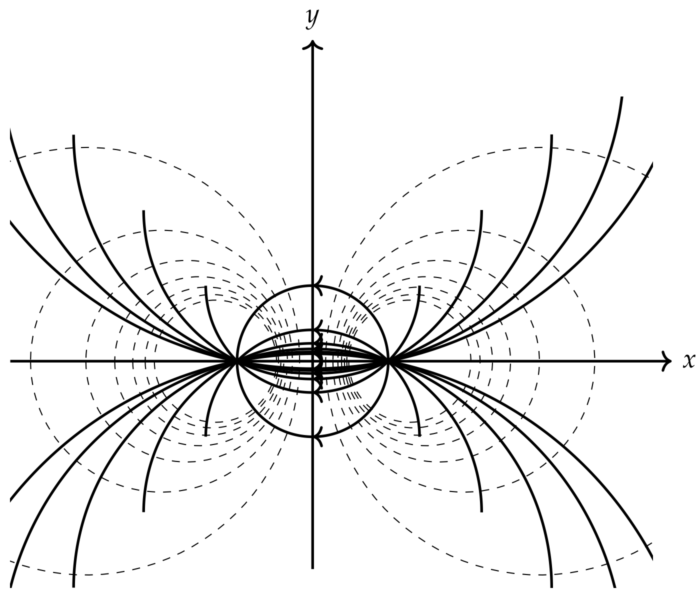

Figure 9.66: The electric field lines (solid) and equipotentials (dashed) for a dipole given by the complex potential $F(z) = \frac{q}{2\pi}\ln\frac{z-a}{z-b}$ for $b=-a$.

In the case that $k=-iq$ we have

$$\begin{aligned} F(z) &= \frac{-iq}{2\pi}\ln\frac{z-a}{z-b} \\ &= \frac{-iq}{2\pi}\left[\ln\sqrt{(x-a)^2+y^2}-\ln\sqrt{(x-a)^2+y^2}\right], \\ &\quad +\frac{q}{2\pi}\left[\tan^{-1}\frac{y}{x-a}-\tan^{-1}\frac{y}{x-b}\right]. \end{aligned} \quad (9.100)$$

So, the roles of the streamlines and potential lines are reversed and the corresponding plots give a flow for a pair of vortices as shown in Figure 9.67.

9.6.2 *Conformal Mappings*

It would be nice if the complex potentials in the last section could be mapped to a region of the complex plane such that the new stream functions and velocity potentials represent new flows. In order for this to be true, we would need the new families to once again be orthogonal families of curves. Thus, the mappings we seek must preserve angles. Such mappings are called conformal mappings.

We let $w=f(z)$ map points in the $z-$plane, (x,y), to points in the w-plane, (u,v) by $f(x+iy)=u+iv$. We have shown this in Figure 9.4.

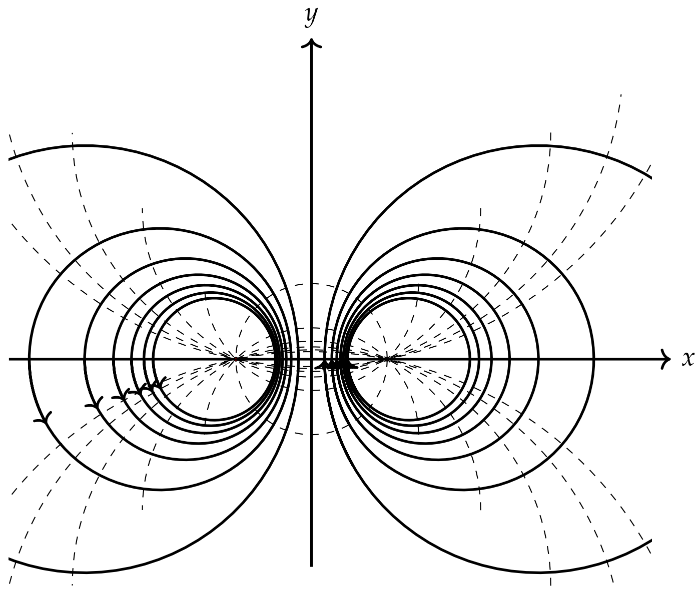

Figure 9.67: The streamlines (solid) and potentials (dashed) for a pair of vortices given by the complex potential $F(z) = \frac{q}{2\pi i} \ln \frac{z-a}{z-b}$ for $b = -a$.

Example 9.56. Map lines in the z-plane to curves in the w-plane under $f(z) = z^2$.

We have seen how grid lines in the z-plane is mapped by $f(z) = z^2$ into the w-plane in Figure 9.5, which is reproduced in Figure 9.68. The horizontal line $x = 1$ is mapped to $u(1, y) = 1 - y^2$ and $v(1, y) = 2y$. Eliminating the "parameter" y between these two equations, we have $u = 1 - v^2/4$. This is a parabolic curve. Similarly, the horizontal line $y = 1$ results in the curve $u = v^2/4 - 1$. These curves intersect at $w = 2i$.

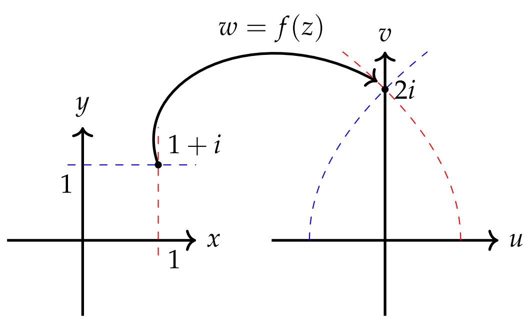

Figure 9.68: 2D plot showing how the function $f(z) = z^2$ maps the lines $x = 1$ and $y = 1$ in the z-plane into parabolae in the w-plane.

The lines in the z-plane intersect at $z = 1 + i$ at right angles. In the w-plane we see that the curves $u = 1 - v^2/4$ and $u = v^2/4 - 1$ intersect at $w = 2i$. The slopes of the tangent lines at $(0, 2)$ are -1 and 1, respectively, as shown in Figure 9.69.

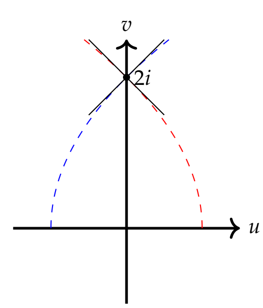

Figure 9.69: The tangents to the images of $x = 1$ and $y = 1$ under $f(z) = z^2$ are orthogonal.

In general, if two curves in the z-plane intersect orthogonally at $z = z_0$ and the corresponding curves in the w-plane under the mapping $w = f(z)$ are orthogonal at $w_0 = f(z_0)$, then the mapping is conformal. As we have seen, holomorphic functions are conformal, but only at points where $f'(z) \neq 0$.

Holomorphic functions are conformal at points where $f'(z) \neq 0$.

Example 9.57. Images of the real and imaginary axes under $f(z) = z^2$.

The line $z = iy$ maps to $w = z^2 = -y^2$ and the line $z = x$ maps to $w = z^2 = x^2$. The point of intersection $z_0 = 0$ maps to $w_0 = 0$. However, the image lines are the same line, the real axis in the w-plane. Obviously, the image lines are not orthogonal at the origin. Note that $f'(0) = 0$.

One special mapping is the inversion mapping, which is given by

$$f(z) = \frac{1}{z}.$$

This mapping maps the interior of the unit circle to the exterior of the unit circle in the w-plane as shown in Figure 9.70.

Let $z = x + iy$, where $x^2 + y^2 < 1$. Then,

$$w = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

Thus, $u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+y^2}$, and

$$\begin{aligned} u^2 + v^2 &= \left(\frac{x}{x^2+y^2}\right)^2 + \left(-\frac{y}{x^2+y^2}\right)^2 \\ &= \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}. \end{aligned} \tag{9.101}$$

Thus, for $x^2 + y^2 < 1$, $u^2 + U^2 > 1$. Furthermore, for $x^2 + y^2 > 1$, $u^2 + U^2 < 1$, and for $x^2 + y^2 = 1$, $u^2 + U^2 = 1$.

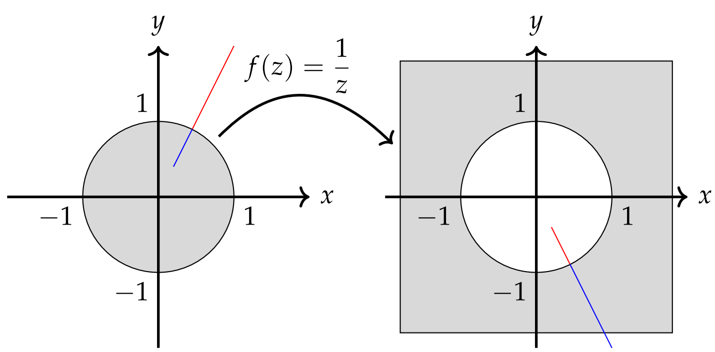

Figure 9.70: The inversion, $f(z) = \frac{1}{z}$, maps the interior of a unit circle to the external of a unit circle. Also, segments of aline through the origin, $y = 2x$, are mapped to the line $u = -2v$.

In fact, an inversion maps circles into circles. Namely, for $z = z_0 + re^{i\theta}$, we have

$$\begin{aligned} w &= \frac{1}{z_0 + re^{i\theta}} \\ &= \frac{\bar{z}_0 + re^{-i\theta}}{|z_0 + re^{i\theta}|^2} \\ &= w_0 + Re^{-i\theta}. \end{aligned} \tag{9.102}$$

Also, lines through the origin in the z-plane map into lines through the origin in the w-plane. Let $z = x + imx$. This corresponds to a line with slope m in the z-plane, $y = mx$. It maps to

$$\begin{aligned} f(z) &= \frac{1}{z} \\ &= \frac{1}{x+imx} \\ &= \frac{x-imx}{(1+m^2)x}. \end{aligned} \tag{9.103}$$

So, $u = \frac{x}{(1+m^2)x}$ and $v = -\frac{mx}{(1+m^2)x} = -mu$. This is a line through the origin in the w-plane with slope $-m$. This is shown in Figure 9.70 Note how the potion of the line $y = 2x$ that is inside the unit disk maps to the outside of the disk in the w-plane.

The bilinear transformation.

Another interesting class of transformation, of which the inversion is contained, is the bilinear transformation. The bilinear transformation is given by

$$w = f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

where a, b, c, and d are complex constants. These transformations were studied by mappings was studied by August Ferdinand Möbius (1790-1868) and are also called Möbius transformations, or linear fractional transformations. We further note that if $ad - bc = 0$, then the transformation reduces to the constat function.

We can seek to invert the transformation. Namely, solving for z, we have

$$z = f^{-1}(w) = \frac{-dw+b}{cw-a}, \quad w \neq \frac{a}{c}.$$

Since $f^{-1}(w)$ is not defined for $w \neq \frac{a}{c}$, we can say that $w \neq \frac{a}{c}$ maps to the point at infinity, or $f^{-1}\left(\frac{a}{c}\right) = \infty$. Similarly, we can let $z \to \infty$ to obtain

$$f(\infty) = \lim_{n\to\infty} f(z) = -\frac{d}{c}.$$

Thus, we have that the bilinear transformation is a one-to-one mapping of the extended complex z-plane to the extended complex w-plane.

The extended complex plane is the union of the complex plane plus the point at infinity. This is usually described in more detail using stereographic projection, which we will not review here.

If $c = 0$, $f(z)$ is easily seen to be a linear transformation. Linear transformations transform lines into lines and circles into circles.

When $c \neq 0$, we can write

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{c(az+b)}{c(cz+d)} \\ &= \frac{acz+ad-ad+bc}{c(cz+d)} \\ &= \frac{a(cz+d)-ad+bc}{c(cz+d)} \\ &= \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}. \end{aligned} \tag{9.104}$$

We note that if $bc - ad = 0$, then $f(z) = \frac{a}{c}$ is a constant, as noted above. The new form for $f(z)$ shows that it is the composition of a linear function $\zeta = cz + d$, an inversion, $g(\zeta) = \frac{1}{\zeta}$, and another linear transformation, $h(\zeta) = \frac{a}{c} + \frac{bc-ad}{c}\zeta$. Since linear transformations and inversions transform the set of circles and lines in the extended complex plane into circles and lines in the extended complex plane, then a bilinear does so as well.

What is important in out applications of complex analysis to the solution of Laplace's equation in the transformation of regions of the complex plane into other regions of the complex plane. Needed transformations can be found using the following property of bilinear transformations:

A given set of three points in the z-plane can be transformed into a given set of points in the w-plane using a bilinear transformation.

This statement is based on the following observation: There are three independent numbers that determine a bilinear transformation. If $a \neq 0$, then

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{z+\frac{b}{a}}{\frac{c}{a}z+\frac{d}{a}} \\ &\equiv \frac{z+\alpha}{\beta z+\gamma}. \end{aligned} \tag{9.105}$$

For $w = \frac{z+\alpha}{\beta z+\gamma}$, we have

$$\begin{aligned} w &= \frac{z+\alpha}{\beta z+\gamma} \\ w(\beta z+\gamma) &= z+\alpha \\ -\alpha + wz\beta + w\gamma &= z. \end{aligned} \tag{9.106}$$

Now, let $w_i = f(z_i)$, $i = 1, 2, 3$. This gives three equations for the three unknowns α, β, and γ. Namely,

$$\begin{aligned} -\alpha + w_1 z_1 \beta + w_1\gamma &= z_1, \\ -\alpha + w_2 z_2 \beta + w_2\gamma &= z_2, \\ -\alpha + w_3 z_3 \beta + w_3\gamma &= z_3. \end{aligned} \tag{9.107}$$

This systems of linear equation can be put into matrix form as

$$\begin{pmatrix} -1 & w_1 z_1 & w_1 \\ -1 & w_2 z_2 & w_2 \\ -1 & w_3 z_3 & w_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

It is only a matter of solving this system for $(\alpha, \beta, \gamma)^T$ in order to find the bilinear transformation.

A quicker method is to use the implicit form of the transformation,

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}.$$

Note that this implicit relation works upon insertion of the values w_i, z_i, for $i = 1, 2, 3$.

Example 9.58. Find the bilinear transformation that maps the points $-1, i, 1$ to the points $-1, 0, 1$.

The implicit form of the transformation becomes

$$\begin{aligned} \frac{(z+1)(i-1)}{(z-1)(i+1)} &= \frac{(w+1)(0-1)}{(w-1)(0+1)} \\ \frac{z+1}{z-1}\frac{i-1}{i+1} &= -\frac{w+1}{w-1}. \end{aligned} \tag{9.108}$$

Solving for w, we have

$$w = f(z) = \frac{(i-1)z+1+i}{(1+i)z-1+i}.$$

We can use the transformation in the last example to map the unit disk containing the points -1, i, and 1 to the half plane $w > 0$. We see that the unit circle gets mapped to the real axis with $z = -i$ mapped to the point at infinity. The point $z = 0$ gets mapped to

$$w = \frac{1+i}{-1+i} = \frac{1+i}{-1+i}\frac{-1-i}{-1-i} = \frac{2}{2} = 1.$$

Thus, interior points of the unit disk get mapped to the upper half plane. This is shown in Figure 9.71.

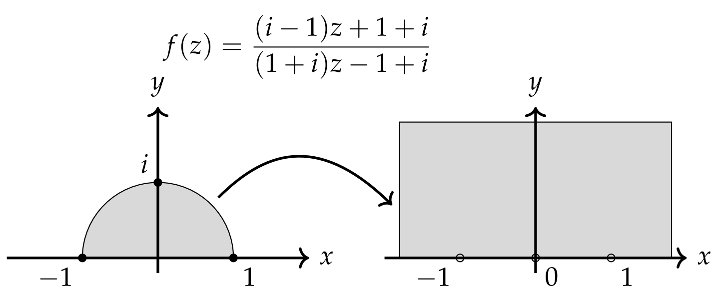

Figure 9.71: The bilinear transformation $f(z) = \frac{(i-1)z+1+i}{(1+i)z-1+i}$ maps the unit disk to the upper half plane.

Problems

1. Write the following in standard form.

a. $(4+5i)(2-3i)$.

b. $(1+i)^3$.

c. $\frac{5+3i}{1-i}$.

2. Write the following in polar form, $z = re^{i\theta}$.

a. $i-1$.

b. $-2i$.

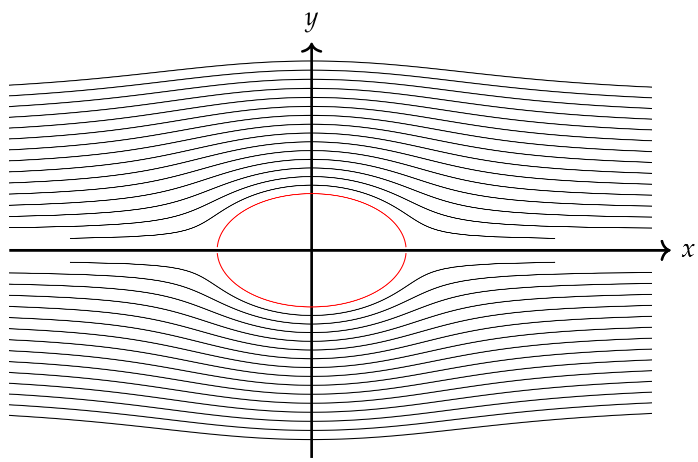

Figure 9.72: Flow about an elliptical cylinder.

c. $\sqrt{3}+3i$.

3. Write the following in rectangular form, $z=a+ib$.

a. $4e^{i\pi/6}$.

b. $\sqrt{2}e^{5i\pi/4}$.

c. $(1-i)^{100}$.

4. Find all z such that $z^4=16i$. Write the solutions in rectangular form, $z=a+ib$, with no decimal approximation or trig functions.

5. Show that $\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y$ using trigonometric identities and the exponential forms of these functions.

6. Find all z such that $\cos z=2$, or explain why there are none. You will need to consider $\cos(x+iy)$ and equate real and imaginary parts of the resulting expression similar to problem 5.

7. Find the principal value of i^i. Rewrite the base, i, as an exponential first.

8. Consider the circle $|z-1|=1$.

a. Rewrite the equation in rectangular coordinates by setting $z=x+iy$.

b. Sketch the resulting circle using part a.

c. Consider the image of the circle under the mapping $f(z)=z^2$, given by $|z^2-1|=1$.

i. By inserting $z=re^{i\theta}=r(\cos\theta+i\sin\theta)$, find the equation of the image curve in polar coordinates.

ii. Sketch the image curve. You may need to refer to your Calculus II text for polar plots. [Maple might help.]

9. Find the real and imaginary parts of the functions:

a. $f(z) = z^3$.

b. $f(z) = \sinh(z)$.

c. $f(z) = \cos \bar{z}$.

10. Find the derivative of each function in Problem 9 when the derivative exists. Otherwise, show that the derivative does not exist.

11. Let $f(z) = u + iv$ be differentiable. Consider the vector field given by $\mathbf{F} = v\mathbf{i} + u\mathbf{j}$. Show that the equations $\nabla \cdot \mathbf{F} = \mathbf{0}$ and $\nabla \times \mathbf{F} = \mathbf{0}$ are equivalent to the Cauchy-Riemann equations. [You will need to recall from multivariable calculus the del operator, $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$.]

12. What parametric curve is described by the function

$$\gamma(t) = (t - 3) + i(2t + 1),$$

$0 \le t \le 2$? [Hint: What would you do if you were instead considering the parametric equations $x = t - 3$ and $y = 2t + 1$?]

13. Write the equation that describes the circle of radius 3 which is centered at $z = 2 - i$ in a) Cartesian form (in terms of x and y); b) polar form (in terms of θ and r); c) complex form (in terms of z, r, and $e^{i\theta}$).

14. Consider the function $u(x, y) = x^3 - 3xy^2$.

a. Show that $u(x, y)$ is harmonic; i.e., $\nabla^2 u = 0$.

b. Find its harmonic conjugate, $v(x, y)$.

c. Find a differentiable function, $f(z)$, for which $u(x, y)$ is the real part.

d. Determine $f'(z)$ for the function in part c. [Use $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and rewrite your answer as a function of z.]

15. Evaluate the following integrals:

a. $\int_C \bar{z}\, dz$, where C is the parabola $y = x^2$ from $z = 0$ to $z = 1 + i$.

b. $\int_C f(z)\, dz$, where $f(z) = 2z - \bar{z}$ and C is the path from $z = 0$ to $z = 2 + i$ consisting of two line segments from $z = 0$ to $z = 2$ and then $z = 2$ to $z = 2 + i$.

c. $\int_C \frac{1}{z^2+4}\, dz$ for C the positively oriented circle, $|z| = 2$. [Hint: Parametrize the circle as $z = 2e^{i\theta}$, multiply numerator and denominator by $e^{-i\theta}$, and put in trigonometric form.]

16. Let C be the positively oriented ellipse $3x^2 + y^2 = 9$. Define

$$F(z_0) = \int_C \frac{z^2 + 2z}{z - z_0}\, dz.$$

Find $F(2i)$ and $F(2)$. [Hint: Sketch the ellipse in the complex plane. Use the Cauchy Integral Theorem with an appropriate $f(z)$, or Cauchy's Theorem if z_0 is outside the contour.]

17. Show that

$$\int_C \frac{dz}{(z-1-i)^{n+1}} = \begin{cases} 0, & n \neq 0, \\ 2\pi i, & n = 0, \end{cases}$$

for C the boundary of the square $0 \leq x \leq 2$, $0 \leq y \leq 2$ taken counterclockwise. [Hint: Use the fact that contours can be deformed into simpler shapes (like a circle) as long as the integrand is analytic in the region between them. After picking a simpler contour, integrate using parametrization.]

18. Show that for g and h analytic functions at z_0, with $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$,

$$\text{Res}\left[\frac{g(z)}{h(z)}; z_0\right] = \frac{g(z_0)}{h'(z_0)}.$$

19. For the following determine if the given point is a removable singularity, an essential singularity, or a pole (indicate its order).

a. $\frac{1-\cos z}{z^2}$, $\quad z = 0$.
b. $\frac{\sin z}{z^2}$, $\quad z = 0$.
c. $\frac{z^2-1}{(z-1)^2}$, $\quad z = 1$.
d. $ze^{1/z}$, $\quad z = 0$.
e. $\cos\frac{\pi}{z-\pi}$, $\quad z = \pi$.

20. Find the Laurent series expansion for $f(z) = \frac{\sinh z}{z^3}$ about $z = 0$. [Hint: You need to first do a MacLaurin series expansion for the hyperbolic sine.]

21. Find series representations for all indicated regions.

a. $f(z) = \frac{z}{z-1}$, $|z| < 1$, $|z| > 1$.
b. $f(z) = \frac{1}{(z-i)(z+2)}$, $|z| < 1$, $1 < |z| < 2$, $|z| > 2$. [Hint: Use partial fractions to write this as a sum of two functions first.]

22. Find the residues at the given points:

a. $\frac{2z^2+3z}{z-1}$ at $z = 1$.
b. $\frac{\ln(1+2z)}{z}$ at $z = 0$.
c. $\frac{\cos z}{(2z-\pi)^3}$ at $z = \frac{\pi}{2}$.

23. Consider the integral $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$.

a. Evaluate this integral by making the substitution $2\cos\theta = z + \frac{1}{z}$, $z = e^{i\theta}$ and using complex integration methods.
b. In the 1800's Weierstrass introduced a method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t = \tan\frac{\theta}{2}$ and converts the integrand into a rational function of t. Note that the integration around the unit circle corresponds to $t \in (-\infty, \infty)$.

   i. Show that

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$

ii. Show that

$$d\theta = \frac{2dt}{1+t^2}$$

iii. Use the Weierstrass substitution to compute the above integral.

24. Do the following integrals.

a.
$$\oint_{|z-i|=3} \frac{e^z}{z^2+\pi^2}\, dz.$$

b.
$$\oint_{|z-i|=3} \frac{z^2-3z+4}{z^2-4z+3}\, dz.$$

c.
$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4}\, dx.$$

[Hint: This is $\operatorname{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\, dx$.]

25. Evaluate the integral $\int_0^\infty \frac{(\ln x)^2}{1+x^2}\, dx$.
[Hint: Replace x with $z = e^t$ and use the rectangular contour in Figure 9.73 with $R \to \infty$.]

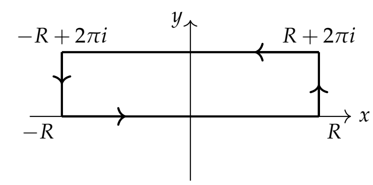

Figure 9.73: Rectangular contour for Problem 25.

26. Do the following integrals for fun!

a. For C the boundary of the square $|x| \le 2$, $|y| \le 2$,

$$\oint_C \frac{dz}{z(z-1)(z-3)^2}.$$

b.
$$\int_0^\pi \frac{\sin^2\theta}{13-12\cos\theta}\, d\theta.$$

c.
$$\int_{-\infty}^{\infty} \frac{dx}{x^2+5x+6}.$$

d.
$$\int_0^\infty \frac{\cos \pi x}{1-9x^2}\, dx.$$

e.
$$\int_0^\infty \frac{dx}{(x^2+9)(1-x)^2}.$$

f.
$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\, dx.$$

g.
$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\, dx.$$

27. Let $f(z) = u(x,y) + iv(x,y)$ be analytic in domain D. Prove that the Jacobian of the transformation from the xy-plane to the uv-plave is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = \left|f'(z)\right|^2.$$

28. Find the bilinear transformation which maps the points $1+i$, i, $2-i$ of the z-plane into the points $0, 1, i$ of the w-plane. Sketch the region in the w-plane to which the triangle formed by the points in the z-plane is mapped.

29. Find stream functions and potential functions for the following fluid motions. Describe the fluid flow from each.

a. $F(z) = z^2 + z$.

b. $F(z) = \cos z$.

c. $F(z) = U_0 \left(z + \frac{1}{z^2} \right)$.

d. $F(z) = \ln \left(1 + \frac{1}{z^2} \right)$.

30. Create a flow function which has two sources at $z = \pm a$ and one sink at $z = 0$ of equal magnitude strength. Verify your choice by finding and plotting the families of steam functions and potential functions.

10
Integral Transforms

"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.", Nikolai Lobatchevsky (1792-1856)

10.1 Introduction

Some of the most powerful tools for solving problems in physics are transform methods. The idea is that one can transform the problem at hand to a new problem in a different space, hoping that the problem in the new space is easier to solve. Such transforms appear in many forms.

