11

More Numerical Methods

If we walk in the woods, we must feed mosquitoes.- Ralph Waldo Emerson (1803-1882), 'Prudence', Essays,

IN THIS CHAPTER WE WILL build on Chapter 3 and also discuss other numerical methods for solving partial differential equations. We begin with the advection equation.

11.1 The Finite Difference Method for the Advection Equation

WE RECALL THE ADVECTION EQUATION FROM SECTION 7.4.2. We will develop a finite difference scheme for solving the linear advection equation $u_t + cu_x = 0$ given $u(x,0) = f(x)$. We approximate the derivatives using the forward difference approximation.

$$\frac{\partial u}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}. \tag{11.1}$$

The domain of the solution is $x \in [a,b]$ and $t \geq 0$ and we seek approximate values of $u(x,t)$ at specific positions and times. We create a grid in the xt-plane by first dividing the interval $[a,b]$ into N subintervals of width $\Delta x = (b-a)/N$. Then, the endpoints of the subintervals are given by

$$x_i = a + i\Delta x, \quad i = 0, 1, \ldots, N.$$

Note that $x_0 = a$ and $x_N = b$. Similarly, we take time steps of Δt, at times

$$t_j = j\Delta t, \quad j = 0, 1, 2, \ldots.$$

This gives a grid of points (x_i, t_j) in the domain as shown in Figure 11.1.

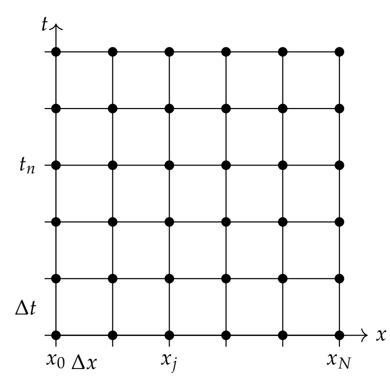

Figure 11.1: The grid of points (x_i, t_j) used to discretize the xt domain, where $x_i = a + i\Delta x, \quad i = 0, 1, \ldots, N$ and $t_j = j\Delta t, \quad j = 0, 1, 2, \ldots$.

Next, we want to approximate the derivatives in the advection equation. We recall the discussion in Chapter 3. Recall that the partial derivative, u_t, is defined by

$$\frac{\partial u}{\partial t} = \lim_{\Delta t \to \infty} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}.$$

Therefore, we can use the approximation

$$\frac{\partial u}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}. \tag{11.2}$$

This is called a forward difference approximation.

Other approximations can be found using Taylor series expansions of $u(x,t)$ about x or t. For example, we know that

$$u(x,t+\Delta t) = u(x,t) + u_t(x,t)\Delta t + \frac{1}{2!}u_{tt}(x,t)\Delta t^2 + \frac{1}{3!}u_{ttt}(x,t)\Delta t^3 + \cdots. \tag{11.3}$$

Then,

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = u_t(x,t) + \frac{1}{2!}u_{tt}(x,t)\Delta t + \frac{1}{3!}u_{ttt}(x,t)\Delta t^2 + \cdots. \tag{11.4}$$

Then, we can write

$$u_t(x,t) = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t). \tag{11.5}$$

Thus, the forward difference approximation in first order in Δt.

Similarly, we have

$$u(x,t-\Delta t) = u(x,t) - u_t(x,t)\Delta t + \frac{1}{2!}u_{tt}(x,t)\Delta t^2 - \frac{1}{3!}u_{ttt}(x,t)\Delta t^3 + \cdots. \tag{11.6}$$

$$u_t(x,t) = \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} + O(\Delta t). \tag{11.7}$$

This the backward difference approximation.

We can use both expansions to find

$$u(x,t+\Delta t) - u(x,t-\Delta t) = 2u_t(x,t)\Delta t + \frac{2}{3!}u_{ttt}\Delta t^3 + \cdots.$$

Solving for u_t, we arrive at the central difference approximation,

$$u_t(x,t) = \frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t} + O(\Delta t^2). \tag{11.8}$$

Notice that it is second order in Δt.

