9
Complex Representations of Functions

"He is not a true man of science who does not bring some sympathy to his studies, and expect to learn something by behavior as well as by application. It is childish to rest in the discovery of mere coincidences, or of partial and extraneous laws. The study of geometry is a petty and idle exercise of the mind, if it is applied to no larger system than the starry one. Mathematics should be mixed not only with physics but with ethics; that is mixed mathematics. The fact which interests us most is the life of the naturalist. The purest science is still biographical." Henry David Thoreau (1817-1862)

9.1 Complex Representations of Waves

We have seen that we can determine the frequency content of a function $f(t)$ defined on an interval $[0, T]$ by looking for the Fourier coefficients in the Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}.$$

The coefficients take forms like

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi nt}{T}\,dt.$$

However, trigonometric functions can be written in a complex exponential form. Using Euler's formula, which was obtained using the Maclaurin expansion of e^x in Example A.36,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

the complex conjugate is found by replacing i with $-i$ to obtain

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding these expressions, we have

$$2\cos\theta = e^{i\theta} + e^{-i\theta}.$$

Subtracting the exponentials leads to an expression for the sine function. Thus, we have the important result that sines and cosines can be written as complex exponentials:

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}, \\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i}. \end{aligned} \tag{9.1}$$

So, we can write

$$\cos\frac{2\pi nt}{T} = \frac{1}{2}(e^{\frac{2\pi int}{T}} + e^{-\frac{2\pi int}{T}}).$$

Later we will see that we can use this information to rewrite the series as a sum over complex exponentials in the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi int}{T}},$$

where the Fourier coefficients now take the form

$$c_n = \int_0^T f(t)e^{-\frac{2\pi int}{T}}\,dt.$$

In fact, when one considers the representation of analogue signals defined over an infinite interval and containing a continuum of frequencies, we will see that Fourier series sums become integrals of complex functions and so do the Fourier coefficients. Thus, we will naturally find ourselves needing to work with functions of complex variables and perform complex integrals.

We can also develop a complex representation for waves. Recall from the discussion in Section 2.6 on finite length strings that a solution to the wave equation was given by

$$u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} A_n \sin k_n(x+ct) + \sum_{n=1}^{\infty} A_n \sin k_n(x-ct)\right]. \tag{9.2}$$

We can replace the sines with their complex forms as

$$\begin{aligned} u(x,t) &= \frac{1}{4i}\left[\sum_{n=1}^{\infty} A_n\left(e^{ik_n(x+ct)} - e^{-ik_n(x+ct)}\right)\right. \\ &\quad \left. + \sum_{n=1}^{\infty} A_n\left(e^{ik_n(x-ct)} - e^{-ik_n(x-ct)}\right)\right]. \end{aligned} \tag{9.3}$$

Defining $k_{-n} = -k_n$, $n = 1, 2, \ldots$, we can rewrite this solution in the form

$$u(x,t) = \sum_{n=-\infty}^{\infty}\left[c_n e^{ik_n(x+ct)} + d_n e^{ik_n(x-ct)}\right]. \tag{9.4}$$

Such representations are also possible for waves propagating over the entire real line. In such cases we are not restricted to discrete frequencies and wave numbers. The sum of the harmonics will then be a sum over a continuous range, which means that the sums become integrals. So, we are lead to the complex representation

$$u(x,t) = \int_{-\infty}^{\infty}\left[c(k)e^{ik(x+ct)} + d(k)e^{ik(x-ct)}\right] dk. \tag{9.5}$$

The forms $e^{ik(x+ct)}$ and $e^{ik(x-ct)}$ are complex representations of what are called plane waves in one dimension. The integral represents a general wave form consisting of a sum over plane waves. The Fourier coefficients in the representation can be complex valued functions and the evaluation of the integral may be done using methods from complex analysis. We would like to be able to compute such integrals.

With the above ideas in mind, we will now take a tour of complex analysis. We will first review some facts about complex numbers and then introduce complex functions. This will lead us to the calculus of functions of a complex variable, including the differentiation and integration complex functions. This will set up the methods needed to explore Fourier transforms in the next chapter.

9.2 *Complex Numbers*

COMPLEX NUMBERS WERE FIRST INTRODUCED in order to solve some simple problems. The history of complex numbers only extends about five hundred years. In essence, it was found that we need to find the roots of equations such as $x^2+1=0$. The solution is $x=\pm\sqrt{-1}$. Due to the usefulness of this concept, which was not realized at first, a special symbol was introduced - the imaginary unit, $i=\sqrt{-1}$. In particular, Girolamo Cardano $(1501-1576)$ was one of the first to use square roots of negative numbers when providing solutions of cubic equations. However, complex numbers did not become an important part of mathematics or science until the late seventh and eighteenth centuries after people like Abraham de Moivre (1667-1754), the Bernoulli[1] family and Euler took them seriously.

[1] The Bernoulli's were a family of Swiss mathematicians spanning three generations. It all started with Jacob Bernoulli (1654-1705) and his brother Johann Bernoulli (1667-1748). Jacob had a son, Nicolaus Bernoulli (1687-1759) and Johann (1667-1748) had three sons, Nicolaus Bernoulli II (1695-1726), Daniel Bernoulli (1700-1872), and Johann Bernoulli II (1710-1790). The last generation consisted of Johann II's sons, Johann Bernoulli III (1747-1807) and Jacob Bernoulli II (1759-1789). Johann, Jacob and Daniel Bernoulli were the most famous of the Bernoulli's. Jacob studied with Leibniz, Johann studied under his older brother and later taught Leonhard Euler and Daniel Bernoulli, who is known for his work in hydrodynamics.

A complex number is a number of the form $z=x+iy$, where x and y are real numbers. x is called the real part of z and y is the imaginary part of z. Examples of such numbers are $3+3i$, $-1i=-i$, $4i$ and 5. Note that $5=5+0i$ and $4i=0+4i$.

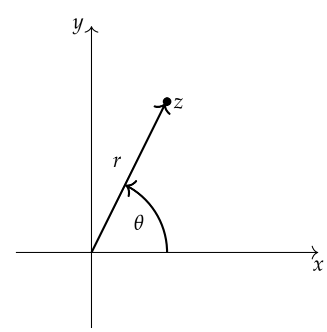


Figure 9.1: The Argand diagram for plotting complex numbers in the complex z-plane.

There is a geometric representation of complex numbers in a two dimensional plane, known as the complex plane C. This is given by the Argand diagram as shown in Figure 9.1. Here we can think of the complex number $z=x+iy$ as a point (x,y) in the z-complex plane or as a vector. The magnitude, or length, of this vector is called the complex modulus of z, denoted by $|z|=\sqrt{x^2+y^2}$. We can also use the geometric picture to develop a polar representation of complex numbers. From Figure 9.1 we can see that in terms of r and θ we have that

The complex modulus, $|z|=\sqrt{x^2+y^2}$.

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned} \tag{9.6}$$

Thus,

$$z=x+iy=r(\cos\theta+i\sin\theta)=re^{i\theta}. \tag{9.7}$$

Complex numbers can be represented in rectangular (Cartesian), $z=x+iy$, or polar form, $z=re^{i\theta}$. Here we define the argument, θ, and modulus, $|z|=r$ of complex numbers.

So, given r and θ we have $z=re^{i\theta}$. However, given the Cartesian form,

$z = x + iy$, we can also determine the polar form, since

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \tan\theta &= \frac{y}{x}. \end{aligned} \tag{9.8}$$

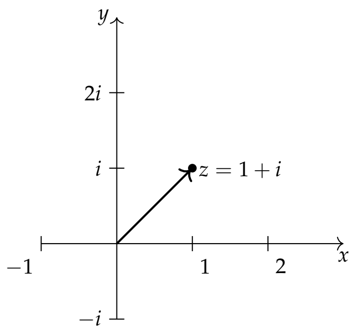


Figure 9.2: Locating $1 + i$ in the complex z-plane.

Note that $r = |z|$.

Locating $1 + i$ in the complex plane, it is possible to immediately determine the polar form from the angle and length of the "complex vector." This is shown in Figure 9.2. It is obvious that $\theta = \frac{\pi}{4}$ and $r = \sqrt{2}$.

Example 9.1. Write $z = 1 + i$ in polar form.

If one did not see the polar form from the plot in the z-plane, then one could systematically determine the results. First, write $z = 1 + i$ in polar form, $z = re^{i\theta}$, for some r and θ.

Using the above relations between polar and Cartesian representations, we have $r = \sqrt{x^2 + y^2} = \sqrt{2}$ and $\tan\theta = \frac{y}{x} = 1$. This gives $\theta = \frac{\pi}{4}$. So, we have found that

$$1 + i = \sqrt{2}e^{i\pi/4}.$$

We can also define binary operations of addition, subtraction, multiplication and division of complex numbers to produce a new complex number.

We can easily add, subtract, multiply and divide complex numbers.

The addition of two complex numbers is simply done by adding the real and imaginary parts of each number. So,

$$(3 + 2i) + (1 - i) = 4 + i.$$

Subtraction is just as easy,

$$(3 + 2i) - (1 - i) = 2 + 3i.$$

We can multiply two complex numbers just like we multiply any binomials, though we now can use the fact that $i^2 = -1$. For example, we have

$$(3 + 2i)(1 - i) = 3 + 2i - 3i + 2i(-i) = 5 - i.$$

We can even divide one complex number into another one and get a complex number as the quotient. Before we do this, we need to introduce the complex conjugate, $\bar{z}$, of a complex number. The complex conjugate of $z = x + iy$, where x and y are real numbers, is given as

The complex conjugate of $z = x + iy$, is given as $\bar{z} = x - iy$.

$$\bar{z} = x - iy.$$

Complex conjugates satisfy the following relations for complex numbers z and w and real number x.

$$\begin{aligned} \overline{z + w} &= \bar{z} + \bar{w} \\ \overline{zw} &= \bar{z}\bar{w} \\ \bar{\bar{z}} &= z \\ \bar{x} &= x. \end{aligned} \tag{9.9}$$

One consequence is that the complex conjugate of $re^{i\theta}$ is

$$\overline{re^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = re^{-i\theta}.$$

Another consequence is that

$$z\bar{z} = re^{i\theta}re^{-i\theta} = r^2.$$

Thus, the product of a complex number with its complex conjugate is a real number. We can also prove this result using the Cartesian form

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2.$$

Now we are in a position to write the quotient of two complex numbers in the standard form of a real plus an imaginary number.

Example 9.2. Simplify the expression $z = \frac{3+2i}{1-i}$.

This simplification is accomplished by multiplying the numerator and denominator of this expression by the complex conjugate of the denominator:

$$z = \frac{3+2i}{1-i} = \frac{3+2i}{1-i}\frac{1+i}{1+i} = \frac{1+5i}{2}.$$

Therefore, the quotient is a complex number and in standard form it is given by $z = \frac{1}{2} + \frac{5}{2}i$.

We can also consider powers of complex numbers. For example,

$$(1+i)^2 = 2i,$$

$$(1+i)^3 = (1+i)(2i) = 2i - 2.$$

But, what is $(1+i)^{1/2} = \sqrt{1+i}$?

In general, we want to find the nth root of a complex number. Let $t = z^{1/n}$. To find t in this case is the same as asking for the solution of

$$z = t^n$$

given z. But, this is the root of an nth degree equation, for which we expect n roots. If we write z in polar form, $z = re^{i\theta}$, then we would naively compute

$$\begin{aligned} z^{1/n} &= \left(re^{i\theta}\right)^{1/n} \\ &= r^{1/n}e^{i\theta/n} \\ &= r^{1/n}\left[\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right]. \end{aligned} \tag{9.10}$$

For example,

$$(1+i)^{1/2} = \left(\sqrt{2}e^{i\pi/4}\right)^{1/2} = 2^{1/4}e^{i\pi/8}.$$

But this is only one solution. We expected two solutions for $n = 2$..

The function $f(z) = z^{1/n}$ is multivalued. $z^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}$, $k = 0, 1, \ldots, n-1$.

The reason we only found one solution is that the polar representation for z is not unique. We note that

$$\boxed{e^{2k\pi i} = 1, \quad k = 0, \pm 1, \pm 2, \ldots.}$$

So, we can rewrite z as $z = re^{i\theta}e^{2k\pi i} = re^{i(\theta+2k\pi)}$. Now, we have that

$$\boxed{z^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1.}$$

Note that these are the only distinct values for the roots. We can see this by considering the case $k = n$. Then, we find that

$$e^{i(\theta+2\pi i n)/n} = e^{i\theta/n}e^{2\pi i} = e^{i\theta/n}.$$

So, we have recovered the $n = 0$ value. Similar results can be shown for the other k values larger than n.

Now, we can finish the example we had started.

Example 9.3. Determine the square roots of $1+i$, or $\sqrt{1+i}$.

As we have seen, we first write $1+i$ in polar form, $1+i = \sqrt{2}e^{i\pi/4}$. Then, introduce $e^{2k\pi i} = 1$ and find the roots:

$$\begin{aligned} (1+i)^{1/2} &= \left(\sqrt{2}e^{i\pi/4}e^{2k\pi i}\right)^{1/2}, \quad k = 0, 1, \\ &= 2^{1/4}e^{i(\pi/8+k\pi)}, \quad k = 0, 1, \\ &= 2^{1/4}e^{i\pi/8}, 2^{1/4}e^{9\pi i/8}. \end{aligned} \tag{9.11}$$

The nth roots of unity, $\sqrt[n]{1}$.

Finally, what is $\sqrt[n]{1}$? Our first guess would be $\sqrt[n]{1} = 1$. But, we now know that there should be n roots. These roots are called the nth roots of unity. Using the above result with $r = 1$ and $\theta = 0$, we have that

$$\sqrt[n]{1} = \left[\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right], \quad k = 0, \ldots, n-1.$$

For example, we have

$$\sqrt[3]{1} = \left[\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right], \quad k = 0, 1, 2.$$

These three roots can be written out as

$$\sqrt[3]{1} = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

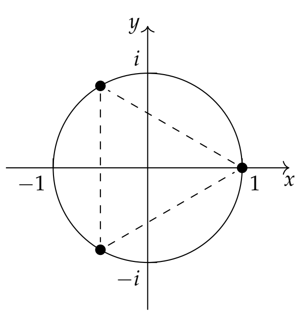


Figure 9.3: Locating the cube roots of unity in the complex z-plane.

We can locate these cube roots of unity in the complex plane. In Figure 9.3 we see that these points lie on the unit circle and are at the vertices of an equilateral triangle. In fact, all nth roots of unity lie on the unit circle and are the vertices of a regular n-gon with one vertex at $z = 1$.

9.3 *Complex Valued Functions*

WE WOULD LIKE TO NEXT EXPLORE complex functions and the calculus of complex functions. We begin by defining a function that takes complex numbers into complex numbers, $f : C \to C$. It is difficult to visualize such functions. For real functions of one variable, $f : R \to R$, we graph these functions by first drawing two intersecting copies of R and then proceed to map the domain into the range of f.

It would be more difficult to do this for complex functions. Imagine placing together two orthogonal copies of the complex plane, C. One would need a four dimensional space in order to complete the visualization. Instead, typically uses two copies of the complex plane side by side in order to indicate how such functions behave. Over the years there have been several ways to visualize complex functions. We will describe a few of these in this chapter.

We will assume that the domain lies in the z-plane and the image lies in the w-plane. We will then write the complex function as $w = f(z)$. We show these planes in Figure 9.4 and the mapping between the planes.

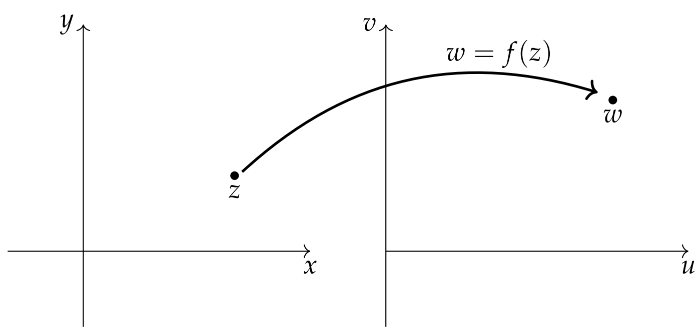


Figure 9.4: Defining a complex valued function, $w = f(z)$, on C for $z = x + iy$ and $w = u + iv$.

Letting $z = x + iy$ and $w = u + iv$, we can write the real and imaginary parts of $f(z)$:

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

We see that one can view this function as a function of z or a function of x and y. Often, we have an interest in writing out the real and imaginary parts of the function, $u(x, y)$ and $v(x, y)$, which are functions of two real variables, x and y. We will look at several functions to determine the real and imaginary parts.

Example 9.4. Find the real and imaginary parts of $f(z) = z^2$.

For example, we can look at the simple function $f(z) = z^2$. It is a simple matter to determine the real and imaginary parts of this function. Namely, we have

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Therefore, we have that

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

In Figure 9.5 we show how a grid in the z-plane is mapped by $f(z) = z^2$ into the w-plane. For example, the horizontal line $x =$

1 is mapped to $u(1, y) = 1 - y^2$ and $v(1, y) = 2y$. Eliminating the "parameter" y between these two equations, we have $u = 1 - v^2/4$. This is a parabolic curve. Similarly, the horizontal line $y = 1$ results in the curve $u = v^2/4 - 1$.

If we look at several curves, x =const and y =const, then we get a family of intersecting parabolae, as shown in Figure 9.5.

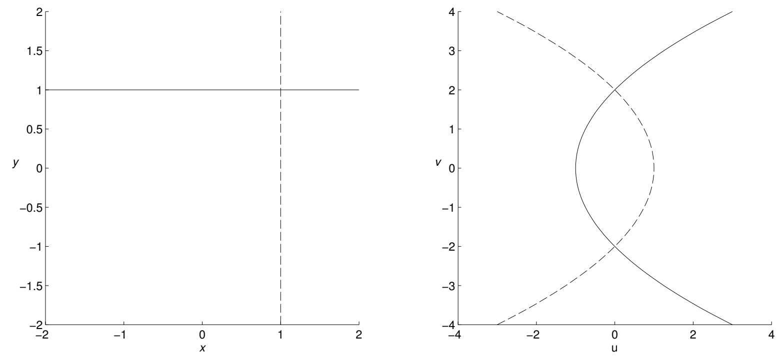


Figure 9.5: 2D plot showing how the function $f(z) = z^2$ maps the lines $x = 1$ and $y = 1$ in the z-plane into parabolae in the w-plane.

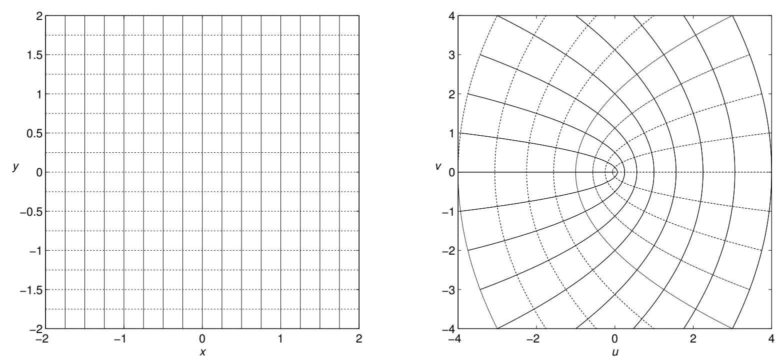


Figure 9.6: 2D plot showing how the function $f(z) = z^2$ maps a grid in the z-plane into the w-plane.

Example 9.5. Find the real and imaginary parts of $f(z) = e^z$.

For this case, we make use of Euler's Formula.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y). \end{aligned} \tag{9.12}$$

Thus, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. In Figure 9.7 we show how a grid in the z-plane is mapped by $f(z) = e^z$ into the w-plane.

Example 9.6. Find the real and imaginary parts of $f(z) = z^{1/2}$.

We have that

$$z^{1/2} = \sqrt{x^2 + y^2} \left(\cos(\theta + k\pi) + i \sin(\theta + k\pi)\right), \quad k = 0, 1. \tag{9.13}$$

Thus,

$$u = |z| \cos(\theta + k\pi), u = |z| \cos(\theta + k\pi),$$

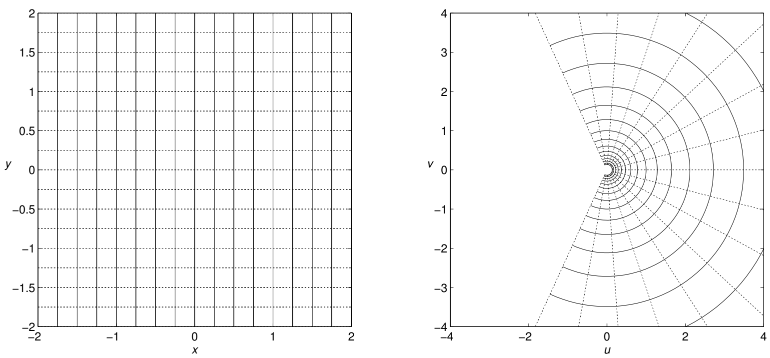


Figure 9.7: 2D plot showing how the function $f(z) = e^z$ maps a grid in the z-plane into the w-plane.

for $|z| = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. For each k-value one has a different surface and curves of constant θ give $u/v = c_1$, and curves of constant nonzero complex modulus give concentric circles, $u^2 + v^2 = c_2$, for c_1 and c_2 constants.

Example 9.7. Find the real and imaginary parts of $f(z) = \ln z$.

In this case we make use of the polar form of a complex number, $z = re^{i\theta}$. Our first thought would be to simply compute

$$\ln z = \ln r + i\theta.$$

However, the natural logarithm is multivalued, just like the square root function. Recalling that $e^{2\pi ik} = 1$ for k an integer, we have $z = re^{i(\theta+2\pi k)}$. Therefore,

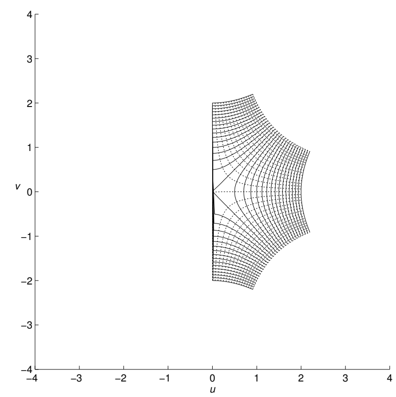


Figure 9.8: 2D plot showing how the function $f(z) = \sqrt{z}$ maps a grid in the z-plane into the w-plane.

$$\boxed{\ln z = \ln r + i(\theta + 2\pi k), \qquad k = \text{ integer.}}$$

The natural logarithm is a multivalued function. In fact there are an infinite number of values for a given z. Of course, this contradicts the definition of a function that you were first taught.