In this chapter we will explore the use of integral transforms. Given a function $f(x)$, we define an integral transform to a new function $F(k)$ as

$$F(k) = \int_a^b f(x)K(x,k)\,dx.$$

Here $K(x,k)$ is called the kernel of the transform. We will concentrate specifically on Fourier transforms,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx,$$

and Laplace transforms

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt.$$

As we had seen in Chapter 1 and will see later in the book, the solutions of linear partial differential equations can be found by using the method of separation of variables to reduce solving partial differential equations (PDEs) to solving ordinary differential equations (ODEs). We can also use transform methods to transform the given PDE into ODEs or algebraic equations. Solving these equations, we then construct solutions of the PDE (or, the ODE) using an inverse transform. A schematic of these processes is shown below and we will describe in this chapter how one can use Fourier and Laplace transforms to this effect.

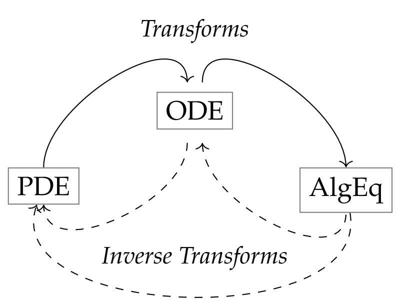

Figure 10.1: Schematic indicating that PDEs and ODEs can be transformed to simpler problems, solved in the new space and transformed back to the original space.

10.1.1 Example 1 - The Linearized KdV Equation

As a relatively simple example, we consider the linearized Korteweg-de Vries (KdV) equation:

$$u_t + cu_x + \beta u_{xxx} = 0, \quad -\infty < x < \infty. \tag{10.1}$$

This equation governs the propagation of some small amplitude water waves. Its nonlinear counterpart has been at the center of attention in the last 40 years as a generic nonlinear wave equation.

The nonlinear counterpart to this equation is the Korteweg-de Vries (KdV) equation: $u_t + 6uu_x + u_{xxx} = 0$. This equation was derived by Diederik Johannes Korteweg (1848-1941) and his student Gustav de Vries (1866-1934). This equation governs the propagation of traveling waves called solitons. These were first observed by John Scott Russell (1808-1882) and were the source of a long debate on the existence of such waves. The history of this debate is interesting and the KdV turned up as a generic equation in many other fields in the latter part of the last century leading to many papers on nonlinear evolution equations.

We seek solutions that oscillate in space. So, we assume a solution of the form

$$u(x,t) = A(t)e^{ikx}. \tag{10.2}$$

Such behavior was seen in Chapters 3 and 6 for the wave equation for vibrating strings. In that case, we found plane wave solutions of the form $e^{ik(x\pm ct)}$, which we could write as $e^{i(kx\pm\omega t)}$ by defining $\omega = kc$. We further note that one often seeks complex solutions as a linear combination of such forms and then takes the real part in order to obtain physical solutions. In this case, we will find plane wave solutions for which the angular frequency $\omega = \omega(k)$ is a function of the wavenumber.

Inserting the guess (10.2) into the linearized KdV equation, we find that

$$\frac{dA}{dt} + i(ck - \beta k^3)A = 0. \tag{10.3}$$

Thus, we have converted the problem of seeking a solution of the partial differential equation into seeking a solution to an ordinary differential equation. This new problem is easier to solve. In fact, given an initial value, $A(0)$, we have

$$A(t) = A(0)e^{-i(ck-\beta k^3)t}. \tag{10.4}$$

Therefore, the solution of the partial differential equation is

$$u(x,t) = A(0)e^{ik(x-(c-\beta k^2)t)}. \tag{10.5}$$

We note that this solution takes the form $e^{i(kx-\omega t)}$, where

$$\omega = ck - \beta k^3.$$

A dispersion relation is an expression giving the angular frequency as a function of the wave number, $\omega = \omega(k)$.

In general, the equation $\omega = \omega(k)$ gives the angular frequency as a function of the wave number, k, and is called a dispersion relation. For $\beta = 0$, we see that c is nothing but the wave speed. For $\beta \neq 0$, the wave speed is given as

$$v = \frac{\omega}{k} = c - \beta k^2.$$

This suggests that waves with different wave numbers will travel at different speeds. Recalling that wave numbers are related to wavelengths, $k = \frac{2\pi}{\lambda}$, this means that waves with different wavelengths will travel at different speeds. For example, an initial localized wave packet will not maintain its shape. It is said to disperse, as the component waves of differing wavelengths will tend to part company.

For a general initial condition, we write the solutions to the linearized KdV as a superposition of plane waves. We can do this since the partial differential equation is linear. This should remind you of what we had done when using separation of variables. We first sought product solutions and then took a linear combination of the product solutions to obtain the general solution.

For this problem, we will sum over all wave numbers. The wave numbers are not restricted to discrete values. We instead have a continuous range of values. Thus, "summing" over k means that we have to integrate over the wave numbers. Thus, we have the general solution[1]

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k,0)e^{ik(x-(c-\beta k^2)t)}\,dk. \tag{10.6}$$

[1] The extra 2π has been introduced to be consistent with the definition of the Fourier transform which is given later in the chapter.

Note that we have indicated that A is a function of k. This is similar to introducing the A_n's and B_n's in the series solution for waves on a string.

How do we determine the $A(k,0)$'s? We introduce as an initial condition the initial wave profile $u(x,0) = f(x)$. Then, we have

$$f(x) = u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k,0)e^{ikx}\,dk. \tag{10.7}$$

Thus, given $f(x)$, we seek $A(k,0)$. In this chapter we will see that

$$A(k,0) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx.$$

This is what is called the Fourier transform of $f(x)$. It is just one of the so-called integral transforms that we will consider in this chapter.

In Figure 10.2 we summarize the transform scheme. One can use methods like separation of variables to solve the partial differential equation directly, evolving the initial condition $u(x,0)$ into the solution $u(x,t)$ at a later time.

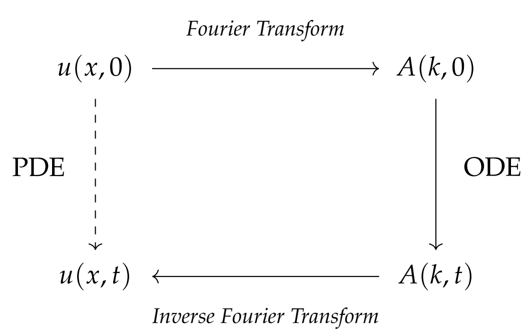

Figure 10.2: Schematic of using Fourier transforms to solve a linear evolution equation.

The transform method works as follows. Starting with the initial condition, one computes its Fourier Transform (FT) as[2]

$$A(k,0) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx.$$

[2] Note: The Fourier transform as used in this section and the next section are defined slightly differently than how we will define them later. The sign of the exponentials has been reversed.

Applying the transform on the partial differential equation, one obtains an ordinary differential equation satisfied by $A(k,t)$ which is simpler to solve than the original partial differential equation. Once $A(k,t)$ has been found, then one applies the Inverse Fourier Transform (IFT) to $A(k,t)$ in order to get the desired solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k,t)e^{ikx}\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k,0)e^{ik(x-(c-\beta k^2)t)}\,dk. \end{aligned} \tag{10.8}$$

10.1.2 *Example 2 - The Free Particle Wave Function*

A MORE FAMILIAR EXAMPLE IN PHYSICS comes from quantum mechanics. The Schrödinger equation gives the wave function $\Psi(x,t)$ for a particle under the influence of forces, represented through the corresponding potential function $V(x)$. The one dimensional time dependent Schrödinger equation is given by

The one dimensional time dependent Schrödinger equation.

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + V\Psi. \tag{10.9}$$

We consider the case of a free particle in which there are no forces, $V = 0$. Then we have

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx}. \tag{10.10}$$

Taking a hint from the study of the linearized KdV equation, we will assume that solutions of Equation (10.10) take the form

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k,t)e^{ikx}\,dk.$$

[Here we have opted to use the more traditional notation, $\phi(k,t)$ instead of $A(k,t)$ as above.]

Inserting the expression for $\Psi(x,t)$ into (10.10), we have

$$i\hbar\int_{-\infty}^{\infty} \frac{d\phi(k,t)}{dt}e^{ikx}\,dk = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \phi(k,t)(ik)^2e^{ikx}\,dk.$$

Since this is true for all t, we can equate the integrands, giving

$$i\hbar\frac{d\phi(k,t)}{dt} = \frac{\hbar^2k^2}{2m}\phi(k,t).$$

As with the last example, we have obtained a simple ordinary differential equation. The solution of this equation is given by

$$\phi(k,t) = \phi(k,0)e^{-i\frac{\hbar k^2}{2m}t}.$$

Applying the inverse Fourier transform, the general solution to the time dependent problem for a free particle is found as

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k,0)e^{ik(x-\frac{\hbar k}{2m}t)}\,dk.$$

We note that this takes the familiar form

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k,0)e^{i(kx-\omega t)}\,dk,$$

where the dispersion relation is found as

$$\omega = \frac{\hbar k^2}{2m}.$$

The wave speed is given as

$$v = \frac{\omega}{k} = \frac{\hbar k}{2m}.$$

As a special note, we see that this is not the particle velocity! Recall that the momentum is given as $p = \hbar k$.[3] So, this wave speed is $v = \frac{p}{2m}$, which is only half the classical particle velocity! A simple manipulation of this result will clarify the "problem."

[3] Since $p = \hbar k$, we also see that the dispersion relation is given by

$$\omega = \frac{\hbar k^2}{2m} = \frac{p^2}{2m\hbar} = \frac{E}{\hbar}.$$

We assume that particles can be represented by a localized wave function. This is the case if the major contributions to the integral are centered about a central wave number, k_0. Thus, we can expand $\omega(k)$ about k_0:

$$\omega(k) = \omega_0 + \omega_0'(k-k_0)t + \ldots. \tag{10.11}$$

Here $\omega_0 = \omega(k_0)$ and $\omega_0' = \omega'(k_0)$. Inserting this expression into the integral representation for $\Psi(x,t)$, we have

$$\Psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k,0)e^{i(kx-\omega_0 t-\omega_0'(k-k_0)t-\ldots)}\,dk,$$

We now make the change of variables, $s = k - k_0$, and rearrange the resulting factors to find

$$\begin{aligned}\Psi(x,t) &\approx \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k_0+s,0)e^{i((k_0+s)x-(\omega_0+\omega_0's)t)}\,ds\\ &= \frac{1}{2\pi}e^{i(-\omega_0 t+k_0\omega_0' t)}\int_{-\infty}^{\infty} \phi(k_0+s,0)e^{i(k_0+s)(x-\omega_0' t)}\,ds\\ &= e^{i(-\omega_0 t+k_0\omega_0' t)}\Psi(x-\omega_0' t,0). \end{aligned}\tag{10.12}$$

Group and phase velocities, $v_g = \frac{d\omega}{dk}$, $v_p = \frac{\omega}{k}$.

Summarizing, for an initially localized wave packet, $\Psi(x,0)$ with wave numbers grouped around k_0 the wave function,$\Psi(x,t)$, is a translated version of the initial wave function up to a phase factor. In quantum mechanics we are more interested in the probability density for locating a particle, so from

$$|\Psi(x,t)|^2 = |\Psi(x-\omega_0' t,0)|^2$$

we see that the "velocity of the wave packet" is found to be

$$\omega_0' = \frac{d\omega}{dk}\Big|_{k=k_0} = \frac{\hbar k}{m}.$$

This corresponds to the classical velocity of the particle ($v_{\text{part}} = p/m$). Thus, one usually defines ω_0' to be the group velocity,

$$v_g = \frac{d\omega}{dk}$$

and the former velocity as the phase velocity,

$$v_p = \frac{\omega}{k}.$$

10.1.3 Transform Schemes

These examples have illustrated one of the features of transform theory. Given a partial differential equation, we can transform the equation from spatial variables to wave number space, or time variables to frequency space. In the new space the time evolution is simpler. In these cases, the evolution was governed by an ordinary differential equation. One solves the problem in the new space and then transforms back to the original space. This is depicted in Figure 10.3 for the Schrödinger equation and was shown in Figure 10.2 for the linearized KdV equation.

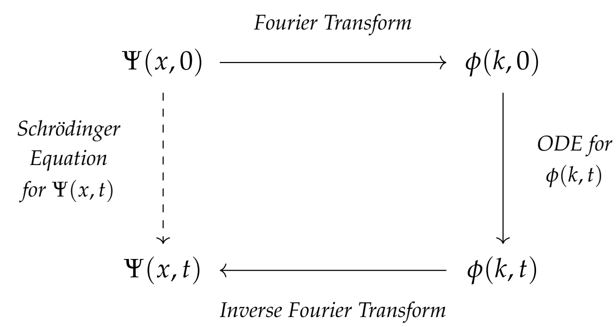

Figure 10.3: The scheme for solving the Schrödinger equation using Fourier transforms. The goal is to solve for $\Psi(x,t)$ given $\Psi(x,0)$. Instead of a direct solution in coordinate space (on the left side), one can first transform the initial condition obtaining $\phi(k,0)$ in wave number space. The governing equation in the new space is found by transforming the PDE to get an ODE. This simpler equation is solved to obtain $\phi(k,t)$. Then an inverse transform yields the solution of the original equation.

This is similar to the solution of the system of ordinary differential equations in Chapter 3, $\dot{\mathbf{x}} = A\mathbf{x}$. In that case we diagonalized the system using the transformation $\mathbf{x} = S\mathbf{y}$. This lead to a simpler system $\dot{\mathbf{y}} = \Lambda\mathbf{y}$, where $\Lambda = S^{-1}AS$. Solving for $\mathbf{y}$, we inverted the solution to obtain $\mathbf{x}$. Similarly, one can apply this diagonalization to the solution of linear algebraic systems of equations. The general scheme is shown in Figure 10.4.

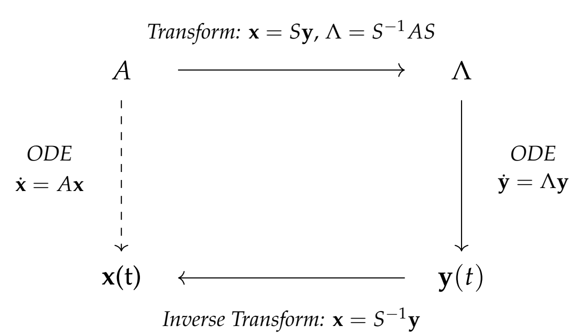

Figure 10.4: This shows the scheme for solving the linear system of ODEs $\dot{\mathbf{x}} = A\mathbf{x}$. One finds a transformation between $\mathbf{x}$ and $\mathbf{y}$ of the form $\mathbf{x} = S\mathbf{y}$ which diagonalizes the system. The resulting system is easier to solve for $\mathbf{y}$. Then, one uses the inverse transformation to obtain the solution to the original problem.

Similar transform constructions occur for many other type of problems. We will end this chapter with a study of Laplace transforms, which are useful in the study of initial value problems, particularly for linear ordinary differential equations with constant coefficients. A similar scheme for using Laplace transforms is depicted in Figure 10.30.

In this chapter we will begin with the study of Fourier transforms. These will provide an integral representation of functions defined on the real line. Such functions can also represent analog signals. Analog signals are continuous signals which can be represented as a sum over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series were used to represent in an earlier chapter. We will then investigate a related transform, the Laplace transform, which is useful in solving initial value problems such as those encountered in ordinary differential equations.

10.2 Complex Exponential Fourier Series

BEFORE DERIVING THE FOURIER TRANSFORM, we will need to rewrite the trigonometric Fourier series representation as a complex exponential Fourier series. We first recall from Chapter 2 the trigonometric Fourier series representation of a function defined on $[-\pi, \pi]$ with period 2π. The Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \tag{10.13}$$

where the Fourier coefficients were found as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 0, 1, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots. \end{aligned} \tag{10.14}$$

In order to derive the exponential Fourier series, we replace the trigonometric functions with exponential functions and collect like exponential terms. This gives

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{inx}+e^{-inx}}{2}\right) + b_n\left(\frac{e^{inx}-e^{-inx}}{2i}\right)\right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{inx} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-inx}. \end{aligned} \tag{10.15}$$

The coefficients of the complex exponentials can be rewritten by defining

$$c_n = \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots. \tag{10.16}$$

This implies that

$$\bar{c}_n = \frac{1}{2}(a_n - ib_n), \quad n = 1, 2, \ldots. \tag{10.17}$$

So far the representation is rewritten as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \bar{c}_n e^{inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Re-indexing the first sum, by introducing $k = -n$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{k=-1}^{-\infty} \bar{c}_{-k} e^{-ikx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

Since k is a dummy index, we replace it with a new n as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=-1}^{-\infty} \bar{c}_{-n} e^{-inx} + \sum_{n=1}^{\infty} c_n e^{-inx}.$$

We can now combine all of the terms into a simple sum. We first define c_n for negative n's by

$$c_n = \bar{c}_{-n}, \quad n = -1, -2, \ldots.$$

Letting $c_0 = \frac{a_0}{2}$, we can write the complex exponential Fourier series representation as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \tag{10.18}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n), \quad n = 1, 2, \ldots \\ c_n &= \frac{1}{2}(a_{-n} - ib_{-n}), \quad n = -1, -2, \ldots \\ c_0 &= \frac{a_0}{2}. \end{aligned} \tag{10.19}$$

Given such a representation, we would like to write out the integral forms of the coefficients, c_n. So, we replace the a_n's and b_n's with their integral representations and replace the trigonometric functions with complex exponential functions. Doing this, we have for $n = 1, 2, \ldots$.

$$\begin{aligned} c_n &= \frac{1}{2}(a_n + ib_n) \\ &= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx}+e^{-inx}}{2}\right)dx + \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{inx}-e^{-inx}}{2i}\right)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx. \end{aligned} \tag{10.20}$$

It is a simple matter to determine the c_n's for other values of n. For $n = 0$, we have that

$$c_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

For $n = -1, -2, \ldots$, we find that

$$c_n = \bar{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{-inx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

Therefore, we have obtained the complex exponential Fourier series coefficients for all n. Now we can define the complex exponential Fourier series for the function $f(x)$ defined on $[-\pi, \pi]$ as shown below.

Complex Exponential Series for $f(x)$ defined on $[-\pi, \pi]$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-inx}, \tag{10.21}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx}\, dx. \tag{10.22}$$

We can easily extend the above analysis to other intervals. For example, for $x \in [-L, L]$ the Fourier trigonometric series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

with Fourier coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \quad n = 0, 1, \ldots,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots.$$

This can be rewritten as an exponential Fourier series of the form

Complex Exponential Series for $f(x)$ defined on $[-L, L]$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \tag{10.23}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L}\, dx. \tag{10.24}$$

We can now use this complex exponential Fourier series for function defined on $[-L, L]$ to derive the Fourier transform by letting L get large. This will lead to a sum over a continuous set of frequencies, as opposed to the sum over discrete frequencies, which Fourier series represent.

10.3 Exponential Fourier Transform

Both the trigonometric and complex exponential Fourier series provide us with representations of a class of functions of finite period in terms of sums over a discrete set of frequencies. In particular, for functions defined on $x \in [-L, L]$, the period of the Fourier series representation is $2L$. We can write the arguments in the exponentials, $e^{-in\pi x/L}$, in terms of the angular frequency, $\omega_n = n\pi/L$, as $e^{-i\omega_n x}$. We note that the frequencies, ν_n, are then defined through $\omega_n = 2\pi\nu_n = \frac{n\pi}{L}$. Therefore, the complex exponential series is seen to be a sum over a discrete, or countable, set of frequencies.

We would now like to extend the finite interval to an infinite interval, $x \in (-\infty, \infty)$, and to extend the discrete set of (angular) frequencies to a continuous range of frequencies, $\omega \in (-\infty, \infty)$. One can do this rigorously. It amounts to letting L and n get large and keeping $\frac{n}{L}$ fixed.

We first define $\Delta\omega = \frac{\pi}{L}$, so that $\omega_n = n\Delta\omega$. Inserting the Fourier coefficients (10.24) into Equation (10.23), we have

$$\begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(\xi) e^{in\pi\xi/L}\, d\xi \right) e^{-in\pi x/L} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(\xi) e^{i\omega_n \xi}\, d\xi \right) e^{-i\omega_n x}. \end{aligned} \tag{10.25}$$

Now, we let L get large, so that $\Delta\omega$ becomes small and ω_n approaches the angular frequency ω. Then,

$$\begin{aligned} f(x) &\sim \lim_{\Delta\omega \to 0, L \to \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-L}^{L} f(\xi) e^{i\omega_n \xi}\, d\xi \right) e^{-i\omega_n x} \Delta\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi) e^{i\omega\xi}\, d\xi \right) e^{-i\omega x}\, d\omega. \end{aligned} \tag{10.26}$$

Definitions of the Fourier transform and the inverse Fourier transform.

Looking at this last result, we formally arrive at the definition of the Fourier transform. It is embodied in the inner integral and can be written as

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx. \tag{10.27}$$

This is a generalization of the Fourier coefficients (10.24).

Once we know the Fourier transform, $\hat{f}(\omega)$, then we can reconstruct the original function, $f(x)$, using the inverse Fourier transform, which is given by the outer integration,

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x}\, d\omega. \tag{10.28}$$

We note that it can be proven that the Fourier transform exists when $f(x)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Such functions are said to be L_1.

We combine these results below, defining the Fourier and inverse Fourier transforms and indicating that they are inverse operations of each other. We will then prove the first of the equations, (10.31). [The second equation, (10.32), follows in a similar way.]

The **Fourier transform** and **inverse Fourier transform** are inverse operations. Defining the Fourier transform as

$$F[f] = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx. \tag{10.29}$$

and the inverse Fourier transform as

$$F^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega x}\,d\omega. \tag{10.30}$$

then

$$F^{-1}[F[f]] = f(x) \tag{10.31}$$

and

$$F[F^{-1}[\hat{f}]] = \hat{f}(\omega). \tag{10.32}$$

Proof. The proof is carried out by inserting the definition of the Fourier transform, (10.29), into the inverse transform definition, (10.30), and then interchanging the orders of integration. Thus, we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F[f]e^{-i\omega x}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)e^{i\omega\xi}\,d\xi\right]e^{-i\omega x}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{i\omega(\xi-x)}\,d\xi d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\,d\omega\right]f(\xi)\,d\xi. \end{aligned} \tag{10.33}$$

In order to complete the proof, we need to evaluate the inside integral, which does not depend upon $f(x)$. This is an improper integral, so we first define

$$D_\Omega(x) = \int_{-\Omega}^{\Omega} e^{i\omega x}\,d\omega$$

and compute the inner integral as

$$\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\,d\omega = \lim_{\Omega\to\infty} D_\Omega(\xi - x).$$

We can compute $D_\Omega(x)$. A simple evaluation yields

$$\begin{aligned} D_\Omega(x) &= \int_{-\Omega}^{\Omega} e^{i\omega x}\,d\omega \\ &= \left.\frac{e^{i\omega x}}{ix}\right|_{-\Omega}^{\Omega} \\ &= \frac{e^{ix\Omega} - e^{-ix\Omega}}{2ix} \\ &= \frac{2\sin x\Omega}{x}. \end{aligned} \tag{10.34}$$

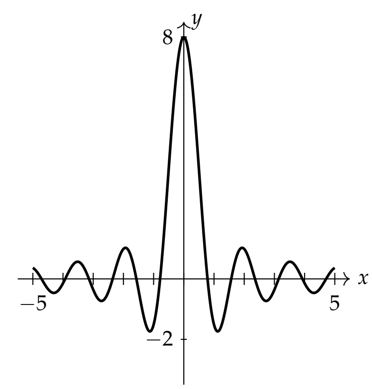

Figure 10.5: A plot of the function $D_\Omega(x)$ for $\Omega = 4$.

A plot of this function is in Figure 10.5 for $\Omega = 4$. For large Ω the peak grows and the values of $D_\Omega(x)$ for $x \neq 0$ tend to zero as shown in Figure

10.6. In fact, as x approaches 0, $D_\Omega(x)$ approaches 2Ω. For $x \neq 0$, the $D_\Omega(x)$ function tends to zero.

We further note that

$$\lim_{\Omega\to\infty} D_\Omega(x) = 0, \quad x \neq 0,$$

and $\lim_{\Omega\to\infty} D_\Omega(x)$ is infinite at $x = 0$. However, the area is constant for each Ω. In fact,

$$\int_{-\infty}^{\infty} D_\Omega(x)\,dx = 2\pi.$$

We can show this by recalling the computation in Example 9.42,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.$$

Then,

$$\begin{aligned}\int_{-\infty}^{\infty} D_\Omega(x)\,dx &= \int_{-\infty}^{\infty} \frac{2\sin x\Omega}{x}\,dx \\ &= \int_{-\infty}^{\infty} 2\frac{\sin y}{y}\,dy \\ &= 2\pi. \qquad (10.35)\end{aligned}$$

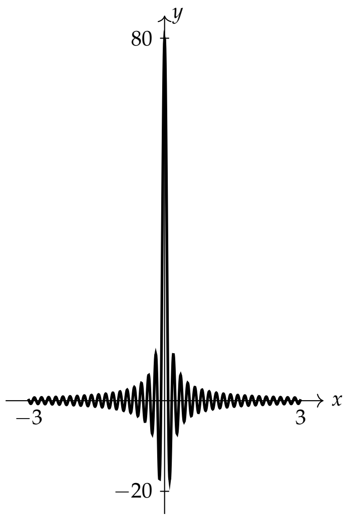

Figure 10.6: A plot of the function $D_\Omega(x)$ for $\Omega = 40$.

Another way to look at $D_\Omega(x)$ is to consider the sequence of functions $f_n(x) = \frac{\sin nx}{\pi x}$, $n = 1, 2, \ldots$. Then we have shown that this sequence of functions satisfies the two properties,

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \neq 0,$$

$$\int_{-\infty}^{\infty} f_n(x)\,dx = 1.$$

This is a key representation of such generalized functions. The limiting value vanishes at all but one point, but the area is finite.

Such behavior can be seen for the limit of other sequences of functions. For example, consider the sequence of functions

$$f_n(x) = \begin{cases} 0, & |x| > \frac{1}{n}, \\ \frac{n}{2}, & |x| <\leq frac1n. \end{cases}$$

This is a sequence of functions as shown in Figure 10.7. As $n \to \infty$, we find the limit is zero for $x \neq 0$ and is infinite for $x = 0$. However, the area under each member of the sequences is one. Thus, the limiting function is zero at most points but has area one.