Now we turn to the advection equation, $u_t + cu_x = 0$. We have a number of choices for how to approximate the derivatives. Let's use forward difference in space and time. Then,

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + c\frac{u(x+\Delta x,t) - u(x,t)}{\Delta x} = 0.$$

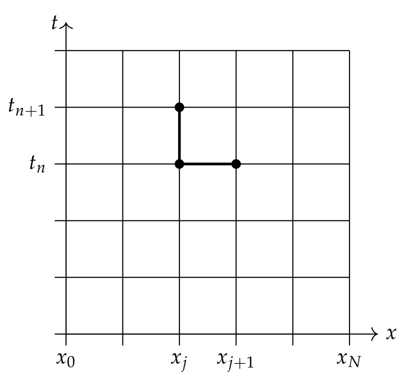

Figure 11.2: Stencil for the scheme for the advection equation in Equation 11.9.

At each grid point in the domain we seek an approximate solution to the advection equation, $u_{i,j} \approx u(x_i, t_j)$. So, we can rewrite the approximations

$$\frac{u_{j,n+1} - u_{j,n}}{\Delta t} + c\frac{u_{j+1,n} - u_{j,n}}{\Delta x}.$$

This is an equation involving only three points, $u_{j,n}$, $u_{j+1,n}$, and $u_{j,n+1}$. In Figure 11.3 we show these points. This provides a stencil that we can use to solve the advection equation. Notice that if we know the solution at the two points at time t_n, then we can approximate the solution $u_{j,n+1}$. So, we solve the scheme for

$$u_{i,j+1} \approx u_{i,j} + \alpha\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right], \tag{11.9}$$

where we have defined $\alpha = \frac{c\Delta t}{\Delta x}$.

Next, we apply the initial condition, $u(x,0) = f(x)$. This provides us with exact values along the x-axis. Name;y, we have

$$u_{j,0} = f(x_j).$$

In Figure **??**(a) we indicate that we know these values using red dots. Laying the stencil on the far left of these dots and sliding to the right points to the row $t = \Delta t$ and we place black dots where we can predict new solution approximations.

Continuing this process results in knowing the solution at the indicated points in Figure **??**(b). Notice how the whole domain is not filled. This means we need more data. One way to provide more data is to provide boundary conditions. In Table 11.1 we provide MATLAB code for implementing this scheme with fixed boundary conditions, $u(0,t) = 0$, $u(b,t) = 0$.

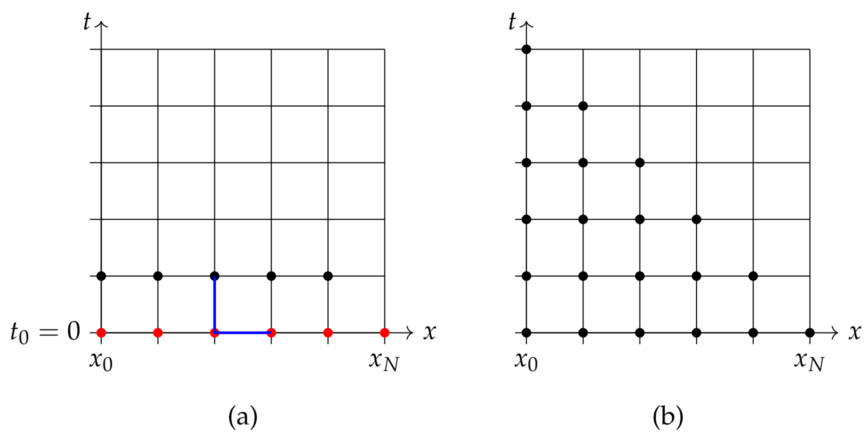

Figure 11.3: Applying the stencil for advection equation in Equation **??**. (a) The solution is known initially (red dots) and the stencil is used to find solutions at the next time step (black dots). (b) Continuing this process results in knowing the solution at the indicated points.

Table 11.1: Numerical solution of the linear advection equation using backward difference in time and forward difference in space.

```
% First Order - Advection Backward Difference in Time

clear
% Specify grid
a=-5;
b=10;
Tmax=2;
Nx=50;
Nt=25;
dx=(b-a)/Nx;
dt=Tmax/Nt;
x=linspace(a,b,Nx);
c=2;
alpha=c*dt/dx;

% Initial profile
u0=exp(-(x-.5).^2);

u=zeros(Nx,Nt);
u(:,1)=u0;

plot(x,u0)
M(1)=getframe;

hold

for n=1:Nt-1;
    for j=2:Nx
        u(j,n+1)=(1-alpha)*u(j,n)+alpha*u(j-1,n);
    end
    u(1,n+1)=0;
    plot(x,u(:,n+1))
    M(n+1)=getframe;
end
hold
```