Thus, one typically will only report the principal value, $\text{Log } z = \ln r + i\theta$, for θ restricted to some interval of length 2π, such as $[0, 2\pi)$. In order to account for the multivaluedness, one introduces a way to extend the complex plane so as to include all of the branches. This is done by assigning a plane to each branch, using (branch) cuts along lines, and then gluing the planes together at the branch cuts to form what is called a Riemann surface. We will not elaborate upon this any further here and refer the interested reader to more advanced texts. Comparing the multivalued logarithm to the principal value logarithm, we have

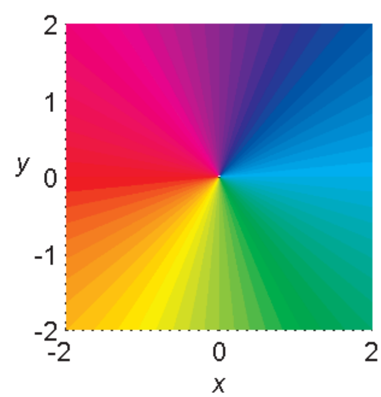


Figure 9.9: Domain coloring of the complex z-plane assigning colors to $\arg(z)$.

$$\ln z = \text{Log } z + 2n\pi i.$$

We should not that some books use $\log z$ instead of $\ln z$. It should not be confused with the common logarithm.

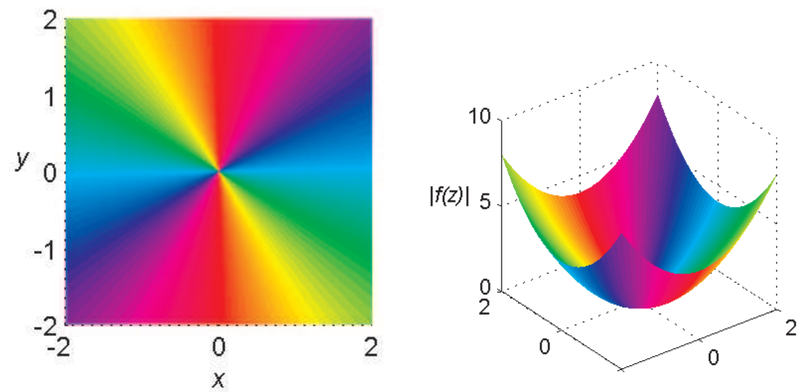


Figure 9.10: Domain coloring for $f(z) = z^2$. The left figure shows the phase coloring. The right figure show the colored surface with height $|f(z)|$.

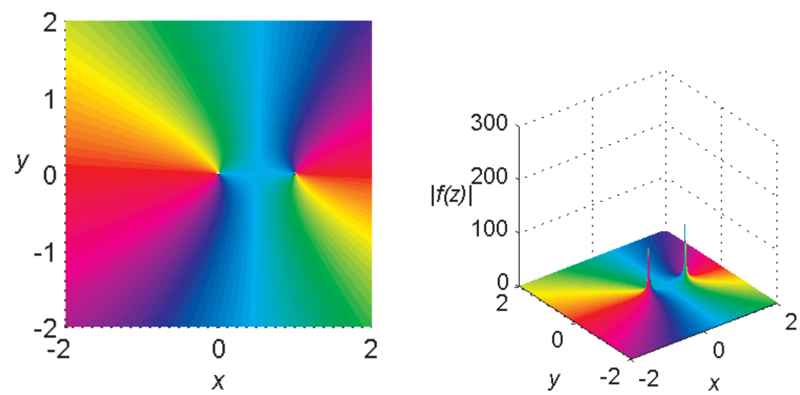


Figure 9.11: Domain coloring for $f(z) = 1/z(1-z)$. The left figure shows the phase coloring. The right figure show the colored surface with height $|f(z)|$.

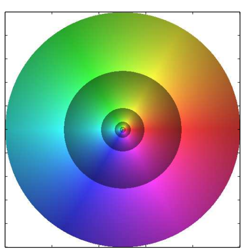


Figure 9.12: Domain coloring for the function $f(z) = z$ showing a coloring for $\arg(z)$ and brightness based on $|f(z)|$.

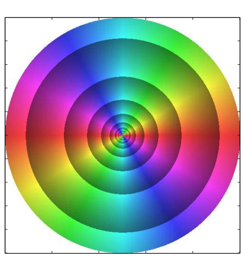


Figure 9.13: Domain coloring for the function $f(z) = z^2$.

9.3.1 *Complex Domain Coloring*

Another method for visualizing complex functions is domain coloring. The idea was described by Frank A. Farris. There are a few approaches to this method. The main idea is that one colors each point of the z-plane (the domain) according to $\arg(z)$ as shown in Figure 9.9. The modulus, $|f(z)|$ is then plotted as a surface. Examples are shown for $f(z) = z^2$ in Figure 9.10 and $f(z) = 1/z(1-z)$ in Figure 9.11.

We would like to put all of this information in one plot. We can do this by adjusting the brightness of the colored domain by using the modulus of the function. In the plots that follow we use the fractional part of $\ln|z|$. In Figure 9.12 we show the effect for the z-plane using $f(z) = z$. In the figures that follow we look at several other functions. In these plots we have chosen to view the functions in a circular window.

One can see the rich behavior hidden in these figures. As you progress in your reading, especially after the next chapter, you should return to these figures and locate the zeros, poles, branch points and branch cuts. A search online will lead you to other colorings and superposition of the uv grid on these figures.

As a final picture, we look at iteration in the complex plane. Consider the function $f(z) = z^2 - 0.75 - 0.2i$. Interesting figures result when studying the iteration in the complex plane. In Figure 9.15 we show $f(z)$ and $f^{20}(z)$, which is the iteration of f twenty times. It leads to an interesting coloring. What happens when one keeps iterating? Such iterations lead to the study of Julia and Mandelbrot sets . In Figure 9.16 we show six iterations of $f(z) = (1 - i/2)\sin x$.

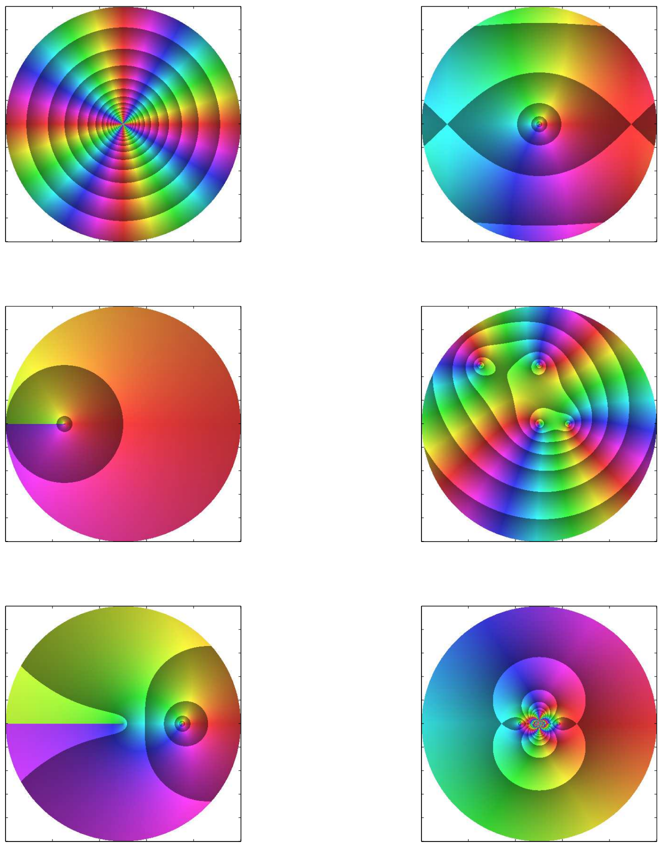


Figure 9.14: Domain coloring for several functions. On the top row the domain coloring is shown for $f(z) = z^4$ and $f(z) = \sin z$. On the second row plots for $f(z) = \sqrt{1+z}$ and $f(z) = \frac{1}{z(1/2-z)(z-i)(z-i+1)}$ are shown. In the last row domain colorings for $f(z) = \ln z$ and $f(z) = \sin(1/z)$ are shown.

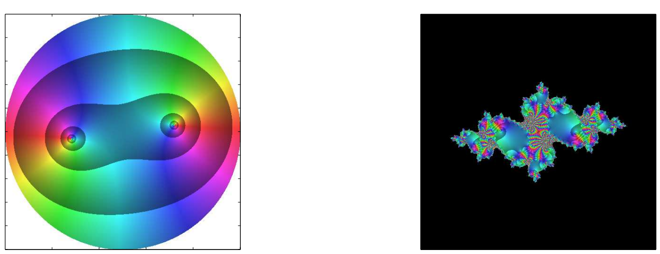


Figure 9.15: Domain coloring for $f(z) = z^2 - 0.75 - 0.2i$. The left figure shows the phase coloring. On the right is the plot for $f^{20}(z)$.

The following code was used in MATLAB to produce these figures.

```
fn = @(x) (1-i/2)*sin(x);
xmin=-2; xmax=2; ymin=-2; ymax=2;
Nx=500;
Ny=500;
x=linspace(xmin,xmax,Nx);
```

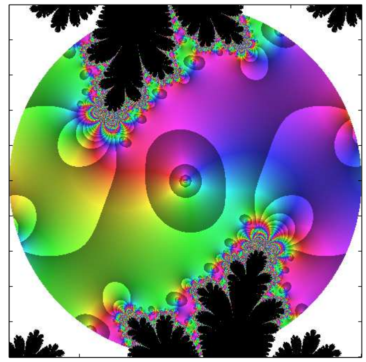


Figure 9.16: Domain coloring for six iterations of $f(z) = (1 - i/2)\sin x$.

```
y=linspace(ymin,ymax,Ny);
[X,Y] = meshgrid(x,y); z = complex(X,Y);
tmp=z; for n=1:6
    tmp = fn(tmp);
end Z=tmp;
XX=real(Z);
YY=imag(Z);
R2=max(max(X.^2));
R=max(max(XX.^2+YY.^2));

circle(:,:,1) = X.^2+Y.^2 < R2;
circle(:,:,2)=circle(:,:,1);
circle(:,:,3)=circle(:,:,1);

addcirc(:,:,1)=circle(:,:,1)==0;
addcirc(:,:,2)=circle(:,:,1)==0;
addcirc(:,:,3)=circle(:,:,1)==0;

warning off MATLAB:divideByZero;
hsvCircle=ones(Nx,Ny,3);
hsvCircle(:,:,1)=atan2(YY,XX)*180/pi+(atan2(YY,XX)*180/pi<0)*360;
hsvCircle(:,:,1)=hsvCircle(:,:,1)/360; lgz=log(XX.^2+YY.^2)/2;
hsvCircle(:,:,2)=0.75; hsvCircle(:,:,3)=1-(lgz-floor(lgz))/2;
hsvCircle(:,:,1) = flipud((hsvCircle(:,:,1)));

hsvCircle(:,:,2) = flipud((hsvCircle(:,:,2)));

hsvCircle(:,:,3) =flipud((hsvCircle(:,:,3)));

rgbCircle=hsv2rgb(hsvCircle);
rgbCircle=rgbCircle.*circle+addcirc;
```

```
image(rgbCircle)
axis square
set(gca,'XTickLabel',{})
set(gca,'YTickLabel',{})
```

9.4 Complex Differentiation

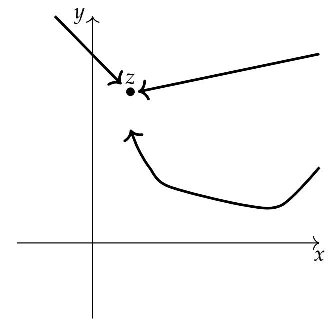


Figure 9.17: There are many paths that approach z as $\Delta z \to 0$.

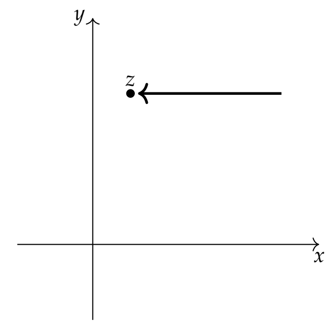


Figure 9.18: A path that approaches z with $y =$ constant.

NEXT WE WANT TO DIFFERENTIATE COMPLEX FUNCTIONS. We generalize the definition from single variable calculus,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}, \tag{9.14}$$

provided this limit exists.

The computation of this limit is similar to what one sees in multivariable calculus for limits of real functions of two variables. Letting $z = x + iy$ and $\delta z = \delta x + i\delta y$, then

$$z + \delta x = (x + \delta x) + i(y + \delta y).$$

Letting $\Delta z \to 0$ means that we get closer to z. There are many paths that one can take that will approach z. [See Figure 9.17.]

It is sufficient to look at two paths in particular. We first consider the path $y =$ constant. This horizontal path is shown in Figure 9.18. For this path, $\Delta z = \Delta x + i\Delta y = \Delta x$, since y does not change along the path. The derivative, if it exists, is then computed as

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - (u(x, y) + iv(x, y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \lim_{\Delta x \to 0} i\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}. \end{aligned} \tag{9.15}$$

The last two limits are easily identified as partial derivatives of real valued functions of two variables. Thus, we have shown that when $f'(z)$ exists,

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.} \tag{9.16}$$

A similar computation can be made if instead we take the vertical path, $x =$ constant, in Figure 9.17). In this case $\Delta z = i\Delta y$ and

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - (u(x, y) + iv(x, y))}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y}. \end{aligned} \tag{9.17}$$

Therefore,

$$\boxed{f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.} \tag{9.18}$$

We have found two different expressions for $f'(z)$ by following two different paths to z. If the derivative exists, then these two expressions must be the same. Equating the real and imaginary parts of these expressions, we have

The Cauchy-Riemann Equations.

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned}} \tag{9.19}$$

These are known as the Cauchy-Riemann equations[2].

[2] Augustin-Louis Cauchy (1789-1857) was a French mathematician well known for his work in analysis. Georg Friedrich Bernhard Riemann (1826-1866) was a German mathematician who made major contributions to geometry and analysis.

Theorem 9.1. *$f(z)$ is holomorphic (differentiable) if and only if the Cauchy-Riemann equations are satisfied.*

Example 9.8. $f(z) = z^2$.

In this case we have already seen that $z^2 = x^2 - y^2 + 2ixy$. Therefore, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We first check the Cauchy-Riemann equations.

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= 2y = -\frac{\partial u}{\partial y}. \end{aligned} \tag{9.20}$$

Therefore, $f(z) = z^2$ is differentiable.

We can further compute the derivative using either Equation (9.16) or Equation (9.18). Thus,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i(2y) = 2z.$$

This result is not surprising.

Example 9.9. $f(z) = \bar{z}$.

In this case we have $f(z) = x - iy$. Therefore, $u(x,y) = x$ and $v(x,y) = -y$. But, $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial y} = -1$. Thus, the Cauchy-Riemann equations are not satisfied and we conclude the $f(z) = \bar{z}$ is not differentiable.

Harmonic functions satisfy Laplace's equation.

Another consequence of the Cauchy-Riemann equations is that both $u(x,y)$ and $v(x,y)$ are harmonic functions. A real-valued function $u(x,y)$ is harmonic if it satisfies Laplace's equation in 2D, $\nabla^2 u = 0$, or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Theorem 9.2. *$f(z) = u(x,y) + iv(x,y)$ is differentiable if and only if u and v are harmonic functions.*

This is easily proven using the Cauchy-Riemann equations.

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \\
&= \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\
&= -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\
&= -\frac{\partial^2 u}{\partial y^2}. \qquad (9.21)
\end{aligned}$$

Example 9.10. Is $u(x,y) = x^2 + y^2$ harmonic?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 \neq 0.$$

No, it is not.

Example 9.11. Is $u(x,y) = x^2 - y^2$ harmonic?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

Yes, it is.

The harmonic conjugate function.

Given a harmonic function $u(x,y)$, can one find a function, $v(x,y)$, such $f(z) = u(x,y) + iv(x,y)$ is differentiable? In this case, v are called the harmonic conjugate of u.

Example 9.12. Find the harmonic conjugate of $u(x,y) = x^2 - y^2$ and determine $f(z) = u + iv$ such that $u + iv$ is differentiable.

The Cauchy-Riemann equations tell us the following about the unknown function, $v(x,y)$:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y,$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x.$$

We can integrate the first of these equations to obtain

$$v(x,y) = \int 2y\, dx = 2xy + c(y).$$

Here $c(y)$ is an arbitrary function of y. One can check to see that this works by simply differentiating the result with respect to x.

However, the second equation must also hold. So, we differentiate the result with respect to y to find that

$$\frac{\partial v}{\partial y} = 2x + c'(y).$$

Since we were supposed to get $2x$, we have that $c'(y) = 0$. Thus, $c(y) = k$ is a constant.

We have just shown that we get an infinite number of functions,

$$v(x, y) = 2xy + k,$$

such that

$$f(z) = x^2 - y^2 + i(2xy + k)$$

is differentiable. In fact, for $k = 0$ this is nothing other than $f(z) = z^2$.

9.5 Complex Integration

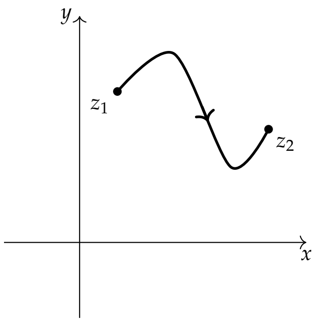


Figure 9.19: We would like to integrate a complex function $f(z)$ over the path Γ in the complex plane.

WE HAVE INTRODUCED FUNCTIONS OF A COMPLEX VARIABLE. We also established when functions are differentiable as complex functions, or holomorphic. In this chapter we will turn to integration in the complex plane. We will learn how to compute complex path integrals, or contour integrals. We will see that contour integral methods are also useful in the computation of some of the real integrals that we will face when exploring Fourier transforms in the next chapter.

9.5.1 Complex Path Integrals

IN THIS SECTION WE WILL INVESTIGATE the computation of complex path integrals. Given two points in the complex plane, connected by a path Γ as shown in Figure 9.19, we would like to define the integral of $f(z)$ along Γ,

$$\int_\Gamma f(z)\,dz.$$

A natural procedure would be to work in real variables, by writing

$$\int_\Gamma f(z)\,dz = \int_\Gamma [u(x, y) + iv(x, y)]\,(dx + idy),$$

since $z = x + iy$ and $dz = dx + idy$.

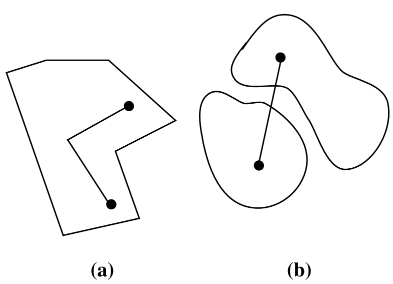


Figure 9.20: Examples of (a) a connected set and (b) a disconnected set.

In order to carry out the integration, we then have to find a parametrization of the path and use methods from a multivariate calculus class. Namely, let u and v be continuous in domain D, and Γ a piecewise smooth curve in D. Let $(x(t), y(t))$ be a parametrization of Γ for $t_0 \leq t \leq t_1$ and $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$. Then

$$\boxed{\int_\Gamma f(z)\,dz = \int_{t_0}^{t_1} [u(x(t), y(t)) + iv(x(t), y(t))]\,(\frac{dx}{dt} + i\frac{dy}{dt})dt. \qquad (9.22)}$$

Here we have used

$$dz = dx + idy = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt.$$

Furthermore, a set D is called a domain if it is both open and connected.

Before continuing, we first define open and connected. A set D is connected if and only if for all z_1, and z_2 in D there exists a piecewise smooth curve connecting z_1 to z_2 and lying in D. Otherwise it is called disconnected. Examples are shown in Figure 9.20

A set D is open if and only if for all z_0 in D there exists an open disk $|z - z_0| < \rho$ in D. In Figure 9.21 we show a region with two disks.

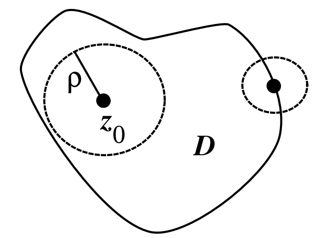


Figure 9.21: Locations of open disks inside and on the boundary of a region.

For all points on the interior of the region one can find at least one disk contained entirely in the region. The closer one is to the boundary, the smaller the radii of such disks. However, for a point on the boundary, every such disk would contain points inside and outside the disk. Thus, an open set in the complex plane would not contain any of its boundary points.

We now have a prescription for computing path integrals. Let's see how this works with a couple of examples.

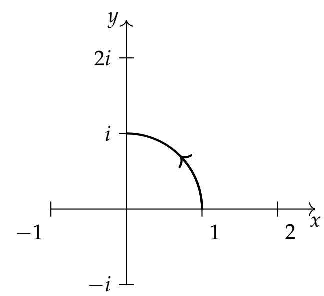


Figure 9.22: Contour for Example 9.13.

Example 9.13. Evaluate $\int_C z^2\,dz$, where C = the arc of the unit circle in the first quadrant as shown in Figure 9.22.

There are two ways we could carry out the parametrization. First, we note that the standard parametrization of the unit circle is

$$(x(\theta), y(\theta)) = (\cos\theta, \sin\theta), \quad 0 \le \theta \le 2\pi.$$

For a quarter circle in the first quadrant, $0 \le \theta \le \frac{\pi}{2}$, we let $z = \cos\theta + i\sin\theta$. Therefore, $dz = (-\sin\theta + i\cos\theta)\,d\theta$ and the path integral becomes

$$\int_C z^2\,dz = \int_0^{\frac{\pi}{2}} (\cos\theta + i\sin\theta)^2(-\sin\theta + i\cos\theta)\,d\theta.$$

We can expand the integrand and integrate, having to perform some trigonometric integrations.

$$\int_0^{\frac{\pi}{2}} [\sin^3\theta - 3\cos^2\theta\sin\theta + i(\cos^3\theta - 3\cos\theta\sin^2\theta)]\,d\theta.$$

The reader should work out these trigonometric integrations and confirm the result. For example, you can use

$$\sin^3\theta = \sin\theta(1 - \cos^2\theta))$$

to write the real part of the integrand as

$$\sin\theta - 4\cos^2\theta\sin\theta.$$

The resulting antiderivative becomes

$$-\cos\theta + \frac{4}{3}\cos^3\theta.$$

The imaginary integrand can be integrated in a similar fashion.