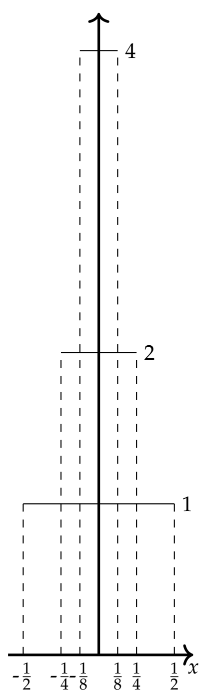

Figure 10.7: A plot of the functions $f_n(x)$ for $n = 2, 4, 8$.

The limit is not really a function. It is a generalized function. It is called the Dirac delta function, which is defined by

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

Before returning to the proof that the inverse Fourier transform of the Fourier transform is the identity, we state one more property of the Dirac delta function, which we will prove in the next section. Namely, we will show that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

Returning to the proof, we now have that

$$\int_{-\infty}^{\infty} e^{i\omega(\xi - x)}\, d\omega = \lim_{\Omega\to\infty} D_\Omega(\xi - x) = 2\pi\delta(\xi - x).$$

Inserting this into (10.33), we have

$$\begin{aligned} F^{-1}[F[f]] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{i\omega(\xi-x)}\, d\omega\right] f(\xi)\, d\xi. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\xi - x) f(\xi)\, d\xi. \\ &= f(x). \end{aligned} \tag{10.36}$$

Thus, we have proven that the inverse transform of the Fourier transform of f is f. □

10.4 The Dirac Delta Function

In the last section we introduced the Dirac delta function, $\delta(x)$. As noted above, this is one example of what is known as a generalized function, or a distribution. Dirac had introduced this function in the 1930's in his study of quantum mechanics as a useful tool. It was later studied in a general theory of distributions and found to be more than a simple tool used by physicists. The Dirac delta function, as any distribution, only makes sense under an integral.

P. A. M. Dirac (1902-1984) introduced the δ function in his book, *The Principles of Quantum Mechanics*, 4th Ed., Oxford University Press, 1958, originally published in 1930, as part of his orthogonality statement for a basis of functions in a Hilbert space, $< \xi'|\xi'' >= c\delta(\xi' - \xi'')$ in the same way we introduced discrete orthogonality using the Kronecker delta.

Two properties were used in the last section. First one has that the area under the delta function is one,

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Integration over more general intervals gives

$$\int_a^b \delta(x)\, dx = \begin{cases} 1, & 0 \in [a,b], \\ 0, & 0 \notin [a,b]. \end{cases} \tag{10.37}$$

The other property that was used was the sifting property:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

This can be seen by noting that the delta function is zero everywhere except at $x = a$. Therefore, the integrand is zero everywhere and the only contribution from $f(x)$ will be from $x = a$. So, we can replace $f(x)$ with $f(a)$ under the integral. Since $f(a)$ is a constant, we have that

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx &= \int_{-\infty}^{\infty} \delta(x-a) f(a)\, dx \\ &= f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a). \qquad (10.38)\end{aligned}$$

Properties of the Dirac δ-function:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a).$$

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \frac{\delta(x-x_j)}{|f'(x_j)|}\, dx.$$

(For n simple roots.)

These and other properties are often written outside the integral:

$$\delta(ax) = \frac{1}{|a|}\delta(x).$$

$$\delta(-x) = \delta(x).$$

$$\delta((x-a)(x-b)) = \frac{[\delta(x-a)+\delta(x-a)]}{|a-b|}.$$

$$\delta(f(x)) = \sum_j \frac{\delta(x-x_j)}{|f'(x_j)|},$$

for $f(x_j) = 0, f'(x_j) \neq 0$.

Another property results from using a scaled argument, ax. In this case we show that

$$\delta(ax) = |a|^{-1}\delta(x). \qquad (10.39)$$

As usual, this only has meaning under an integral sign. So, we place $\delta(ax)$ inside an integral and make a substitution $y = ax$:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(ax)\, dx &= \lim_{L\to\infty} \int_{-L}^{L} \delta(ax)\, dx \\ &= \lim_{L\to\infty} \frac{1}{a} \int_{-aL}^{aL} \delta(y)\, dy. \qquad (10.40)\end{aligned}$$

If $a > 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

However, if $a < 0$ then

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{a} \int_{\infty}^{-\infty} \delta(y)\, dy = -\frac{1}{a} \int_{-\infty}^{\infty} \delta(y)\, dy.$$

The overall difference in a multiplicative minus sign can be absorbed into one expression by changing the factor $1/a$ to $1/|a|$. Thus,

$$\int_{-\infty}^{\infty} \delta(ax)\, dx = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(y)\, dy. \qquad (10.41)$$

Example 10.1. Evaluate $\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx$. This is a straight forward integration:

$$\int_{-\infty}^{\infty} (5x+1)\delta(4(x-2))\, dx = \frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\, dx = \frac{11}{4}.$$

The first strep is to write $\delta(4(x-2)) = \frac{1}{4}\delta(x-2)$. Then, the final evaluation is given by

$$\frac{1}{4} \int_{-\infty}^{\infty} (5x+1)\delta(x-2)\, dx = \frac{1}{4}(5(2)+1) = \frac{11}{4}.$$

Even more general than $\delta(ax)$ is the delta function $\delta(f(x))$. The integral of $\delta(f(x))$ can be evaluated depending upon the number of zeros of $f(x)$. If there is only one zero, $f(x_1) = 0$, then one has that

$$\int_{-\infty}^{\infty} \delta(f(x))\, dx = \int_{-\infty}^{\infty} \frac{1}{|f'(x_1)|} \delta(x-x_1)\, dx.$$

This can be proven using the substitution $y = f(x)$ and is left as an exercise for the reader. This result is often written as

$$\delta(f(x)) = \frac{1}{|f'(x_1)|} \delta(x-x_1),$$

again keeping in mind that this only has meaning when placed under an integral.

Example 10.2. Evaluate $\int_{-\infty}^{\infty} \delta(3x-2)x^2\,dx$.

This is not a simple $\delta(x-a)$. So, we need to find the zeros of $f(x) = 3x-2$. There is only one, $x = \frac{2}{3}$. Also, $|f'(x)| = 3$. Therefore, we have

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\,dx = \int_{-\infty}^{\infty} \frac{1}{3}\delta(x-\frac{2}{3})x^2\,dx = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}.$$

Note that this integral can be evaluated the long way by using the substitution $y = 3x-2$. Then, $dy = 3\,dx$ and $x = (y+2)/3$. This gives

$$\int_{-\infty}^{\infty} \delta(3x-2)x^2\,dx = \frac{1}{3}\int_{-\infty}^{\infty} \delta(y)\left(\frac{y+2}{3}\right)^2\,dy = \frac{1}{3}\left(\frac{4}{9}\right) = \frac{4}{27}.$$

More generally, one can show that when $f(x_j) = 0$ and $f'(x_j) \neq 0$ for $j = 1, 2, \ldots, n$, (i.e.; when one has n simple zeros), then

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{1}{|f'(x_j)|}\delta(x-x_j).$$

Example 10.3. Evaluate $\int_0^{2\pi} \cos x\,\delta(x^2-\pi^2)\,dx$.

In this case the argument of the delta function has two simple roots. Namely, $f(x) = x^2-\pi^2 = 0$ when $x = \pm\pi$. Furthermore, $f'(x) = 2x$. Therefore, $|f'(\pm\pi)| = 2\pi$. This gives

$$\delta(x^2-\pi^2) = \frac{1}{2\pi}[\delta(x-\pi)+\delta(x+\pi)].$$

Inserting this expression into the integral and noting that $x = -\pi$ is not in the integration interval, we have

$$\begin{aligned}\int_0^{2\pi} \cos x\,\delta(x^2-\pi^2)\,dx &= \frac{1}{2\pi}\int_0^{2\pi} \cos x\,[\delta(x-\pi)+\delta(x+\pi)]\,dx \\ &= \frac{1}{2\pi}\cos\pi = -\frac{1}{2\pi}. \end{aligned} \tag{10.42}$$

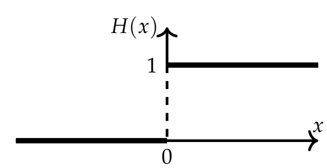

Figure 10.8: The Heaviside step function, $H(x)$.

Example 10.4. Show $H'(x) = \delta(x)$, where the Heaviside function (or, step function) is defined as

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

and is shown in Figure 10.8.

Looking at the plot, it is easy to see that $H'(x) = 0$ for $x \neq 0$. In order to check that this gives the delta function, we need to compute the area integral. Therefore, we have

$$\int_{-\infty}^{\infty} H'(x)\,dx = H(x)\Big|_{-\infty}^{\infty} = 1-0 = 1.$$

Thus, $H'(x)$ satisfies the two properties of the Dirac delta function.

10.5 Properties of the Fourier Transform

We now return to the Fourier transform. Before actually computing the Fourier transform of some functions, we prove a few of the properties of the Fourier transform.

First we note that there are several forms that one may encounter for the Fourier transform. In applications functions can either be functions of time, $f(t)$, or space, $f(x)$. The corresponding Fourier transforms are then written as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt, \tag{10.43}$$

or

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx. \tag{10.44}$$

ω is called the angular frequency and is related to the frequency ν by $\omega = 2\pi\nu$. The units of frequency are typically given in Hertz (Hz). Sometimes the frequency is denoted by f when there is no confusion. k is called the wavenumber. It has units of inverse length and is related to the wavelength, λ, by $k = \frac{2\pi}{\lambda}$.

We explore a few basic properties of the Fourier transform and use them in examples in the next section.

1. **Linearity:** For any functions $f(x)$ and $g(x)$ for which the Fourier transform exists and constant a, we have

   $$F[f+g] = F[f] + F[g]$$

   and

   $$F[af] = aF[f].$$

   These simply follow from the properties of integration and establish the linearity of the Fourier transform.

2. **Transform of a Derivative:** $F\left[\frac{df}{dx}\right] = -ik\hat{f}(k)$

   Here we compute the Fourier transform (10.29) of the derivative by inserting the derivative in the Fourier integral and using integration by parts.

   $$\begin{aligned} F\left[\frac{df}{dx}\right] &= \int_{-\infty}^{\infty} \frac{df}{dx} e^{ikx}\, dx \\ &= \lim_{L\to\infty} \left[f(x)e^{ikx}\right]_{-L}^{L} - ik \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx. \end{aligned} \tag{10.45}$$

   The limit will vanish if we assume that $\lim_{x\to\pm\infty} f(x) = 0$. The last integral is recognized as the Fourier transform of f, proving the given property.

3. **Higher Order Derivatives:** $F\left[\dfrac{d^n f}{dx^n}\right] = (-ik)^n \hat{f}(k)$

   The proof of this property follows from the last result, or doing several integration by parts. We will consider the case when $n = 2$. Noting that the second derivative is the derivative of $f'(x)$ and applying the last result, we have

$$\begin{aligned} F\left[\frac{d^2 f}{dx^2}\right] &= F\left[\frac{d}{dx} f'\right] \\ &= -ikF\left[\frac{df}{dx}\right] = (-ik)^2 \hat{f}(k). \end{aligned} \tag{10.46}$$

   This result will be true if

$$\lim_{x\to\pm\infty} f(x) = 0 \text{ and } \lim_{x\to\pm\infty} f'(x) = 0.$$

   The generalization to the transform of the nth derivative easily follows.

4. **Multiplication by x:** $F[xf(x)] = -i\frac{d}{dk}\hat{f}(k)$

   This property can be shown by using the fact that $\frac{d}{dk}e^{ikx} = ixe^{ikx}$ and the ability to differentiate an integral with respect to a parameter.

$$\begin{aligned} F[xf(x)] &= \int_{-\infty}^{\infty} xf(x)e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{d}{dk}\left(\frac{1}{i}e^{ikx}\right)dx \\ &= -i\frac{d}{dk}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= -i\frac{d}{dk}\hat{f}(k). \end{aligned} \tag{10.47}$$

   This result can be generalized to $F[x^n f(x)]$ as an exercise.

5. **Shifting Properties:** For constant a, we have the following shifting properties:

$$f(x-a) \leftrightarrow e^{ika}\hat{f}(k), \tag{10.48}$$

$$f(x)e^{-iax} \leftrightarrow \hat{f}(k-a). \tag{10.49}$$

   Here we have denoted the Fourier transform pairs using a double arrow as $f(x) \leftrightarrow \hat{f}(k)$. These are easily proven by inserting the desired forms into the definition of the Fourier transform (10.29), or inverse Fourier transform (10.30). The first shift property (10.48) is shown by the following argument. We evaluate the Fourier transform.

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{ikx}\,dx.$$

   Now perform the substitution $y = x - a$. Then,

$$\begin{aligned} F[f(x-a)] &= \int_{-\infty}^{\infty} f(y)e^{ik(y+a)}\,dy \\ &= e^{ika}\int_{-\infty}^{\infty} f(y)e^{iky}\,dy \\ &= e^{ika}\hat{f}(k). \end{aligned} \tag{10.50}$$

The second shift property (10.49) follows in a similar way.

6. **Convolution of Functions:** We define the convolution of two functions $f(x)$ and $g(x)$ as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dx. \tag{10.51}$$

Then, the Fourier transform of the convolution is the product of the Fourier transforms of the individual functions:

$$F[f * g] = \hat{f}(k)\hat{g}(k). \tag{10.52}$$

We will return to the proof of this property in Section 10.6.

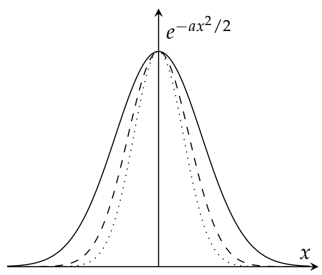

Figure 10.9: Plots of the Gaussian function $f(x) = e^{-ax^2/2}$ for $a = 1, 2, 3$.

10.5.1 Fourier Transform Examples

In this section we will compute the Fourier transforms of several functions.

Example 10.5. Find the Fourier transform of a Gaussian, $f(x) = e^{-ax^2/2}$.

This function, shown in Figure 10.9 is called the Gaussian function. It has many applications in areas such as quantum mechanics, molecular theory, probability and heat diffusion. We will compute the Fourier transform of this function and show that the Fourier transform of a Gaussian is a Gaussian. In the derivation we will introduce classic techniques for computing such integrals.

We begin by applying the definition of the Fourier transform,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx. \tag{10.53}$$

The first step in computing this integral is to complete the square in the argument of the exponential. Our goal is to rewrite this integral so that a simple substitution will lead to a classic integral of the form $\int_{-\infty}^{\infty} e^{\beta y^2}\,dy$, which we can integrate. The completion of the square follows as usual:

$$\begin{aligned}
-\frac{a}{2}x^2 + ikx &= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x\right] \\
&= -\frac{a}{2}\left[x^2 - \frac{2ik}{a}x + \left(-\frac{ik}{a}\right)^2 - \left(-\frac{ik}{a}\right)^2\right] \\
&= -\frac{a}{2}\left(x - \frac{ik}{a}\right)^2 - \frac{k^2}{2a}.
\end{aligned} \tag{10.54}$$

We now put this expression into the integral and make the substitutions $y = x - \frac{ik}{a}$ and $\beta = \frac{a}{2}$.

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ax^2/2+ikx}\,dx$$

$$= e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x-\frac{ik}{a}\right)^2} dx$$

$$= e^{-\frac{k^2}{2a}} \int_{-\infty-\frac{ik}{a}}^{\infty-\frac{ik}{a}} e^{-\beta y^2} dy. \quad (10.55)$$

One would be tempted to absorb the $-\frac{ik}{a}$ terms in the limits of integration. However, we know from our previous study that the integration takes place over a contour in the complex plane as shown in Figure 10.10.

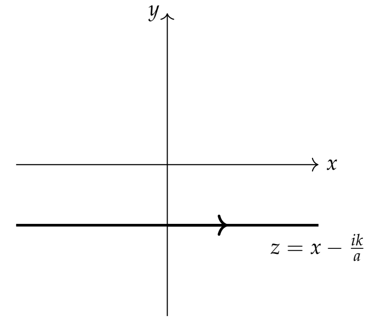

Figure 10.10: Simple horizontal contour.

In this case we can deform this horizontal contour to a contour along the real axis since we will not cross any singularities of the integrand. So, we now safely write

$$\hat{f}(k) = e^{-\frac{k^2}{2a}} \int_{-\infty}^{\infty} e^{-\beta y^2} dy.$$

The resulting integral is a classic integral and can be performed using a standard trick. Define I by[4]

$$I = \int_{-\infty}^{\infty} e^{-\beta y^2} dy.$$

Then,

$$I^2 = \int_{-\infty}^{\infty} e^{-\beta y^2} dy \int_{-\infty}^{\infty} e^{-\beta x^2} dx.$$

Note that we needed to change the integration variable so that we can write this product as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta(x^2+y^2)} dxdy.$$

This is an integral over the entire xy-plane. We now transform to polar coordinates to obtain

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-\beta r^2} rdrd\theta \\ &= 2\pi \int_0^{\infty} e^{-\beta r^2} rdr \\ &= -\frac{\pi}{\beta}\left[e^{-\beta r^2}\right]_0^{\infty} = \frac{\pi}{\beta}. \quad (10.56) \end{aligned}$$

The final result is gotten by taking the square root, yielding

$$I = \sqrt{\frac{\pi}{\beta}}.$$

We can now insert this result to give the Fourier transform of the Gaussian function:

$$\hat{f}(k) = \sqrt{\frac{2\pi}{a}} e^{-k^2/2a}. \quad (10.57)$$

Therefore, we have shown that the Fourier transform of a Gaussian is a Gaussian.

The Fourier transform of a Gaussian is a Gaussian.

[4] Here we show

$$\int_{-\infty}^{\infty} e^{-\beta y^2} dy = \sqrt{\frac{\pi}{\beta}}.$$

Note that we solved the $\beta = 1$ case in Example 5.11, so a simple variable transformation $z = \sqrt{\beta} y$ is all that is needed to get the answer. However, it cannot hurt to see this classic derivation again.

Example 10.6. Find the Fourier transform of the Box, or Gate, Function,

$$f(x) = \begin{cases} b, & |x| \leq a \\ 0, & |x| > a \end{cases}.$$

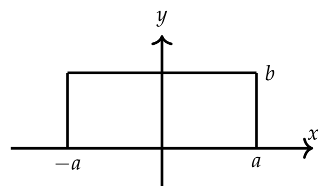

Figure 10.11: A plot of the box function in Example 10.6.

This function is called the box function, or gate function. It is shown in Figure 10.11. The Fourier transform of the box function is relatively easy to compute. It is given by

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{ikx}\,dx \\ &= \int_{-a}^{a} b e^{ikx}\,dx \\ &= \left.\frac{b}{ik} e^{ikx}\right|_{-a}^{a} \\ &= \frac{2b}{k}\sin ka. \end{aligned} \tag{10.58}$$

We can rewrite this as

$$\hat{f}(k) = 2ab\frac{\sin ka}{ka} \equiv 2ab \operatorname{sinc} ka.$$

Here we introduced the sinc function,

$$\operatorname{sinc} x = \frac{\sin x}{x}.$$

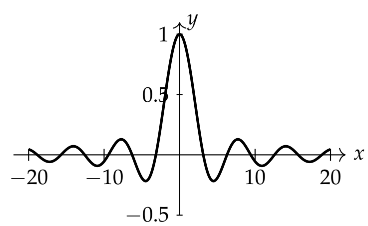

Figure 10.12: A plot of the Fourier transform of the box function in Example 10.6. This is the general shape of the sinc function.

A plot of this function is shown in Figure 10.12. This function appears often in signal analysis and it plays a role in the study of diffraction.

We will now consider special limiting values for the box function and its transform. This will lead us to the Uncertainty Principle for signals, connecting the relationship between the localization properties of a signal and its transform.

1. $a \to \infty$ and b fixed.

   In this case, as a gets large the box function approaches the constant function $f(x) = b$. At the same time, we see that the Fourier transform approaches a Dirac delta function. We had seen this function earlier when we first defined the Dirac delta function. Compare Figure 10.12 with Figure 10.5. In fact, $\hat{f}(k) = bD_a(k)$. [Recall the definition of $D_\Omega(x)$ in Equation (10.34).] So, in the limit we obtain $\hat{f}(k) = 2\pi b\delta(k)$. This limit implies fact that the Fourier transform of $f(x) = 1$ is $\hat{f}(k) = 2\pi\delta(k)$. As the width of the box becomes wider, the Fourier transform becomes more localized. In fact, we have arrived at the important result that

$$\int_{-\infty}^{\infty} e^{ikx} = 2\pi\delta(k). \tag{10.59}$$

2. $b \to \infty$, $a \to 0$, and $2ab = 1$.

   In this case the box narrows and becomes steeper while maintaining a constant area of one. This is the way we had found a representation of the Dirac delta function previously. The Fourier transform approaches a constant in this limit. As a approaches zero, the sinc function approaches one, leaving $\hat{f}(k) \to 2ab = 1$. Thus, the Fourier transform of the Dirac delta function is one. Namely, we have

$$\int_{-\infty}^{\infty} \delta(x) e^{ikx} = 1. \tag{10.60}$$

   In this case we have that the more localized the function $f(x)$ is, the more spread out the Fourier transform, $\hat{f}(k)$, is. We will summarize these notions in the next item by relating the widths of the function and its Fourier transform.

3. *The Uncertainty Principle,* $\Delta x \Delta k = 4\pi$.

   The widths of the box function and its Fourier transform are related as we have seen in the last two limiting cases. It is natural to define the width, Δx of the box function as

$$\Delta x = 2a.$$

   The width of the Fourier transform is a little trickier. This function actually extends along the entire k-axis. However, as $\hat{f}(k)$ became more localized, the central peak in Figure 10.12 became narrower. So, we define the width of this function, Δk as the distance between the first zeros on either side of the main lobe as shown in Figure 10.13. This gives

$$\Delta k = \frac{2\pi}{a}.$$

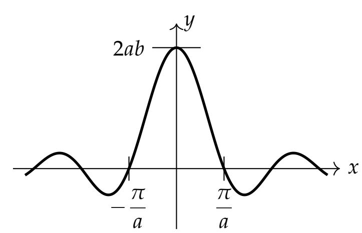

Figure 10.13: The width of the function $2ab\frac{\sin ka}{ka}$ is defined as the distance between the smallest magnitude zeros.

   Combining these two relations, we find that

$$\Delta x \Delta k = 4\pi.$$

   Thus, the more localized a signal, the less localized its transform and vice versa. This notion is referred to as the Uncertainty Principle. For general signals, one needs to define the effective widths more carefully, but the main idea holds:

$$\Delta x \Delta k \geq c > 0.$$

More formally, the uncertainty principle for signals is about the relation between duration and bandwidth, which are defined by $\Delta t = \frac{\|tf\|_2}{\|f\|_2}$ and $\Delta \omega = \frac{\|\omega \hat{f}\|_2}{\|\hat{f}\|_2}$, respectively, where $\|f\|_2 = \int_{-\infty}^{\infty} |f(t)|^2\,dt$ and $\|\hat{f}\|_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$. Under appropriate conditions, one can prove that $\Delta t \Delta \omega \geq \frac{1}{2}$. Equality holds for Gaussian signals. Werner Heisenberg (1901-1976) introduced the uncertainty principle into quantum physics in 1926, relating uncertainties in the position (Δx) and momentum (Δp_x) of particles. In this case, $\Delta x \Delta p_x \geq \frac{1}{2}\hbar$. Here, the uncertainties are defined as the positive square roots of the quantum mechanical variances of the position and momentum.

We now turn to other examples of Fourier transforms.

Example 10.7. Find the Fourier transform of $f(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $a > 0$.

The Fourier transform of this function is

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx \\ &= \int_{0}^{\infty} e^{ikx-ax}\,dx \\ &= \frac{1}{a-ik}. \end{aligned} \tag{10.61}$$

Next, we will compute the inverse Fourier transform of this result and recover the original function.

Example 10.8. Find the inverse Fourier transform of $\hat{f}(k) = \frac{1}{a-ik}$.

The inverse Fourier transform of this function is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{-ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\,dk.$$

This integral can be evaluated using contour integral methods. We evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-ixz}}{a-iz}\,dz,$$

using Jordan's Lemma from Section 9.5.8. According to Jordan's Lemma, we need to enclose the contour with a semicircle in the upper half plane for $x < 0$ and in the lower half plane for $x > 0$ as shown in Figure 10.14.

The integrations along the semicircles will vanish and we will have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{a-ik}\,dk \\ &= \pm\frac{1}{2\pi}\oint_C \frac{e^{-ixz}}{a-iz}\,dz \\ &= \begin{cases} 0, & x<0 \\ -\frac{1}{2\pi}2\pi i \text{ Res } [z=-ia], & x>0 \end{cases} \\ &= \begin{cases} 0, & x<0 \\ e^{-ax}, & x>0 \end{cases}. \end{aligned} \tag{10.62}$$

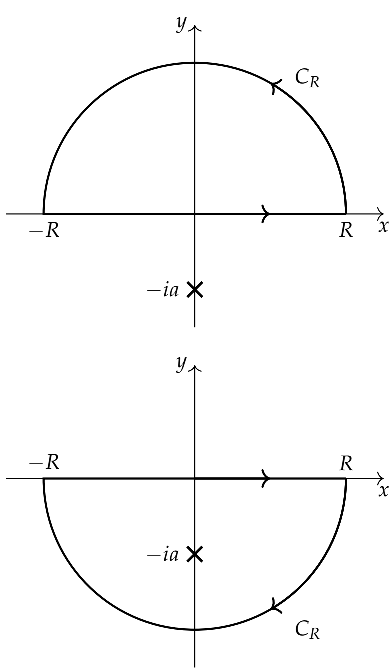

Figure 10.14: Contours for inverting $\hat{f}(k) = \frac{1}{a-ik}$.

Note that without paying careful attention to Jordan's Lemma one might not retrieve the function from the last example.

Example 10.9. Find the inverse Fourier transform of $\hat{f}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

We would like to find the inverse Fourier transform of this function. Instead of carrying out any integration, we will make use of the properties of Fourier transforms. Since the transforms of sums are the sums of transforms, we can look at each term individually. Consider $\delta(\omega - \omega_0)$. This is a shifted function. From the shift theorems in Equations (10.48)-(10.49) we have the Fourier transform pair

$$e^{i\omega_0 t}f(t) \leftrightarrow \hat{f}(\omega - \omega_0).$$

Recalling from Example 10.6 that

$$\int_{-\infty}^{\infty} e^{i\omega t}\, dt = 2\pi\delta(\omega),$$

we have from the shift property that

$$F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} e^{-i\omega_0 t}.$$

The second term can be transformed similarly. Therefore, we have

$$F^{-1}[\pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0] = \frac{1}{2}e^{i\omega_0 t} + \frac{1}{2}e^{-i\omega_0 t} = \cos\omega_0 t.$$

Example 10.10. Find the Fourier transform of the finite wave train.

$$f(t) = \begin{cases} \cos\omega_0 t, & |t| \le a \\ 0, & |t| > a \end{cases}.$$

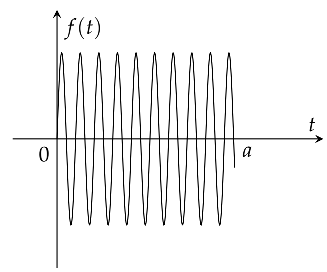

Figure 10.15: A plot of the finite wave train.

For the last example, we consider the finite wave train, which will reappear in the last chapter on signal analysis. In Figure 10.15 we show a plot of this function.

A straight forward computation gives

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt \\ &= \int_{-a}^{a} [\cos\omega_0 t + i\sin\omega_0 t]e^{i\omega t}\, dt \\ &= \int_{-a}^{a} \cos\omega_0 t \cos\omega t\, dt + i\int_{-a}^{a} \sin\omega_0 t \sin\omega t\, dt \\ &= \frac{1}{2}\int_{-a}^{a} [\cos((\omega+\omega_0)t) + \cos((\omega-\omega_0)t)]\; dt \\ &= \frac{\sin((\omega+\omega_0)a)}{\omega+\omega_0} + \frac{\sin((\omega-\omega_0)a)}{\omega-\omega_0}. \end{aligned} \tag{10.63}$$

10.6 The Convolution Operation

IN THE LIST OF PROPERTIES OF THE FOURIER TRANSFORM, we defined the convolution of two functions, $f(x)$ and $g(x)$ to be the integral

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\, dt. \tag{10.64}$$

In some sense one is looking at a sum of the overlaps of one of the functions and all of the shifted versions of the other function. The German word for convolution is faltung, which means "folding" and in old texts this is referred to as the Faltung Theorem. In this section we will look into the convolution operation and its Fourier transform.

Before we get too involved with the convolution operation, it should be noted that there are really two things you need to take away from this discussion. The rest is detail. First, the convolution of two functions is a new functions as defined by 10.64 when dealing wit the Fourier transform. The second and most relevant is that the Fourier transform of the convolution of two functions is the product of the transforms of each function. The rest is all about the use and consequences of these two statements. In this section we will show how the convolution works and how it is useful.

First, we note that the convolution is commutative: $f * g = g * f$. This is easily shown by replacing $x - t$ with a new variable, $y = x - t$ and $dy = -dt$.

$$\begin{aligned} (g * f)(x) &= \int_{-\infty}^{\infty} g(t) f(x-t)\, dt \\ &= -\int_{\infty}^{-\infty} g(x-y) f(y)\, dy \\ &= \int_{-\infty}^{\infty} f(y) g(x-y)\, dy \\ &= (f * g)(x). \end{aligned} \tag{10.65}$$

The convolution is commutative.

The best way to understand the folding of the functions in the convolution is to take two functions and convolve them. The next example gives a graphical rendition followed by a direct computation of the convolution. The reader is encouraged to carry out these analyses for other functions.

Example 10.11. Graphical Convolution of the box function and a triangle function.

In order to understand the convolution operation, we need to apply it to specific functions. We will first do this graphically for the box function

$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1 \end{cases}$$

and the triangular function

$$g(x) = \begin{cases} x, & 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases}$$

as shown in Figure 10.16.

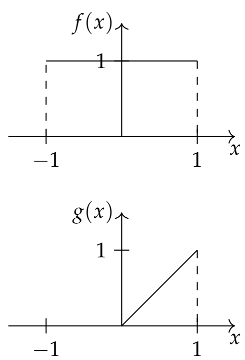

Figure 10.16: A plot of the box function $f(x)$ and the triangle function $g(x)$.

Next, we determine the contributions to the integrand. We consider the shifted and reflected function $g(t - x)$ in Equation 10.64 for various values of t. For $t = 0$, we have $g(x - 0) = g(-x)$. This function is a reflection of the triangle function, $g(x)$, as shown in Figure 10.17.

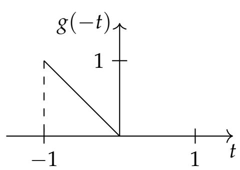

Figure 10.17: A plot of the reflected triangle function, $g(-t)$.

We then translate the triangle function performing horizontal shifts by t. In Figure 10.18 we show such a shifted and reflected $g(x)$ for $t = 2$, or $g(2 - x)$.

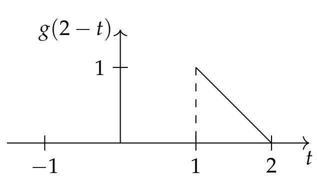

Figure 10.18: A plot of the reflected triangle function shifted by 2 units, $g(2 - t)$.

In Figure 10.18 we show several plots of other shifts, $g(x - t)$, superimposed on $f(x)$.

The integrand is the product of $f(t)$ and $g(x - t)$ and the integral of the product $f(t)g(x - t)$ is given by the sum of the shaded areas for each value of x.

In the first plot of Figure 10.19 the area is zero, as there is no overlap of the functions. Intermediate shift values are displayed in the other plots in Figure 10.19. The value of the convolution at x is shown by the area under the product of the two functions for each value of x.

Plots of the areas of the convolution of the box and triangle functions for several values of x are given in Figure 10.18. We see that the value of the convolution integral builds up and then quickly drops to zero as a function of x. In Figure 10.20 the values of these areas is shown as a function of x.

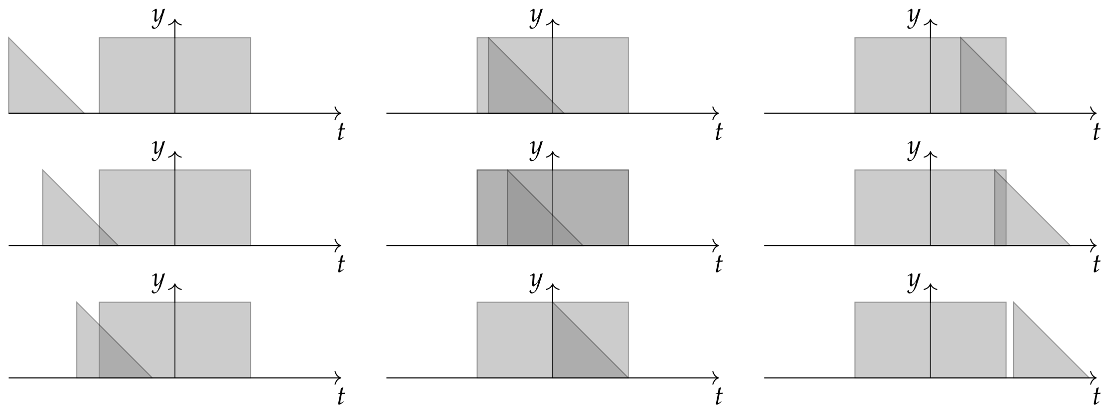

Figure 10.19: A plot of the box and triangle functions with the overlap indicated by the shaded area.

The plot of the convolution in Figure 10.20 is not easily determined using the graphical method. However, we can directly compute the convolution as shown in the next example.

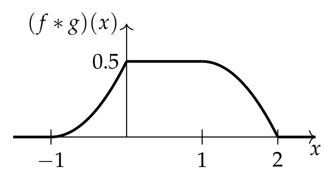

Figure 10.20: A plot of the convolution of the box and triangle functions.

Example 10.12. Analytically find the convolution of the box function and the triangle function.

The nonvanishing contributions to the convolution integral are when both $f(t)$ and $g(x-t)$ do not vanish. $f(t)$ is nonzero for $|t| \leq 1$, or $-1 \leq t \leq 1$. $g(x-t)$ is nonzero for $0 \leq x-t \leq 1$, or $x-1 \leq t \leq x$. These two regions are shown in Figure 10.21. On this region, $f(t)g(x-t) = x-t$.

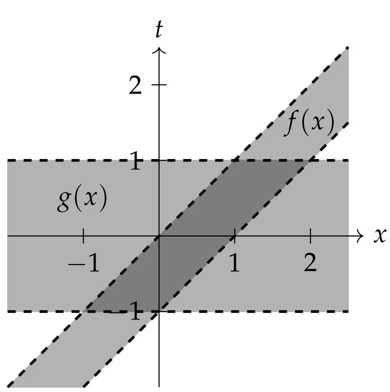

Figure 10.21: Intersection of the support of $g(x)$ and $f(x)$.

Isolating the intersection in Figure 10.22, we see in Figure 10.22 that there are three regions as shown by different shadings. These regions lead to a piecewise defined function with three different branches of nonzero values for $-1 < x < 0$, $0 < x < 1$, and $1 < x < 2$.

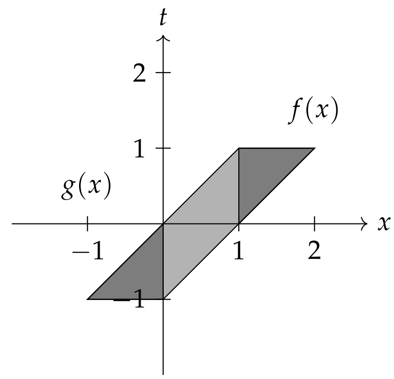

Figure 10.22: Intersection of the support of $g(x)$ and $f(x)$ showing the integration regions.

The values of the convolution can be determined through careful integration. The resulting integrals are given as

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(t) g(x-t)\, dt \\
&= \begin{cases} \int_{-1}^{x} (x-t)\, dt, & -1 < x < 0 \\ \int_{x-1}^{x} (x-t)\, dt, & 0 < x < 1 \\ \int_{x-1}^{1} (x-t)\, dt, & 1 < x < 2 \end{cases} \\
&= \begin{cases} \frac{1}{2}(x+1)^2, & -1 < x < 0 \\ \frac{1}{2}, & 0 < x < 1 \\ \frac{1}{2}\left[1-(x-1)^2\right] & 1 < x < 2 \end{cases}
\end{aligned} \tag{10.66}$$

A plot of this function is shown in Figure 10.20.

10.6.1 Convolution Theorem for Fourier Transforms

IN THIS SECTION WE COMPUTE the Fourier transform of the convolution integral and show that the Fourier transform of the convolution is the product of the transforms of each function,

$$F[f * g] = \hat{f}(k)\hat{g}(k). \tag{10.67}$$

First, we use the definitions of the Fourier transform and the convolution to write the transform as

$$\begin{aligned}
F[f * g] &= \int_{-\infty}^{\infty} (f * g)(x) e^{ikx}\, dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) g(x-t)\, dt \right) e^{ikx}\, dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t) e^{ikx}\, dx \right) f(t)\, dt.
\end{aligned} \tag{10.68}$$

We now substitute $y = x - t$ on the inside integral and separate the integrals:

$$\begin{aligned} F[f * g] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-t) e^{ikx}\, dx \right) f(t)\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{ik(y+t)}\, dy \right) f(t)\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(y) e^{iky}\, dy \right) f(t) e^{ikt}\, dt. \\ &= \left(\int_{-\infty}^{\infty} f(t) e^{ikt}\, dt \right) \left(\int_{-\infty}^{\infty} g(y) e^{iky}\, dy \right). \end{aligned} \qquad (10.69)$$

We see that the two integrals are just the Fourier transforms of f and g. Therefore, the Fourier transform of a convolution is the product of the Fourier transforms of the functions involved:

$$F[f * g] = \hat{f}(k)\hat{g}(k).$$

Example 10.13. Compute the convolution of the box function of height one and width two with itself.

Let $\hat{f}(k)$ be the Fourier transform of $f(x)$. Then, the Convolution Theorem says that $F[f * f](k) = \hat{f}^2(k)$, or

$$(f * f)(x) = F^{-1}[\hat{f}^2(k)].$$

For the box function, we have already found that

$$\hat{f}(k) = \frac{2}{k} \sin k.$$

So, we need to compute

$$\begin{aligned} (f * f)(x) &= F^{-1}[\frac{4}{k^2} \sin^2 k] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{4}{k^2} \sin^2 k \right) e^{-ikx}\, dk. \end{aligned} \qquad (10.70)$$

One way to compute this integral is to extend the computation into the complex k-plane. We first need to rewrite the integrand. Thus,

$$\begin{aligned} (f * f)(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{k^2} \sin^2 k e^{-ikx}\, dk \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{k^2} [1 - \cos 2k] e^{-ikx}\, dk \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{k^2} \left[1 - \frac{1}{2}(e^{ik} + e^{-ik}) \right] e^{-ikx}\, dk \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{k^2} \left[e^{-ikx} - \frac{1}{2}(e^{-i(1-k)} + e^{-i(1+k)}) \right] dk. \end{aligned} \qquad (10.71)$$

We can compute the above integrals if we know how to compute the integral

$$I(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-iky}}{k^2}\, dk.$$

Then, the result can be found in terms of $I(y)$ as

$$(f*f)(x) = I(x) - \frac{1}{2}[I(1-k) + I(1+k)].$$

We consider the integral

$$\oint_C \frac{e^{-iyz}}{\pi z^2}\, dz$$

over the contour in Figure 10.23. We can see that there is a double pole at $z = 0$. The pole is on the real axis. So, we will need to cut out the pole as we seek the value of the principal value integral.

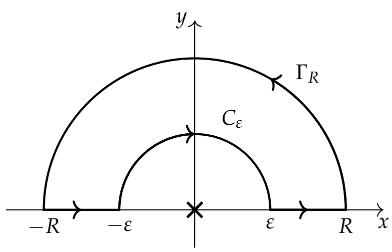

Figure 10.23: Contour for computing $P\int_{-\infty}^{\infty} \frac{e^{-iyz}}{\pi z^2}\, dz$.

Recall from Chapter 9 that

$$\oint_{C_R} \frac{e^{-iyz}}{\pi z^2}\, dz = \int_{\Gamma_R} \frac{e^{-iyz}}{\pi z^2}\, dz + \int_{-R}^{-\epsilon} \frac{e^{-iyz}}{\pi z^2}\, dz + \int_{C_\epsilon} \frac{e^{-iyz}}{\pi z^2}\, dz + \int_{\epsilon}^{R} \frac{e^{-iyz}}{\pi z^2}\, dz.$$

The integral $\oint_{C_R} \frac{e^{-iyz}}{\pi z^2}\, dz$ vanishes since there are no poles enclosed in the contour! The sum of the second and fourth integrals gives the integral we seek as $\epsilon \to 0$ and $R \to \infty$. The integral over Γ_R will vanish as R gets large according to Jordan's Lemma provided $y < 0$. That leaves the integral over the small semicircle.

As before, we can show that

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)\, dz = -\pi i \operatorname{Res}[f(z); z = 0].$$

Therefore, we find

$$I(y) = P\int_{-\infty}^{\infty} \frac{e^{-iyz}}{\pi z^2}\, dz = \pi i \operatorname{Res}\left[\frac{e^{-iyz}}{\pi z^2}; z = 0\right].$$

A simple computation of the reside gives $I(y) = -y$, for $y < 0$.

When $y > 0$, we need to close the contour in the lower half plane in order to apply Jordan's Lemma. Carrying out the computation, one finds $I(y) = y$, for $y > 0$. Thus,

$$I(y) = \begin{cases} -y, & y > 0, \\ y, & y < 0, \end{cases} \tag{10.72}$$

We are now ready to finish the computation of the convolution. We have to combine the integrals $I(y)$, $I(y+1)$, and $I(y-1)$, since $(f*f)(x) = I(x) - \frac{1}{2}[I(1-k) + I(1+k)]$. This gives different results in four intervals:

$$\begin{aligned} (f*f)(x) &= x - \frac{1}{2}[(x-2) + (x+2)] = 0, \quad x < -2, \\ &= x - \frac{1}{2}[(x-2) - (x+2)] = 2 + x \quad -2 < x < 0, \\ &= -x - \frac{1}{2}[(x-2) - (x+2)] = 2 - x, \quad 0 < x < 2, \\ &= -x - \frac{1}{2}[-(x-2) - (x+2)] = 0, \quad x > 2. \end{aligned} \tag{10.73}$$

A plot of this solution is the triangle function,

$$(f * f)(x) = \begin{cases} 0, & x < -2 \\ 2 + x, & -2 < x < 0 \\ 2 - x, & 0 < x < 2 \\ 0, & x > 2, \end{cases} \tag{10.74}$$

which was shown in the last example.

Example 10.14. Find the convolution of the box function of height one and width two with itself using a direct computation of the convolution integral.

The nonvanishing contributions to the convolution integral are when both $f(t)$ and $f(x-t)$ do not vanish. $f(t)$ is nonzero for $|t| \leq 1$, or $-1 \leq t \leq 1$. $f(x-t)$ is nonzero for $|x-t| \leq 1$, or $x-1 \leq t \leq x+1$. These two regions are shown in Figure 10.25. On this region, $f(t)g(x-t) = 1$.

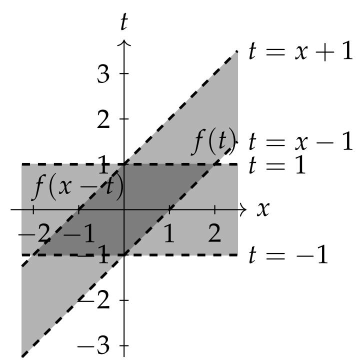

Figure 10.24: Plot of the regions of support for $f(t)$ and $f(x-t)$..

Thus, the nonzero contributions to the convolution are

$$(f * f)(x) = \begin{cases} \int_{-1}^{x+1} dt, & 0 \leq x \leq 2, \\ \int_{x-1}^{1} dt, & -2 \leq x \leq 0, \end{cases} = \begin{cases} 2 + x, & 0 \leq x \leq 2, \\ 2 - x, & -2 \leq x \leq 0. \end{cases}$$

Once again, we arrive at the triangle function.

In the last section we showed the graphical convolution. For completeness, we do the same for this example. In figure 10.25 we show the results. We see that the convolution of two box functions is a triangle function.

Example 10.15. Show the graphical convolution of the box function of height one and width two with itself.

Let's consider a slightly more complicated example, the convolution of two Gaussian functions.

Example 10.16. Convolution of two Gaussian functions $f(x) = e^{-ax^2}$.

In this example we will compute the convolution of two Gaussian functions with different widths. Let $f(x) = e^{-ax^2}$ and $g(x) = e^{-bx^2}$. A direct evaluation of the integral would be to compute

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\, dt = \int_{-\infty}^{\infty} e^{-at^2 - b(x-t)^2}\, dt.$$

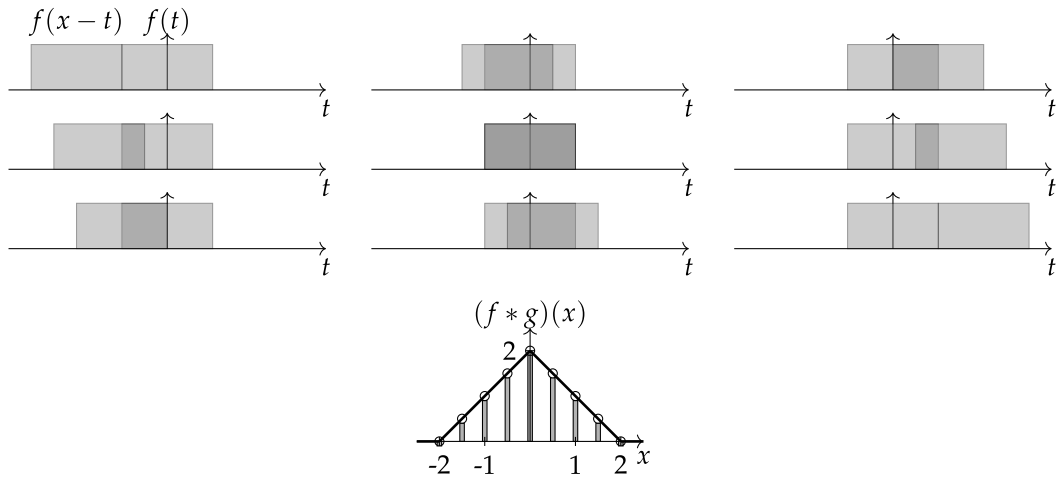

Figure 10.25: A plot of the convolution of a box function with itself. The areas of the overlaps of as $f(x-t)$ is translated across $f(t)$ are shown as well. The result is the triangular function.

This integral can be rewritten as

$$(f*g)(x) = e^{-bx^2}\int_{-\infty}^{\infty} e^{-(a+b)t^2+2bxt}\,dt.$$

One could proceed to complete the square and finish carrying out the integration. However, we will use the Convolution Theorem to evaluate the convolution and leave the evaluation of this integral to Problem 12.

Recalling the Fourier transform of a Gaussian from Example 10.5, we have

$$\hat{f}(k) = F[e^{-ax^2}] = \sqrt{\frac{\pi}{a}}e^{-k^2/4a} \tag{10.75}$$

and

$$\hat{g}(k) = F[e^{-bx^2}] = \sqrt{\frac{\pi}{b}}e^{-k^2/4b}.$$

Denoting the convolution function by $h(x) = (f*g)(x)$, the Convolution Theorem gives

$$\hat{h}(k) = \hat{f}(k)\hat{g}(k) = \frac{\pi}{\sqrt{ab}}e^{-k^2/4a}e^{-k^2/4b}.$$

This is another Gaussian function, as seen by rewriting the Fourier transform of $h(x)$ as

$$\hat{h}(k) = \frac{\pi}{\sqrt{ab}}e^{-\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)k^2} = \frac{\pi}{\sqrt{ab}}e^{-\frac{a+b}{4ab}k^2}. \tag{10.76}$$

In order to complete the evaluation of the convolution of these two Gaussian functions, we need to find the inverse transform of the Gaussian in Equation (10.76). We can do this by looking at Equation (10.75). We have first that

$$F^{-1}\left[\sqrt{\frac{\pi}{a}}e^{-k^2/4a}\right] = e^{-ax^2}.$$

Moving the constants, we then obtain

$$F^{-1}[e^{-k^2/4a}] = \sqrt{\frac{a}{\pi}} e^{-ax^2}.$$

We now make the substitution $\alpha = \frac{1}{4a}$,

$$F^{-1}[e^{-\alpha k^2}] = \sqrt{\frac{1}{4\pi\alpha}} e^{-x^2/4\alpha}.$$

This is in the form needed to invert (10.76). Thus, for $\alpha = \frac{a+b}{4ab}$ we find

$$(f * g)(x) = h(x) = \sqrt{\frac{\pi}{a+b}} e^{-\frac{ab}{a+b}x^2}.$$

10.6.2 Application to Signal Analysis

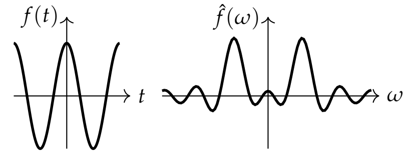

Figure 10.26: Schematic plot of a signal $f(t)$ and its Fourier transform $\hat{f}(\omega)$.

THERE ARE MANY APPLICATIONS of the convolution operation. One of these areas is the study of analog signals. An analog signal is a continuous signal and may contain either a finite, or continuous, set of frequencies. Fourier transforms can be used to represent such signals as a sum over the frequency content of these signals. In this section we will describe how convolutions can be used in studying signal analysis.

Filtering signals.

The first application is filtering. For a given signal there might be some noise in the signal, or some undesirable high frequencies. For example, a device used for recording an analog signal might naturally not be able to record high frequencies. Let $f(t)$ denote the amplitude of a given analog signal and $\hat{f}(\omega)$ be the Fourier transform of this signal such the example provided in Figure 10.26. Recall that the Fourier transform gives the frequency content of the signal.

There are many ways to filter out unwanted frequencies. The simplest would be to just drop all of the high (angular) frequencies. For example, for some cutoff frequency ω_0 frequencies $|\omega| > \omega_0$ will be removed. The Fourier transform of the filtered signal would then be zero for $|\omega| > \omega_0$. This could be accomplished by multiplying the Fourier transform of the signal by a function that vanishes for $|\omega| > \omega_0$. For example, we could use the gate function

$$p_{\omega_0}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \\ 0, & |\omega| > \omega_0 \end{cases}, \tag{10.77}$$

as shown in Figure 10.27.

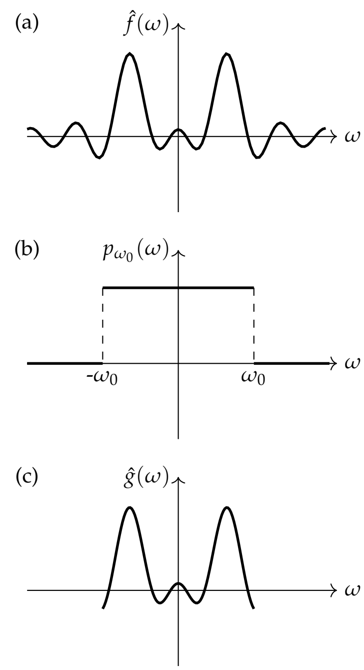

Figure 10.27: (a) Plot of the Fourier transform $\hat{f}(\omega)$ of a signal. (b) The gate function $p_{\omega_0}(\omega)$ used to filter out high frequencies. (c) The product of the functions, $\hat{g}(\omega) = \hat{f}(\omega)p_{\omega_0}(\omega)$, in (a) and (b) shows how the filters cuts out high frequencies, $|\omega| > \omega_0$.