While this integral is doable, there is a simpler procedure. We first note that $z = e^{i\theta}$ on C. So, $dz = ie^{i\theta}d\theta$. The integration then becomes

$$\int_C z^2\,dz = \int_0^{\frac{\pi}{2}} (e^{i\theta})^2 ie^{i\theta}\,d\theta$$

$$
\begin{aligned}
&= i \int_0^{\frac{\pi}{2}} e^{3i\theta}\, d\theta \\
&= \left.\frac{ie^{3i\theta}}{3i}\right|_0^{\pi/2} \\
&= -\frac{1+i}{3}. \qquad (9.23)
\end{aligned}
$$

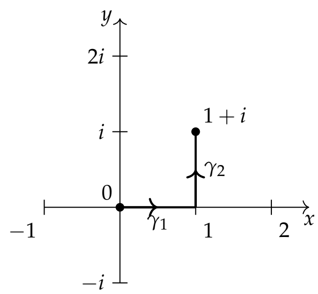


Figure 9.23: Contour for Example 9.14 with $\Gamma = \gamma_1 \cup \gamma_2$.

Example 9.14. Evaluate $\int_\Gamma z\,dz$, for the path $\Gamma = \gamma_1 \cup \gamma_2$ shown in Figure 9.23.

In this problem we have a path that is a piecewise smooth curve. We can compute the path integral by computing the values along the two segments of the path and adding the results. Let the two segments be called γ_1 and γ_2 as shown in Figure 9.23 and parametrize each path separately.

Over γ_1 we note that $y = 0$. Thus, $z = x$ for $x \in [0,1]$. It is natural to take x as the parameter. So, we let $dz = dx$ to find

$$\int_{\gamma_1} z\,dz = \int_0^1 x\,dx = \frac{1}{2}.$$

For path γ_2 we have that $z = 1 + iy$ for $y \in [0,1]$ and $dz = i\,dy$. Inserting this parametrization into the integral, the integral becomes

$$\int_{\gamma_2} z\,dz = \int_0^1 (1+iy)\, idy = i - \frac{1}{2}.$$

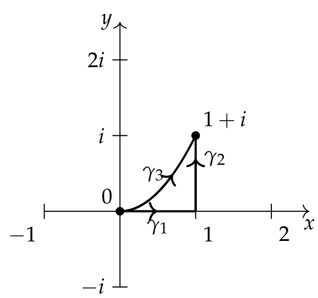


Figure 9.24: Contour for Example 9.15.

Combining the results for the paths γ_1 and γ_2, we have $\int_\Gamma z\,dz = \frac{1}{2} + (i - \frac{1}{2}) = i$.

Example 9.15. Evaluate $\int_{\gamma_3} z\,dz$, where γ_3, is the path shown in Figure 9.24.

In this case we take a path from $z = 0$ to $z = 1 + i$ along a different path than in the last example. Let $\gamma_3 = \{(x,y) | y = x^2, x \in [0,1]\} = \{z | z = x + ix^2, x \in [0,1]\}$. Then, $dz = (1 + 2ix)\,dx$.

The integral becomes

$$
\begin{aligned}
\int_{\gamma_3} z\,dz &= \int_0^1 (x + ix^2)(1 + 2ix)\,dx \\
&= \int_0^1 (x + 3ix^2 - 2x^3)\,dx = \\
&= \left[\frac{1}{2}x^2 + ix^3 - \frac{1}{2}x^4\right]_0^1 = i. \qquad (9.24)
\end{aligned}
$$

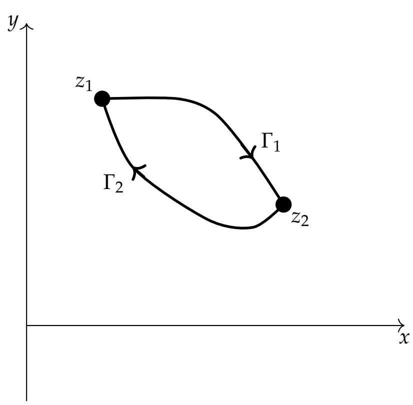


Figure 9.25: $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$ for all paths from z_1 to z_2 when the integral of $f(z)$ is path independent.

In the last case we found the same answer as we had obtained in Example 9.14. But we should not take this as a general rule for all complex path integrals. In fact, it is not true that integrating over different paths always yields the same results. However, when this is true, then we refer to this property as path independence. In particular, the integral $\int f(z)\,dz$ is path independent if

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

for all paths from z_1 to z_2 as shown in Figure 9.25.

We can show that if $\int f(z)\,dz$ is path independent, then the integral of $f(z)$ over all closed loops is zero,

$$\int_{\text{closed loops}} f(z)\,dz = 0.$$

A common notation for integrating over closed loops is $\oint_C f(z)\,dz$. But first we have to define what we mean by a closed loop. A simple closed contour is a path satisfying

A simple closed contour.

a The end point is the same as the beginning point. (This makes the loop closed.)

b The are no self-intersections. (This makes the loop simple.)

A loop in the shape of a figure eight is closed, but it is not simple.

Now, consider an integral over the closed loop C shown in Figure 9.26. We pick two points on the loop breaking it into two contours, C_1 and C_2. Then we make use of the path independence by defining C_2^- to be the path along C_2 but in the opposite direction. Then,

$\oint_C f(z)\,dz = 0$ if the integral is path independent.

$$\begin{aligned}\oint_C f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{C_2^-} f(z)\,dz. \qquad (9.25)\end{aligned}$$

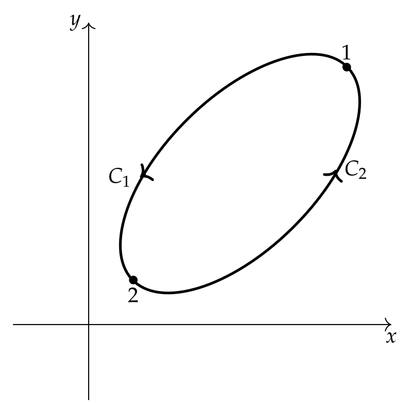


Figure 9.26: The integral $\oint_C f(z)\,dz$ around C is zero if the integral $\int_\Gamma f(z)\,dz$ is path independent.

Assuming that the integrals from point 1 to point 2 are path independent, then the integrals over C_1 and C_2^- are equal. Therefore, we have $\oint_C f(z)\,dz = 0$.

Example 9.16. Consider the integral $\oint_C z\,dz$ for C the closed contour shown in Figure 9.24 starting at $z = 0$ following path γ_1, then γ_2 and returning to $z = 0$. Based on the earlier examples and the fact that going backwards on γ_3 introduces a negative sign, we have

$$\oint_C z\,dz = \int_{\gamma_1} z\,dz + \int_{\gamma_2} z\,dz - \int_{\gamma_3} z\,dz = \frac{1}{2} + \left(i - \frac{1}{2}\right) - i = 0.$$

9.5.2 *Cauchy's Theorem*

NEXT WE WANT TO INVESTIGATE if we can determine that integrals over simple closed contours vanish without doing all the work of parametrizing the contour. First, we need to establish the direction about which we traverse the contour. We can define the orientation of a curve by referring to the normal of the curve.

A curve with parametrization $(x(t), y(t))$ has a normal $(n_x, n_y) = (-\frac{dx}{dt}, \frac{dy}{dt})$.

Recall that the normal is a perpendicular to the curve. There are two such perpendiculars. The above normal points outward and the other normal points towards the interior of a closed curve. We will define a positively oriented contour as one that is traversed with the outward normal pointing to the right. As one follows loops, the interior would then be on the left.

We now consider $\oint_C (u+iv)\,dz$ over a simple closed contour. This can be written in terms of two real integrals in the xy-plane.

$$\begin{aligned}\oint_C (u+iv)\,dz &= \int_C (u+iv)(dx+i\,dy) \\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy. \qquad (9.26)\end{aligned}$$

These integrals in the plane can be evaluated using Green's Theorem in the Plane. Recall this theorem from your last semester of calculus:

Green's Theorem in the Plane.

Theorem 9.3. *Let $P(x,y)$ and $Q(x,y)$ be continuously differentiable functions on and inside the simple closed curve C as shown in Figure 9.27. Denoting the enclosed region S, we have*

$$\int_C P\,dx + Q\,dy = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dxdy. \qquad (9.27)$$

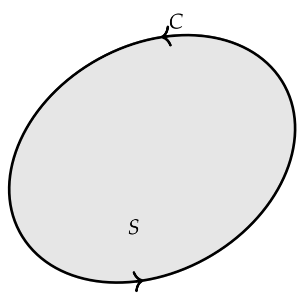


Figure 9.27: Region used in Green's Theorem.

Green's Theorem in the Plane is one of the major integral theorems of vector calculus. It was discovered by George Green (1793-1841) and published in 1828, about four years before he entered Cambridge as an undergraduate.

Using Green's Theorem to rewrite the first integral in (9.26), we have

$$\int_C u\,dx - v\,dy = \int\int_S \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dxdy.$$

If u and v satisfy the Cauchy-Riemann equations (9.19), then the integrand in the double integral vanishes. Therefore,

$$\int_C u\,dx - v\,dy = 0.$$

In a similar fashion, one can show that

$$\int_C v\,dx + u\,dy = 0.$$

We have thus proven the following theorem:

Cauchy's Theorem

Theorem 9.4. *If u and v satisfy the Cauchy-Riemann equations (9.19) inside and on the simple closed contour C, then*

$$\oint_C (u+iv)\,dz = 0. \qquad (9.28)$$

Corollary $\oint_C f(z)\,dz = 0$ *when f is differentiable in domain D with $C \subset D$.*

Either one of these is referred to as **Cauchy's Theorem**.

Example 9.17. Evaluate $\oint_{|z-1|=3} z^4\,dz$.

Since $f(z) = z^4$ is differentiable inside the circle $|z-1| = 3$, this integral vanishes.

We can use Cauchy's Theorem to show that we can deform one contour into another, perhaps simpler, contour.

Theorem 9.5. *If $f(z)$ is holomorphic between two simple closed contours, C and C', then $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$.*

One can deform contours into simpler ones.

Proof. We consider the two curves C and C' as shown in Figure 9.28. Connecting the two contours with contours Γ_1 and Γ_2 (as shown in the figure), C is seen to split into contours C_1 and C_2 and C' into contours C_1' and C_2'. Note that $f(z)$ is differentiable inside the newly formed regions between the curves. Also, the boundaries of these regions are now simple closed curves. Therefore, Cauchy's Theorem tells us that the integrals of $f(z)$ over these regions are zero.

Noting that integrations over contours opposite to the positive orientation are the negative of integrals that are positively oriented, we have from Cauchy's Theorem that

$$\int_{C_1} f(z)\,dz + \int_{\Gamma_1} f(z)\,dz - \int_{C_1'} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0$$

and

$$\int_{C_2} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz - \int_{C_2'} f(z)\,dz - \int_{\Gamma_1} f(z)\,dz = 0.$$

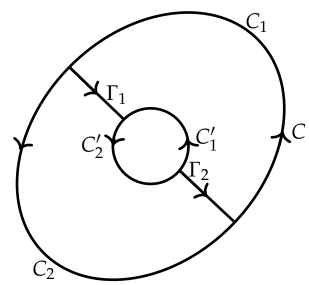


Figure 9.28: The contours needed to prove that $\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz$ when $f(z)$ is holomorphic between the contours C and C'.

In the first integral we have traversed the contours in the following order: C_1, Γ_1, C_1' backwards, and Γ_2. The second integral denotes the integration over the lower region, but going backwards over all contours except for C_2.

Combining these results by adding the two equations above, we have

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz - \int_{C_1'} f(z)\,dz - \int_{C_2'} f(z)\,dz = 0.$$

Noting that $C = C_1 + C_2$ and $C' = C_1' + C_2'$, we have

$$\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz,$$

as was to be proven. □

Example 9.18. Compute $\oint_R \frac{dz}{z}$ for R the rectangle $[-2, 2] \times [-2i, 2i]$.

We can compute this integral by looking at four separate integrals over the sides of the rectangle in the complex plane. One simply parametrizes each line segment, perform the integration and sum the four separate results. From the last theorem, we can instead integrate over a simpler contour by deforming the rectangle into a circle as long as $f(z) = \frac{1}{z}$ is differentiable in the region bounded by the rectangle and the circle. So, using the unit circle, as shown in Figure 9.29, the integration might be easier to perform.

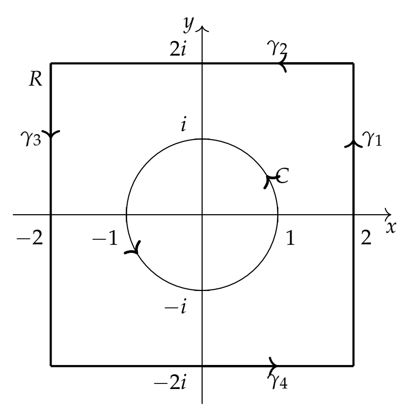


Figure 9.29: The contours used to compute $\oint_R \frac{dz}{z}$. Note that to compute the integral around R we can deform the contour to the circle C since $f(z)$ is differentiable in the region between the contours.

More specifically, the last theorem tells us that

$$\oint_R \frac{dz}{z} = \oint_{|z|=1} \frac{dz}{z}$$

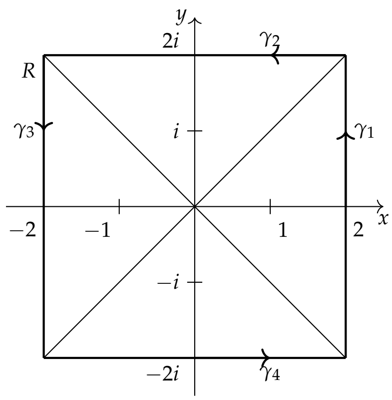


Figure 9.30: The contours used to compute $\oint_R \frac{dz}{z}$. The added diagonals are for the reader to easily see the arguments used in the evaluation of the limits when integrating over the segments of the square R.

The latter integral can be computed using the parametrization $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Thus,

$$\begin{aligned} \oint_{|z|=1} \frac{dz}{z} &= \int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{e^{i\theta}} \\ &= i\int_0^{2\pi} d\theta = 2\pi i. \end{aligned} \tag{9.29}$$

Therefore, we have found that $\oint_R \frac{dz}{z} = 2\pi i$ by deforming the original simple closed contour.

For fun, let's do this the long way to see how much effort was saved. We will label the contour as shown in Figure 9.30. The lower segment, γ_4 of the square can be simple parametrized by noting that along this segment $z = x - 2i$ for $x \in [-2, 2]$. Then, we have

$$\begin{aligned} \oint_{\gamma_4} \frac{dz}{z} &= \int_{-2}^{2} \frac{dx}{x-2i} \\ &= \ln\left|x-2i\right|_{-2}^{2} \\ &= \left(\ln(2\sqrt{2}) - \frac{\pi i}{4}\right) - \left(\ln(2\sqrt{2}) - \frac{3\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{9.30}$$

We note that the arguments of the logarithms are determined from the angles made by the diagonals provided in Figure 9.30.

Similarly, the integral along the top segment, $z = x + 2i$, $x \in [-2, 2]$, is computed as

$$\begin{aligned} \oint_{\gamma_2} \frac{dz}{z} &= \int_{2}^{-2} \frac{dx}{x+2i} \\ &= \ln\left|x+2i\right|_{2}^{-2} \\ &= \left(\ln(2\sqrt{2}) + \frac{3\pi i}{4}\right) - \left(\ln(2\sqrt{2}) + \frac{\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{9.31}$$

The integral over the right side, $z = 2 + iy$, $y \in [-2, 2]$, is

$$\begin{aligned} \oint_{\gamma_1} \frac{dz}{z} &= \int_{-2}^{2} \frac{i\,dy}{2+iy} \\ &= \ln\left|2+iy\right|_{-2}^{2} \\ &= \left(\ln(2\sqrt{2}) + \frac{\pi i}{4}\right) - \left(\ln(2\sqrt{2}) - \frac{\pi i}{4}\right) \\ &= \frac{\pi i}{2}. \end{aligned} \tag{9.32}$$

Finally, the integral over the left side, $z = -2 + iy$, $y \in [-2, 2]$, is

$$\oint_{\gamma_3} \frac{dz}{z} = \int_{2}^{-2} \frac{i\,dy}{-2+iy}$$

$$
\begin{aligned}
&= \ln \left| -2+iy \right|_{-2}^{2} \\
&= \left(\ln(2\sqrt{2}) + \frac{5\pi i}{4}\right) - \left(\ln(2\sqrt{2}) + \frac{3\pi i}{4}\right) \\
&= \frac{\pi i}{2}. \qquad (9.33)
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
\oint_R \frac{dz}{z} &= \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} + \int_{\gamma_3} \frac{dz}{z} + \int_{\gamma_4} \frac{dz}{z} \\
&= \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2} + \frac{\pi i}{2} \\
&= 4\left(\frac{\pi i}{2}\right) = 2\pi i. \qquad (9.34)
\end{aligned}
$$

This gives the same answer we had found using a simple contour deformation.

The converse of Cauchy's Theorem is not true, namely $\oint_C f(z)\,dz = 0$ does not always imply that $f(z)$ is differentiable. What we do have is Morera's Theorem(Giacinto Morera, 1856-1909):

Morera's Theorem.

Theorem 9.6. *Let f be continuous in a domain D. Suppose that for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$. Then f is differentiable in D.*

The proof is a bit more detailed than we need to go into here. However, this theorem is useful in the next section.

9.5.3 *Analytic Functions and Cauchy's Integral Formula*

IN THE PREVIOUS SECTION WE SAW that Cauchy's Theorem was useful for computing particular integrals without having to parametrize the contours or for deforming contours into simpler contours. The integrand needs to possess certain differentiability properties. In this section, we will generalize the functions that we can integrate slightly so that we can integrate a larger family of complex functions. This will lead us to the Cauchy's Integral Formula, which extends Cauchy's Theorem to functions analytic in an annulus. However, first we need to explore the concept of analytic functions.

There are various types of complex-valued functions.

A holomorphic function is (complex) differentiable in a neighborhood of every point in its domain.

An analytic function has a convergent Taylor series expansion in a neighborhood of each point in its domain. We see here that analytic functions are holomorphic and vice versa.

If a function is holomorphic throughout the complex plane, then it is called an entire function.

Finally, a function which is holomorphic on all of its domain except at a set of isolated poles (to be defined later), then it is called a meromorphic function.

A function $f(z)$ is analytic in domain D if for every open disk $|z - z_0| < \rho$ lying in D, $f(z)$ can be represented as a power series in z_0. Namely,

$$
f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.
$$

This series converges uniformly and absolutely inside the circle of convergence, $|z - z_0| < R$, with radius of convergence R. [See the Appendix for a review of convergence.]

Since $f(z)$ can be written as a uniformly convergent power series, we can integrate it term by term over any simple closed contour in D containing z_0. In particular, we have to compute integrals like $\oint_C (z - z_0)^n\,dz$. As we will see in the homework exercises, these integrals evaluate to zero for most n. Thus, we can show that for $f(z)$ analytic in D and on any closed contour C lying in D, $\oint_C f(z)\,dz = 0$. Also, f is a uniformly convergent sum of continuous functions, so $f(z)$ is also continuous. Thus, by Morera's Theorem, we have that $f(z)$ is differentiable if it is analytic. Often terms like analytic, differentiable and holomorphic are used interchangeably, though there is a subtle distinction due to their definitions.

As examples of series expansions about a given point, we will consider series expansions and regions of convergence for $f(z) = \frac{1}{1+z}$.

Example 9.19. Find the series expansion of $f(z) = \frac{1}{1+z}$ about $z_0 = 0$.

This case is simple. From Chapter 1 we recall that $f(z)$ is the sum of a geometric series for $|z| < 1$. We have

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty}(-z)^n.$$

Thus, this series expansion converges inside the unit circle ($|z| < 1$) in the complex plane.

Example 9.20. Find the series expansion of $f(z) = \frac{1}{1+z}$ about $z_0 = \frac{1}{2}$.

We now look into an expansion about a different point. We could compute the expansion coefficients using Taylor's formula for the coefficients. However, we can also make use of the formula for geometric series after rearranging the function. We seek an expansion in powers of $z - \frac{1}{2}$. So, we rewrite the function in a form that has is a function of $z - \frac{1}{2}$. Thus,

$$f(z) = \frac{1}{1+z} = \frac{1}{1 + (z - \frac{1}{2} + \frac{1}{2})} = \frac{1}{\frac{3}{2} + (z - \frac{1}{2})}.$$

This is not quite in the form we need. It would be nice if the denominator were of the form of one plus something. [Note: This is similar to what we had seen in Example A.35.] We can get the denominator into such a form by factoring out the $\frac{3}{2}$. Then we would have

$$f(z) = \frac{2}{3}\frac{1}{1 + \frac{2}{3}(z - \frac{1}{2})}.$$

The second factor now has the form $\frac{1}{1-r}$, which would be the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{2}{3}(z - \frac{1}{2})$ provided that |r|<1. Therefore, we have found that

$$f(z) = \frac{2}{3}\sum_{n=0}^{\infty}\left[-\frac{2}{3}(z - \frac{1}{2})\right]^n$$

for

$$\left| -\frac{2}{3}(z - \frac{1}{2})\right| < 1.$$

This convergence interval can be rewritten as

$$\left|z - \frac{1}{2}\right| < \frac{3}{2},$$

which is a circle centered at $z = \frac{1}{2}$ with radius $\frac{3}{2}$.

In Figure 9.31 we show the regions of convergence for the power series expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. We note that the first expansion gives that $f(z)$ is at least analytic inside the region $|z| < 1$. The second expansion shows that $f(z)$ is analytic in a larger region, $|z - \frac{1}{2}| < \frac{3}{2}$. We will see later that there are expansions which converge outside of these regions and that some yield expansions involving negative powers of $z - z_0$.

We now present the main theorem of this section:

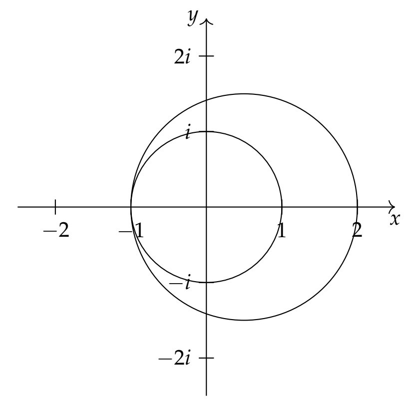


Figure 9.31: Regions of convergence for expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$.