In general, we multiply the Fourier transform of the signal by some filtering function $\hat{h}(t)$ to get the Fourier transform of the filtered signal,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{h}(\omega).$$

The new signal, $g(t)$ is then the inverse Fourier transform of this product, giving the new signal as a convolution:

$$g(t) = F^{-1}[\hat{f}(\omega)\hat{h}(\omega)] = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)\,d\tau. \tag{10.78}$$

Such processes occur often in systems theory as well. One thinks of $f(t)$ as the input signal into some filtering device which in turn produces the output, $g(t)$. The function $h(t)$ is called the impulse response. This is because it is a response to the impulse function, $\delta(t)$. In this case, one has

$$\int_{-\infty}^{\infty} h(t-\tau)\delta(\tau)\, d\tau = h(t).$$

Windowing signals.

Another application of the convolution is in windowing. This represents what happens when one measures a real signal. Real signals cannot be recorded for all values of time. Instead data is collected over a finite time interval. If the length of time the data is collected is T, then the resulting signal is zero outside this time interval. This can be modeled in the same way as with filtering, except the new signal will be the product of the old signal with the windowing function. The resulting Fourier transform of the new signal will be a convolution of the Fourier transforms of the original signal and the windowing function.

Example 10.17. Finite Wave Train, Revisited.

We return to the finite wave train in Example 10.10 given by

$$h(t) = \begin{cases} \cos\omega_0 t, & |t| \le a \\ 0, & |t| > a \end{cases}.$$

We can view this as a windowed version of $f(t) = \cos\omega_0 t$ obtained by multiplying $f(t)$ by the gate function

$$g_a(t) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}. \tag{10.79}$$

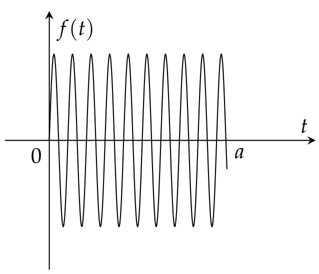

Figure 10.28: A plot of the finite wave train.

This is shown in Figure 10.28. Then, the Fourier transform is given as a convolution,

The convolution in spectral space is defined with an extra factor of $1/2\pi$ so as to preserve the idea that the inverse Fourier transform of a convolution is the product of the corresponding signals.

$$\begin{aligned} \hat{h}(\omega) &= (\hat{f} * \hat{g}_a)(\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega-\nu)\hat{g}_a(\nu)\, d\nu. \end{aligned} \tag{10.80}$$

Note that the convolution in frequency space requires the extra factor of $1/(2\pi)$.

We need the Fourier transforms of f and g_a in order to finish the computation. The Fourier transform of the box function was found in Example 10.6 as

$$\hat{g}_a(\omega) = \frac{2}{\omega}\sin\omega a.$$

The Fourier transform of the cosine function, $f(t) = \cos\omega_0 t$, is

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{i\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) e^{i\omega t}\, dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{i(\omega+\omega_0)t} + e^{i(\omega-\omega_0)t}\right)\, dt \\ &= \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]. \end{aligned} \tag{10.81}$$

Note that we had earlier computed the inverse Fourier transform of this function in Example 10.9.

Inserting these results in the convolution integral, we have

$$\begin{aligned}\hat{h}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega-\nu)\hat{g}_a(\nu)\,d\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\left[\delta(\omega-\nu+\omega_0)+\delta(\omega-\nu-\omega_0)\right]\frac{2}{\nu}\sin\nu a\,d\nu \\ &= \frac{\sin(\omega+\omega_0)a}{\omega+\omega_0}+\frac{\sin(\omega-\omega_0)a}{\omega-\omega_0}. \qquad (10.82)\end{aligned}$$

This is the same result we had obtained in Example 10.10.

10.6.3 Parseval's Equality

The integral/sum of the (modulus) square of a function is the integral/sum of the (modulus) square of the transform.

As another example of the convolution theorem, we derive Parseval's Equality (named after Marc-Antoine Parseval (1755-1836)):

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega. \qquad (10.83)$$

This equality has a physical meaning for signals. The integral on the left side is a measure of the energy content of the signal in the time domain. The right side provides a measure of the energy content of the transform of the signal. Parseval's equality, is simply a statement that the energy is invariant under the Fourier transform. Parseval's equality is a special case of Plancherel's formula (named after Michel Plancherel, 1885-1967).

Let's rewrite the Convolution Theorem in its inverse form

$$F^{-1}[\hat{f}(k)\hat{g}(k)] = (f*g)(t). \qquad (10.84)$$

Then, by the definition of the inverse Fourier transform, we have

$$\int_{-\infty}^{\infty}f(t-u)g(u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\hat{g}(\omega)e^{-i\omega t}\,d\omega.$$

Setting $t=0$,

$$\int_{-\infty}^{\infty}f(-u)g(u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\hat{g}(\omega)\,d\omega. \qquad (10.85)$$

Now, let $g(t)=\overline{f(-t)}$, or $f(-t)=\overline{g(t)}$. We note that the Fourier transform of $g(t)$ is related to the Fourier transform of $f(t)$:

$$\begin{aligned}\hat{g}(\omega) &= \int_{-\infty}^{\infty}\overline{f(-t)}e^{i\omega t}\,dt \\ &= -\int_{\infty}^{-\infty}\overline{f(\tau)}e^{-i\omega\tau}\,d\tau \\ &= \overline{\int_{-\infty}^{\infty}f(\tau)e^{i\omega\tau}\,d\tau} = \overline{\hat{f}(\omega)}. \qquad (10.86)\end{aligned}$$

So, inserting this result into Equation (10.85), we find that

$$\int_{-\infty}^{\infty}f(-u)\overline{f(-u)}\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega$$

which yields Parseval's Equality in the form (10.83) after substituting $t = -u$ on the left.

As noted above, the forms in Equations (10.83) and (10.85) are often referred to as the Plancherel formula or Parseval formula. A more commonly defined Parseval equation is that given for Fourier series. For example, for a function $f(x)$ defined on $[-\pi, \pi]$, which has a Fourier series representation, we have

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx.$$

In general, there is a Parseval identity for functions that can be expanded in a complete sets of orthonormal functions, $\{\phi_n(x)\}$, $n = 1, 2, \ldots,$ which is given by

$$\sum_{n=1}^{\infty} < f, \phi_n >^2 = \|f\|^2.$$

Here $\|f\|^2 =< f, f >$. The Fourier series example is just a special case of this formula.

10.7 *The Laplace Transform*

The Laplace transform is named after Pierre-Simon de Laplace (1749-1827). Laplace made major contributions, especially to celestial mechanics, tidal analysis, and probability.

Up to this point we have only explored Fourier exponential transforms as one type of integral transform. The Fourier transform is useful on infinite domains. However, students are often introduced to another integral transform, called the Laplace transform, in their introductory differential equations class. These transforms are defined over semi-infinite domains and are useful for solving initial value problems for ordinary differential equations.

Integral transform on $[a, b]$ with respect to the integral kernel, $K(x, k)$.

The Fourier and Laplace transforms are examples of a broader class of transforms known as integral transforms . For a function $f(x)$ defined on an interval (a, b), we define the integral transform

$$F(k) = \int_a^b K(x, k) f(x)\,dx,$$

where $K(x, k)$ is a specified kernel of the transform. Looking at the Fourier transform, we see that the interval is stretched over the entire real axis and the kernel is of the form, $K(x, k) = e^{ikx}$. In Table 10.1 we show several types of integral transforms.

Table 10.1: A table of common integral transforms.

Laplace Transform	$F(s) = \int_0^\infty e^{-sx} f(x)\,dx$
Fourier Transform	$F(k) = \int_{-\infty}^\infty e^{ikx} f(x)\,dx$
Fourier Cosine Transform	$F(k) = \int_0^\infty \cos(kx) f(x)\,dx$
Fourier Sine Transform	$F(k) = \int_0^\infty \sin(kx) f(x)\,dx$
Mellin Transform	$F(k) = \int_0^\infty x^{k-1} f(x)\,dx$
Hankel Transform	$F(k) = \int_0^\infty x J_n(kx) f(x)\,dx$

It should be noted that these integral transforms inherit the linearity of integration. Namely. let $h(x) = \alpha f(x) + \beta g(x)$, where α and β are constants. Then,

$$\begin{aligned} H(k) &= \int_a^b K(x,k)h(x)\,dx, \\ &= \int_a^b K(x,k)(\alpha f(x) + \beta g(x))\,dx, \\ &= \alpha \int_a^b K(x,k)f(x)\,dx + \beta \int_a^b K(x,k)g(x)\,dx, \\ &= \alpha F(x) + \beta G(x). \end{aligned} \tag{10.87}$$

Therefore, we have shown linearity of the integral transforms. We have seen the linearity property used for Fourier transforms and we will use linearity in the study of Laplace transforms.

The Laplace transform of f, $F = \mathcal{L}[f]$.

We now turn to Laplace transforms. The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0. \tag{10.88}$$

This is an improper integral and one needs

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

to guarantee convergence.

Laplace transforms also have proven useful in engineering for solving circuit problems and doing systems analysis. In Figure 10.29 it is shown that a signal $x(t)$ is provided as input to a linear system, indicated by $h(t)$. One is interested in the system output, $y(t)$, which is given by a convolution of the input and system functions. By considering the transforms of $x(t)$ and $h(t)$, the transform of the output is given as a product of the Laplace transforms in the s-domain. In order to obtain the output, one needs to compute a convolution product for Laplace transforms similar to the convolution operation we had seen for Fourier transforms earlier in the chapter. Of course, for us to do this in practice, we have to know how to compute Laplace transforms.

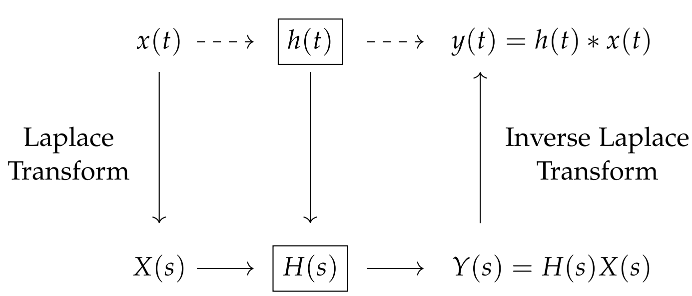

Figure 10.29: A schematic depicting the use of Laplace transforms in systems theory.

10.7.1 Properties and Examples of Laplace Transforms

It is typical that one makes use of Laplace transforms by referring to a Table of transform pairs. A sample of such pairs is given in Table 10.2. Combining some of these simple Laplace transforms with the properties of the Laplace transform, as shown in Table 10.3, we can deal with many applications of the Laplace transform. We will first prove a few of the given Laplace transforms and show how they can be used to obtain new transform pairs. In the next section we will show how these transforms can be used to sum infinite series and to solve initial value problems for ordinary differential equations.

Table 10.2: Table of selected Laplace transform pairs.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
c	$\frac{c}{s}$	e^{at}	$\frac{1}{s-a}, \quad s>a$
t^n	$\frac{n!}{s^{n+1}}, \quad s>0$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s>0$	$\delta(t-a)$	$e^{-as}, \quad a\ge 0, s>0$

We begin with some simple transforms. These are found by simply using the definition of the Laplace transform.

Example 10.18. Show that $\mathcal{L}[1]=\frac{1}{s}$.

For this example, we insert $f(t)=1$ into the definition of the Laplace transform:

$$\mathcal{L}[1]=\int_0^\infty e^{-st}\,dt.$$

This is an improper integral and the computation is understood by introducing an upper limit of a and then letting $a\to\infty$. We will not always write this limit, but it will be understood that this is how one computes such improper integrals. Proceeding with the computation, we have

$$\begin{aligned}
\mathcal{L}[1] &= \int_0^\infty e^{-st}\,dt \\
&= \lim_{a\to\infty}\int_0^a e^{-st}\,dt \\
&= \lim_{a\to\infty}\left(-\frac{1}{s}e^{-st}\right)_0^a \\
&= \lim_{a\to\infty}\left(-\frac{1}{s}e^{-sa}+\frac{1}{s}\right)=\frac{1}{s}. \qquad (10.89)
\end{aligned}$$

Thus, we have found that the Laplace transform of 1 is $\frac{1}{s}$. This result can be extended to any constant c, using the linearity of the transform, $\mathcal{L}[c] = c\mathcal{L}[1]$. Therefore,

$$\mathcal{L}[c] = \frac{c}{s}.$$

Example 10.19. Show that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, for $s > a$.

For this example, we can easily compute the transform. Again, we only need to compute the integral of an exponential function.

$$\begin{aligned} \mathcal{L}[e^{at}] &= \int_0^\infty e^{at} e^{-st}\, dt \\ &= \int_0^\infty e^{(a-s)t}\, dt \\ &= \left(\frac{1}{a-s} e^{(a-s)t} \right)_0^\infty \\ &= \lim_{t\to\infty} \frac{1}{a-s} e^{(a-s)t} - \frac{1}{a-s} = \frac{1}{s-a}. \end{aligned} \tag{10.90}$$

Note that the last limit was computed as $\lim_{t\to\infty} e^{(a-s)t} = 0$. This is only true if $a - s < 0$, or $s > a$. [Actually, a could be complex. In this case we would only need s to be greater than the real part of a, $s > Re(a)$.]

Example 10.20. Show that $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$ and $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$.

For these examples, we could again insert the trigonometric functions directly into the transform and integrate. For example,

$$\mathcal{L}[\cos at] = \int_0^\infty e^{-st} \cos at\, dt.$$

Recall how one evaluates integrals involving the product of a trigonometric function and the exponential function. One integrates by parts two times and then obtains an integral of the original unknown integral. Rearranging the resulting integral expressions, one arrives at the desired result. However, there is a much simpler way to compute these transforms.

Recall that $e^{iat} = \cos at + i \sin at$. Making use of the linearity of the Laplace transform, we have

$$\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at] + i\mathcal{L}[\sin at].$$

Thus, transforming this complex exponential will simultaneously provide the Laplace transforms for the sine and cosine functions!

The transform is simply computed as

$$\mathcal{L}[e^{iat}] = \int_0^\infty e^{iat} e^{-st}\, dt = \int_0^\infty e^{-(s-ia)t}\, dt = \frac{1}{s-ia}.$$

Note that we could easily have used the result for the transform of an exponential, which was already proven. In this case $s > Re(ia) = 0$.

We now extract the real and imaginary parts of the result using the complex conjugate of the denominator:

$$\frac{1}{s-ia} = \frac{1}{s-ia}\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}.$$

Reading off the real and imaginary parts, we find the sought transforms,

$$\begin{aligned} \mathcal{L}[\cos at] &= \frac{s}{s^2+a^2} \\ \mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}. \end{aligned} \tag{10.91}$$

Example 10.21. Show that $\mathcal{L}[t] = \frac{1}{s^2}$.

For this example we evaluate

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt.$$

This integral can be evaluated using the method of integration by parts:

$$\begin{aligned} \int_0^\infty te^{-st}\,dt &= -t\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= \frac{1}{s^2}. \end{aligned} \tag{10.92}$$

Example 10.22. Show that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ for nonnegative integer n.

We have seen the $n = 0$ and $n = 1$ cases: $\mathcal{L}[1] = \frac{1}{s}$ and $\mathcal{L}[t] = \frac{1}{s^2}$. We now generalize these results to nonnegative integer powers, $n > 1$, of t. We consider the integral

$$\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt.$$

Following the previous example, we again integrate by parts:[5]

$$\begin{aligned} \int_0^\infty t^n e^{-st}\,dt &= -t^n\frac{1}{s}e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\,dt \\ &= \frac{n}{s}\int_0^\infty t^{-n}e^{-st}\,dt. \end{aligned} \tag{10.93}$$

[5] This integral can just as easily be done using differentiation. We note that

$$\left(-\frac{d}{ds}\right)^n \int_0^\infty e^{-st}\,dt = \int_0^\infty t^n e^{-st}\,dt.$$

Since

$$\int_0^\infty e^{-st}\,dt = \frac{1}{s},$$

$$\int_0^\infty t^n e^{-st}\,dt = \left(-\frac{d}{ds}\right)^n \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

We could continue to integrate by parts until the final integral is computed. However, look at the integral that resulted after one integration by parts. It is just the Laplace transform of t^{n-1}. So, we can write the result as

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}].$$

We compute $\int_0^\infty t^n e^{-st}\,dt$ by turning it into an initial value problem for a first order difference equation and finding the solution using an iterative method.

This is an example of a recursive definition of a sequence. In this case we have a sequence of integrals. Denoting

$$I_n = \mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$$

and noting that $I_0 = \mathcal{L}[1] = \frac{1}{s}$, we have the following:

$$I_n = \frac{n}{s}I_{n-1}, \quad I_0 = \frac{1}{s}. \tag{10.94}$$

This is also what is called a difference equation. It is a first order difference equation with an "initial condition," I_0. The next step is to solve this difference equation.

Finding the solution of this first order difference equation is easy to do using simple iteration. Note that replacing n with $n-1$, we have

$$I_{n-1} = \frac{n-1}{s} I_{n-2}.$$

Repeating the process, we find

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n}{s}\left(\frac{n-1}{s} I_{n-2}\right) \\ &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3}. \end{aligned} \tag{10.95}$$

We can repeat this process until we get to I_0, which we know. We have to carefully count the number of iterations. We do this by iterating k times and then figure out how many steps will get us to the known initial value. A list of iterates is easily written out:

$$\begin{aligned} I_n &= \frac{n}{s} I_{n-1} \\ &= \frac{n(n-1)}{s^2} I_{n-2} \\ &= \frac{n(n-1)(n-2)}{s^3} I_{n-3} \\ &= \ldots \\ &= \frac{n(n-1)(n-2)\ldots(n-k+1)}{s^k} I_{n-k}. \end{aligned} \tag{10.96}$$

Since we know $I_0 = \frac{1}{s}$, we choose to stop at $k = n$ obtaining

$$I_n = \frac{n(n-1)(n-2)\ldots(2)(1)}{s^n} I_0 = \frac{n!}{s^{n+1}}.$$

Therefore, we have shown that $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$.

Such iterative techniques are useful in obtaining a variety of integrals, such as $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$.

As a final note, one can extend this result to cases when n is not an integer. To do this, we use the Gamma function, which was discussed in Section 5.4. Recall that the Gamma function is the generalization of the factorial function and is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{10.97}$$

Note the similarity to the Laplace transform of t^{x-1} :

$$\mathcal{L}[t^{x-1}] = \int_0^\infty t^{x-1} e^{-st}\, dt.$$

For $x-1$ an integer and $s=1$, we have that

$$\Gamma(x) = (x-1)!.$$

Thus, the Gamma function can be viewed as a generalization of the factorial and we have shown that

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

for $p > -1$.

Now we are ready to introduce additional properties of the Laplace transform in Table 10.3. We have already discussed the first property, which is a consequence of linearity of the integral transforms. We will prove the other properties in this and the following sections.

Table 10.3: Table of selected Laplace transform properties.

Laplace Transform Properties
$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$
$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$
$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0)$
$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s)-sf(0)-f'(0)$
$\mathcal{L}[e^{at}f(t)] = F(s-a)$
$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$
$\mathcal{L}[(f*g)(t)] = \mathcal{L}[\int_0^t f(t-u)g(u)\,du] = F(s)G(s)$

Example 10.23. Show that $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0)$.

We have to compute

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{-st}\,dt.$$

We can move the derivative off f by integrating by parts. This is similar to what we had done when finding the Fourier transform of the derivative of a function. Letting $u=e^{-st}$ and $v=f(t)$, we have

$$\begin{aligned}
\mathcal{L}\left[\frac{df}{dt}\right] &= \int_0^\infty \frac{df}{dt}e^{-st}\,dt \\
&= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\
&= -f(0)+sF(s). \qquad (10.98)
\end{aligned}$$

Here we have assumed that $f(t)e^{-st}$ vanishes for large t.

The final result is that

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s)-f(0).$$

Example 6: Show that $\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s)-sf(0)-f'(0)$.

We can compute this Laplace transform using two integrations by parts, or we could make use of the last result. Letting $g(t) = \frac{df(t)}{dt}$, we have

$$\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = \mathcal{L}\left[\frac{dg}{dt}\right] = sG(s) - g(0) = sG(s) - f'(0).$$

But,

$$G(s) = \mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0).$$

So,

$$\begin{aligned}\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] &= sG(s) - f'(0)\\ &= s\left[sF(s) - f(0)\right] - f'(0)\\ &= s^2F(s) - sf(0) - f'(0). \qquad (10.99)\end{aligned}$$

We will return to the other properties in Table 10.3 after looking at a few applications.

10.8 Applications of Laplace Transforms

Although the Laplace transform is a very useful transform, it is often encountered only as a method for solving initial value problems in introductory differential equations. In this section we will show how to solve simple differential equations. Along the way we will introduce step and impulse functions and show how the Convolution Theorem for Laplace transforms plays a role in finding solutions. However, we will first explore an unrelated application of Laplace transforms. We will see that the Laplace transform is useful in finding sums of infinite series.

10.8.1 Series Summation Using Laplace Transforms

We saw in Chapter 2 that Fourier series can be used to sum series. For example, in Problem 2.13, one proves that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In this section we will show how Laplace transforms can be used to sum series.[6] There is an interesting history of using integral transforms to sum series. For example, Richard Feynman[7] (1918-1988) described how one can use the convolution theorem for Laplace transforms to sum series with denominators that involved products. We will describe this and simpler sums in this section.

We begin by considering the Laplace transform of a known function,

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt.$$

[6] Albert D. Wheelon, *Tables of Summable Series and Integrals Involving Bessel Functions*, Holden-Day, 1968.

[7] R. P. Feynman, 1949, *Phys. Rev.* **76**, p. 769

Inserting this expression into the sum $\sum_n F(n)$ and interchanging the sum and integral, we find

$$\begin{aligned} \sum_{n=0}^{\infty} F(n) &= \sum_{n=0}^{\infty} \int_0^{\infty} f(t) e^{-nt}\, dt \\ &= \int_0^{\infty} f(t) \sum_{n=0}^{\infty} \left(e^{-t}\right)^n \, dt \\ &= \int_0^{\infty} f(t) \frac{1}{1-e^{-t}}\, dt. \end{aligned} \tag{10.100}$$

The last step was obtained using the sum of a geometric series. The key is being able to carry out the final integral as we show in the next example.

Example 10.24. Evaluate the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Since, $\mathcal{L}[1] = 1/s$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= \sum_{n=1}^{\infty} \int_0^{\infty} (-1)^{n+1} e^{-nt}\, dt \\ &= \int_0^{\infty} \frac{e^{-t}}{1+e^{-t}}\, dt \\ &= \int_1^2 \frac{du}{u} = \ln 2. \end{aligned} \tag{10.101}$$

Example 10.25. Evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

This is a special case of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{10.102}$$

The Riemann zeta function[8] is important in the study of prime numbers and more recently has seen applications in the study of dynamical systems. The series in this example is $\zeta(2)$. We have already seen in 2.13 that

$$\zeta(2) = \frac{\pi^2}{6}.$$

Using Laplace transforms, we can provide an integral representation of $\zeta(2)$.

The first step is to find the correct Laplace transform pair. The sum involves the function $F(n) = 1/n^2$. So, we look for a function $f(t)$ whose Laplace transform is $F(s) = 1/s^2$. We know by now that the inverse Laplace transform of $F(s) = 1/s^2$ is $f(t) = t$. As before, we replace each term in the series by a Laplace transform, exchange the summation and integration, and sum the resulting geometric series:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \int_0^{\infty} t e^{-nt}\, dt \\ &= \int_0^{\infty} \frac{t}{e^t - 1}\, dt. \end{aligned} \tag{10.103}$$

So, we have that

$$\int_0^{\infty} \frac{t}{e^t-1}\, dt = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

[8] A translation of Riemann, Bernhard (1859), "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" is in H. M. Edwards (1974). *Riemann's Zeta Function.* Academic Press. Riemann had shown that the Riemann zeta function can be obtained through contour integral representation, $2\sin(\pi s)\Gamma\zeta(s) = i\oint_C \frac{(-x)^{s-1}}{e^x-1}\, dx$, for a specific contour C.

Integrals of this type occur often in statistical mechanics in the form of Bose-Einstein integrals. These are of the form

$$G_n(z) = \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1}\, dx.$$

Note that $G_n(1) = \Gamma(n)\zeta(n)$.

In general the Riemann zeta function has to be tabulated through other means. In some special cases, one can closed form expressions. For example,

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_n,$$

where the B_n's are the Bernoulli numbers. Bernoulli numbers are defined through the Maclaurin series expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

The first few Riemann zeta functions are

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

We can extend this method of using Laplace transforms to summing series whose terms take special general forms. For example, from Feynman's 1949 paper we note that

$$\frac{1}{(a+bn)^2} = -\frac{\partial}{\partial a} \int_0^\infty e^{-s(a+bn)}\, ds.$$

This identity can be shown easily by first noting

$$\int_0^\infty e^{-s(a+bn)}\, ds = \left[\frac{-e^{-s(a+bn)}}{a+bn} \right]_0^\infty = \frac{1}{a+bn}.$$

Now, differentiate the result with respect to a and the result follows.

The latter identity can be generalized further as

$$\frac{1}{(a+bn)^{k+1}} = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial a^k} \int_0^\infty e^{-s(a+bn)}\, ds.$$

In Feynman's 1949 paper, he develops methods for handling several other general sums using the convolution theorem. Wheelon gives more examples of these. We will just provide one such result and an example. First, we note that

$$\frac{1}{ab} = \int_0^1 \frac{du}{[a(1-u)+bu]^2}.$$

However,

$$\frac{1}{[a(1-u)+bu]^2} = \int_0^\infty t e^{-t[a(1-u)+bu]}\, dt.$$

So, we have

$$\frac{1}{ab} = \int_0^1 du \int_0^\infty t e^{-t[a(1-u)+bu]}\, dt.$$

We see in the next example how this representation can be useful.

Example 10.26. Evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)}$.

We sum this series by first letting $a = 2n+1$ and $b = 2n+2$ in the formula for $1/ab$. Collecting the n-dependent terms, we can sum the series leaving a double integral computation in ut-space. The details are as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} &= \sum_{n=0}^{\infty} \int_0^1 \frac{du}{[(2n+1)(1-u)+(2n+2)u]^2} \\
&= \sum_{n=0}^{\infty} \int_0^1 du \int_0^{\infty} te^{-t(2n+1+u)}\, dt \\
&= \int_0^1 du \int_0^{\infty} te^{-t(1+u)} \sum_{n=0}^{\infty} e^{-2nt}\, dt \\
&= \int_0^{\infty} \frac{te^{-t}}{1-e^{-2t}} \int_0^1 e^{-tu}\, du\, dt \\
&= \int_0^{\infty} \frac{te^{-t}}{1-e^{-2t}} \frac{1-e^{-t}}{t}\, dt \\
&= \int_0^{\infty} \frac{e^{-t}}{1+e^{-t}}\, dt \\
&= -\ln(1+e^{-t})\Big|_0^{\infty} = \ln 2. \qquad (10.104)
\end{aligned}$$

10.8.2 Solution of ODEs Using Laplace Transforms

One of the typical applications of Laplace transforms is the solution of nonhomogeneous linear constant coefficient differential equations. In the following examples we will show how this works.

The general idea is that one transforms the equation for an unknown function $y(t)$ into an algebraic equation for its transform, $Y(t)$. Typically, the algebraic equation is easy to solve for $Y(s)$ as a function of s. Then, one transforms back into t-space using Laplace transform tables and the properties of Laplace transforms. The scheme is shown in Figure 10.30.

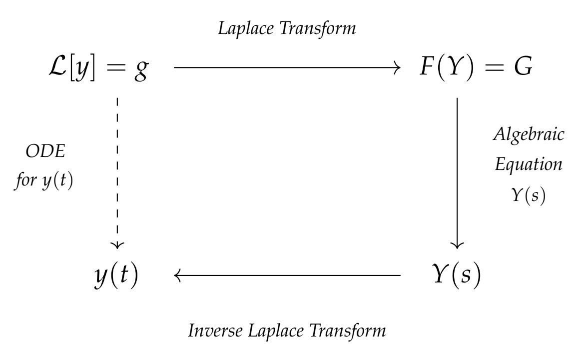

Figure 10.30: The scheme for solving an ordinary differential equation using Laplace transforms. One transforms the initial value problem for $y(t)$ and obtains an algebraic equation for $Y(s)$. Solve for $Y(s)$ and the inverse transform give the solution to the initial value problem.

Example 10.27. Solve the initial value problem $y' + 3y = e^{2t}$, $y(0) = 1$.

The first step is to perform a Laplace transform of the initial value problem. The transform of the left side of the equation is

$$\mathcal{L}[y' + 3y] = sY - y(0) + 3Y = (s+3)Y - 1.$$

Transforming the right hand side, we have

$$\mathcal{L}[e^{2t}] = \frac{1}{s-2}.$$

Combining these two results, we obtain

$$(s+3)Y - 1 = \frac{1}{s-2}.$$

The next step is to solve for $Y(s)$:

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s-2)(s+3)}.$$

Now, we need to find the inverse Laplace transform. Namely, we need to figure out what function has a Laplace transform of the above form. We will use the tables of Laplace transform pairs. Later we will show that there are other methods for carrying out the Laplace transform inversion.

The inverse transform of the first term is e^{-3t}. However, we have not seen anything that looks like the second form in the table of transforms that we have compiled; but, we can rewrite the second term by using a partial fraction decomposition. Let's recall how to do this.

The goal is to find constants, A and B, such that

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}. \tag{10.105}$$

This is an example of carrying out a partial fraction decomposition.

We picked this form because we know that recombining the two terms into one term will have the same denominator. We just need to make sure the numerators agree afterwards. So, adding the two terms, we have

$$\frac{1}{(s-2)(s+3)} = \frac{A(s+3) + B(s-2)}{(s-2)(s+3)}.$$

Equating numerators,

$$1 = A(s+3) + B(s-2).$$

There are several ways to proceed at this point.

a. Method 1.

We can rewrite the equation by gathering terms with common powers of s, we have

$$(A+B)s + 3A - 2B = 1.$$

The only way that this can be true for all s is that the coefficients of the different powers of s agree on both sides. This leads to two equations for A and B:

$$\begin{aligned} A + B &= 0 \\ 3A - 2B &= 1. \end{aligned} \tag{10.106}$$

The first equation gives $A = -B$, so the second equation becomes $-5B = 1$. The solution is then $A = -B = \frac{1}{5}$.

b. Method 2.

Since the equation $\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$ is true for all s, we can pick specific values. For $s = 2$, we find $1 = 5A$, or $A = \frac{1}{5}$. For $s = -3$, we find $1 = -5B$, or $B = -\frac{1}{5}$. Thus, we obtain the same result as Method 1, but much quicker.

c. Method 3.

We could just inspect the original partial fraction problem. Since the numerator has no s terms, we might guess the form

$$\frac{1}{(s-2)(s+3)} = \frac{1}{s-2} - \frac{1}{s+3}.$$

But, recombining the terms on the right hand side, we see that

$$\frac{1}{s-2} - \frac{1}{s+3} = \frac{5}{(s-2)(s+3)}.$$

Since we were off by 5, we divide the partial fractions by 5 to obtain

$$\frac{1}{(s-2)(s+3)} = \frac{1}{5}\left[\frac{1}{s-2} - \frac{1}{s+3}\right],$$

which once again gives the desired form.

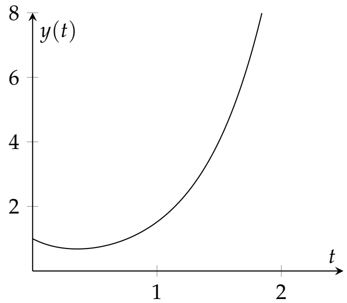

Figure 10.31: A plot of the solution to Example 10.27.

Returning to the problem, we have found that

$$Y(s) = \frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right).$$

We can now see that the function with this Laplace transform is given by

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+3} + \frac{1}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right)\right] = e^{-3t} + \frac{1}{5}\left(e^{2t} - e^{-3t}\right)$$

works. Simplifying, we have the solution of the initial value problem

$$y(t) = \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}.$$

We can verify that we have solved the initial value problem.

$$y' + 3y = \frac{2}{5}e^{2t} - \frac{12}{5}e^{-3t} + 3(\frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t}) = e^{2t}$$

and $y(0) = \frac{1}{5} + \frac{4}{5} = 1$.

Example 10.28. Solve the initial value problem $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$.

We can probably solve this without Laplace transforms, but it is a simple exercise. Transforming the equation, we have

$$\begin{aligned} 0 &= s^2Y - sy(0) - y'(0) + 4Y \\ &= (s^2+4)Y - s - 3. \end{aligned} \quad (10.107)$$

Solving for Y, we have

$$Y(s) = \frac{s+3}{s^2+4}.$$

We now ask if we recognize the transform pair needed. The denominator looks like the type needed for the transform of a sine or cosine. We just need to play with the numerator. Splitting the expression into two terms, we have

$$Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4}.$$

The first term is now recognizable as the transform of $\cos 2t$. The second term is not the transform of $\sin 2t$. It would be if the numerator were a 2. This can be corrected by multiplying and dividing by 2:

$$\frac{3}{s^2+4} = \frac{3}{2}\left(\frac{2}{s^2+4}\right).$$

The solution is then found as

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+4} + \frac{3}{2}\left(\frac{2}{s^2+4}\right)\right] = \cos 2t + \frac{3}{2}\sin 2t.$$

The reader can verify that this is the solution of the initial value problem. The solution is shown in Figure 10.32.

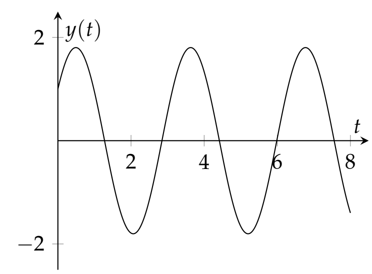

Figure 10.32: A plot of the solution to Example 10.28.

10.8.3 Step and Impulse Functions

Often the initial value problems that one faces in differential equations courses can be solved using either the Method of Undetermined Coefficients or the Method of Variation of Parameters. However, using the latter can be messy and involves some skill with integration. Many circuit designs can be modeled with systems of differential equations using Kirchoff's Rules. Such systems can get fairly complicated. However, Laplace transforms can be used to solve such systems and electrical engineers have long used such methods in circuit analysis.

In this section we add a couple of more transform pairs and transform properties that are useful in accounting for things like turning on a driving force, using periodic functions like a square wave, or introducing impulse forces.

We first recall the Heaviside step function, given by

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{10.108}$$

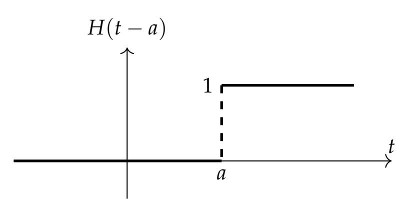

Figure 10.33: A shifted Heaviside function, $H(t-a)$.

A more general version of the step function is the horizontally shifted step function, $H(t-a)$. This function is shown in Figure 10.33. The Laplace transform of this function is found for $a > 0$ as

$$\mathcal{L}[H(t-a)] = \int_0^\infty H(t-a)e^{-st}\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$
$$= \left.\frac{e^{-st}}{s}\right|_a^\infty = \frac{e^{-as}}{s}. \qquad (10.109)$$

Just like the Fourier transform, the Laplace transform has two shift theorems involving the multiplication of the function, $f(t)$, or its transform, $F(s)$, by exponentials. The first and second shifting properties/theorems are given by

$$\mathcal{L}[e^{at}f(t)] = F(s-a) \qquad (10.110)$$
$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s). \qquad (10.111)$$

The Shift Theorems.

We prove the First Shift Theorem and leave the other proof as an exercise for the reader. Namely,

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{at}f(t)e^{-st}\,dt$$
$$= \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a). \qquad (10.112)$$

Example 10.29. Compute the Laplace transform of $e^{-at}\sin\omega t$.

This function arises as the solution of the underdamped harmonic oscillator. We first note that the exponential multiplies a sine function. The shift theorem tells us that we first need the transform of the sine function. So, for $f(t) = \sin\omega t$, we have

$$F(s) = \frac{\omega}{s^2+\omega^2}.$$

Using this transform, we can obtain the solution to this problem as

$$\mathcal{L}[e^{-at}\sin\omega t] = F(s+a) = \frac{\omega}{(s+a)^2+\omega^2}.$$

More interesting examples can be found using piecewise defined functions. First we consider the function $H(t) - H(t-a)$. For $t < 0$ both terms are zero. In the interval $[0,a]$ the function $H(t) = 1$ and $H(t-a) = 0$. Therefore, $H(t) - H(t-a) = 1$ for $t \in [0,a]$. Finally, for $t > a$, both functions are one and therefore the difference is zero. The graph of $H(t) - H(t-a)$ is shown in Figure 10.34.

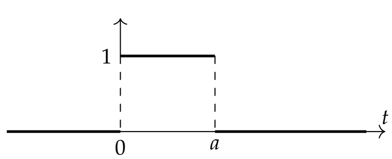

Figure 10.34: The box function, $H(t) - H(t-a)$.

We now consider the piecewise defined function

$$g(t) = \begin{cases} f(t), & 0 \le t \le a, \\ 0, & t < 0, t > a. \end{cases}$$

This function can be rewritten in terms of step functions. We only need to multiply $f(t)$ by the above box function,

$$g(t) = f(t)[H(t) - H(t-a)].$$

We depict this in Figure 10.35.

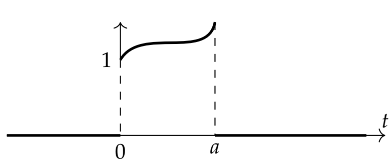

Figure 10.35: Formation of a piecewise function, $f(t)[H(t) - H(t-a)]$.

Even more complicated functions can be written in terms of step functions. We only need to look at sums of functions of the form $f(t)[H(t-a)-H(t-b)]$ for $b>a$. This is similar to a box function. It is nonzero between a and b and has height $f(t)$.

We show as an example the square wave function in Figure 10.36. It can be represented as a sum of an infinite number of boxes,

$$f(t)=\sum_{n=-\infty}^{\infty}[H(t-2na)-H(t-(2n+1)a)],$$

for $a>0$.

Example 10.30. Find the Laplace Transform of a square wave "turned on" at $t=0$. .

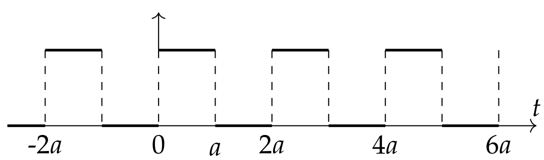

Figure 10.36: A square wave, $f(t)=\sum_{n=-\infty}^{\infty}[H(t-2na)-H(t-(2n+1)a)]$.

We let

$$f(t)=\sum_{n=0}^{\infty}[H(t-2na)-H(t-(2n+1)a)],\quad a>0.$$

Using the properties of the Heaviside function, we have

$$\begin{aligned}
\mathcal{L}[f(t)] &= \sum_{n=0}^{\infty}[\mathcal{L}[H(t-2na)]-\mathcal{L}[H(t-(2n+1)a)]] \\
&= \sum_{n=0}^{\infty}\left[\frac{e^{-2nas}}{s}-\frac{e^{-(2n+1)as}}{s}\right] \\
&= \frac{1-e^{-as}}{s}\sum_{n=0}^{\infty}\left(e^{-2as}\right)^n \\
&= \frac{1-e^{-as}}{s}\left(\frac{1}{1-e^{-2as}}\right) \\
&= \frac{1-e^{-as}}{s(1-e^{-2as})}. \qquad (10.113)
\end{aligned}$$

Note that the third line in the derivation is a geometric series. We summed this series to get the answer in a compact form since $e^{-2as}<1$.

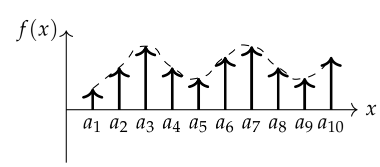

Figure 10.37: Plot representing impulse forces of height $f(a_i)$. The sum $\sum_{i=1}^{n}f(a_i)\delta(x-a_i)$ describes a general impulse function.

Other interesting examples are provided by the delta function. The Dirac delta function can be used to represent a unit impulse. Summing over a number of impulses, or point sources, we can describe a general function as shown in Figure 10.37. The sum of impulses located at points a_i, $i=1,\ldots,n$ with strengths $f(a_i)$ would be given by

$$f(x)=\sum_{i=1}^{n}f(a_i)\delta(x-a_i).$$

A continuous sum could be written as

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\,d\xi.$$

This is simply an application of the sifting property of the delta function. We will investigate a case when one would use a single impulse. While a mass on a spring is undergoing simple harmonic motion, we hit it for an instant at time $t = a$. In such a case, we could represent the force as a multiple of $\delta(t-a)$.

$\mathcal{L}[\delta(t-a)] = e^{-as}$.

One would then need the Laplace transform of the delta function to solve the associated initial value problem. Inserting the delta function into the Laplace transform, we find that for $a > 0$

$$\begin{aligned} \mathcal{L}[\delta(t-a)] &= \int_0^{\infty} \delta(t-a)e^{-st}\,dt \\ &= \int_{-\infty}^{\infty} \delta(t-a)e^{-st}\,dt \\ &= e^{-as}. \end{aligned} \tag{10.114}$$

Example 10.31. Solve the initial value problem $y'' + 4\pi^2 y = \delta(t-2)$, $y(0) = y'(0) = 0$.

This initial value problem models a spring oscillation with an impulse force. Without the forcing term, given by the delta function, this spring is initially at rest and not stretched. The delta function models a unit impulse at $t = 2$. Of course, we anticipate that at this time the spring will begin to oscillate. We will solve this problem using Laplace transforms.

First, we transform the differential equation:

$$s^2 Y - sy(0) - y'(0) + 4\pi^2 Y = e^{-2s}.$$

Inserting the initial conditions, we have

$$(s^2 + 4\pi^2)Y = e^{-2s}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-2s}}{s^2 + 4\pi^2}.$$

We now seek the function for which this is the Laplace transform. The form of this function is an exponential times some Laplace transform, $F(s)$. Thus, we need the Second Shift Theorem since the solution is of the form $Y(s) = e^{-2s}F(s)$ for

$$F(s) = \frac{1}{s^2 + 4\pi^2}.$$

We need to find the corresponding $f(t)$ of the Laplace transform pair. The denominator in $F(s)$ suggests a sine or cosine. Since the numerator is constant, we pick sine. From the tables of transforms, we have

$$\mathcal{L}[\sin 2\pi t] = \frac{2\pi}{s^2 + 4\pi^2}.$$

So, we write

$$F(s) = \frac{1}{2\pi} \frac{2\pi}{s^2 + 4\pi^2}.$$

This gives $f(t) = (2\pi)^{-1} \sin 2\pi t$.

We now apply the Second Shift Theorem, $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$, or

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[e^{-2s}F(s)\right] \\ &= H(t-2)f(t-2) \\ &= \frac{1}{2\pi}H(t-2)\sin 2\pi(t-2). \end{aligned} \quad (10.115)$$

This solution tells us that the mass is at rest until $t = 2$ and then begins to oscillate at its natural frequency. A plot of this solution is shown in Figure 10.38

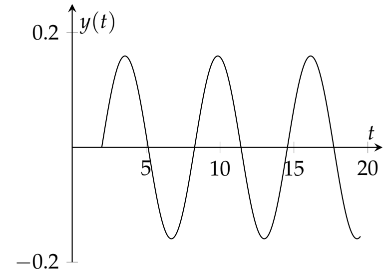

Figure 10.38: A plot of the solution to Example 10.31 in which a spring at rest experiences an impulse force at $t = 2$.

Example 10.32. Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, y'(0) = 0,$$

where

$$f(t) = \begin{cases} cos\pi t, & 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

We need the Laplace transform of $f(t)$. This function can be written in terms of a Heaviside function, $f(t) = \cos\pi t H(t-2)$. In order to apply the Second Shift Theorem, we need a shifted version of the cosine function. We find the shifted version by noting that $\cos\pi(t-2) = \cos\pi t$. Thus, we have

$$\begin{aligned} f(t) &= \cos\pi t\,[H(t) - H(t-2)] \\ &= \cos\pi t - \cos\pi(t-2)H(t-2), \quad t \ge 0. \end{aligned} \quad (10.116)$$

The Laplace transform of this driving term is

$$F(s) = (1 - e^{-2s})\mathcal{L}[\cos\pi t] = (1 - e^{-2s})\frac{s}{s^2 + \pi^2}.$$

Now we can proceed to solve the initial value problem. The Laplace transform of the initial value problem yields

$$(s^2 + 1)Y(s) = (1 - e^{-2s})\frac{s}{s^2 + \pi^2}.$$

Therefore,

$$Y(s) = (1 - e^{-2s})\frac{s}{(s^2 + \pi^2)(s^2 + 1)}.$$

We can retrieve the solution to the initial value problem using the Second Shift Theorem. The solution is of the form $Y(s) = (1 - e^{-2s})G(s)$ for

$$G(s) = \frac{s}{(s^2 + \pi^2)(s^2 + 1)}.$$

Then, the final solution takes the form

$$y(t) = g(t) - g(t-2)H(t-2).$$

We only need to find $g(t)$ in order to finish the problem. This is easily done by using the partial fraction decomposition

$$G(s) = \frac{s}{(s^2+\pi^2)(s^2+1)} = \frac{1}{\pi^2-1}\left[\frac{s}{s^2+1} - \frac{s}{s^2+\pi^2}\right].$$

Then,

$$g(t) = \mathcal{L}^{-1}\left[\frac{s}{(s^2+\pi^2)(s^2+1)}\right] = \frac{1}{\pi^2-1}\left(\cos t - \cos \pi t\right).$$

The final solution is then given by

$$y(t) = \frac{1}{\pi^2-1}\left[\cos t - \cos \pi t - H(t-2)(\cos(t-2) - \cos \pi t)\right].$$

A plot of this solution is shown in Figure 10.39

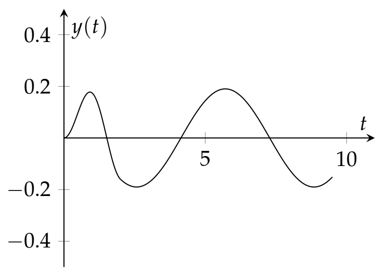

Figure 10.39: A plot of the solution to Example 10.32 in which a spring at rest experiences an piecewise defined force.

10.9 The Convolution Theorem

Finally, we consider the convolution of two functions. Often we are faced with having the product of two Laplace transforms that we know and we seek the inverse transform of the product. For example, let's say we have obtained $Y(s) = \frac{1}{(s-1)(s-2)}$ while trying to solve an initial value problem. In this case we could find a partial fraction decomposition. But, are other ways to find the inverse transform, especially if we cannot perform a partial fraction decomposition. We could use the Convolution Theorem for Laplace transforms or we could compute the inverse transform directly. We will look into these methods in the next two sections.We begin with defining the convolution.

We define the convolution of two functions defined on $[0,\infty)$ much the same way as we had done for the Fourier transform. The convolution $f*g$ is defined as

$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du. \tag{10.117}$$

Note that the convolution integral has finite limits as opposed to the Fourier transform case.

The convolution operation has two important properties:

The convolution is commutative.

1. The convolution is commutative: $f*g = g*f$

   Proof. The key is to make a substitution $y = t-u$ in the integral. This makes f a simple function of the integration variable.

$$\begin{aligned}(g*f)(t) &= \int_0^t g(u)f(t-u)\,du \\ &= -\int_t^0 g(t-y)f(y)\,dy\end{aligned}$$

$$\begin{aligned} &= \int_0^t f(y)g(t-y)\,dy \\ &= (f*g)(t). \end{aligned} \tag{10.118}$$

□

The Convolution Theorem for Laplace transforms.

2. The Convolution Theorem: The Laplace transform of a convolution is the product of the Laplace transforms of the individual functions:

$$\mathcal{L}[f*g] = F(s)G(s)$$

Proof. Proving this theorem takes a bit more work. We will make some assumptions that will work in many cases. First, we assume that the functions are causal, $f(t) = 0$ and $g(t) = 0$ for $t < 0$. Secondly, we will assume that we can interchange integrals, which needs more rigorous attention than will be provided here. The first assumption will allow us to write the finite integral as an infinite integral. Then a change of variables will allow us to split the integral into the product of two integrals that are recognized as a product of two Laplace transforms.

Carrying out the computation, we have

$$\begin{aligned} \mathcal{L}[f*g] &= \int_0^\infty \left(\int_0^t f(u)g(t-u)\,du \right) e^{-st}\,dt \\ &= \int_0^\infty \left(\int_0^\infty f(u)g(t-u)\,du \right) e^{-st}\,dt \\ &= \int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\,dt \right) du \end{aligned} \tag{10.119}$$

Now, make the substitution $\tau = t - u$. We note that

$$int_0^\infty f(u) \left(\int_0^\infty g(t-u)e^{-st}\,dt \right) du = \int_0^\infty f(u) \left(\int_{-u}^\infty g(\tau)e^{-s(\tau+u)}\,d\tau \right) du$$

However, since $g(\tau)$ is a causal function, we have that it vanishes for $\tau < 0$ and we can change the integration interval to $[0, \infty)$. So, after a little rearranging, we can proceed to the result.

$$\begin{aligned} \mathcal{L}[f*g] &= \int_0^\infty f(u) \left(\int_0^\infty g(\tau)e^{-s(\tau+u)}\,d\tau \right) du \\ &= \int_0^\infty f(u)e^{-su} \left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau \right) du \\ &= \left(\int_0^\infty f(u)e^{-su}\,du \right) \left(\int_0^\infty g(\tau)e^{-s\tau}\,d\tau \right) \\ &= F(s)G(s). \end{aligned} \tag{10.120}$$

□

We make use of the Convolution Theorem to do the following examples.

Example 10.33. Find $y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)(s-2)}\right]$.

We note that this is a product of two functions

$$Y(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-1}\frac{1}{s-2} = F(s)G(s).$$

We know the inverse transforms of the factors: $f(t) = e^t$ and $g(t) = e^{2t}$.

Using the Convolution Theorem, we find $y(t) = (f * g)(t)$. We compute the convolution:

$$\begin{aligned} y(t) &= \int_0^t f(u)g(t-u)\,du \\ &= \int_0^t e^u e^{2(t-u)}\,du \\ &= e^{2t}\int_0^t e^{-u}\,du \\ &= e^{2t}[-e^t + 1] = e^{2t} - e^t. \end{aligned} \quad (10.121)$$

One can also confirm this by carrying out a partial fraction decomposition.

Example 10.34. Consider the initial value problem, $y'' + 9y = 2\sin 3t$, $y(0) = 1$, $y'(0) = 0$.

The Laplace transform of this problem is given by

$$(s^2+9)Y - s = \frac{6}{s^2+9}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{6}{(s^2+9)^2} + \frac{s}{s^2+9}.$$

The inverse Laplace transform of the second term is easily found as $cos(3t)$; however, the first term is more complicated.

We can use the Convolution Theorem to find the Laplace transform of the first term. We note that

$$\frac{6}{(s^2+9)^2} = \frac{2}{3}\frac{3}{(s^2+9)}\frac{3}{(s^2+9)}$$

is a product of two Laplace transforms (up to the constant factor). Thus,

$$\mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right] = \frac{2}{3}(f * g)(t),$$

where $f(t) = g(t) = sin 3t$. Evaluating this convolution product, we have

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{6}{(s^2+9)^2}\right] &= \frac{2}{3}(f * g)(t) \\ &= \frac{2}{3}\int_0^t \sin 3u \sin 3(t-u)\,du \end{aligned}$$

$$= \frac{1}{3}\int_0^t [\cos 3(2u-t) - \cos 3t]\,du$$
$$= \frac{1}{3}\left[\frac{1}{6}\sin(6u-3t) - u\cos 3t\right]_0^t$$
$$= \frac{1}{9}\sin 3t - \frac{1}{3}t\cos 3t. \qquad (10.122)$$

Combining this with the inverse transform of the second term of $Y(s)$, the solution to the initial value problem is

$$y(t) = -\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t + \cos 3t.$$

Note that the amplitude of the solution will grow in time from the first term. You can see this in Figure 10.40. This is known as a resonance.

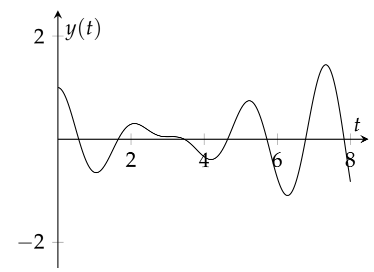

Figure 10.40: Plot of the solution to Example 10.34 showing a resonance.