Cauchy Integral Formula

Theorem 9.7. *Let $f(z)$ be analytic in $|z - z_0| < \rho$ and let C be the boundary (circle) of this disk. Then,*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz. \tag{9.35}$$

Proof. In order to prove this, we first make use of the analyticity of $f(z)$. We insert the power series expansion of $f(z)$ about z_0 into the integrand. Then we have

$$\begin{aligned} \frac{f(z)}{z - z_0} &= \frac{1}{z - z_0}\left[\sum_{n=0}^{\infty} c_n (z - z_0)^n\right] \\ &= \frac{1}{z - z_0}\left[c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots\right] \\ &= \frac{c_0}{z - z_0} + \underbrace{c_1 + c_2(z - z_0) + \ldots}_{\text{analytic function}}. \end{aligned} \tag{9.36}$$

As noted the integrand can be written as

$$\frac{f(z)}{z - z_0} = \frac{c_0}{z - z_0} + h(z),$$

where $h(z)$ is an analytic function, since $h(z)$ is representable as a series expansion about z_0. We have already shown that analytic functions are differentiable, so by Cauchy's Theorem $\oint_C h(z)\, dz = 0$.

Noting also that $c_0 = f(z_0)$ is the first term of a Taylor series expansion about $z = z_0$, we have

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_C \left[\frac{c_0}{z - z_0} + h(z)\right] dz = f(z_0) \oint_C \frac{1}{z - z_0}\, dz.$$

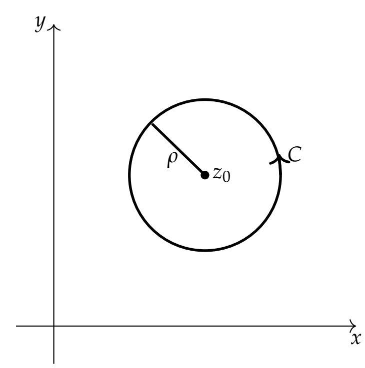


Figure 9.32: Circular contour used in proving the Cauchy Integral Formula.

We need only compute the integral $\oint_C \frac{1}{z - z_0}\, dz$ to finish the proof of Cauchy's Integral Formula. This is done by parametrizing the circle, $|z - z_0| = \rho$, as shown in Figure 9.32. This is simply done by letting

$$z - z_0 = \rho e^{i\theta}.$$

(Note that this has the right complex modulus since $|e^{i\theta}| = 1$. Then $dz = i\rho e^{i\theta} d\theta$. Using this parametrization, we have

$$\oint_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\, d\theta}{\rho e^{i\theta}} = i \int_0^{2\pi} d\theta = 2\pi i.$$

Therefore,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{1}{z-z_0}\,dz = 2\pi i f(z_0),$$

as was to be shown. $\square$

Example 9.21. Compute $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\,dz$.

In order to apply the Cauchy Integral Formula, we need to factor the denominator, $z^2-6z+5=(z-1)(z-5)$. We next locate the zeros of the denominator. In Figure 9.33 we show the contour and the points $z=1$ and $z=5$. The only point inside the region bounded by the contour is $z=1$. Therefore, we can apply the Cauchy Integral Formula for $f(z)=\frac{\cos z}{z-5}$ to the integral

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\,dz = \int_{|z|=4} \frac{f(z)}{(z-1)}\,dz = 2\pi i f(1).$$

Therefore, we have

$$\int_{|z|=4} \frac{\cos z}{(z-1)(z-5)}\,dz = -\frac{\pi i \cos(1)}{2}.$$

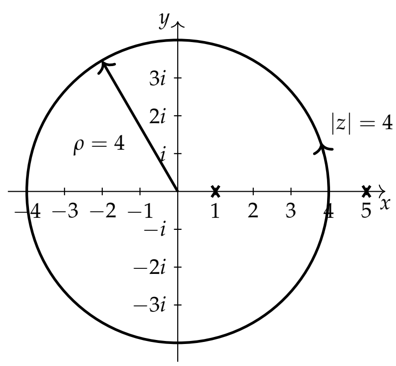


Figure 9.33: Circular contour used in computing $\oint_{|z|=4} \frac{\cos z}{z^2-6z+5}\,dz$.

We have shown that $f(z_0)$ has an integral representation for $f(z)$ analytic in $|z-z_0|<\rho$. In fact, all derivatives of an analytic function have an integral representation. This is given by

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.} \tag{9.37}$$

This can be proven following a derivation similar to that for the Cauchy Integral Formula. Inserting the Taylor series expansion for $f(z)$ into the integral on the right hand side, we have

$$\begin{aligned}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz &= \sum_{m=0}^{\infty} c_m \oint_C \frac{(z-z_0)^m}{(z-z_0)^{n+1}}\,dz \\ &= \sum_{m=0}^{\infty} c_m \oint_C \frac{dz}{(z-z_0)^{n-m+1}}.\end{aligned} \tag{9.38}$$

Picking $k=n-m$, the integrals in the sum can be computed by using the following result:

$$\oint_C \frac{dz}{(z-z_0)^{k+1}} = \begin{cases} 0, & k\neq 0 \\ 2\pi i, & k=0. \end{cases} \tag{9.39}$$

The proof is left for the exercises.

The only nonvanishing integrals, $\oint_C \frac{dz}{(z-z_0)^{n-m+1}}$, occur when $k=n-m=0$, or $m=n$. Therefore, the series of integrals collapses to one term and we have

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i c_n.$$

We finish the proof by recalling that the coefficients of the Taylor series expansion for $f(z)$ are given by

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Then,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

and the result follows.

9.5.4 *Laurent Series*

Until this point we have only talked about series whose terms have nonnegative powers of $z - z_0$. It is possible to have series representations in which there are negative powers. In the last section we investigated expansions of $f(z) = \frac{1}{1+z}$ about $z = 0$ and $z = \frac{1}{2}$. The regions of convergence for each series was shown in Figure 9.31. Let us reconsider each of these expansions, but for values of z outside the region of convergence previously found.

Example 9.22. $f(z) = \frac{1}{1+z}$ for $|z| > 1$.

As before, we make use of the geometric series . Since $|z| > 1$, we instead rewrite the function as

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\frac{1}{1+\frac{1}{z}}.$$

We now have the function in a form of the sum of a geometric series with first term $a = 1$ and ratio $r = -\frac{1}{z}$. We note that $|z| > 1$ implies that $|r| < 1$. Thus, we have the geometric series

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n.$$

This can be re-indexed[3] as

$$f(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n-1} = \sum_{j=1}^{\infty}(-1)^{j-1} z^{-j}.$$

Note that this series, which converges outside the unit circle, $|z| > 1$, has negative powers of z.

Example 9.23. $f(z) = \frac{1}{1+z}$ for $|z - \frac{1}{2}| > \frac{3}{2}$.

As before, we express this in a form in which we can use a geometric series expansion. We seek powers of $z - \frac{1}{2}$. So, we add and subtract $\frac{1}{2}$ to the z to obtain:

$$f(z) = \frac{1}{1+z} = \frac{1}{1+(z-\frac{1}{2}+\frac{1}{2})} = \frac{1}{\frac{3}{2}+(z-\frac{1}{2})}.$$

[3] Re-indexing a series is often useful in series manipulations. In this case, we have the series

$$\sum_{n=0}^{\infty}(-1)^n z^{-n-1} = z^{-1} - z^{-2} + z^{-3} + \ldots.$$

The index is n. You can see that the index does not appear when the sum is expanded showing the terms. The summation index is sometimes referred to as a dummy index for this reason. Reindexing allows one to rewrite the shorthand summation notation while capturing the same terms. In this example, the exponents are $-n-1$. We can simplify the notation by letting $-n-1 = -j$, or $j = n+1$. Noting that $j = 1$ when $n = 0$, we get the sum $\sum_{j=1}^{\infty}(-1)^{j-1}z^{-j}$.

Instead of factoring out the $\frac{3}{2}$ as we had done in Example 9.20, we factor out the $(z-\frac{1}{2})$ term. Then, we obtain

$$f(z)=\frac{1}{1+z}=\frac{1}{(z-\frac{1}{2})}\frac{1}{\left[1+\frac{3}{2}(z-\frac{1}{2})^{-1}\right]}.$$

Now we identify $a=1$ and $r=-\frac{3}{2}(z-\frac{1}{2})^{-1}$. This leads to the series

$$\begin{aligned} f(z) &= \frac{1}{z-\frac{1}{2}}\sum_{n=0}^{\infty}\left(-\frac{3}{2}(z-\frac{1}{2})^{-1}\right)^{n} \\ &= \sum_{n=0}^{\infty}\left(-\frac{3}{2}\right)^{n}\left(z-\frac{1}{2}\right)^{-n-1}. \end{aligned} \tag{9.40}$$

This converges for $|z-\frac{1}{2}|>\frac{3}{2}$ and can also be re-indexed to verify that this series involves negative powers of $z-\frac{1}{2}$.

This leads to the following theorem:

Theorem 9.8. *Let $f(z)$ be analytic in an annulus, $R_1<|z-z_0|<R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus as shown in Figure 9.34. Then,*

$$f(z)=\sum_{j=0}^{\infty}a_j(z-z_0)^j+\sum_{j=1}^{\infty}b_j(z-z_0)^{-j},$$

with

$$a_j=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^{j+1}}\,dz,$$

and

$$b_j=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^{-j+1}}\,dz.$$

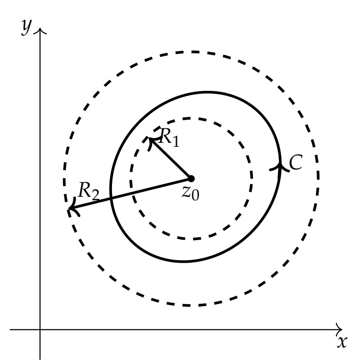


Figure 9.34: This figure shows an annulus, $R_1<|z-z_0|<R_2$, with C a positively oriented simple closed curve around z_0 and inside the annulus.

The above series can be written in the more compact form

$$f(z)=\sum_{j=-\infty}^{\infty}c_j(z-z_0)^j.$$

Such a series expansion is called a Laurent series expansion named after its discoverer Pierre Alphonse Laurent (1813-1854).

Example 9.24. Expand $f(z)=\frac{1}{(1-z)(2+z)}$ in the annulus $1<|z|<2$.

Using partial fractions , we can write this as

$$f(z)=\frac{1}{3}\left[\frac{1}{1-z}+\frac{1}{2+z}\right].$$

We can expand the first fraction, $\frac{1}{1-z}$, as an analytic function in the region $|z|>1$ and the second fraction, $\frac{1}{2+z}$, as an analytic function in $|z|<2$. This is done as follows. First, we write

$$\frac{1}{2+z}=\frac{1}{2[1-(-\frac{z}{2})]}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^{n}.$$

Then, we write

$$\frac{1}{1-z} = -\frac{1}{z[1-\frac{1}{z}]} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}.$$

Therefore, in the common region, $1 < |z| < 2$, we have that

$$\begin{aligned}\frac{1}{(1-z)(2+z)} &= \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right] \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{6(2^n)}z^n + \sum_{n=1}^{\infty}\frac{(-1)}{3}z^{-n}. \qquad (9.41)\end{aligned}$$

We note that this is not a Taylor series expansion due to the existence of terms with negative powers in the second sum.

Example 9.25. Find series representations of $f(z) = \frac{1}{(1-z)(2+z)}$ throughout the complex plane.

In the last example we found series representations of $f(z) = \frac{1}{(1-z)(2+z)}$ in the annulus $1 < |z| < 2$. However, we can also find expansions which converge for other regions. We first write

$$f(z) = \frac{1}{3}\left[\frac{1}{1-z} + \frac{1}{2+z}\right].$$

We then expand each term separately.

The first fraction, $\frac{1}{1-z}$, can be written as the sum of the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

This series converges inside the unit circle. We indicate this by region 1 in Figure 9.35.

In the last example, we showed that the second fraction, $\frac{1}{2+z}$, has the series expansion

$$\frac{1}{2+z} = \frac{1}{2[1-(-\frac{z}{2})]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n.$$

which converges in the circle $|z| < 2$. This is labeled as region 2 in Figure 9.35.

Regions 1 and 2 intersect for $|z| < 1$, so, we can combine these two series representations to obtain

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n\right], \quad |z| < 1.$$

In the annulus, $1 < |z| < 2$, we had already seen in the last example that we needed a different expansion for the fraction $\frac{1}{1-z}$. We looked for an expansion in powers of $1/z$ which would converge for large values of z. We had found that

$$\frac{1}{1-z} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}, \quad |z| > 1.$$

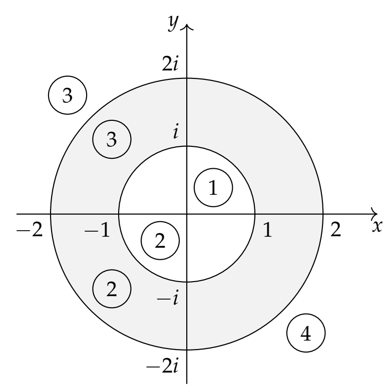


Figure 9.35: Regions of convergence for Laurent expansions of $f(z) = \frac{1}{1+z}$.

This series converges in region 3 in Figure 9.35. Combining this series with the one for the second fraction, we obtain a series representation that converges in the overlap of regions 2 and 3. Thus, in the annulus $1 < |z| < 2$ we have

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right].$$

So far, we have series representations for $|z| < 2$. The only region not covered yet is outside this disk, $|z| > 2$. In in Figure 9.35 we see that series 3, which converges in region 3, will converge in the last section of the complex plane. We just need one more series expansion for $1/(2+z)$ for large z. Factoring out a z in the denominator, we can write this as a geometric series with $r = 2/z$,

$$\frac{1}{2+z} = \frac{1}{z[\frac{2}{z}+1]} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n.$$

This series converges for $|z| > 2$. Therefore, it converges in region 4 and the final series representation is

$$\frac{1}{(1-z)(2+z)} = \frac{1}{3}\left[\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right].$$

9.5.5 *Singularities and The Residue Theorem*

In the last section we found that we could integrate functions satisfying some analyticity properties along contours without using detailed parametrizations around the contours. We can deform contours if the function is analytic in the region between the original and new contour. In this section we will extend our tools for performing contour integrals.

Singularities of complex functions.

The integrand in the Cauchy Integral Formula was of the form $g(z) = \frac{f(z)}{z-z_0}$, where $f(z)$ is well behaved at z_0. The point $z = z_0$ is called a singularity of $g(z)$, as $g(z)$ is not defined there. More specifically, a singularity of $f(z)$ is a point at which $f(z)$ fails to be analytic.

We can also classify these singularities. Typically these are isolated singularities. As we saw from the proof of the Cauchy Integral Formula, $g(z) = \frac{f(z)}{z-z_0}$ has a Laurent series expansion about $z = z_0$, given by

$$g(z) = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{1}{2}f''(z_0)(z-z_0) + \ldots.$$

It is the nature of the first term that gives information about the type of singularity that $g(z)$ has. Namely, in order to classify the singularities of $f(z)$, we look at the principal part of the Laurent series of $f(z)$ about $z = z_0$, $\sum_{j-1}^{\infty} b_j(z-z_0)^{-j}$, which consists of the negative powers of $z - z_0$.

Classification of singularities.

There are three types of singularities, removable, poles, and essential singularities. They are defined as follows:

1. If $f(z)$ is bounded near z_0, then z_0 is a removable singularity.
2. If there are a finite number of terms in the principal part of the Laurent series of $f(z)$ about $z = z_0$, then z_0 is called a pole.
3. If there are an infinite number of terms in the principal part of the Laurent series of $f(z)$ about $z = z_0$, then z_0 is called an essential singularity.

Example 9.26. $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$.

At first it looks like there is a possible singularity at $z = 0$, since the denominator is zero at $z = 0$. However, we know from the first semester of calculus that $\lim_{z\to 0} \frac{\sin z}{z} = 1$. Furthermore, we can expand $\sin z$ about $z = 0$ and see that

$$\frac{\sin z}{z} = \frac{1}{z}(z - \frac{z^3}{3!} + \ldots) = 1 - \frac{z^2}{3!} + \ldots.$$

Thus, there are only nonnegative powers in the series expansion. So, $z = 0$ is a removable singularity.

Example 9.27. $f(z) = \frac{e^z}{(z-1)^n}$ has poles at $z = 1$ for n a positive integer.

For $n = 1$ we have $f(z) = \frac{e^z}{z-1}$. This function has a singularity at $z = 1$. The series expansion is found by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{z-1}e^{z-1} = \frac{e}{z-1} + e + \frac{e}{2!}(z-1) + \ldots.$$

Note that the principal part of the Laurent series expansion about $z = 1$ only has one term, $\frac{e}{z-1}$. Therefore, $z = 1$ is a pole. Since the leading term has an exponent of -1, $z = 1$ is called a pole of order one, or a simple pole.

Simple pole.

For $n = 2$ we have $f(z) = \frac{e^z}{(z-1)^2}$. The series expansion is found again by expanding e^z about $z = 1$:

$$f(z) = \frac{e}{(z-1)^2}e^{z-1} = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2!} + \frac{e}{3!}(z-1) + \ldots.$$

Note that the principal part of the Laurent series has two terms involving $(z-1)^{-2}$ and $(z-1)^{-1}$. Since the leading term has an exponent of -2, $z = 1$ is called a pole of order 2, or a double pole.

Double pole.

Example 9.28. $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$.

In this case we have the series expansion about $z = 0$ given by

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}.$$

We see that there are an infinite number of terms in the principal part of the Laurent series. So, this function has an essential singularity at $z = 0$.

In the above examples we have seen poles of order one (a simple pole) and two (a double pole). In general, we can say that $f(z)$ has a pole of order k at z_0 if and only if $(z - z_0)^k f(z)$ has a removable singularity at z_0, but $(z - z_0)^{k-1} f(z)$ for $k > 0$ does not.

Poles of order k.

Example 9.29. Determine the order of the pole of $f(z) = \cot z \csc z$ at $z = 0$.

First we rewrite $f(z)$ in terms of sines and cosines.

$$f(z) = \cot z \csc z = \frac{\cos z}{\sin^2 z}.$$

We note that the denominator vanishes at $z = 0$.

How do we know that the pole is not a simple pole? Well, we check to see if $(z-0)f(z)$ has a removable singularity at $z = 0$:

$$\begin{aligned} \lim_{z\to 0}(z-0)f(z) &= \lim_{z\to 0}\frac{z\cos z}{\sin^2 z} \\ &= \left(\lim_{z\to 0}\frac{z}{\sin z}\right)\left(\lim_{z\to 0}\frac{\cos z}{\sin z}\right) \\ &= \lim_{z\to 0}\frac{\cos z}{\sin z}. \end{aligned} \tag{9.42}$$

We see that this limit is undefined. So, now we check to see if $(z-0)^2 f(z)$ has a removable singularity at $z = 0$:

$$\begin{aligned} \lim_{z\to 0}(z-0)^2 f(z) &= \lim_{z\to 0}\frac{z^2\cos z}{\sin^2 z} \\ &= \left(\lim_{z\to 0}\frac{z}{\sin z}\right)\left(\lim_{z\to 0}\frac{z\cos z}{\sin z}\right) \\ &= \lim_{z\to 0}\frac{z}{\sin z}\cos(0) = 1. \end{aligned} \tag{9.43}$$

In this case, we have obtained a finite, nonzero, result. So, $z = 0$ is a pole of order 2.

We could have also relied on series expansions. Expanding both the sine and cosine functions in a Taylor series expansion, we have

$$f(z) = \frac{\cos z}{\sin^2 z} = \frac{1 - \frac{1}{2!}z^2 + \ldots}{(z - \frac{1}{3!}z^3 + \ldots)^2}.$$

Factoring a z from the expansion in the denominator,

$$f(z) = \frac{1}{z^2}\frac{1 - \frac{1}{2!}z^2 + \ldots}{(1 - \frac{1}{3!}z + \ldots)^2} = \frac{1}{z^2}\left(1 + O(z^2)\right),$$

we can see that the leading term will be a $1/z^2$, indicating a pole of order 2.

We will see how knowledge of the poles of a function can aid in the computation of contour integrals. We now show that if a function, $f(z)$, has a pole of order k, then

Integral of a function with a simple pole inside C.

Residues of a function with poles of order k.

$$\boxed{\oint_C f(z)\,dz = 2\pi i \operatorname{Res}[f(z); z_0],}$$

where we have defined $\operatorname{Res}[f(z); z_0]$ as the residue of $f(z)$ at $z = z_0$. In particular, for a pole of order k the residue is given by

Residues - Poles of order k

$$\text{Res}[f(z); z_0] = \lim_{z\to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z) \right]. \tag{9.44}$$

Proof. Let $\phi(z) = (z-z_0)^k f(z)$ be an analytic function. Then $\phi(z)$ has a Taylor series expansion about z_0. As we had seen in the last section, we can write the integral representation of any derivative of ϕ as

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i} \oint_C \frac{\phi(z)}{(z-z_0)^k}\, dz.$$

Inserting the definition of $\phi(z)$, we then have

$$\phi^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi i} \oint_C f(z)\, dz.$$

Solving for the integral, we have the result

$$\begin{aligned} \oint_C f(z)\, dz &= \frac{2\pi i}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z) \right]_{z=z_0} \\ &\equiv 2\pi i \;\text{Res}[f(z); z_0] \end{aligned} \tag{9.45}$$

□

Note: If z_0 is a simple pole, the residue is easily computed as

$$Res[f(z); z_0] = \lim_{z\to z_0} (z-z_0) f(z).$$

The residue for a simple pole.

In fact, one can show (Problem 18) that for g and h analytic functions at z_0, with $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$,

$$\text{Res}\left[\frac{g(z)}{h(z)}; z_0\right] = \frac{g(z_0)}{h'(z_0)}.$$

Example 9.30. Find the residues of $f(z) = \frac{z-1}{(z+1)^2(z^2+4)}$.