Example 10.35. Find $\mathcal{L}^{-1}[\frac{6}{(s^2+9)^2}]$ using partial fraction decomposition.

If we look at Table 10.2, we see that the Laplace transform pairs with the denominator $(s^2+\omega^2)^2$ are

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2},$$

and

$$\mathcal{L}[t\cos\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

So, we might consider rewriting a partial fraction decomposition as

$$\frac{6}{(s^2+9)^2} = \frac{A6s}{(s^2+9)^2} + \frac{B(s^2-9)}{(s^2+9)^2} + \frac{Cs+D}{s^2+9}.$$

Combining the terms on the right over a common denominator, we find

$$6 = 6As + B(s^2-9) + (Cs+D)(s^2+9).$$

Collecting like powers of s, we have

$$Cs^3 + (D+B)s^2 + 6As + (D-B) = 6.$$

Therefore, $C=0$, $A=0$, $D+B=0$, and $D-B=\frac{2}{3}$. Solving the last two equations, we find $D=-B=\frac{1}{3}$.

Using these results, we find

$$\frac{6}{(s^2+9)^2} = -\frac{1}{3}\frac{(s^2-9)}{(s^2+9)^2} + \frac{1}{3}\frac{1}{s^2+9}.$$

This is the result we had obtained in the last example using the Convolution Theorem.

10.10 *The Inverse Laplace Transform*

UNTIL THIS POINT WE HAVE SEEN that the inverse Laplace transform can be found by making use of Laplace transform tables and properties of Laplace transforms. This is typically the way Laplace transforms are taught and used in a differential equations course. One can do the same for Fourier transforms. However, in the case of Fourier transforms we introduced an inverse transform in the form of an integral. Does such an inverse integral transform exist for the Laplace transform? Yes, it does! In this section we will derive the inverse Laplace transform integral and show how it is used.

We begin by considering a causal function $f(t)$ which vanishes for $t<0$ and define the function $g(t) = f(t)e^{-ct}$ with $c>0$. For $g(t)$ absolutely integrable,

A function $f(t)$ is said to be of exponential order if $\int_0^\infty |f(t)|e^{-ct}\,dt < \infty$

$$\int_{-\infty}^{\infty} |g(t)|\,dt = \int_0^{\infty} |f(t)|e^{-ct}\,dt < \infty,$$

we can write the Fourier transform,

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}dt = \int_0^{\infty} f(t)e^{i\omega t - ct}dt$$

and the inverse Fourier transform,

$$g(t) = f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)e^{-i\omega t}\,d\omega.$$

Multiplying by e^{ct} and inserting $\hat{g}(\omega)$ into the integral for $g(t)$, we find

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} f(\tau)e^{(i\omega-c)\tau}d\tau e^{-(i\omega-c)t}\,d\omega.$$

Letting $s = c - i\omega$ (so $d\omega = ids$), we have

$$f(t) = \frac{i}{2\pi}\int_{c+i\infty}^{c-i\infty}\int_0^{\infty} f(\tau)e^{-s\tau}d\tau e^{st}\,ds.$$

Note that the inside integral is simply $F(s)$. So, we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds. \tag{10.123}$$

The integral in the last equation is the inverse Laplace transform, called the Bromwich integral and is named after Thomas John I'Anson Bromwich (1875-1929) . This inverse transform is not usually covered in differential equations courses because the integration takes place in the complex plane. This integral is evaluated along a path in the complex plane called the Bromwich contour. The typical way to compute this integral is to first chose c so that all poles are to the left of the contour. This guarantees that $f(t)$ is of exponential type. The contour is closed a semicircle enclosing all of the poles. One then relies on a generalization of Jordan's lemma to the second and third quadrants.[9]

[9] Closing the contour to the left of the contour can be reasoned in a manner similar to what we saw in Jordan's Lemma. Write the exponential as $e^{st} = e^{(s_R+is_I)t} = e^{s_R t}e^{is_I t}$. The second factor is an oscillating factor and the growth in the exponential can only come from the first factor. In order for the exponential to decay as the radius of the semicircle grows, $s_R t < 0$. Since $t>0$, we need $s<0$ which is done by closing the contour to the left. If $t<0$, then the contour to the right would enclose no singularities and preserve the causality of $f(t)$.

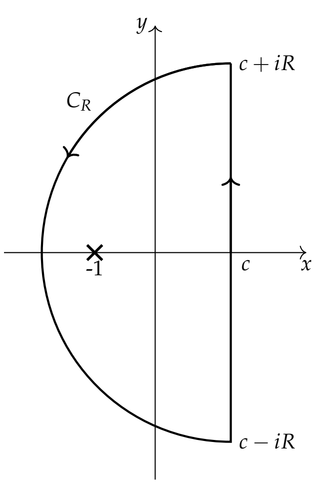

Figure 10.41: The contour used for applying the Bromwich integral to the Laplace transform $F(s) = \frac{1}{s(s+1)}$.

Example 10.36. Find the inverse Laplace transform of $F(s) = \frac{1}{s(s+1)}$.

The integral we have to compute is

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(s+1)}\,ds.$$

This integral has poles at $s=0$ and $s=-1$. The contour we will use is shown in Figure 10.41. We enclose the contour with a semicircle to the left of the path in the complex s-plane. One has to verify that the integral over the semicircle vanishes as the radius goes to infinity. Assuming that we have done this, then the result is simply obtained as $2\pi i$ times the sum of the residues. The residues in this case are:

$$\text{Res}\left[\frac{e^{zt}}{z(z+1)}; z=0\right] = \lim_{z\to 0}\frac{e^{zt}}{(z+1)} = 1$$

and

$$\text{Res}\left[\frac{e^{zt}}{z(z+1)}; z=-1\right] = \lim_{z\to -1}\frac{e^{zt}}{z} = -e^{-t}.$$

Therefore, we have

$$f(t) = 2\pi i\left[\frac{1}{2\pi i}(1) + \frac{1}{2\pi i}(-e^{-t})\right] = 1 - e^{-t}.$$

We can verify this result using the Convolution Theorem or using a partial fraction decomposition. The latter method is simplest. We note that

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

The first term leads to an inverse transform of 1 and the second term gives e^{-t}. So,

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t}.$$

Thus, we have verified the result from doing contour integration.

Example 10.37. Find the inverse Laplace transform of $F(s) = \frac{1}{s(1+e^s)}$.

In this case, we need to compute

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{s(1+e^s)}\,ds.$$

This integral has poles at complex values of s such that $1+e^s = 0$, or $e^s = -1$. Letting $s = x+iy$, we see that

$$e^s = e^{x+iy} = e^x(\cos y + i\sin y) = -1.$$

We see $x = 0$ and y satisfies $\cos y = -1$ and $\sin y = 0$. Therefore, $y = n\pi$ for n an odd integer. Therefore, the integrand has an infinite number of simple poles at $s = n\pi i$, $n = \pm 1, \pm 3, \ldots$. It also has a simple pole at $s = 0$.

In Figure 10.42 we indicate the poles. We need to compute the resides at each pole. At $s = n\pi i$ we have

$$\begin{aligned}\text{Res}\left[\frac{e^{st}}{s(1+e^s)}; s=n\pi i\right] &= \lim_{s\to n\pi i}(s-n\pi i)\frac{e^{st}}{s(1+e^s)}\\ &= \lim_{s\to n\pi i}\frac{e^{st}}{se^s}\\ &= -\frac{e^{n\pi it}}{n\pi i}, \quad n \text{ odd}. \qquad (10.124)\end{aligned}$$

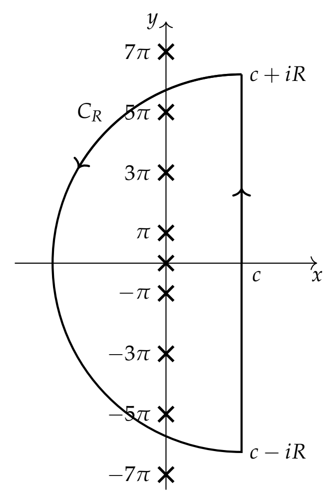

Figure 10.42: The contour used for applying the Bromwich integral to the Laplace transform $F(s) = \frac{1}{1+e^s}$.

At $s = 0$, the residue is

$$\text{Res}\left[\frac{e^{st}}{s(1+e^s)}; s = 0\right] = \lim_{s\to 0} \frac{e^{st}}{1+e^s} = \frac{1}{2}.$$

Summing the residues and noting the exponentials for $\pm n$ can be combined to form sine functions, we arrive at the inverse transform.

$$\begin{aligned} f(t) &= \frac{1}{2} - \sum_{n \text{ odd}} \frac{e^{n\pi i t}}{n\pi i} \\ &= \frac{1}{2} - 2\sum_{k=1}^{\infty} \frac{\sin(2k-1)\pi t}{(2k-1)\pi}. \end{aligned} \tag{10.125}$$

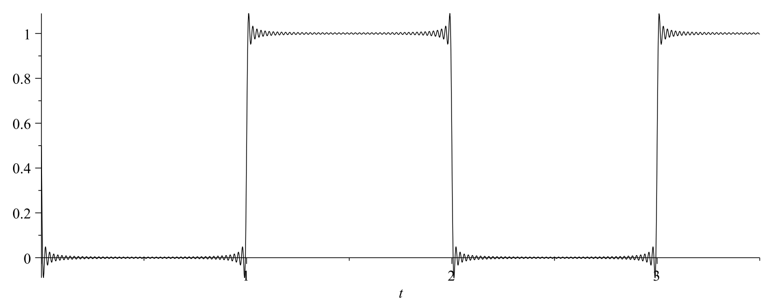

Figure 10.43: Plot of the square wave result as the inverse Laplace transform of $F(s) = \frac{1}{s(1+e^s}$ with 50 terms.

The series in this example might look familiar. It is a Fourier sine series with odd harmonics whose amplitudes decay like $1/n$. It is a vertically shifted square wave. In fact, we had computed the Laplace transform of a general square wave in Example 10.30.

In that example we found

$$\begin{aligned} \mathcal{L}\left[\sum_{n=0}^{\infty}[H(t-2na) - H(t-(2n+1)a)]\right] &= \frac{1-e^{-as}}{s(1-e^{-2as})} \\ &= \frac{1}{s(1+e^{-as})}. \end{aligned} \tag{10.126}$$

In this example, one can show that

$$f(t) = \sum_{n=0}^{\infty}[H(t-2n+1) - H(t-2n)].$$

The reader should verify that this result is indeed the square wave shown in Figure 10.43.

10.11 Transforms and Partial Differential Equations

As another application of the transforms, we will see that we can use transforms to solve some linear partial differential equations. We will first solve the one dimensional heat equation and the two dimensional

Laplace equations using Fourier transforms. The transforms of the partial differential equations lead to ordinary differential equations which are easier to solve. The final solutions are then obtained using inverse transforms.

We could go further by applying a Fourier transform in space and a Laplace transform in time to convert the heat equation into an algebraic equation. We will also show that we can use a finite sine transform to solve nonhomogeneous problems on finite intervals. Along the way we will identify several Green's functions.

10.11.1 Fourier Transform and the Heat Equation

WE WILL FIRST CONSIDER THE SOLUTION OF THE HEAT EQUATION on an infinite interval using Fourier transforms. The basic scheme has been discussed earlier and is outlined in Figure 10.44.

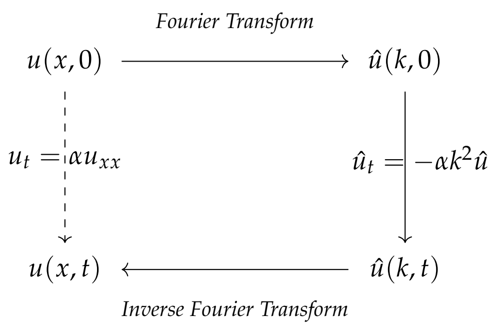

Figure 10.44: Using Fourier transforms to solve a linear partial differential equation.

Consider the heat equation on the infinite line,

$$\begin{aligned} u_t = \alpha u_{xx}, &\quad -\infty < x < \infty, t > 0. \\ u(x,0) = f(x), &\quad -\infty < x < \infty. \end{aligned} \tag{10.127}$$

We can Fourier transform the heat equation using the Fourier transform of $u(x,t)$,

$$\mathcal{F}[u(x,t)] = \hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{ikx}\,dx.$$

We need to transform the derivatives in the equation. First we note that

$$\begin{aligned} \mathcal{F}[u_t] &= \int_{-\infty}^{\infty} \frac{\partial u(x,t)}{\partial t} e^{ikx}\,dx \\ &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{ikx}\,dx \\ &= \frac{\partial \hat{u}(k,t)}{\partial t}. \end{aligned} \tag{10.128}$$

Assuming that $\lim_{|x|\to\infty} u(x,t) = 0$ and $\lim_{|x|\to\infty} u_x(x,t) = 0$, then we also have that

$$\begin{aligned} \mathcal{F}[u_{xx}] &= \int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2} e^{ikx}\,dx \\ &= -k^2\hat{u}(k,t). \end{aligned} \tag{10.129}$$

Therefore, the heat equation becomes

$$\frac{\partial \hat{u}(k,t)}{\partial t} = -\alpha k^2 \hat{u}(k,t).$$

The transformed heat equation.

This is a first order differential equation which is readily solved as

$$\hat{u}(k,t) = A(k)e^{-\alpha k^2 t},$$

where $A(k)$ is an arbitrary function of k. The inverse Fourier transform is

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)e^{-ikx}\,dk. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{A}(k)e^{-\alpha k^2 t}e^{-ikx}\,dk. \end{aligned} \tag{10.130}$$

We can determine $A(k)$ using the initial condition. Note that

$$\mathcal{F}[u(x,0)] = \hat{u}(k,0) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx.$$

But we also have from the solution,

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{A}(k)e^{-ikx}\,dk.$$

Comparing these two expressions for $\hat{u}(k,0)$, we see that

$$A(k) = \mathcal{F}[f(x)].$$

We note that $\hat{u}(k,t)$ is given by the product of two Fourier transforms, $\hat{u}(k,t) = A(k)e^{-\alpha k^2 t}$. So, by the Convolution Theorem, we expect that $u(x,t)$ is the convolution of the inverse transforms,

$$u(x,t) = (f * g)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi,t)g(x-\xi,t)\,d\xi,$$

where

$$g(x,t) = \mathcal{F}^{-1}[e^{-\alpha k^2 t}].$$

In order to determine $g(x,t)$, we need only recall example 10.5. In that example we saw that the Fourier transform of a Gaussian is a Gaussian. Namely, we found that

$$\mathcal{F}[e^{-ax^2/2}] = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a},$$

or,

$$\mathcal{F}^{-1}[\sqrt{\frac{2\pi}{a}}e^{-k^2/2a}] = e^{-ax^2/2}.$$

Applying this to the current problem, we have

$$g(x) = \mathcal{F}^{-1}[e^{-\alpha k^2 t}] = \sqrt{\frac{\pi}{\alpha t}}e^{-x^2/4t}.$$

Finally, we can write down the solution to the problem:

$$u(x,t) = (f * g)(x,t) = \int_{-\infty}^{\infty} f(\xi,t)\frac{e^{-(x-\xi)^2/4t}}{\sqrt{4\pi\alpha t}}\,d\xi,$$

The function in the integrand,

$$K(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi\alpha t}}$$

is called the heat kernel.

$K(x,t)$ is called the heat kernel.

10.11.2 *Laplace's Equation on the Half Plane*

WE CONSIDER A STEADY STATE SOLUTION in two dimensions. In particular, we look for the steady state solution, $u(x,y)$, satisfying the two-dimensional Laplace equation on a semi-infinite slab with given boundary conditions as shown in Figure 10.45. The boundary value problem is given as

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad -\infty < x < \infty, \quad y > 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty \\ \lim_{y\to\infty} u(x,y) &= 0, \quad \lim_{|x|\to\infty} u(x,y) = 0. \end{aligned} \tag{10.131}$$

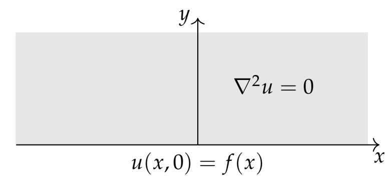

Figure 10.45: This is the domain for a semi-infinite slab with boundary value $u(x,0) = f(x)$ and governed by Laplace's equation.

This problem can be solved using a Fourier transform of $u(x,y)$ with respect to x. The transform scheme for doing this is shown in Figure 10.46. We begin by defining the Fourier transform

$$\hat{u}(k,y) = \mathcal{F}[u] = \int_{-\infty}^{\infty} u(x,y)e^{ikx}\,dx.$$

We can transform Laplace's equation. We first note from the properties of Fourier transforms that

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -k^2\hat{u}(k,y),$$

if $\lim_{|x|\to\infty} u(x,y) = 0$ and $\lim_{|x|\to\infty} u_x(x,y) = 0$. Also,

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial^2\hat{u}(k,y)}{\partial y^2}.$$

Thus, the transform of Laplace's equation gives $\hat{u}_{yy} = k^2\hat{u}$.

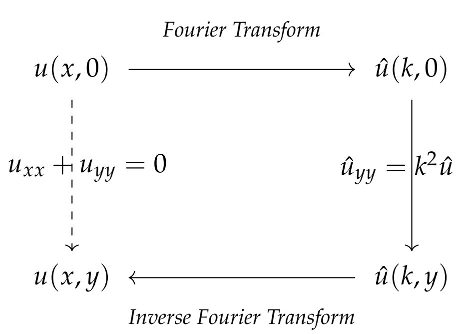

Figure 10.46: The transform scheme used to convert Laplace's equation to an ordinary differential equation which is easier to solve.

This is a simple ordinary differential equation. We can solve this equation using the boundary conditions. The general solution is

The transformed Laplace equation.

$$\hat{u}(k,y) = a(k)e^{ky} + b(k)e^{-ky}.$$

Since $\lim_{y\to\infty} u(x,y) = 0$ and k can be positive or negative, we have that $\hat{u}(k,y) = a(k)e^{-|k|y}$. The coefficient $a(k)$ can be determined using the remaining boundary condition, $u(x,0) = f(x)$. We find that $a(k) = \hat{f}(k)$ since

$$a(k) = \hat{u}(k,0) = \int_{-\infty}^{\infty} u(x,0)e^{ikx}\,dx = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = \hat{f}(k).$$

We have found that $\hat{u}(k,y) = \hat{f}(k)e^{-|k|y}$. We can obtain the solution using the inverse Fourier transform,

$$u(x,t) = \mathcal{F}^{-1}[\hat{f}(k)e^{-|k|y}].$$

We note that this is a product of Fourier transforms and use the Convolution Theorem for Fourier transforms. Namely, we have that $a(k) = \mathcal{F}[f]$ and $e^{-|k|y} = \mathcal{F}[g]$ for $g(x,y) = \frac{2y}{x^2+y^2}$. This last result is essentially proven in Problem 6.

Then, the Convolution Theorem gives the solution

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\frac{2y}{(x-\xi)^2+y^2}\,d\xi. \end{aligned} \tag{10.132}$$

We note for future use, that this solution is in the form

$$u(x,y) = \int_{-\infty}^{\infty} f(\xi)G(x,\xi;y,0)\,d\xi,$$

where

$$G(x,\xi;y,0) = \frac{2y}{\pi((x-\xi)^2+y^2)}$$

is the Green's function for this problem.

The Green's function for the Laplace equation.

10.11.3 Heat Equation on Infinite Interval, Revisited

We will reconsider the initial value problem for the heat equation on an infinite interval,

$$\begin{aligned} u_t = u_{xx}, &\quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), &\quad -\infty < x < \infty. \end{aligned} \tag{10.133}$$

We can apply both a Fourier and a Laplace transform to convert this to an algebraic problem. The general solution will then be written in terms of an initial value Green's function as

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t;\xi,0)f(\xi)\,d\xi.$$

For the time dependence we can use the Laplace transform and for the spatial dependence we use the Fourier transform. These combined transforms lead us to define

$$\hat{u}(k,s) = \mathcal{F}[\mathcal{L}[u]] = \int_{-\infty}^{\infty}\int_{0}^{\infty} u(x,t)e^{-st}e^{ikx}\,dtdx.$$

Applying this to the terms in the heat equation, we have

$$\begin{aligned}\mathcal{F}[\mathcal{L}[u_t]] &= s\hat{u}(k,s) - \mathcal{F}[u(x,0)] \\ &= s\hat{u}(k,s) - \hat{f}(k) \\ \mathcal{F}[\mathcal{L}[u_{xx}]] &= -k^2\hat{u}(k,s). \end{aligned} \tag{10.134}$$

Here we have assumed that

$$\lim_{t\to\infty} u(x,t)e^{-st} = 0, \quad \lim_{|x|\to\infty} u(x,t) = 0, \quad \lim_{|x|\to\infty} u_x(x,t) = 0.$$

Therefore, the heat equation can be turned into an algebraic equation for the transformed solution,

$$(s+k^2)\hat{u}(k,s) = \hat{f}(k),$$

The transformed heat equation.

or

$$\hat{u}(k,s) = \frac{\hat{f}(k)}{s+k^2}.$$

The solution to the heat equation is obtained using the inverse transforms for both the Fourier and Laplace transform. Thus, we have

$$\begin{aligned}u(x,t) &= \mathcal{F}^{-1}[\mathcal{L}^{-1}[\hat{u}]] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\frac{\hat{f}(k)}{s+k^2}e^{st}\,ds\right)e^{-ikx}\,dk. \end{aligned} \tag{10.135}$$

Since the inside integral has a simple pole at $s=-k^2$, we can compute the Bromwich integral by choosing $c > -k^2$. Thus,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\frac{\hat{f}(k)}{s+k^2}e^{st}\,ds = \operatorname{Res}\left[\frac{\hat{f}(k)}{s+k^2}e^{st}; s=-k^2\right] = e^{-k^2t}\hat{f}(k).$$

Inserting this result into the solution, we have

$$\begin{aligned}u(x,t) &= \mathcal{F}^{-1}[\mathcal{L}^{-1}[\hat{u}]] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{-k^2t}e^{-ikx}\,dk. \end{aligned} \tag{10.136}$$

This solution is of the form

$$u(x,t) = \mathcal{F}^{-1}[\hat{f}\hat{g}]$$

for $\hat{g}(k) = e^{-k^2t}$. So, by the Convolution Theorem for Fourier transforms,the solution is a convolution,

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi.$$

All we need is the inverse transform of $\hat{g}(k)$.

We note that $\hat{g}(k) = e^{-k^2t}$ is a Gaussian. Since the Fourier transform of a Gaussian is a Gaussian, we need only recall Example 10.5,

$$\mathcal{F}[e^{-ax^2/2}] = \sqrt{\frac{2\pi}{a}}e^{-k^2/2a}.$$

Setting $a = 1/2t$, this becomes

$$\mathcal{F}[e^{-x^2/4t}] = \sqrt{4\pi t}e^{-k^2 t}.$$

So,

$$g(x) = \mathcal{F}^{-1}[e^{-k^2 t}] = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}.$$

Inserting $g(x)$ into the solution, we have

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4t}\, d\xi \\ &= \int_{-\infty}^{\infty} G(x,t;\xi,0) f(\xi)\, d\xi. \end{aligned} \tag{10.137}$$

Here we have identified the initial value Green's function

$$G(x,t;\xi,0) = \frac{1}{\sqrt{4\pi t}} e^{-(x-\xi)^2/4t}.$$

The initial value Green's function for the heat equation.

10.11.4 Nonhomogeneous Heat Equation

We now consider the nonhomogeneous heat equation with homogeneous boundary conditions defined on a finite interval.

$$\begin{aligned} u_t - k u_{xx} = h(x,t), &\quad 0 \le x \le L, \quad t > 0. \\ u(0,t) = 0, \quad u(L,t) = 0, &\quad t > 0, \\ u(x,0) = f(x), &\quad 0 \le x \le . \end{aligned} \tag{10.138}$$

We know that when $h(x,t) \equiv 0$ the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

So, when $h(x,t) \neq 0$, we might assume that the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}$$

where the b_n's are the finite Fourier sine transform of the desired solution,

$$b_n(t) = \mathcal{F}_s[u] = \frac{2}{L} \int_0^L u(x,t) \sin \frac{n\pi x}{L}\, dx$$

Note that the finite Fourier sine transform is essentially the Fourier sine transform which we encountered in Section 2.4.

The idea behind using the finite Fourier Sine Transform is to solve the given heat equation by transforming the heat equation to a simpler equation for the transform, $b_n(t)$, solve for $bn(t)$, and then do an inverse transform, i.e., insert the $b_n(t)$'s back into the series representation. This is depicted

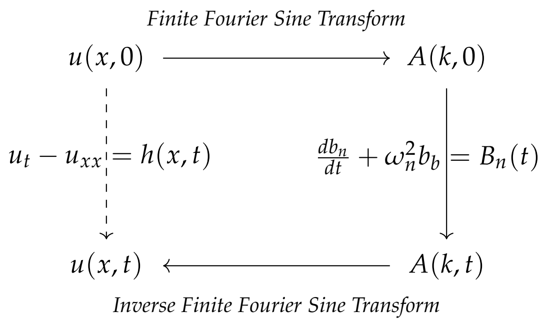

Figure 10.47: Using finite Fourier transforms to solve the heat equation by solving an ODE instead of a PDE.

in Figure 10.47. Note that we had explored similar diagram earlier when discussing the use of transforms to solve differential equations.

First, we need to transform the partial differential equation. The finite transforms of the derivative terms are given by

$$\begin{aligned} \mathcal{F}_s[u_t] &= \frac{2}{L}\int_0^L \frac{\partial u}{\partial t}(x,t)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{d}{dt}\left(\frac{2}{L}\int_0^L u(x,t)\sin\frac{n\pi x}{L}\,dx\right) \\ &= \frac{db_n}{dt}. \end{aligned} \tag{10.139}$$

$$\begin{aligned} \mathcal{F}_s[u_{xx}] &= \frac{2}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)\sin\frac{n\pi x}{L}\,dx \\ &= \left[u_x \sin\frac{n\pi x}{L}\right]_0^L - \left(\frac{n\pi}{L}\right)\frac{2}{L}\int_0^L \frac{\partial u}{\partial x}(x,t)\cos\frac{n\pi x}{L}\,dx \\ &= -\left[\frac{n\pi}{L}u\cos\frac{n\pi x}{L}\right]_0^L - \left(\frac{n\pi}{L}\right)^2\frac{2}{L}\int_0^L u(x,t)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{n\pi}{L}[u(0,t) - u(L,0)\cos n\pi] - \left(\frac{n\pi}{L}\right)^2 b_n^2 \\ &= -\omega_n^2 b_n^2, \end{aligned} \tag{10.140}$$

where $\omega_n = \frac{n\pi}{L}$.