$f(z)$ has poles at $z = -1$, $z = 2i$, and $z = -2i$. The pole at $z = -1$ is a double pole (pole of order 2). The other poles are simple poles. We compute those residues first:

$$\begin{aligned} \text{Res}[f(z); 2i] &= \lim_{z\to 2i} (z-2i) \frac{z-1}{(z+1)^2(z+2i)(z-2i)} \\ &= \lim_{z\to 2i} \frac{z-1}{(z+1)^2(z+2i)} \\ &= \frac{2i-1}{(2i+1)^2(4i)} = -\frac{1}{50} - \frac{11}{100}i. \end{aligned} \tag{9.46}$$

$$\begin{aligned} \text{Res}[f(z); -2i] &= \lim_{z\to -2i} (z+2i) \frac{z-1}{(z+1)^2(z+2i)(z-2i)} \\ &= \lim_{z\to -2i} \frac{z-1}{(z+1)^2(z-2i)} \\ &= \frac{-2i-1}{(-2i+1)^2(-4i)} = -\frac{1}{50} + \frac{11}{100}i. \end{aligned} \tag{9.47}$$

For the double pole, we have to do a little more work.

$$\begin{aligned}
\operatorname{Res}[f(z);-1] &= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z-1}{(z+1)^2(z^2+4)}\right]\\
&= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z-1}{z^2+4}\right]\\
&= \lim_{z\to -1}\frac{d}{dz}\left[\frac{z^2+4-2z(z-1)}{(z^2+4)^2}\right]\\
&= \lim_{z\to -1}\frac{d}{dz}\left[\frac{-z^2+2z+4}{(z^2+4)^2}\right]\\
&= \frac{1}{25}. \qquad (9.48)
\end{aligned}$$

Example 9.31. Find the residue of $f(z) = \cot z$ at $z = 0$.

We write $f(z) = \cot z = \frac{\cos z}{\sin z}$ and note that $z = 0$ is a simple pole. Thus,

$$\operatorname{Res}[\cot z; z=0] = \lim_{z\to 0}\frac{z\cos z}{\sin z} = \cos(0) = 1.$$

The residue of $f(z)$ at z_0 is the coefficient of the $(z-z_0)^{-1}$ term, $c_{-1} = b_1$, of the Laurent series expansion about z_0.

Another way to find the residue of a function $f(z)$ at a singularity z_0 is to look at the Laurent series expansion about the singularity. This is because the residue of $f(z)$ at z_0 is the coefficient of the $(z-z_0)^{-1}$ term, or $c_{-1} = b_1$.

Example 9.32. Find the residue of $f(z) = \frac{1}{z(3-z)}$ at $z = 0$ using a Laurent series expansion.

First, we need the Laurent series expansion about $z = 0$ of the form $\sum_{-\infty}^{\infty} c_n z^n$. A partial fraction expansion gives

$$f(z) = \frac{1}{z(3-z)} = \frac{1}{3}\left(\frac{1}{z} + \frac{1}{3-z}\right).$$

The first term is a power of z. The second term needs to be written as a convergent series for small z. This is given by

$$\begin{aligned}
\frac{1}{3-z} &= \frac{1}{3(1-z/3)}\\
&= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n. \qquad (9.49)
\end{aligned}$$

Thus, we have found

$$f(z) = \frac{1}{3}\left(\frac{1}{z} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right).$$

The coefficient of z^{-1} can be read off to give $\operatorname{Res}[f(z); z=0] = \frac{1}{3}$.

Example 9.33. Find the residue of $f(z) = z\cos\frac{1}{z}$ at $z = 0$ using a Laurent series expansion.

Finding the residue at an essential singularity.

In this case $z = 0$ is an essential singularity. The only way to find residues at essential singularities is to use Laurent series. Since

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \ldots,$$

then we have

$$\begin{aligned} f(z) &= z\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \dots\right) \\ &= z - \frac{1}{2!z} + \frac{1}{4!z^3} - \frac{1}{6!z^5} + \dots. \end{aligned} \tag{9.50}$$

From the second term we have that $\operatorname{Res}[f(z); z = 0] = -\frac{1}{2}$.

We are now ready to use residues in order to evaluate integrals.

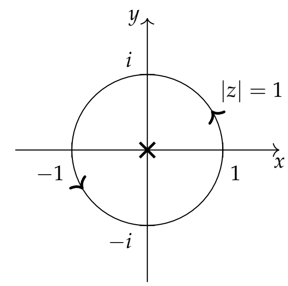


Figure 9.36: Contour for computing $\oint_{|z|=1} \frac{dz}{\sin z}$.

Example 9.34. Evaluate $\oint_{|z|=1} \frac{dz}{\sin z}$.

We begin by looking for the singularities of the integrand. These are located at values of z for which $\sin z = 0$. Thus, $z = 0, \pm\pi, \pm 2\pi, \dots,$ are the singularities. However, only $z = 0$ lies inside the contour, as shown in Figure 9.36. We note further that $z = 0$ is a simple pole, since

$$\lim_{z \to 0} (z - 0) \frac{1}{\sin z} = 1.$$

Therefore, the residue is one and we have

$$\oint_{|z|=1} \frac{dz}{\sin z} = 2\pi i.$$

In general, we could have several poles of different orders. For example, we will be computing

$$\oint_{|z|=2} \frac{dz}{z^2 - 1}.$$

The integrand has singularities at $z^2 - 1 = 0$, or $z = \pm 1$. Both poles are inside the contour, as seen in Figure 9.38. One could do a partial fraction decomposition and have two integrals with one pole each integral. Then, the result could be found by adding the residues from each pole.

In general, when there are several poles, we can use the Residue Theorem.

The Residue Theorem.

The Residue Theorem

Theorem 9.9. *Let $f(z)$ be a function which has poles z_j, $j = 1, \dots, N$ inside a simple closed contour C and no other singularities in this region. Then,*

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}[f(z); z_j], \tag{9.51}$$

where the residues are computed using Equation (9.44),

$$\operatorname{Res}[f(z); z_0] = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right].$$

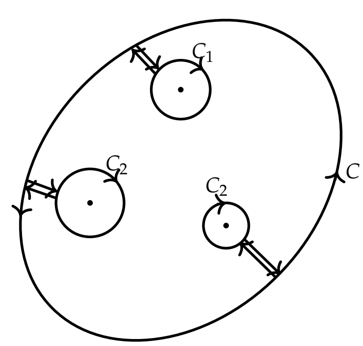


Figure 9.37: A depiction of how one cuts out poles to prove that the integral around C is the sum of the integrals around circles with the poles at the center of each.

The proof of this theorem is based upon the contours shown in Figure 9.37. One constructs a new contour C' by encircling each pole, as show in the figure. Then one connects a path from C to each circle. In the figure two separated paths along the cut are shown only to indicate the direction followed on the cut. The new contour is then obtained by following C and crossing each cut as it is encountered. Then one goes around a circle in the negative sense and returns along the cut to proceed around C. The sum of the contributions to the contour integration involve two integrals for each cut, which will cancel due to the opposing directions. Thus, we are left with

$$\oint_{C'} f(z)\,dz = \oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \oint_{C_3} f(z)\,dz = 0.$$

Of course, the sum is zero because $f(z)$ is analytic in the enclosed region, since all singularities have been cut out. Solving for $\oint_C f(z)\,dz$, one has that this integral is the sum of the integrals around the separate poles, which can be evaluated with single residue computations. Thus, the result is that $\oint_C f(z)\,dz$ is $2\pi i$ times the sum of the residues.

Example 9.35. Evaluate $\oint_{|z|=2} \frac{dz}{z^2-1}$.

We first note that there are two poles in this integral since

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}.$$

In Figure 9.38 we plot the contour and the two poles, denoted by an "x." Since both poles are inside the contour, we need to compute the residues for each one. They are each simple poles, so we have

$$\begin{aligned} \operatorname{Res}\left[\frac{1}{z^2-1}; z=1\right] &= \lim_{z\to 1}(z-1)\frac{1}{z^2-1} \\ &= \lim_{z\to 1}\frac{1}{z+1} = \frac{1}{2}, \end{aligned} \tag{9.52}$$

and

$$\begin{aligned} \operatorname{Res}\left[\frac{1}{z^2-1}; z=-1\right] &= \lim_{z\to -1}(z+1)\frac{1}{z^2-1} \\ &= \lim_{z\to -1}\frac{1}{z-1} = -\frac{1}{2}. \end{aligned} \tag{9.53}$$

Then,

$$\oint_{|z|=2} \frac{dz}{z^2-1} = 2\pi i\left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

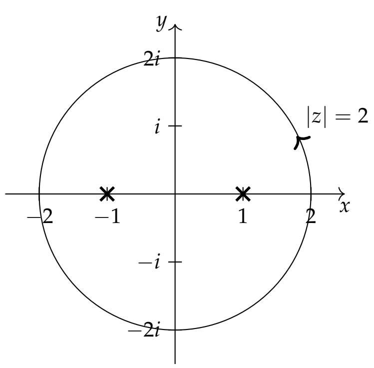


Figure 9.38: Contour for computing $\oint_{|z|=2} \frac{dz}{z^2-1}$.

Example 9.36. Evaluate $\oint_{|z|=3} \frac{z^2+1}{(z-1)^2(z+2)}\,dz$.

In this example there are two poles $z = 1, -2$ inside the contour. [See Figure 9.39.] $z = 1$ is a second order pole and $z = -2$ is a simple pole. Therefore, we need to compute the residues at each pole of $f(z) = \frac{z^2+1}{(z-1)^2(z+2)}$:

$$\begin{aligned} \operatorname{Res}[f(z); z=1] &= \lim_{z\to 1}\frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\frac{z^2+1}{(z-1)^2(z+2)}\right] \\ &= \lim_{z\to 1}\left(\frac{z^2+4z-1}{(z+2)^2}\right) \\ &= \frac{4}{9}. \end{aligned} \tag{9.54}$$

$$\begin{aligned}\operatorname{Res}[f(z); z=-2] &= \lim_{z\to -2}(z+2)\frac{z^2+1}{(z-1)^2(z+2)} \\ &= \lim_{z\to -2}\frac{z^2+1}{(z-1)^2} \\ &= \frac{5}{9}.\end{aligned} \tag{9.55}$$

The evaluation of the integral is found by computing $2\pi i$ times the sum of the residues:

$$\oint_{|z|=3}\frac{z^2+1}{(z-1)^2(z+2)}\,dz = 2\pi i\left(\frac{4}{9}+\frac{5}{9}\right) = 2\pi i.$$

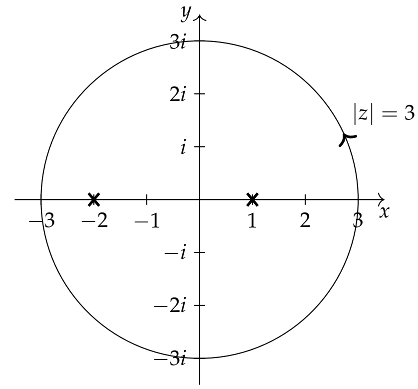


Figure 9.39: Contour for computing $\oint_{|z|=3}\frac{z^2+1}{(z-1)^2(z+2)}\,dz$.

Example 9.37. Compute $\oint_{|z|=2} z^3 e^{2/z}\,dz$.

In this case, $z=0$ is an essential singularity and is inside the contour. A Laurent series expansion about $z=0$ gives

$$\begin{aligned} z^3e^{2/z} &= z^3\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2}{z}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{2^n}{n!}z^{3-n} \\ &= z^3+\frac{2}{2!}z^2+\frac{4}{3!}z+\frac{8}{4!}+\frac{16}{5!z}+\ldots.\end{aligned} \tag{9.56}$$

The residue is the coefficient of z^{-1}, or $\operatorname{Res}[z^3e^{2/z}; z=0] = -\frac{2}{15}$. Therefore,

$$\oint_{|z|=2} z^3e^{2/z}\,dz = \frac{4}{15}\pi i.$$

Example 9.38. Evaluate $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$.

Here we have a real integral in which there are no signs of complex functions. In fact, we could apply simpler methods from a calculus course to do this integral, attempting to write $1+\cos\theta = 2\cos^2\frac{\theta}{2}$. However, we do not get very far.

Computation of integrals of functions of sines and cosines, $f(\cos\theta,\sin\theta)$.

One trick, useful in computing integrals whose integrand is in the form $f(\cos\theta,\sin\theta)$, is to transform the integration to the complex plane through the transformation $z=e^{i\theta}$. Then,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = -\frac{i}{2}\left(z-\frac{1}{z}\right).$$

Under this transformation, $z=e^{i\theta}$, the integration now takes place around the unit circle in the complex plane. Noting that $dz = ie^{i\theta}\,d\theta = iz\,d\theta$, we have

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \oint_{|z|=1}\frac{\frac{dz}{iz}}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

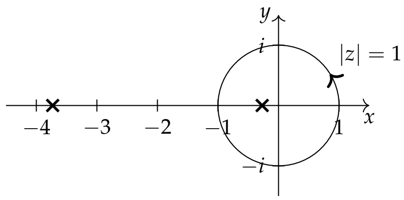


Figure 9.40: Contour for computing $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

$$\begin{aligned} &= -i \oint_{|z|=1} \frac{dz}{2z + \frac{1}{2}(z^2+1)} \\ &= -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1}. \end{aligned} \tag{9.57}$$

We can apply the Residue Theorem to the resulting integral. The singularities occur at the roots of $z^2 + 4z + 1 = 0$. Using the quadratic formula, we have the roots $z = -2 \pm \sqrt{3}$.

The location of these poles are shown in Figure 9.40. Only $z = -2 + \sqrt{3}$ lies inside the integration contour. We will therefore need the residue of $f(z) = \frac{-2i}{z^2+4z+1}$ at this simple pole:

$$\begin{aligned} \operatorname{Res}[f(z); z = -2+\sqrt{3}] &= \lim_{z \to -2+\sqrt{3}} (z - (-2+\sqrt{3})) \frac{-2i}{z^2+4z+1} \\ &= -2i \lim_{z \to -2+\sqrt{3}} \frac{z - (-2+\sqrt{3})}{(z - (-2+\sqrt{3}))(z - (-2-\sqrt{3}))} \\ &= -2i \lim_{z \to -2+\sqrt{3}} \frac{1}{z - (-2-\sqrt{3})} \\ &= \frac{-2i}{-2+\sqrt{3} - (-2-\sqrt{3})} \\ &= \frac{-i}{\sqrt{3}} \\ &= \frac{-i\sqrt{3}}{3}. \end{aligned} \tag{9.58}$$

Therefore, we have

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i \oint_{|z|=1} \frac{dz}{z^2+4z+1} = 2\pi i \left(\frac{-i\sqrt{3}}{3} \right) = \frac{2\pi\sqrt{3}}{3}. \tag{9.59}$$

Before moving on to further applications, we note that there is another way to compute the integral in the last example. Karl Theodor Wilhelm Weierstraß (1815-1897) introduced a substitution method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t = \tan\frac{\theta}{2}$ and converts the integrand into a rational function of t. One can show that this substitution implies that

The Weierstraß substitution method.

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2},$$

and

$$d\theta = \frac{2dt}{1+t^2}.$$

The details are left for Problem 8 and apply the method. In order to see how it works, we will redo the last problem.

Example 9.39. Apply the Weierstraß substitution method to compute $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{-\infty}^{\infty} \frac{1}{2+\frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2}$$

$$\begin{aligned} &= 2\int_{-\infty}^{\infty} \frac{dt}{t^2+3} \\ &= \frac{2}{3}\sqrt{3}\left[\tan^{-1}\left(\frac{\sqrt{3}}{3}t\right)\right]_{-\infty}^{\infty} = \frac{2\pi\sqrt{3}}{3}. \end{aligned} \tag{9.60}$$

9.5.6 *Infinite Integrals*

INFINITE INTEGRALS OF THE FORM $\int_{-\infty}^{\infty} f(x)\,dx$ occur often in physics. They can represent wave packets, wave diffraction, Fourier transforms, and arise in other applications. In this section we will see that such integrals may be computed by extending the integration to a contour in the complex plane.

Recall from your calculus experience that these integrals are improper integrals and the way that one determines if improper integrals exist, or converge, is to carefully compute these integrals using limits such as

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$$

For example, we evaluate the integral of $f(x) = x$ as

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left(\frac{R^2}{2} - \frac{(-R)^2}{2}\right) = 0.$$

One might also be tempted to carry out this integration by splitting the integration interval, $(-\infty, 0] \cup [0, \infty)$. However, the integrals $\int_0^{\infty} x\,dx$ and $\int_{-\infty}^{0} x\,dx$ do not exist. A simple computation confirms this.

$$\int_{0}^{\infty} x\,dx = \lim_{R\to\infty}\int_{0}^{R} x\,dx = \lim_{R\to\infty}\left(\frac{R^2}{2}\right) = \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

does not exist while $\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$ does exist. We will be interested in computing the latter type of integral. Such an integral is called the Cauchy Principal Value Integral and is denoted with either a P, PV, or a bar through the integral:

The Cauchy principal value integral.

$$P\int_{-\infty}^{\infty} f(x)\,dx = PV\int_{-\infty}^{\infty} f(x)\,dx = ⨍_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \tag{9.61}$$

If there is a discontinuity in the integral, one can further modify this definition of principal value integral to bypass the singularity. For example, if $f(x)$ is continuous on $a \le x \le b$ and not defined at $x = x_0 \in [a, b]$, then

$$\int_{a}^{b} f(x)\,dx = \lim_{\epsilon\to 0}\left(\int_{a}^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^{b} f(x)\,dx\right).$$

In our discussions we will be computing integrals over the real line in the Cauchy principal value sense.

Example 9.40. Compute $\int_{-1}^{1} \frac{dx}{x^3}$ in the Cauchy Principal Value sense.

In this case, $f(x) = \frac{1}{x^3}$ is not defined at $x = 0$. So, we have

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{x^3} &= \lim_{\epsilon \to 0} \left(\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3} \right) \\
&= \lim_{\epsilon \to 0} \left(-\frac{1}{2x^2}\bigg|_{-1}^{-\epsilon} - \frac{1}{2x^2}\bigg|_{\epsilon}^{1} \right) = 0. \qquad (9.62)
\end{aligned}$$

Computation of real integrals by embedding the problem in the complex plane.

We now proceed to the evaluation of principal value integrals using complex integration methods. We want to evaluate the integral $\int_{-\infty}^{\infty} f(x)\, dx$. We will extend this into an integration in the complex plane. We extend $f(x)$ to $f(z)$ and assume that $f(z)$ is analytic in the upper half plane ($\text{Im}(z) > 0$) except at isolated poles. We then consider the integral $\int_{-R}^{R} f(x)\, dx$ as an integral over the interval $(-R, R)$. We view this interval as a piece of a larger contour C_R obtained by completing the contour with a semicircle Γ_R of radius R extending into the upper half plane as shown in Figure 9.41. Note, a similar construction is sometimes needed extending the integration into the lower half plane ($\text{Im}(z) < 0$) as we will later see.

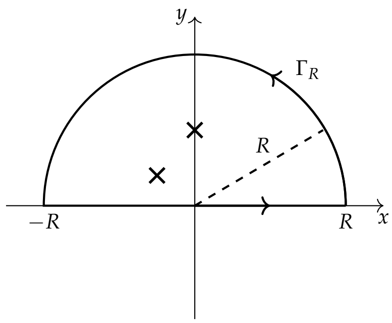


Figure 9.41: Contours for computing $P\int_{-\infty}^{\infty} f(x)\, dx$.

The integral around the entire contour C_R can be computed using the Residue Theorem and is related to integrations over the pieces of the contour by

$$\oint_{C_R} f(z)\, dz = \int_{\Gamma_R} f(z)\, dz + \int_{-R}^{R} f(z)\, dz. \qquad (9.63)$$

Taking the limit $R \to \infty$ and noting that the integral over $(-R, R)$ is the desired integral, we have

$$P\int_{-\infty}^{\infty} f(x)\, dx = \oint_C f(z)\, dz - \lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz, \qquad (9.64)$$

where we have identified C as the limiting contour as R gets large.

Now the key to carrying out the integration is that the second integral vanishes in the limit. This is true if $R|f(z)| \to 0$ along Γ_R as $R \to \infty$. This can be seen by the following argument. We parametrize the contour Γ_R using $z = Re^{i\theta}$. Then, when $|f(z)| < M(R)$,

$$\begin{aligned}
\left| \int_{\Gamma_R} f(z)\, dz \right| &= \left| \int_0^{2\pi} f(Re^{i\theta}) Re^{i\theta}\, d\theta \right| \\
&\le R \int_0^{2\pi} \left| f(Re^{i\theta}) \right| d\theta \\
&< RM(R) \int_0^{2\pi} d\theta \\
&= 2\pi RM(R). \qquad (9.65)
\end{aligned}$$

So, if $\lim_{R\to\infty} RM(R) = 0$, then $\lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz = 0$.

We now demonstrate how to use complex integration methods in evaluating integrals over real valued functions.

Example 9.41. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

We already know how to do this integral using calculus without complex analysis. We have that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \left(2\tan^{-1} R \right) = 2\left(\frac{\pi}{2}\right) = \pi.$$

We will apply the methods of this section and confirm this result. The needed contours are shown in Figure 9.42 and the poles of the integrand are at $z = \pm i$. We first write the integral over the bounded contour C_R as the sum of an integral from $-R$ to R along the real axis plus the integral over the semicircular arc in the upper half complex plane,

$$\int_{C_R} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\Gamma_R} \frac{dz}{1+z^2}.$$

Next, we let R get large.

We first note that $f(z) = \frac{1}{1+z^2}$ goes to zero fast enough on Γ_R as R gets large.

$$R|f(z)| = \frac{R}{|1+R^2 e^{2i\theta}|} = \frac{R}{\sqrt{1+2R^2\cos\theta + R^4}}.$$

Thus, as $R \to \infty$, $R|f(z)| \to 0$ and $C_R \to C$. So,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint_C \frac{dz}{1+z^2}.$$

We need only compute the residue at the enclosed pole, $z = i$.

$$\operatorname{Res}[f(z); z = i] = \lim_{z\to i}(z-i)\frac{1}{1+z^2} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}.$$

Then, using the Residue Theorem, we have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \left(\frac{1}{2i}\right) = \pi.$$

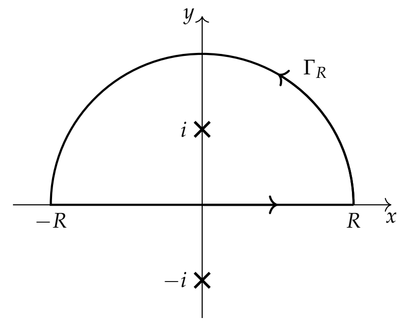


Figure 9.42: Contour for computing $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Example 9.42. Evaluate $P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$.

For this example the integral is unbounded at $z = 0$. Constructing the contours as before we are faced for the first time with a pole lying on the contour. We cannot ignore this fact. We can proceed with the computation by carefully going around the pole with a small semicircle of radius ϵ, as shown in Figure 9.43. Then the principal value integral computation becomes

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{\epsilon\to 0, R\to\infty}\left(\int_{-R}^{-\epsilon} \frac{\sin x}{x}\, dx + \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx\right). \tag{9.66}$$

We will also need to rewrite the sine function in term of exponentials in this integral. There are two approaches that we could take. First, we could employ the definition of the sine function in terms of complex exponentials. This gives two integrals to compute:

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2i}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx\right). \tag{9.67}$$

The other approach would be to realize that the sine function is the imaginary part of an exponential, $\operatorname{Im} e^{ix} = sinx$. Then, we would have

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \operatorname{Im}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx\right). \tag{9.68}$$

We first consider $P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx$, which is common to both approaches. We use the contour in Figure 9.43. Then we have

$$\oint_{C_R}\frac{e^{iz}}{z}\,dz=\int_{\Gamma_R}\frac{e^{iz}}{z}\,dz+\int_{-R}^{-\epsilon}\frac{e^{iz}}{z}\,dz+\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz+\int_{\epsilon}^{R}\frac{e^{iz}}{z}\,dz.$$

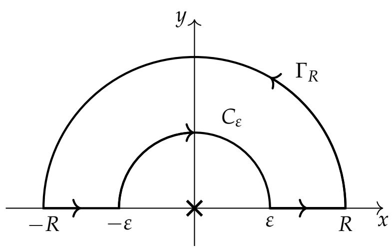


Figure 9.43: Contour for computing $P\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$.

The integral $\oint_{C_R}\frac{e^{iz}}{z}\,dz$ vanishes since there are no poles enclosed in the contour! The sum of the second and fourth integrals gives the integral we seek as $\epsilon\to 0$ and $R\to\infty$. The integral over Γ_R will vanish as R gets large according to Jordan's Lemma.

Jordan's Lemma give conditions as when integrals over Γ_R will vanish as R gets large. We state a version of Jordan's Lemma here for reference and give a proof is at the end of this chapter.

Jordan's Lemma

If $f(z)$ converges uniformly to zero as $z\to\infty$, then

$$\lim_{R\to\infty}\int_{C_R}f(z)e^{ikz}\,dz=0$$

where $k>0$ and C_R is the upper half of the circle $|z|=R$.

A similar result applies for $k<0$, but one closes the contour in the lower half plane. [See Section 9.5.8 for the proof of Jordan's Lemma.]

The remaining integral around the small semicircular arc has to be done separately. We have

$$\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz=\int_{\pi}^{0}\frac{\exp(i\epsilon e^{i\theta})}{\epsilon e^{i\theta}}i\epsilon e^{i\theta}\,d\theta=-\int_0^{\pi}i\exp(i\epsilon e^{i\theta})\,d\theta.$$

Taking the limit as ϵ goes to zero, the integrand goes to i and we have

$$\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz=-\pi i.$$

Note that we have not previously done integrals in which a singularity lies on the contour. One can show, as in this example, that points on the contour can be accounted for by using half of a residue (times $2\pi i$). For the semicircle C_ϵ you can verify this. The negative sign comes from going clockwise around the semicircle.

So far, we have that

$$P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx=-\lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz=\pi i.$$

At this point we can get the answer using the second approach in Equation (9.68). Namely,

$$P\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx=\operatorname{Im}\left(P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx\right)=\operatorname{Im}(\pi i)=\pi. \tag{9.69}$$

It is instructive to carry out the first approach in Equation (9.67). We will need to compute $P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx$. This is done in a similar to the above computation, being careful with the sign changes due to the orientations of the contours as shown in Figure 9.44.

We note that the contour is closed in the lower half plane. This is because $k < 0$ in the application of Jordan's Lemma. One can understand why this is the case from the following observation. Consider the exponential in Jordan's Lemma. Let $z = z_R + iz_I$. Then,

$$e^{ikz} = e^{ik(z_R+iz_I)} = e^{-kz_I}e^{ikz_R}.$$

As $|z|$ gets large, the second factor just oscillates. The first factor would go to zero if $kz_I > 0$. So, if $k > 0$, we would close the contour in the upper half plane. If $k < 0$, then we would close the contour in the lower half plane. In the current computation, $k = -1$, so we use the lower half plane.

Working out the details, we find the same value for

$$P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx = \pi i.$$

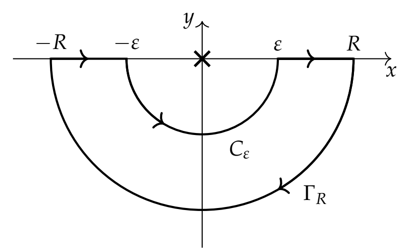


Figure 9.44: Contour in the lower half plane for computing $P\int_{-\infty}^{\infty}\frac{e^{-ix}}{x}\,dx$.

Finally, we can compute the original integral as

$$\begin{aligned} P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx &= \frac{1}{2i}\left(P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - P\int_{-\infty}^{\infty} \frac{e^{-ix}}{x}\, dx\right) \\ &= \frac{1}{2i}\left(\pi i + \pi i\right) \\ &= \pi. \end{aligned} \tag{9.70}$$

This is the same result as we obtained using Equation(9.68).

Example 9.43. Evaluate $\oint_{|z|=1} \frac{dz}{z^2+1}$.

In this example there are two simple poles, $z = \pm i$ lying on the contour, as seen in Figure 9.45. This problem is similar to Problem 1c, except we will do it using contour integration instead of a parametrization. We bypass the two poles by drawing small semicircles around them. Since the poles are not included in the closed contour, then the Residue Theorem tells us that the integral over the path vanishes. We can write the full integration as a sum over three paths, $C_\pm$ for the semicircles and C for the original contour with the poles cut out. Then we take the limit as the semicircle radii go to zero. So,

$$0 = \int_C \frac{dz}{z^2+1} + \int_{C_+} \frac{dz}{z^2+1} + \int_{C_-} \frac{dz}{z^2+1}.$$

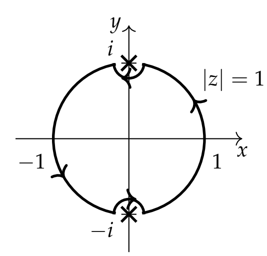


Figure 9.45: Example with poles on contour.

The integral over the semicircle around i can be done using the parametrization $z = i + \epsilon e^{i\theta}$. Then $z^2 + 1 = 2i\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}$. This gives

$$\int_{C_+} \frac{dz}{z^2+1} = \lim_{\epsilon\to 0}\int_0^{-\pi} \frac{i\epsilon e^{i\theta}}{2i\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}}\, d\theta = \frac{1}{2}\int_0^{-\pi} d\theta = -\frac{\pi}{2}.$$

As in the last example, we note that this is just πi times the residue, $Res\left[\frac{1}{z^2+1}; z = i\right] = \frac{1}{2i}$. Since the path is traced clockwise, we find the contribution is $-\pi i Res = -\frac{\pi}{2}$, which is what we obtained above. A Similar computation will give the contribution from $z = -i$ as $\frac{\pi}{2}$.

Adding these values gives the total contribution from $C_\pm$ as zero. So, the final result is that

$$\oint_{|z|=1} \frac{dz}{z^2+1} = 0.$$

Example 9.44. Evaluate $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$, for $0 < a < 1$.

In dealing with integrals involving exponentials or hyperbolic functions it is sometimes useful to use different types of contours. This example is one such case. We will replace x with z and integrate over the contour in Figure 9.46. Letting $R \to \infty$, the integral along the real axis is the integral that we desire. The integral along the path for $y = 2\pi$ leads to a multiple of this integral since $z = x + 2\pi i$ along this path. Integration along the vertical paths vanish as $R \to \infty$. This is captured in the following integrals:

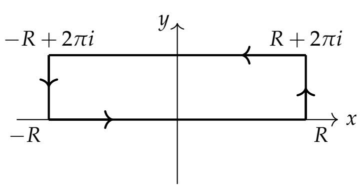


Figure 9.46: Example using a rectangular contour.

$$\begin{aligned}\oint_{C_R} \frac{e^{az}}{1+e^z}dz &= \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx + \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}}\,dy \\ &\quad + \int_R^{-R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx + \int_{2\pi}^{0} \frac{e^{a(-R+iy)}}{1+e^{-R+iy}}\,dy \qquad (9.71)\end{aligned}$$

We can now let $R \to \infty$. For large R the second integral decays as $e^{(a-1)R}$ and the fourth integral decays as e^{-aR}. Thus, we are left with

$$\begin{aligned}\oint_C \frac{e^{az}}{1+e^z}dz &= \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{e^{ax}}{1+e^x}dx - e^{2\pi i a}\int_{-R}^{R} \frac{e^{ax}}{1+e^x}dx\right) \\ &= (1-e^{2\pi i a})\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx. \qquad (9.72)\end{aligned}$$

We need only evaluate the left contour integral using the Residue Theorem. The poles are found from

$$1 + e^z = 0.$$

Within the contour, this is satisfied by $z = i\pi$. So,

$$Res\left[\frac{e^{az}}{1+e^z}; z = i\pi\right] = \lim_{z\to i\pi}(z - i\pi)\frac{e^{az}}{1+e^z} = -e^{i\pi a}.$$

Applying the Residue Theorem, we have

$$(1-e^{2\pi i a})\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = -2\pi i e^{i\pi a}.$$

Therefore, we have found that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = \frac{-2\pi i e^{i\pi a}}{1-e^{2\pi i a}} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1.$$

9.5.7 *Integration Over Multivalued Functions*

We have seen that some complex functions inherently possess multivaluedness; i.e., such "functions" do not evaluate to a single value, but have many values. The key examples were $f(z) = z^{1/n}$ and $f(z) = \ln z$. The nth roots have n distinct values and logarithms have an infinite number of values as determined by the range of the resulting arguments. We mentioned that the way to handle multivaluedness is to assign different branches to these functions, introduce a branch cut and glue them together at the branch cuts to form Riemann surfaces. In this way we can draw continuous paths along the Riemann surfaces as we move from one Riemann sheet to another.

Before we do examples of contour integration involving multivalued functions, lets first try to get a handle on multivaluedness in a simple case. We will consider the square root function,

$$w = z^{1/2} = r^{1/2} e^{i(\frac{\theta}{2}+k\pi)}, \quad k = 0, 1.$$

There are two branches, corresponding to each k value. If we follow a path not containing the origin, then we stay in the same branch, so the final argument (θ) will be equal to the initial argument. However, if we follow a path that encloses the origin, this will not be true. In particular, for an initial point on the unit circle, $z_0 = e^{i\theta_0}$, we have its image as $w_0 = e^{i\theta_0/2}$. However, if we go around a full revolution, $\theta = \theta_0 + 2\pi$, then

$$z_1 = e^{i\theta_0 + 2\pi i} = e^{i\theta_0},$$

but

$$w_1 = e^{(i\theta_0 + 2\pi i)/2} = e^{i\theta_0/2} e^{\pi i} \neq w_0.$$

Here we obtain a final argument (θ) that is not equal to the initial argument! Somewhere, we have crossed from one branch to another. Points, such as the origin in this example, are called branch points. Actually, there are two branch points, because we can view the closed path around the origin as a closed path around complex infinity in the compactified complex plane. However, we will not go into that at this time.

We can demonstrate this in the following figures. In Figure 9.47 we show how the points A-E are mapped from the z-plane into the w-plane under the square root function for the principal branch, $k = 0$. As we trace out the unit circle in the z-plane, we only trace out a semicircle in the w-plane. If we consider the branch $k = 1$, we then trace out a semicircle in the lower half plane, as shown in Figure 9.48 following the points from F to J.

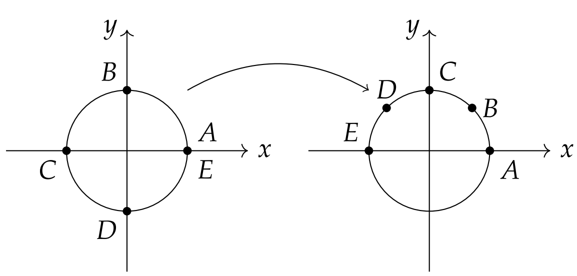


Figure 9.47: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the principal square root function.

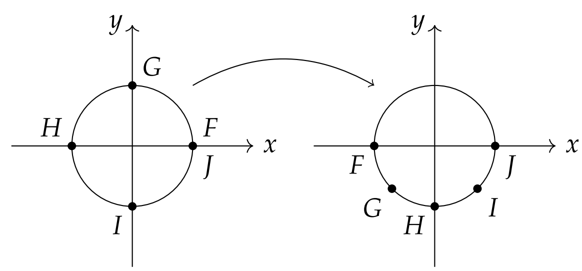


Figure 9.48: In this figure we show how points on the unit circle in the z-plane are mapped to points in the w-plane under the square root function for the second branch, $k = 1$.

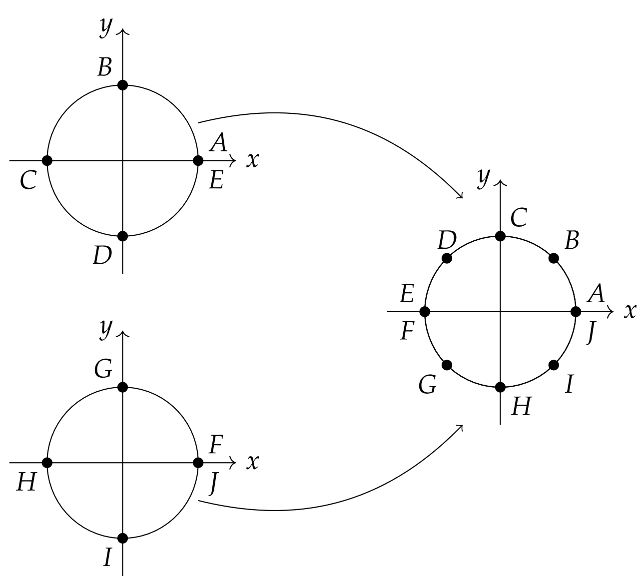


Figure 9.49: In this figure we show the combined mapping using two branches of the square root function.

We can combine these into one mapping depicting how the two complex planes corresponding to each branch provide a mapping to the w-plane. This is shown in Figure 9.49.

A common way to draw this domain, which looks like two separate complex planes, would be to glue them together. Imagine cutting each plane along the positive x-axis, extending between the two branch points, $z = 0$ and $z = \infty$. As one approaches the cut on the principal branch, then one can move onto the glued second branch. Then one continues around the origin on this branch until one once again reaches the cut. This cut is glued to the principal branch in such a way that the path returns to its starting point. The resulting surface we obtain is the Riemann surface shown in Figure 9.50. Note that there is nothing that forces us to place the branch cut at a particular place. For example, the branch cut could be along the positive real axis, the negative real axis, or any path connecting the origin and complex infinity.

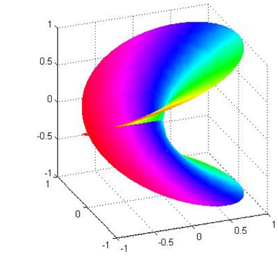


Figure 9.50: Riemann surface for $f(z) = z^{1/2}$.

We now look at examples involving integrals of multivalued functions.

Example 9.45. Evaluate $\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$.

We consider the contour integral $\oint_C \frac{\sqrt{z}}{1+z^2}\,dz$.

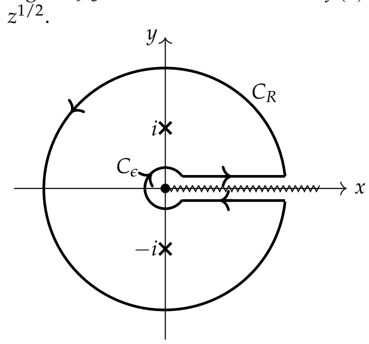


Figure 9.51: An example of a contour which accounts for a branch cut.

The first thing we can see in this problem is the square root function in the integrand. Being there is a multivalued function, we locate the branch point and determine where to draw the branch cut. In Figure 9.51 we show the contour that we will use in this problem. Note that we picked the branch cut along the positive x-axis.

We take the contour C to be positively oriented, being careful to enclose the two poles and to hug the branch cut. It consists of two circles. The outer circle C_R is a circle of radius R and the inner circle C_ϵ will have a radius of ϵ. The sought answer will be obtained by letting $R \to \infty$ and $\epsilon \to 0$. On the large circle we have that the integrand goes to zero fast enough as $R \to \infty$. The integral around the small circle vanishes as $\epsilon \to 0$. We can see this by parametrizing the circle as $z = \epsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$:

$$\begin{aligned}
\oint_{C_\epsilon} \frac{\sqrt{z}}{1+z^2}\,dz &= \int_0^{2\pi} \frac{\sqrt{\epsilon e^{i\theta}}}{1+(\epsilon e^{i\theta})^2} i\epsilon e^{i\theta}\,d\theta \\
&= i\epsilon^{3/2} \int_0^{2\pi} \frac{e^{3i\theta/2}}{1+(\epsilon^2 e^{2i\theta})}\,d\theta. \qquad (9.73)
\end{aligned}$$

It should now be easy to see that as $\epsilon \to 0$ this integral vanishes.

The integral above the branch cut is the one we are seeking, $\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$. The integral under the branch cut, where $z = re^{2\pi i}$, is

$$\begin{aligned}
\int \frac{\sqrt{z}}{1+z^2}\,dz &= \int_\infty^0 \frac{\sqrt{re^{2\pi i}}}{1+r^2 e^{4\pi i}}\,dr \\
&= \int_0^\infty \frac{\sqrt{r}}{1+r^2}\,dr. \qquad (9.74)
\end{aligned}$$

We note that this is the same as that above the cut.

Up to this point, we have that the contour integral, as $R \to \infty$ and $\epsilon \to 0$ is

$$\oint_C \frac{\sqrt{z}}{1+z^2}dz = 2\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx.$$

In order to finish this problem, we need the residues at the two simple poles.

$$Res\left[\frac{\sqrt{z}}{1+z^2}; z=i\right] = \frac{\sqrt{i}}{2i} = \frac{\sqrt{2}}{4}(1+i),$$

$$Res\left[\frac{\sqrt{z}}{1+z^2}; z=-i\right] = \frac{\sqrt{-i}}{-2i} = \frac{\sqrt{2}}{4}(1-i).$$

So,

$$2\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = 2\pi i\left(\frac{\sqrt{2}}{4}(1+i)+\frac{\sqrt{2}}{4}(1-i)\right) = \pi\sqrt{2}.$$

Finally, we have the value of the integral that we were seeking,

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \frac{\pi\sqrt{2}}{2}.$$

Example 9.46. Compute $\int_a^\infty f(x)\,dx$ using contour integration involving logarithms.[4]

In this example we will apply contour integration to the integral

$$\oint_C f(z)\ln(a-z)\,dz$$

for the contour shown in Figure 9.52.

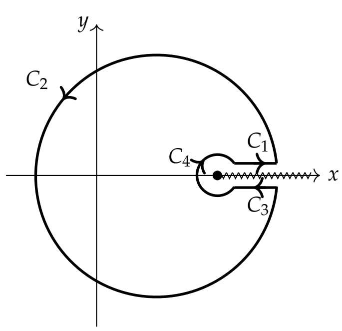


Figure 9.52: Contour needed to compute $\oint_C f(z)\ln(a-z)\,dz$.

We will assume that $f(z)$ is single valued and vanishes as $|z| \to \infty$. We will choose the branch cut to span from the origin along the positive real axis. Employing the Residue Theorem and breaking up the integrals over the pieces of the contour in Figure 9.52, we have schematically that

$$2\pi i\sum Res[f(z)\ln(a-z)] = \left(\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right) f(z)\ln(a-z)\,dz.$$

First of all, we assume that $f(z)$ is well behaved at $z=a$ and vanishes fast enough as $|z| = R \to \infty$. Then, the integrals over C_2 and C_4 will vanish. For example, for the path C_4, we let $z = a+\epsilon e^{i\theta}$, $0<\theta<2\pi$. Then,

$$\int_{C_4} f(z)\ln(a-z)\,dz. = \lim_{\epsilon\to 0}\int_{2\pi}^0 f(a+\epsilon e^{i\theta})\ln(\epsilon e^{i\theta}) i\epsilon e^{i\theta}\,d\theta.$$

If $f(a)$ is well behaved, then we only need to show that $\lim_{\epsilon\to 0}\epsilon\ln\epsilon = 0$. This is left to the reader.

Similarly, we consider the integral over C_2 as R gets large,

$$\int_{C_2} f(z)\ln(a-z)\,dz = \lim_{R\to\infty}\int_0^{2\pi} f(Re^{i\theta})\ln(Re^{i\theta}) iRe^{i\theta}\,d\theta.$$

[4] This approach was originally published in Neville, E. H., 1945, Indefinite integration by means of residues. *The Mathematical Student*, **13**, 16-35, and discussed in Duffy, D. G., *Transform Methods for Solving Partial Differential Equations*, 1994.

Thus, we need only require that

$$\lim_{R\to\infty} R\ln R|f(Re^{i\theta})| = 0.$$

Next, we consider the two straight line pieces. For C_1, the integration along the real axis occurs for $z = x$, so

$$\int_{C_1} f(z)\ln(a-z)\,dz = \int_a^\infty f(x)\ln(a-x)\,dz.$$

However, integration over C_3 requires noting that we need the branch for the logarithm such that $\ln z = \ln(a-x) + 2\pi i$. Then,

$$\int_{C_3} f(z)\ln(a-z)\,dz = \int_\infty^a f(x)[\ln(a-x)+2\pi i]\,dz.$$

Combining these results, we have

$$\begin{aligned} 2\pi i \sum Res[f(z)\ln(a-z)] &= \int_a^\infty f(x)\ln(a-x)\,dz \\ &\quad + \int_\infty^a f(x)[\ln(a-x)+2\pi i]\,dz. \\ &= -2\pi i \int_a^\infty f(x)\,dz. \end{aligned} \tag{9.75}$$

Therefore,

$$\int_a^\infty f(x)\,dx = -\sum Res[f(z)\ln(a-z)].$$

Example 9.47. Compute $\int_1^\infty \frac{dx}{4x^2-1}$.