Furthermore, we define

$$H_n(t) = \mathcal{F}_s[h] = \frac{2}{L}\int_0^L h(x,t)\sin\frac{n\pi x}{L}\,dx.$$

Then, the heat equation is transformed to

$$\frac{db_n}{dt} + \omega_n^2 b_n = H_n(t), \quad n = 1,2,3,\ldots.$$

This is a simple linear first order differential equation. We can supplement this equation with the initial condition

$$b_n(0) = \frac{2}{L}\int_0^L u(x,0)\sin\frac{n\pi x}{L}\,dx.$$

The differential equation for b_n is easily solved using the integrating factor, $\mu(t) = e^{\omega_n^2 t}$. Thus,

$$\frac{d}{dt}\left(e^{\omega_n^2 t} b_n(t)\right) = H_n(t)e^{\omega_n^2 t}$$

and the solution is

$$b_n(t) = b_n(0)e^{-\omega_n^2 t} + \int_0^t H_n(\tau)e^{-\omega_n^2(t-\tau)}\,d\tau.$$

The final step is to insert these coefficients (finite Fourier sine transform) into the series expansion (inverse finite Fourier sine transform) for $u(x,t)$. The result is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(0)e^{-\omega_n^2 t}\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\left[\int_0^t H_n(\tau)e^{-\omega_n^2(t-\tau)}\,d\tau\right]\sin\frac{n\pi x}{L}.$$

This solution can be written in a more compact form in order to identify the Green's function. We insert the expressions for $b_n(0)$ and $H_n(t)$ in terms of the initial profile and source term and interchange sums and integrals. This leads to

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L u(\xi,0)\sin\frac{n\pi\xi}{L}\,d\xi\right)e^{-\omega_n^2 t}\sin\frac{n\pi x}{L} \\
&\quad + \sum_{n=1}^{\infty}\left[\int_0^t\left(\frac{2}{L}\int_0^L h(\xi,\tau)\sin\frac{n\pi\xi}{L}\,d\xi\right)e^{-\omega_n^2(t-\tau)}\,d\tau\right]\sin\frac{n\pi x}{L} \\
&= \int_0^L u(\xi,0)\left[\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2 t}\right]d\xi \\
&\quad + \int_0^t\int_0^L h(\xi,\tau)\left[\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2(t-\tau)}\right] \\
&= \int_0^L u(\xi,0)G(x,\xi;t,0)d\xi + \int_0^t\int_0^L h(\xi,\tau)G(x,\xi;t,\tau)\,d\xi d\tau.
\end{aligned} \tag{10.141}$$

Here we have defined the Green's function

$$G(x,\xi;t,\tau) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2(t-\tau)}.$$

We note that $G(x,\xi;t,0)$ gives the initial value Green's function.

Note that at $t=\tau$,

$$G(x,\xi;t,t) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}.$$

This is actually the series representation of the Dirac delta function. The Fourier sine transform of the delta function is

$$\mathcal{F}_s[\delta(x-\xi)] = \frac{2}{L}\int_0^L \delta(x-\xi)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\sin\frac{n\pi\xi}{L}.$$

Then, the representation becomes

$$\delta(x-\xi) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}.$$

Also, we note that

$$\frac{\partial G}{\partial t} = -\omega_n^2 G$$

$$\frac{\partial^2 G}{\partial x^2} = -\left(\frac{n\pi}{L}\right)^2 G.$$

Therefore, $G_t = G_{xx}$ at least for $\tau \neq t$ and $\xi \neq x$.

We can modify this problem by adding nonhomogeneous boundary conditions.

$$\begin{aligned} u_t - k u_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= A, \quad u(L,t) = B, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{10.142}$$

One way to treat these conditions is to assume $u(x,t) = w(x) + v(x,t)$ where $v_t - k v_{xx} = h(x,t)$ and $w_{xx} = 0$. Then, $u(x,t) = w(x) + v(x,t)$ satisfies the original nonhomogeneous heat equation.

If $v(x,t)$ satisfies $v(0,t) = v(L,t) = 0$ and $w(x)$ satisfies $w(0) = A$ and $w(L) = B$, then $u(0,t) = w(0) + v(0,t) = A$ $u(L,t) = w(L) + v(L,t) = B$

Finally, we note that

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x).$$

Therefore, $u(x,t) = w(x) + v(x,t)$ satisfies the original problem if

$$\begin{aligned} v_t - k v_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= f(x) - w(x), \quad 0 \le x \le L. \end{aligned} \tag{10.143}$$

and

$$\begin{aligned} w_{xx} &= 0, \quad 0 \le x \le L, \\ w(0) &= A, \quad w(L) = B. \end{aligned} \tag{10.144}$$

We can solve the last problem to obtain $w(x) = A + \frac{B-A}{L}x$. The solution to the problem for $v(x,t)$ is simply the problem we had solved already in terms of Green's functions with the new initial condition, $f(x) - A - \frac{B-A}{L}x$.

10.11.5 *Solution of the 3D Poisson Equation*

WE RECALL FROM ELECTROSTATICS THAT THE GRADIENT OF THE ELECTRIC POTENTIAL gives the electric field, $\mathbf{E} = -\nabla\phi$. However, we also have from Gauss' Law for electric fields $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$, where $\rho(\mathbf{r})$ is the charge distribution at position $\mathbf{r}$. Combining these equations, we arrive at Poisson's equation for the electric potential,

Poisson's equation for the electric potential.

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}.$$

We note that Poisson's equation also arises in Newton's theory of gravitation for the gravitational potential in the form $\nabla^2\phi = -4\pi G\rho$ where ρ is the matter density.

Poisson's equation for the gravitational potential.

We consider Poisson's equation in the form

$$\nabla^2 \phi(\mathbf{r}) = -4\pi f(\mathbf{r})$$

for $\mathbf{r}$ defined throughout all space. We will seek a solution for the potential function using a three dimensional Fourier transform. In the electrostatic problem $f = \rho(\mathbf{r})/4\pi\epsilon_0$ and the gravitational problem has $f = G\rho(\mathbf{r})$

The Fourier transform can be generalized to three dimensions as

$$\hat{\phi}(\mathbf{k}) = \int_V \phi(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r,$$

where the integration is over all space, V, $d^3r = dxdydz$, and $\mathbf{k}$ is a three dimensional wavenumber, $\mathbf{k} = k_x\mathbf{i} + k_y\mathbf{j} + k_z\mathbf{k}$. The inverse Fourier transform can then be written as

$$\phi(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{V_k} \hat{\phi}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3k,$$

where $d^3k = dk_x dk_y dk_z$ and V_k is all of k-space.

Three dimensional Fourier transform.

The Fourier transform of the Laplacian follows from computing Fourier transforms of any derivatives that are present. Assuming that ϕ and its gradient vanish for large distances, then

$$\mathcal{F}[\nabla^2\phi] = -(k_x^2 + k_y^2 + k_z^2)\hat{\phi}(\mathbf{k}).$$

Defining $k^2 = k_x^2 + k_y^2 + k_z^2$, then Poisson's equation becomes the algebraic equation

$$k^2\hat{\phi}(\mathbf{k}) = 4\pi \hat{f}(\mathbf{k}).$$

Solving for $\hat{\phi}(\mathbf{k})$, we have

$$\phi(\mathbf{k}) = \frac{4\pi}{k^2}\hat{f}(\mathbf{k}).$$

The solution to Poisson's equation is then determined from the inverse Fourier transform,

$$\phi(\mathbf{r}) = \frac{4\pi}{(2\pi)^3} \int_{V_k} \hat{f}(\mathbf{k}) \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2}\, d^3k. \tag{10.145}$$

First we will consider an example of a point charge (or mass in the gravitational case) at the origin. We will set $f(\mathbf{r}) = f_0\delta^3(\mathbf{r})$ in order to represent a point source. For a unit point charge, $f_0 = 1/4\pi\epsilon_0$.

The three dimensional Dirac delta function, $\delta^3(\mathbf{r} - \mathbf{r}_0)$.

Here we have introduced the three dimensional Dirac delta function which, like the one dimensional case, vanishes outside the origin and satisfies a unit volume condition,

$$\int_V \delta^3(\mathbf{r})\, d^3r = 1.$$

Also, there is a sifting property, which takes the form

$$\int_V \delta^3(\mathbf{r} - \mathbf{r}_0) f(\mathbf{r})\, d^3r = f(\mathbf{r}_0).$$

In Cartesian coordinates,

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z),$$

$$\int_V \delta^3(\mathbf{r})\, d^3r = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\, dxdydz = 1,$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) f(x,y,z)\, dxdydz = f(x_0,y_0,z_0).$$

One can define similar delta functions operating in two dimensions and n dimensions.

We can also transform the Cartesian form into curvilinear coordinates. From Section 6.9 we have that the volume element in curvilinear coordinates is

$$d^3r = dxdydz = h_1h_2h_3du_1du_2du_3,$$

where .

This gives

$$\int_V \delta^3(\mathbf{r})\, d^3r = \int_V \delta^3(\mathbf{r})\, h_1h_2h_3du_1du_2du_3 = 1.$$

Therefore,

$$\begin{aligned} \delta^3(\mathbf{r}) &= \frac{\delta(u_1)}{\left|\frac{\partial \mathbf{r}}{\partial u_1}\right|}\frac{\delta(u_2)}{\left|\frac{\partial \mathbf{r}}{\partial u_2}\right|}\frac{\delta(u_3)}{\left|\frac{\partial \mathbf{r}}{\partial u_2}\right|} \\ &= \frac{1}{h_1h_2h_3}\delta(u_1)\delta(u_2)\delta(u_3). \end{aligned} \tag{10.146}$$

So, for cylindrical coordinates,

$$\delta^3(\mathbf{r}) = \frac{1}{r}\delta(r)\delta(\theta)\delta(z).$$

Example 10.38. Find the solution of Poisson's equation for a point source of the form $f(\mathbf{r}) = f_0\delta^3(\mathbf{r})$.

The solution is found by inserting the Fourier transform of this source into Equation (10.145) and carrying out the integration. The transform of $f(\mathbf{r})$ is found as

$$\hat{f}(\mathbf{k}) = \int_V f_0\delta^3(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r = f_0.$$

Inserting $\hat{f}(\mathbf{k})$ into the inverse transform in Equation (10.145) and carrying out the integration using spherical coordinates in k-space, we find

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{4\pi}{(2\pi)^3}\int_{V_k} f_0 \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2}\, d^3k \\ &= \frac{f_0}{2\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{-ikx\cos\theta}}{k^2}\, k^2\sin\theta\, dkd\theta d\phi \\ &= \frac{f_0}{\pi}\int_0^{\pi}\int_0^{\infty} e^{-ikx\cos\theta}\sin\theta\, dkd\theta \\ &= \frac{f_0}{\pi}\int_0^{\infty}\int_{-1}^{1} e^{-ikxy}\, dkdy, \quad y = \cos\theta, \\ &= \frac{2f_0}{\pi r}\int_0^{\infty}\frac{\sin z}{z}\, dz = \frac{f_0}{r}. \end{aligned} \tag{10.147}$$

If the last example is applied to a unit point charge, then $f_0 = 1/4\pi\epsilon_0$. So, the electric potential outside a unit point charge located at the origin becomes

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r}.$$

This is the form familiar from introductory physics.

Also, by setting $f_0 = 1$, we have also shown in the last example that

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r}).$$

Since $\nabla \left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$, then we have also shown that

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = 4\pi\delta^3(\mathbf{r}).$$

Problems

1. In this problem you will show that the sequence of functions

$$f_n(x) = \frac{n}{\pi}\left(\frac{1}{1+n^2x^2}\right)$$

approaches $\delta(x)$ as $n \to \infty$. Use the following to support your argument:

a. Show that $\lim_{n\to\infty} f_n(x) = 0$ for $x \neq 0$.

b. Show that the area under each function is one.

2. Verify that the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$, defined by $f_n(x) = \frac{n}{2}e^{-n|x|}$, approaches a delta function.

3. Evaluate the following integrals:

a. $\int_0^{\pi} \sin x\delta\left(x - \frac{\pi}{2}\right)\, dx$.

b. $\int_{-\infty}^{\infty} \delta\left(\frac{x-5}{3}e^{2x}\right)\left(3x^2 - 7x + 2\right)\, dx$.

c. $\int_0^{\pi} x^2\delta\left(x + \frac{\pi}{2}\right)\, dx$.

d. $\int_0^{\infty} e^{-2x}\delta(x^2 - 5x + 6)\, dx$. [See Problem 4.]

e. $\int_{-\infty}^{\infty}(x^2 - 2x + 3)\delta(x^2 - 9)\, dx$. [See Problem 4.]

4. For the case that a function has multiple roots, $f(x_i) = 0$, $i = 1, 2, \ldots$, it can be shown that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}.$$

Use this result to evaluate $\int_{-\infty}^{\infty} \delta(x^2 - 5x - 6)(3x^2 - 7x + 2)\, dx$.

5. Find a Fourier series representation of the Dirac delta function, $\delta(x)$, on $[-L, L]$.

6. For $a > 0$, find the Fourier transform, $\hat{f}(k)$, of $f(x) = e^{-a|x|}$.

7. Use the result from the last problem plus properties of the Fourier transform to find the Fourier transform, of $f(x) = x^2e^{-a|x|}$ for $a > 0$.

8. Find the Fourier transform, $\hat{f}(k)$, of $f(x) = e^{-2x^2+x}$.

9. Prove the second shift property in the form

$$F\left[e^{i\beta x} f(x)\right] = \hat{f}(k+\beta).$$

10. A damped harmonic oscillator is given by

$$f(t) = \begin{cases} Ae^{-\alpha t}e^{i\omega_0 t}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

.

a. Find $\hat{f}(\omega)$ and

b. the frequency distribution $|\hat{f}(\omega)|^2$.

c. Sketch the frequency distribution.

11. Show that the convolution operation is associative: $(f * (g * h))(t) = ((f * g) * h)(t)$.

12. In this problem you will directly compute the convolution of two Gaussian functions in two steps.

a. Use completing the square to evaluate

$$\int_{-\infty}^{\infty} e^{-\alpha t^2 + \beta t}\, dt.$$

b. Use the result from part a to directly compute the convolution in example 10.16:

$$(f * g)(x) = e^{-bx^2} \int_{-\infty}^{\infty} e^{-(a+b)t^2 + 2bxt}\, dt.$$

13. You will compute the (Fourier) convolution of two box functions of the same width. Recall the box function is given by

$$f_a(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a. \end{cases}$$

Consider $(f_a * f_a)(x)$ for different intervals of x. A few preliminary sketches would help. In Figure 10.48 the factors in the convolution integrand are show for one value of x. The integrand is the product of the first two functions. The convolution at x is the area of the overlap in the third figure. Think about how these pictures change as you vary x. Plot the resulting areas as a function of x. This is the graph of the desired convolution.

14. Define the integrals $I_n = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx$. Noting that $I_0 = \sqrt{\pi}$,

a. Find a recursive relation between I_n and I_{n-1}.

b. Use this relation to determine I_1, I_2 and I_3.

c. Find an expression in terms of n for I_n.

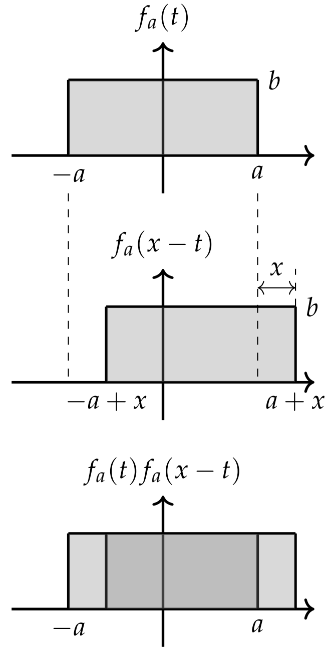

Figure 10.48: Sketch used to compute the convolution of the box function with itself. In the top figure is the box function. The second figure shows the box shifted by x. The last figure indicates the overlap of the functions.

15. Find the Laplace transform of the following functions.

a. $f(t) = 9t^2 - 7$.

b. $f(t) = e^{5t-3}$.

c. $f(t) = \cos 7t$.

d. $f(t) = e^{4t} \sin 2t$.

e. $f(t) = e^{2t}(t + \cosh t)$.

f. $f(t) = t^2 H(t-1)$.

g. $f(t) = \begin{cases} \sin t, & t < 4\pi, \\ \sin t + \cos t, & t > 4\pi \end{cases}$.

h. $f(t) = \int_0^t (t-u)^2 \sin u \, du$.

i. $f(t) = (t+5)^2 + te^{2t} \cos 3t$ and write the answer in the simplest form.

16. Find the inverse Laplace transform of the following functions using the properties of Laplace transforms and the table of Laplace transform pairs.

a. $F(s) = \dfrac{18}{s^3} + \dfrac{7}{s}$.

b. $F(s) = \dfrac{1}{s-5} - \dfrac{2}{s^2+4}$.

c. $F(s) = \dfrac{s+1}{s^2+1}$.

d. $F(s) = \dfrac{3}{s^2+2s+2}$.

e. $F(s) = \dfrac{1}{(s-1)^2}$.

f. $F(s) = \dfrac{e^{-3s}}{s^2 - 1}$.

g. $F(s) = \dfrac{1}{s^2 + 4s - 5}$.

h. $F(s) = \dfrac{s + 3}{s^2 + 8s + 17}$.

17. Compute the convolution $(f * g)(t)$ (in the Laplace transform sense) and its corresponding Laplace transform $\mathcal{L}[f * g]$ for the following functions:

a. $f(t) = t^2$, $g(t) = t^3$.

b. $f(t) = t^2$, $g(t) = \cos 2t$.

c. $f(t) = 3t^2 - 2t + 1$, $g(t) = e^{-3t}$.

d. $f(t) = \delta\left(t - \frac{\pi}{4}\right)$, $g(t) = \sin 5t$.

18. For the following problems draw the given function and find the Laplace transform in closed form.

a. $f(t) = 1 + \sum_{n=1}^{\infty}(-1)^n H(t - n)$.

b. $f(t) = \sum_{n=0}^{\infty}[H(t - 2n + 1) - H(t - 2n)]$.

c. $f(t) = \sum_{n=0}^{\infty}(t - 2n)[H(t - 2n) - H(t - 2n - 1)] + (2n + 2 - t)[H(t - 2n - 1) - H(t - 2n - 2)]$.

19. Use the convolution theorem to compute the inverse transform of the following:

a. $F(s) = \dfrac{2}{s^2(s^2 + 1)}$.

b. $F(s) = \dfrac{e^{-3s}}{s^2}$.

c. $F(s) = \dfrac{1}{s(s^2 + 2s + 5)}$.

20. Find the inverse Laplace transform two different ways: i) Use Tables. ii) Use the Bromwich Integral.

a. $F(s) = \dfrac{1}{s^3(s + 4)^2}$.

b. $F(s) = \dfrac{1}{s^2 - 4s - 5}$.

c. $F(s) = \dfrac{s + 3}{s^2 + 8s + 17}$.

d. $F(s) = \dfrac{s + 1}{(s - 2)^2(s + 4)}$.

e. $F(s) = \dfrac{s^2 + 8s - 3}{(s^2 + 2s + 1)(s^2 + 1)}$.

21. Use Laplace transforms to solve the following initial value problems. Where possible, describe the solution behavior in terms of oscillation and decay.

a. $y'' - 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 0$.

b. $y'' - y = te^{2t}$, $y(0) = 0$, $y'(0) = 1$.

c. $y'' + 4y = \delta(t-1)$, $y(0) = 3$, $y'(0) = 0$.

d. $y'' + 6y' + 18y = 2H(\pi - t)$, $y(0) = 0$, $y'(0) = 0$.

22. Use Laplace transforms to convert the following system of differential equations into an algebraic system and find the solution of the differential equations.

$$\begin{aligned} x'' &= 3x - 6y, \quad x(0) = 1, \quad x'(0) = 0, \\ y'' &= x + y, \quad\quad y(0) = 0, \quad y'(0) = 0. \end{aligned}$$

23. Use Laplace transforms to convert the following nonhomogeneous systems of differential equations into an algebraic system and find the solutions of the differential equations.

a.

$$\begin{aligned} x' &= 2x + 3y + 2\sin 2t, \quad x(0) = 1, \\ y' &= -3x + 2y, \quad y(0) = 0. \end{aligned}$$

b.

$$\begin{aligned} x' &= -4x - y + e^{-t}, \quad x(0) = 2, \\ y' &= x - 2y + 2e^{-3t}, \quad y(0) = -1. \end{aligned}$$

c.

$$\begin{aligned} x' &= x - y + 2\cos t, \quad x(0) = 3, \\ y' &= x + y - 3\sin t, \quad y(0) = 2. \end{aligned}$$

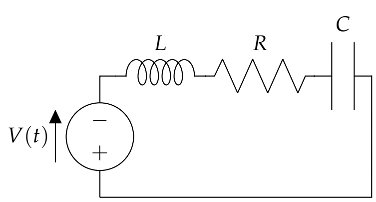

Figure 10.49: Series LRC Circuit.

24. Consider the series circuit in Problem 2.20 and in Figure 10.49 with $L = 1.00$ H, $R = 1.00 \times 10^2\ \Omega$, $C = 1.00 \times 10^{-4}$ F, and $V_0 = 1.00 \times 10^3$ V.

a. Write the second order differential equation for this circuit.

b. Suppose that no charge is present and no current is flowing at time $t = 0$ when V_0 is applied. Use Laplace transforms to find the current and the charge on the capacitor as functions of time.

b. Replace the battery with the alternating source $V(t) = V_0 \sin 2\pi f t$ with $V_0 = 1.00 \times 10^3$ V and $f = 150Hz$. Again, suppose that no charge is present and no current is flowing at time $t = 0$ when the AC source is applied. Use Laplace transforms to find the current and the charge on the capacitor as functions of time.

d. Plot your solutions and describe how the system behaves over time.

25. Use Laplace transforms to sum the following series.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{1+2n}$.

b. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

c. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+3)}$.

d. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2-a^2}$.

e. $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2-a^2}$.

f. $\sum_{n=1}^{\infty} \frac{1}{n} e^{-an}$.

26. Use Laplace transforms to prove

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{b-a} \int_0^1 \frac{u^a - u^b}{1-u}\, du.$$

Use this result to evaluate the sums

a. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

b. $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$.

27. Do the following.

a. Find the first four nonvanishing terms of the Maclaurin series expansion of $f(x) = \frac{x}{e^x - 1}$.

b. Use the result in part a. to determine the first four nonvanishing Bernoulli numbers, B_n.

c. Use these results to compute $\zeta(2n)$ for $n = 1, 2, 3, 4$.

28. Given the following Laplace transforms, $F(s)$, find the function $f(t)$. Note that in each case there are an infinite number of poles, resulting in an infinite series representation.

a. $F(s) = \frac{1}{s^2(1+e^{-s})}$.

b. $F(s) = \frac{1}{s \sinh s}$.

c. $F(s) = \frac{\sinh s}{s^2 \cosh s}$.

d. $F(s) = \frac{\sinh(\beta\sqrt{s}x)}{s \sinh(\beta\sqrt{s}L)}$.

29. Consider the initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= 2u_{xx}, & 0 < t, \quad 0 \le x \le 1, \\ u(x,0) &= x(1-x), & 0 < x < 1, \\ u(0,t) &= 0, & t > 0, \\ u(1,t) &= 0, & t > 0. \end{aligned}$$

Use the finite transform method to solve this problem. Namely, assume that the solution takes the form $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin n\pi x$ and obtain an ordinary differential equation for b_n and solve for the b_n's for each n.

30. The telegraph equation is given by

$$u_{tt} + 2\beta u_t + \alpha u = c^2 u_{xx}, \quad |x| < \infty, t > 0.$$

Use the Fourier transform to solve this problem for $\alpha = \beta^2$, satisfying the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$.

31. Use Fourier transforms to express the solution of the following problem as a simple integral involving the initial condition.

$$\begin{aligned} u_t &= t^2 u_{xx}, & |x| < \infty, t > 0, \\ u(x,0) &= f(x), & |x| < \infty. \end{aligned}$$

32. Consider the linear first order problem

$$u_t + u_x + u = 0, \quad x, t > 0$$

with the conditions $u(0,t) = 0$, $t > 0$, and $u(x,0) = \sin x$, $x > 0$.

a. Solve this problem using the Laplace transform $U(x,s) = \mathcal{L}[u(x,t)]$.
b. In Example 1.2 we used the Method of Characteristics to solve a similar problem. By modifying that example, show that the general solution is given by $u(x,y) = G(y-x)e^{-x}$. Use this solution to find the particular solution satisfying the given conditions. Show that these solutions are the same.

33. The wave equation for a flat profile, semi-infinite string that is at rest is given by

$$\begin{aligned} u_{tt} &= u_{xx}, & 0 < x < \infty, \quad t > 0, \\ u(x,0) &= 0, & u_t(x,0) = 0. \end{aligned}$$

Now, send a pulse down the string by imposing the time dependent boundary condition ($t > 0$)

$$u(0,t) = \begin{cases} \sin t, & 0 \le t \le \pi \\ 0, & \text{otherwise.} \end{cases}$$

Assuming that the solution remains bounded, use Laplace transforms to find the solution. For $c = 1$, plot the solution at several times to show the evolution of the pulse.

34. Simultaneously apply the Fourier and Laplace transforms to solve the inhomogeneous heat equation

$$u_t - ku_{xx} = f(x)\delta(t), \quad |x| < \infty, t > 0,$$

with the boundary conditions $u(x,0) = 0$, $|x| < \infty$, and $\lim_{|x|\to 0} u(x,t) = 0$. First obtain an algebraic equation for

$$\hat{U}(k,s) = \int_0^\infty \int_{-\infty}^\infty u(x,t) e^{ikx-st}\, dxdt.$$

Solve for $\hat{U}(k,s)$, and invert the transform of the solution, using the Convolution Theorem for Fourier transforms, to obtain a solution in a form in which one can identify a Green's function.

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