We can apply the last example to this case. We see from Figure 9.53 that the two poles at $z = \pm\frac{1}{2}$ are inside contour C. So, we compute the residues of $\frac{\ln(1-z)}{4z^2-1}$ at these poles and find that

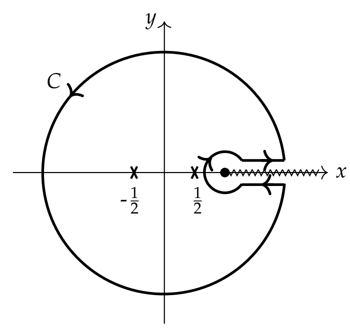


Figure 9.53: Contour needed to compute $\int_1^\infty \frac{dx}{4x^2-1}$.

$$\begin{aligned} \int_1^\infty \frac{dx}{4x^2-1} &= -Res\left[\frac{\ln(1-z)}{4z^2-1};\frac{1}{2}\right] - Res\left[\frac{\ln(1-z)}{4z^2-1};-\frac{1}{2}\right] \\ &= -\frac{\ln\frac{1}{2}}{4} + \frac{\ln\frac{3}{2}}{4} = \frac{\ln 3}{4}. \end{aligned} \tag{9.76}$$

9.5.8 *Appendix: Jordan's Lemma*

FOR COMPLETENESS, WE PROVE JORDAN'S LEMMA.

Theorem 9.10. *If $f(z)$ converges uniformly to zero as $z\to\infty$, then*

$$\lim_{R\to\infty}\int_{C_R} f(z)e^{ikz}\,dz = 0$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$.

Proof. We consider the integral

$$I_R = \int_{C_R} f(z)e^{ikz}\,dz,$$

where $k > 0$ and C_R is the upper half of the circle $|z| = R$ in the complex plane. Let $z = Re^{i\theta}$ be a parametrization of C_R. Then,

$$I_R = \int_0^{\pi} f(Re^{i\theta})e^{ikR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta.$$

Since

$$\lim_{|z|\to\infty} f(z) = 0, \quad 0 \le \arg z \le \pi,$$

then for large $|R|$, $|f(z)| < \epsilon$ for some $\epsilon > 0$. Then,

$$\begin{aligned}
|I_R| &= \left| \int_0^{\pi} f(Re^{i\theta})e^{ikR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta \right| \\
&\le \int_0^{\pi} \left| f(Re^{i\theta}) \right| \left| e^{ikR\cos\theta} \right| \left| e^{-aR\sin\theta} \right| \left| iRe^{i\theta} \right| d\theta \\
&\le \epsilon R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta \\
&= 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta. \qquad (9.77)
\end{aligned}$$

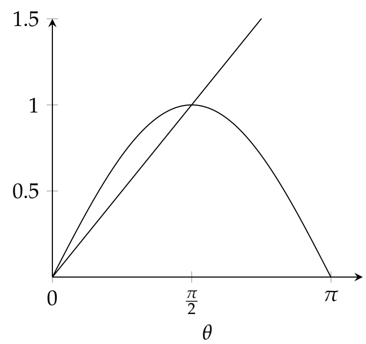


Figure 9.54: Plots of $y = \sin\theta$ and $y = \frac{2}{\pi}\theta$ to show where $\sin\theta \ge \frac{2}{\pi}\theta$.

The last integral still cannot be computed, but we can get a bound on it over the range $\theta \in [0, \pi/2]$. Note from Figure 9.54 that

$$\sin\theta \ge \frac{2}{\pi}\theta, \quad \theta \in [0, \pi/2].$$

Therefore, we have

$$|I_R| \le 2\epsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\, d\theta = \frac{2\epsilon R}{2aR/\pi}(1 - e^{-aR}).$$

For large R we have

$$\lim_{R\to\infty} |I_R| \le \frac{\pi\epsilon}{a}.$$

So, as $\epsilon \to 0$, the integral vanishes.

$\square$

9.6 Laplace's Equation in 2D, Revisited

Harmonic functions are solutions of Laplace's equation. We have seen that the real and imaginary parts of a holomorphic function are harmonic. So, there must be a connection between complex functions and solutions of the two-dimensional Laplace equation. In this section we will describe how conformal mapping can be used to find solutions of Laplace's equation in two dimensional regions.

In Section 1.8 we had first seen applications in two-dimensional steady-state heat flow (or, diffusion), electrostatics, and fluid flow. For example, letting $\phi(\mathbf{r})$ be the electric potential, one has for a static charge distribution, $\rho(\mathbf{r})$, that the electric field, $\mathbf{E} = \nabla\phi$, satisfies

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0.$$

In regions devoid of charge, these equations yield the Laplace equation, $\nabla^2\phi = 0$.

Similarly, we can derive Laplace's equation for an incompressible, $\nabla \cdot \mathbf{v} = 0$, irrotational, , $\nabla \times \mathbf{v} = 0$, fluid flow. From well-known vector identities, we know that $\nabla \times \nabla\phi = 0$ for a scalar function, ϕ. Therefore, we can introduce a velocity potential, ϕ, such that $\mathbf{v} = \nabla\phi$. Thus, $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, the velocity potential satisfies Laplace's equation.

Fluid flow is probably the simplest and most interesting application of complex variable techniques for solving Laplace's equation. So, we will spend some time discussing how conformal mappings have been used to study two-dimensional ideal fluid flow, leading to the study of airfoil design.

9.6.1 Fluid Flow

The study of fluid flow and conformal mappings dates back to Euler, Riemann, and others.[5] The method was further elaborated upon by physicists like Lord Rayleigh (1877) and applications to airfoil theory we presented in papers by Kutta (1902) and Joukowski (1906) on later to be improved upon by others.

[5] "On the Use of Conformal Mapping in Shaping Wing Profiles," MAA lecture by R. S. Burington, 1939, published (1940) in ... 362-373

The physics behind flight and the dynamics of wing theory relies on the ideas of drag and lift. Namely, as the the cross section of a wing, the airfoil, goes through the air, it will experience several forces. The air speed above and belong the wing will differ due to the distance the air has to travel across the top and bottom of the wing. According to Bernoulli's Principle, steady fluid flow satisfies the conservation of energy in the form

$$P + \frac{1}{2}\rho U^2 + \rho g h = \text{ constant}$$

at points on either side of the wing profile. Here P is the pressure, ρ is the air density, U is the fluid speed, h is a reference height, and g is the acceleration due to gravity. The gravitational potential energy, $\rho g h$, is roughly constant on either side of the wing. So, this reduces to

$$P + \frac{1}{2}\rho U^2 = \text{ constant.}$$

Therefore, if the speed of the air below the wing is lower that than above the wing, the pressure below the wing will be higher, resulting in a net upward pressure. Since the pressure is the force per area, this will result in an upward force, a lift force, acting on the wing. This is the simplified version for the lift force. There is also a drag force acting in the direction of the flow. In general, we want to use complex variable methods to model the streamlines of the airflow as the air flows around an airfoil.

We begin by considering the fluid flow across a curve, C as shown in Figure 9.55. We assume that it is an ideal fluid with zero viscosity (i.e., does not flow like molasses) and is incompressible. It is a continuous, homogeneous flow with a constant thickness and represented by a velocity

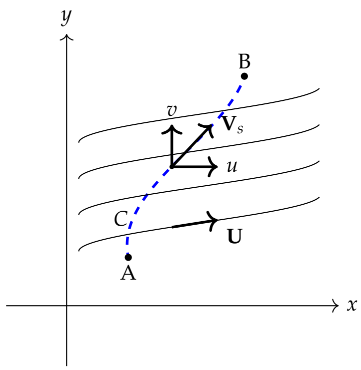


Figure 9.55: Fluid flow U across curve C between the points A and B.

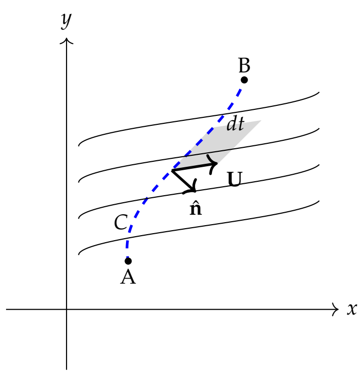


Figure 9.56: An amount of fluid crossing curve c in unit time.

$\mathbf{U} = (u(x,y), v(x,y))$, where u and v are the horizontal components of the flow as shown in Figure 9.55.

We are interested in the flow of fluid across a given curve which crosses several streamlines. The mass that flows over C per unit thickness in time dt can be given by

$$dm = \rho \mathbf{U} \cdot \hat{\mathbf{n}}\, dA dt.$$

Here $\hat{\mathbf{n}}\, dA$ is the normal area to the flow and for unit thickness can be written as $\hat{\mathbf{n}}\, dA = \mathbf{i}\, dy - \mathbf{i}\, dx$. Therefore, for a unit thickness the mass flow rate is given by

$$\frac{dm}{dt} = \rho(u\, dy - v\, dx).$$

Since the total mass flowing across ds in time dt is given by $dm = \rho dV$, for constant density, this also gives the volume flow rate,

$$\frac{dV}{dt} = u\, dy - v\, dx,$$

over a section of the curve. The total volume flow over C is therefore

$$\frac{dV}{dt}\Big|_{\text{total}} = \int_C u\, dy - v\, dx.$$

If this flow is independent of the curve, i.e., the path, then we have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}.$$

[This is just a consequence of Green's Theorem in the Plane. See Equation (9.3).] Another way to say this is that there exists a function, $\psi(x,t)$, such that $d\psi = u\, dy - v\, dx$. Then,

$$\int_C u\, dy - v\, dx = \int_A^B d\psi = \psi_B - \psi_A.$$

However, from basic calculus of several variables, we know that

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy = u\, dy - v\, dx.$$

Therefore,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

Streamline functions.

It follows that if $\psi(x,y)$ has continuous second derivatives, then $u_x = -v_y$. This function is called the streamline function.

Furthermore, for constant density, we have

$$\begin{aligned} \nabla \cdot (\rho \mathbf{U}) &= \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &= \rho \left(\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial^2 \psi}{\partial x \partial y} \right) = 0. \end{aligned} \tag{9.78}$$

This is the conservation of mass formula for constant density fluid flow.

We can also assume that the flow is irrotational. This means that the vorticity of the flow vanishes; i.e., $\nabla \times \mathbf{U} = \mathbf{0}$. Since the curl of the velocity field is zero, we can assume that the velocity is the gradient of a scalar function, $\mathbf{U} = \nabla\phi$. Then, a standard vector identity automatically gives

$$\nabla \times \mathbf{U} = \nabla \times \nabla\phi = 0.$$

For the two-dimensional flow with $\mathbf{U} = (u, v)$, we have

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}.$$

This is the velocity potential function for the flow.

Velocity potential curves.

Let's place the two-dimensional flow in the complex plane. Let an arbitrary point be $z = (x, y)$. Then, we have found two real-valued functions, $\psi(x, y)$ and $\psi(x, y)$, satisfying the relations

$$\begin{aligned} u &= \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \\ v &= \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \end{aligned} \tag{9.79}$$

These are the Cauchy-Riemann relations for the real and imaginary parts of a complex differentiable function,

$$F(z(x, y) = \phi(x, y) + i\psi(x, y).$$

Furthermore, we have

$$\frac{dF}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv.$$

From its form, $\frac{dF}{dz}$ is called the complex velocity and $\sqrt{\left|\frac{dF}{dz}\right|} = \sqrt{u^2 + v^2}$ is the flow speed.

Integrating, we have

$$\begin{aligned} F &= \int_C (u - iv)\, dz \\ \phi(x, y) + i\psi(x, y) &= \int_{(x_0,y_0)}^{(x,y)} [u(x, y)\, dx + v(x, y)\, dy] \\ &\quad + i\int_{(x_0,y_0)}^{(x,y)} [-v(x, y)\, dx + u(x, y)\, dy]. \end{aligned} \tag{9.80}$$

Therefore, the streamline and potential functions are given by the integral forms

$$\begin{aligned} \phi(x, y) &= \int_{(x_0,y_0)}^{(x,y)} [u(x, y)\, dx + v(x, y)\, dy], \\ \psi(x, y) &= \int_{(x_0,y_0)}^{(x,y)} [-v(x, y)\, dx + u(x, y)\, dy]. \end{aligned} \tag{9.81}$$

These integrals give the circulation $\int_C V_s\, ds = \int_C u\, dx + v\, dy$ and the fluid flow per time, $\int_C -v\, dx + u\, dy$.

Streamline and potential curves are orthogonal families of curves.

The streamlines for the flow are given by the level curves $\psi(x, y) = c_1$ and the potential lines are given by the level curves $\phi(x, y) = c_2$. These are two orthogonal families of curves; i.e., these families of curves intersect each other orthogonally at each point as we will see in the examples. Note that these families of curves also provide the field lines and equipotential curves for electrostatic problems.

Example 9.48. Show that $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ are an orthogonal family of curves when $F(z) = \phi(x,y) + i\psi(x,y)$ is holomorphic.

In order to show that these curves are orthogonal, we need to find the slopes of the curves at an arbitrary point, (x,y). For $\phi(x,y) = c_1$, we recall from multivaribale calculus that

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = 0.$$

So, the slope is found as

$$\frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}.$$

Similarly, we have

$$\frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}}.$$

Since $F(z)$ is differentiable, we can use the Cauchy-Riemann equations to find the product of the slopes satisfy

$$\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = -\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial x}}\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = -1.$$

Therefore, $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ form an orthogonal family of curves.

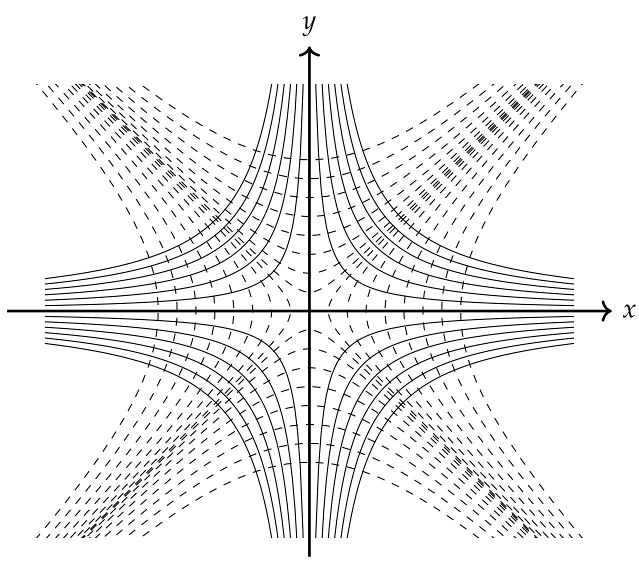


Figure 9.57: Plot of the orthogonal families $\phi = x^2 - y^2 = c_1$ (dashed) and $\phi(x,y) = 2xy = c_2$.

As an example, consider $F(z) = z^2 = x^2 - y^2 + 2ixy$. Then, $\phi(x,y) = x^2 - y^2$ and $\psi(x,y) = 2xy$. The slopes of the families of curves are given by

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}} \\
&= -\frac{2x}{-2y} = \frac{x}{y}. \\
\frac{dy}{dx} &= -\frac{\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}} \\
&= -\frac{2y}{2x} = -\frac{y}{x}. \qquad (9.82)
\end{aligned}$$

The products of these slopes is -1. The orthogonal families are depicted in Figure 9.57.

We will now turn to some typical examples by writing down some differentiable functions, $F(z)$, and determining the types of flows that result from these examples. We will then turn in the next section to using these basic forms to solve problems in slightly different domains through the use of conformal mappings.

Example 9.49. Describe the fluid flow associated with $F(z) = U_0 e^{-i\alpha} z$, where U_0 and α are real.

For this example, we have

$$\frac{dF}{dz} = U_0 e^{-i\alpha} = u - iv.$$

Thus, the velocity is constant,

$$\mathbf{U} = (U_0 \cos\alpha, U_0 \sin\alpha).$$

Thus, the velocity is a uniform flow at an angle of α.

Since

$$F(z) = U_0 e^{-i\alpha} z = U_0(x\cos\alpha + y\sin\alpha) + iU_0(y\cos\alpha - x\sin\alpha).$$

Thus, we have

$$\begin{aligned}
\phi(x,y) &= U_0(x\cos\alpha + y\sin\alpha), \\
\psi(x,y) &= U_0(y\cos\alpha - x\sin\alpha). \qquad (9.83)
\end{aligned}$$

An example of this family of curves is shown in Figure 9.58.

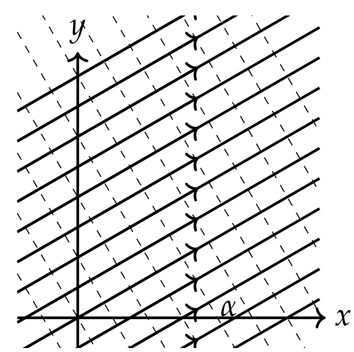


Figure 9.58: Stream lines (solid) and potential lines (dashed) for uniform flow at an angle of α, given by $F(z) = U_0 e^{-i\alpha} z$.

Example 9.50. Describe the flow given by $F(z) = \frac{U_0 e^{-i\alpha}}{z - z_0}$.

We write

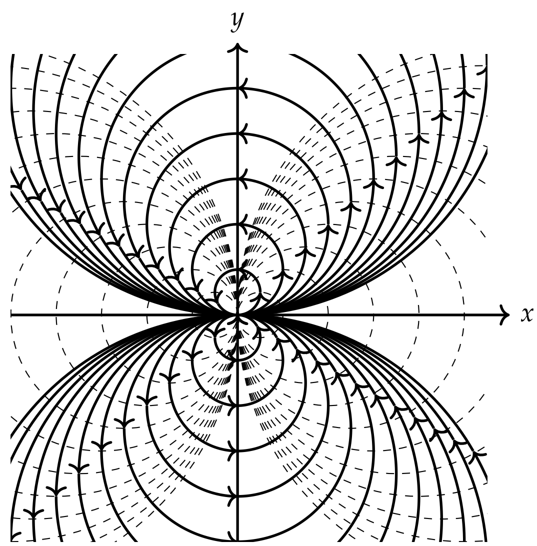
$$\begin{aligned}
F(z) &= \frac{U_0 e^{-i\alpha}}{z - z_0} \\
&= \frac{U_0(\cos\alpha + i\sin\alpha)}{(x-x_0)^2 + (y-y_0)^2}[(x-x_0) - i(y-y_0)]
\end{aligned}$$

$$\begin{aligned}
&= \frac{U_0}{(x-x_0)^2+(y-y_0)^2}[(x-x_0)\cos\alpha+(y-y_0)\sin\alpha] \\
&\quad +i\frac{U_0}{(x-x_0)^2+(y-y_0)^2}[-(y-y_0)\cos\alpha+(x-x_0)\sin\alpha].
\end{aligned} \tag{9.84}$$

The level curves become

$$\begin{aligned}
\phi(x,y) &= \frac{U_0}{(x-x_0)^2+(y-y_0)^2}[(x-x_0)\cos\alpha+(y-y_0)\sin\alpha]=c_1, \\
\psi(x,y) &= \frac{U_0}{(x-x_0)^2+(y-y_0)^2}[-(y-y_0)\cos\alpha+(x-x_0)\sin\alpha]=c_2.
\end{aligned} \tag{9.85}$$

Figure 9.59: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z)=\frac{U_0e^{-i\alpha}}{z}$ for $\alpha=0$.



The level curves for the stream and potential functions satisfy equations of the form

$$\beta_i(\Delta x^2+\Delta y^2)-\cos(\alpha+\delta_i)\Delta x-\sin(\alpha+\delta_i)\Delta y=0,$$

where $\Delta x = x-x_0$, $\Delta y = y-y_0$, $\beta_i=\frac{c_i}{U_0}$, $\delta_1=0$, and $\delta_2=\pi/2.$, These can be written in the more suggestive form

$$(\Delta x-\gamma_i\cos(\alpha-\delta_i))^2+(\Delta y-\gamma_i\sin(\alpha-\delta_i))^2=\gamma_i^2$$

for $\gamma_i=\frac{c_i}{2U_0}$, $i=1,2$. Thus, the stream and potential curves are circles with varying radii (γ_i) and centers $((x_0+\gamma_i\cos(\alpha-\delta_i), y_0+\gamma_i\sin(\alpha-\delta_i)))$. Examples of this family of curves is shown for $\alpha=0$ in in Figure 9.59 and for $\alpha=\pi/6$ in in Figure 9.60.

The components of the velocity field for $\alpha=0$ are found from

$$\begin{aligned}
\frac{dF}{dz} &= \frac{d}{dz}\left(\frac{U_0}{z-z_0}\right) \\
&= -\frac{U_0}{(z-z_0)^2}
\end{aligned}$$

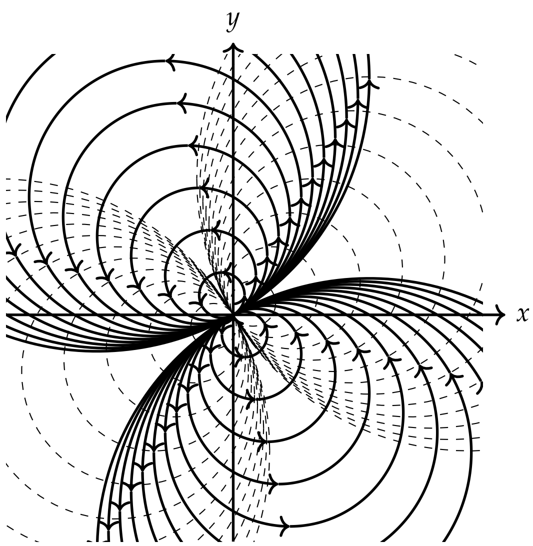


Figure 9.60: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{U_0 e^{-i\alpha}}{z}$ for $\alpha = \pi/6$.

$$
\begin{aligned}
&= -\frac{U_0[(x-x_0) - i(y-y_0)]^2}{[(x-x_0)^2+(y-y_0)^2]^2} \\
&= -\frac{U_0[(x-x_0)^2+(y-y_0)^2 - 2i(x-x_0)(y-y_0)]}{[(x-x_0)^2+(y-y_0)^2]^2} \\
&= -\frac{U_0[(x-x_0)^2+(y-y_0)^2]}{[(x-x_0)^2+(y-y_0)^2]^2} + i\frac{U_0[2(x-x_0)(y-y_0)]}{[(x-x_0)^2+(y-y_0)^2]^2} \\
&= -\frac{U_0}{[(x-x_0)^2+(y-y_0)^2]} + i\frac{U_0[2(x-x_0)(y-y_0)]}{[(x-x_0)^2+(y-y_0)^2]^2}. \qquad (9.86)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
u &= -\frac{U_0}{[(x-x_0)^2+(y-y_0)^2]}, \\
v &= \frac{U_0[2(x-x_0)(y-y_0)]}{[(x-x_0)^2+(y-y_0)^2]^2}. \qquad (9.87)
\end{aligned}
$$

Example 9.51. Describe the flow given by $F(z) = \frac{m}{2\pi}\ln(z-z_0)$.

We write $F(z)$ in terms of its real and imaginary parts:

$$
\begin{aligned}
F(z) &= \frac{m}{2\pi}\ln(z-z_0) \\
&= \frac{m}{2\pi}\left[\ln\sqrt{(x-x_0)^2+(y-y_0)^2} + i\tan^{-1}\frac{y-y_0}{x-x_0}\right]. \qquad (9.88)
\end{aligned}
$$

The level curves become

$$
\begin{aligned}
\phi(x,y) &= \frac{m}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2} = c_1, \\
\psi(x,y) &= \frac{m}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} = c_2. \qquad (9.89)
\end{aligned}
$$

Rewriting these equations, we have

$$
(x-x_0)^2+(y-y_0)^2 = e^{4\pi c_1/m},
$$

$$y - y_0 = (x - x_0) \tan \frac{2\pi c_2}{m}. \tag{9.90}$$

In Figure 9.61 we see that the stream lines are those for a source or sink depending if $m > 0$ or $m < 0$, respectively.

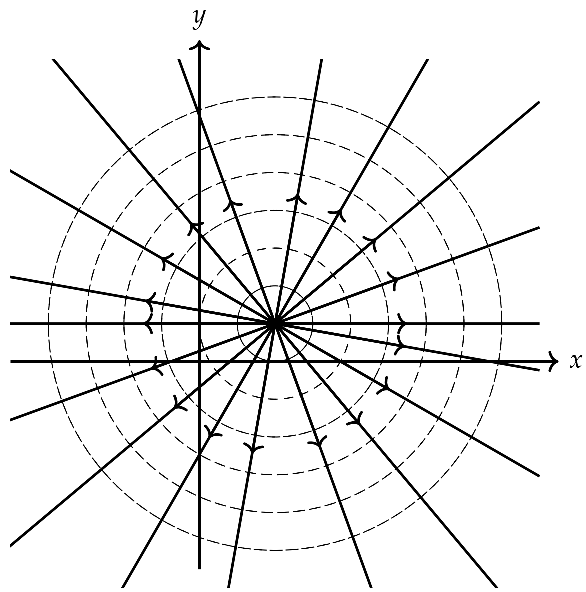


Figure 9.61: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{m}{2\pi} \ln(z - z_0)$ for $(x_0, y_0) = (2, 1)$.

Example 9.52. Describe the flow given by $F(z) = -\frac{i\Gamma}{2\pi} \ln \frac{z-z_0}{a}$.

We write $F(z)$ in terms of its real and imaginary parts:

$$\begin{aligned} F(z) &= -\frac{i\Gamma}{2\pi} \ln \frac{z - z_0}{a} \\ &= -i\frac{\Gamma}{2\pi} \ln \sqrt{\left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{a}\right)^2} + \frac{\Gamma}{2\pi} \tan^{-1} \frac{y - y_0}{x - x_0} \end{aligned} \tag{9.91}$$

The level curves become

$$\begin{aligned} \phi(x, y) &= \frac{\Gamma}{2\pi} \tan^{-1} \frac{y - y_0}{x - x_0} = c_1, \\ \psi(x, y) &= -\frac{\Gamma}{2\pi} \ln \sqrt{\left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{a}\right)^2} = c_2. \end{aligned} \tag{9.92}$$

Rewriting these equations, we have

$$\begin{aligned} y - y_0 &= (x - x_0) \tan \frac{2\pi c_1}{\Gamma}, \\ \left(\frac{x - x_0}{a}\right)^2 + \left(\frac{y - y_0}{a}\right)^2 &= e^{-2\pi c_2/\Gamma}. \end{aligned} \tag{9.93}$$

In Figure 9.62 we see that the stream lines circles, indicating rotational motion. Therefore, we have a vortex of counterclockwise, or clockwise flow, depending if $\Gamma > 0$ or $\Gamma < 0$, respectively.

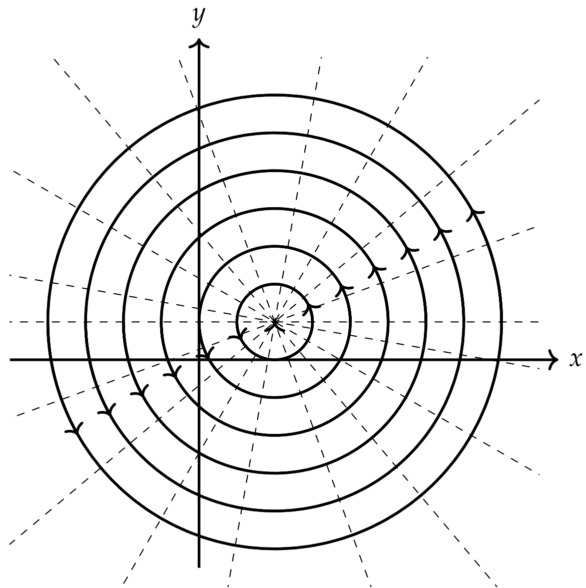


Figure 9.62: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$ for $(x_0, y_0) = (2, 1)$.

Example 9.53. Flow around a cylinder, $F(z) = U_0\left(z + \frac{a^2}{z}\right)$, $a, U_0 \in R$.

For this example, we have

$$\begin{aligned}
F(z) &= U_0\left(z + \frac{a^2}{z}\right) \\
&= U_0\left(x + iy + \frac{a^2}{x+iy}\right) \\
&= U_0\left(x + iy + \frac{a^2}{x^2+y^2}(x - iy)\right) \\
&= U_0 x\left(1 + \frac{a^2}{x^2+y^2}\right) + iU_0 y\left(1 - \frac{a^2}{x^2+y^2}\right). \qquad (9.94)
\end{aligned}$$

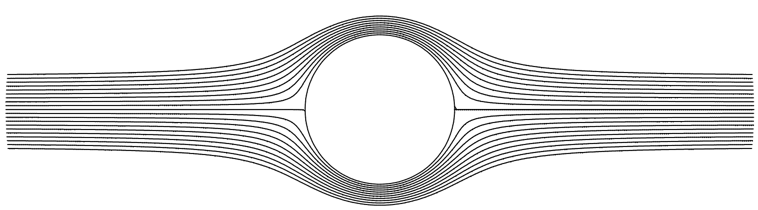


Figure 9.63: Stream lines for the flow given by $F(z) = U_0\left(z + \frac{a^2}{z}\right)$.

The level curves become

$$\begin{aligned}
\phi(x,y) &= U_0 x\left(1 + \frac{a^2}{x^2+y^2}\right) = c_1, \\
\psi(x,y) &= U_0 y\left(1 - \frac{a^2}{x^2+y^2}\right) = c_2.
\end{aligned} \qquad (9.95)$$

Note that for the streamlines when $|z|$ is large, then $\psi \sim Vy$, or horizontal lines. For $x^2 + y^2 = a^2$, we have $\psi = 0$. This behavior is shown in Figure 9.63 where we have graphed the solution for $r \geq a$.

The level curves in Figure 9.63 can be obtained using the implicitplot feature of Maple. An example is shown below:

```
restart: with(plots):
k0:=20:
for k from 0 to k0 do
  P[k]:=implicitplot(sin(t)*(r-1/r)*1=(k0/2-k)/20, r=1..5,
  t=0..2*Pi, coords=polar,view=[-2..2, -1..1], axes=none,
  grid=[150,150],color=black):
  od:
display({seq(P[k],k=1..k0)},scaling=constrained);
```

A slight modification of the last example is if a circulation term is added:

$$F(z) = U_0 \left(z + \frac{a^2}{z} \right) - \frac{i\Gamma}{2\pi} \ln \frac{r}{a}.$$

The combination of the two terms gives the streamlines,

$$\psi(x, y) = U_0 y \left(1 - \frac{a^2}{x^2 + y^2} \right) - \frac{\Gamma}{2\pi} \ln \frac{r}{a},$$

which are seen in Figure 9.64. We can see interesting features in this flow including what is called a stagnation point. A stagnation point is a point where the flow speed, $\left| \frac{dF}{dz} \right| = 0$.

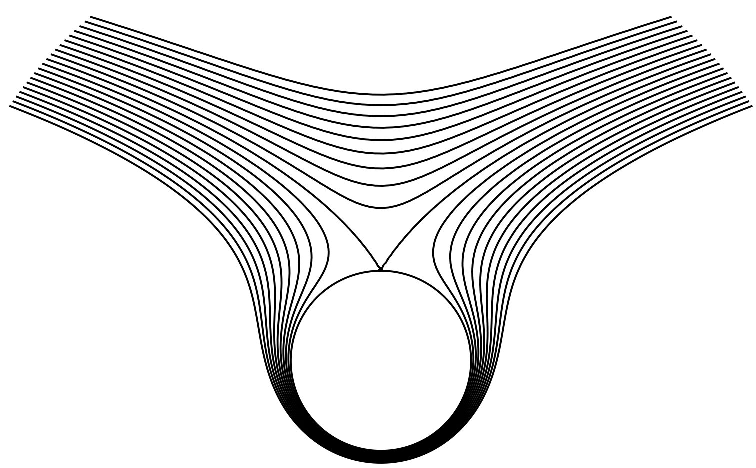


Figure 9.64: Stream lines for the flow given by $F(z) = U_0 \left(z + \frac{a^2}{z} \right) - \frac{\Gamma}{2\pi} \ln \frac{z}{a}$.

Example 9.54. Find the stagnation point for the flow $F(z) = \left(z + \frac{1}{z} \right) - i \ln z$.

Since the flow speed vanishes at the stagnation points, we consider

$$\frac{dF}{dz} = 1 - \frac{1}{z^2} - \frac{i}{z} = 0.$$

This can be rewritten as

$$z^2 - iz - 1 = 0.$$

The solutions are $z = \frac{1}{2}(i \pm \sqrt{3})$. Thus, there are two stagnation points on the cylinder about which the flow is circulating. These are shown in Figure 9.65.

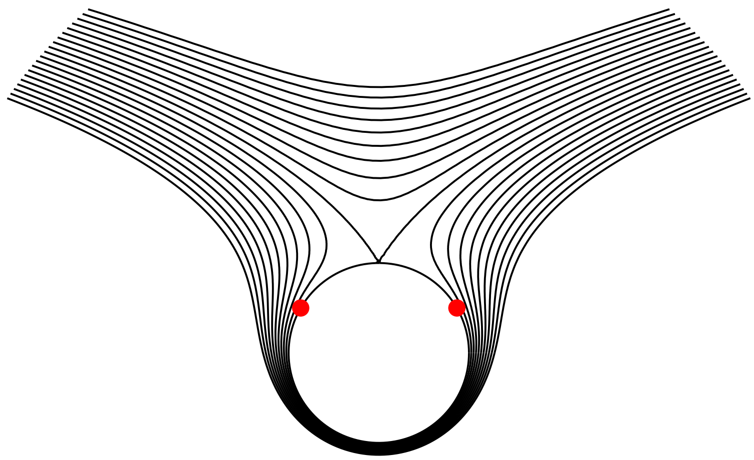


Figure 9.65: Stagnation points (red) on the cylinder are shown for the flow given by $F(z) = \left(z + \frac{1}{z}\right) - i \ln z$.

Example 9.55. Consider the complex potentials $F(z) = \frac{k}{2\pi} \ln \frac{z-a}{z-b}$, where $k = q$ and $k = -iq$ for q real.

We first note that for $z = x + iy$,

$$\begin{aligned} \ln \frac{z-a}{z-b} &= \ln \sqrt{(x-a)^2 + y^2} - \ln \sqrt{(x-a)^2 + y^2}, \\ & \quad + i \tan^{-1} \frac{y}{x-a} - i \tan^{-1} \frac{y}{x-b}. \end{aligned} \tag{9.96}$$

For $k = q$, we have

$$\begin{aligned} \psi(x,y) &= \frac{q}{2\pi} \left[\ln \sqrt{(x-a)^2 + y^2} - \ln \sqrt{(x-a)^2 + y^2} \right] = c_1, \\ \phi(x,y) &= \frac{q}{2\pi} \left[\tan^{-1} \frac{y}{x-a} - \tan^{-1} \frac{y}{x-b} \right] = c_2. \end{aligned} \tag{9.97}$$

The potential lines are circles and the streamlines are circular arcs as shown in Figure 9.66. These correspond to a source at $z = a$ and a sink at $z = b$. One can also view these as the electric field lines and equipotentials for an electric dipole consisting of two point charges of opposite sign at the points $z = a$ and $z = b$.

The equations for the curves are found from[6]

$$\begin{aligned} (x-a)^2 + y^2 &= C_1[(x-b)^2 + y^2], \\ (x-a)(x-b) + y^2 &= C_2 y(a-b), \end{aligned} \tag{9.98}$$

where these can be rewritten, respectively, in the more suggestive forms

$$\left(x - \frac{a - bC_1}{1 - C_1} \right)^2 + y^2 = \frac{C_1(a-b)^2}{(1-C_1)^2},$$

[6] The streamlines are found using the identity

$$\tan^{-1} \alpha - \tan^{-1} \beta = \tan^{-1} \frac{\alpha - \beta}{1 + \alpha\beta}.$$

$$\left(x-\frac{a+b}{2}\right)^2+\left(y-\frac{C_2(a-b)}{2}\right)^2 \quad = \quad (1+C_2^2)\left(\frac{a-b}{2}\right)^2. \qquad (9.99)$$

Note that the first family of curves are the potential curves and the second give the streamlines.

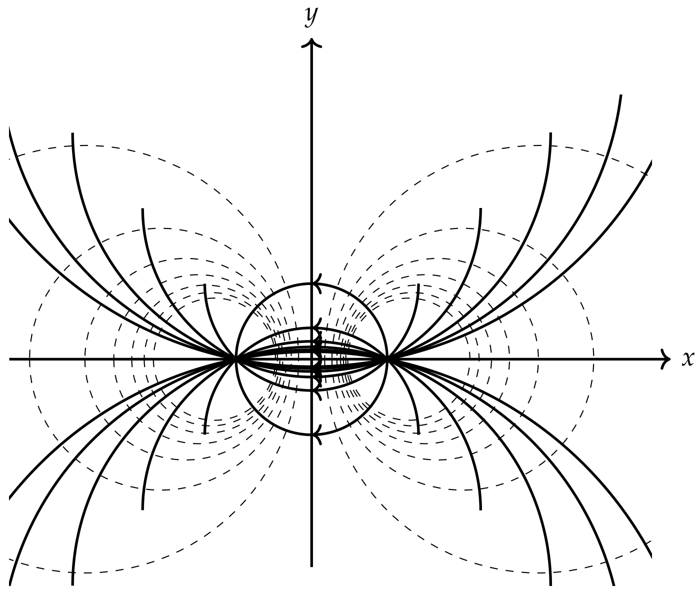


Figure 9.66: The electric field lines (solid) and equipotentials (dashed) for a dipole given by the complex potential $F(z)=\frac{q}{2\pi}\ln\frac{z-a}{z-b}$ for $b=-a$.

In the case that $k=-iq$ we have

$$\begin{aligned} F(z) &= \frac{-iq}{2\pi}\ln\frac{z-a}{z-b} \\ &= \frac{-iq}{2\pi}\left[\ln\sqrt{(x-a)^2+y^2}-\ln\sqrt{(x-a)^2+y^2}\right], \\ &\quad +\frac{q}{2\pi}\left[\tan^{-1}\frac{y}{x-a}-\tan^{-1}\frac{y}{x-b}\right]. \qquad (9.100) \end{aligned}$$

So, the roles of the streamlines and potential lines are reversed and the corresponding plots give a flow for a pair of vortices as shown in Figure 9.67.

9.6.2 *Conformal Mappings*

IT WOULD BE NICE IF THE COMPLEX POTENTIALS in the last section could be mapped to a region of the complex plane such that the new stream functions and velocity potentials represent new flows. In order for this to be true, we would need the new families to once again be orthogonal families of curves. Thus, the mappings we seek must preserve angles. Such mappings are called conformal mappings.

We let $w=f(z)$ map points in the $z-$plane, (x,y), to points in the w-plane, (u,v) by $f(x+iy)=u+iv$. We have shown this in Figure 9.4.

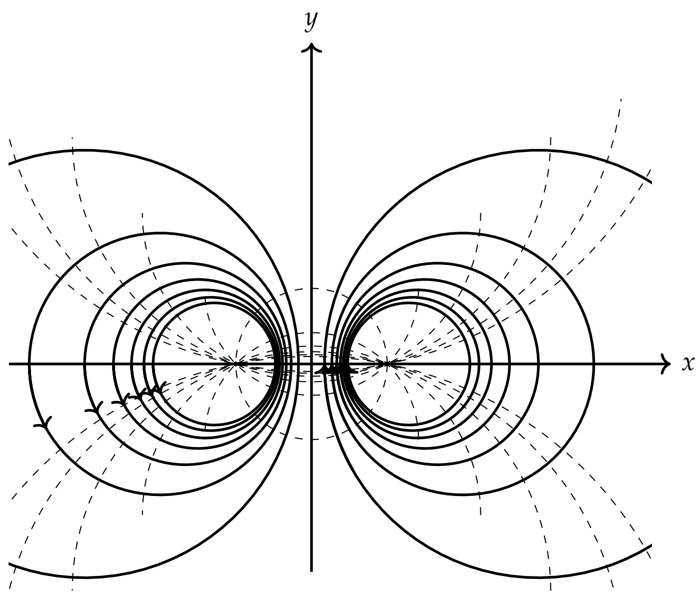


Figure 9.67: The streamlines (solid) and potentials (dashed) for a pair of vortices given by the complex potential $F(z) = \frac{q}{2\pi i} \ln \frac{z-a}{z-b}$ for $b = -a$.

Example 9.56. Map lines in the z-plane to curves in the w-plane under $f(z) = z^2$.

We have seen how grid lines in the z-plane is mapped by $f(z) = z^2$ into the w-plane in Figure 9.5, which is reproduced in Figure 9.68. The horizontal line $x = 1$ is mapped to $u(1, y) = 1 - y^2$ and $v(1, y) = 2y$. Eliminating the "parameter" y between these two equations, we have $u = 1 - v^2/4$. This is a parabolic curve. Similarly, the horizontal line $y = 1$ results in the curve $u = v^2/4 - 1$. These curves intersect at $w = 2i$.

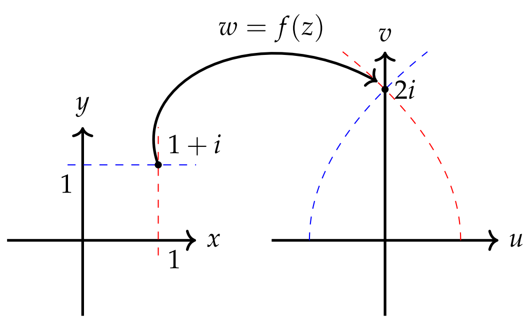


Figure 9.68: 2D plot showing how the function $f(z) = z^2$ maps the lines $x = 1$ and $y = 1$ in the z-plane into parabolae in the w-plane.

The lines in the z-plane intersect at $z = 1 + i$ at right angles. In the w-plane we see that the curves $u = 1 - v^2/4$ and $u = v^2/4 - 1$ intersect at $w = 2i$. The slopes of the tangent lines at $(0, 2)$ are -1 and 1, respectively, as shown in Figure 9.69.

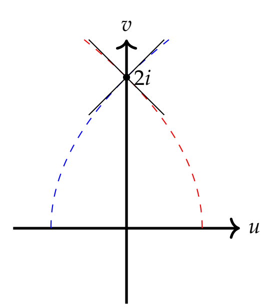


Figure 9.69: The tangents to the images of $x = 1$ and $y = 1$ under $f(z) = z^2$ are orthogonal.

In general, if two curves in the z-plane intersect orthogonally at $z = z_0$ and the corresponding curves in the w-plane under the mapping $w = f(z)$ are orthogonal at $w_0 = f(z_0)$, then the mapping is conformal. As we have seen, holomorphic functions are conformal, but only at points where $f'(z) \neq 0$.

Holomorphic functions are conformal at points where $f'(z) \neq 0$.

Example 9.57. Images of the real and imaginary axes under $f(z) = z^2$.

The line $z = iy$ maps to $w = z^2 = -y^2$ and the line $z = x$ maps to $w = z^2 = x^2$. The point of intersection $z_0 = 0$ maps to $w_0 = 0$. However, the image lines are the same line, the real axis in the w-plane. Obviously, the image lines are not orthogonal at the origin. Note that $f'(0) = 0$.

One special mapping is the inversion mapping, which is given by

$$f(z) = \frac{1}{z}.$$

This mapping maps the interior of the unit circle to the exterior of the unit circle in the w-plane as shown in Figure 9.70.

Let $z = x + iy$, where $x^2 + y^2 < 1$. Then,

$$w = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

Thus, $u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+y^2}$, and

$$\begin{aligned} u^2 + v^2 &= \left(\frac{x}{x^2+y^2}\right)^2 + \left(-\frac{y}{x^2+y^2}\right)^2 \\ &= \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}. \end{aligned} \tag{9.101}$$

Thus, for $x^2 + y^2 < 1$, $u^2 + U^2 > 1$. Furthermore, for $x^2 + y^2 > 1$, $u^2 + U^2 < 1$, and for $x^2 + y^2 = 1$, $u^2 + U^2 = 1$.

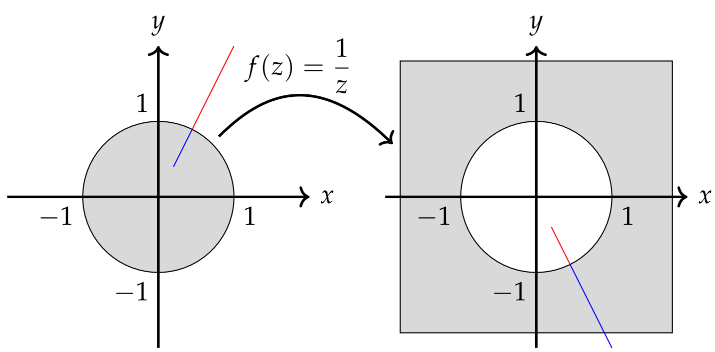


Figure 9.70: The inversion, $f(z) = \frac{1}{z}$, maps the interior of a unit circle to the external of a unit circle. Also, segments of aline through the origin, $y = 2x$, are mapped to the line $u = -2v$.

In fact, an inversion maps circles into circles. Namely, for $z = z_0 + re^{i\theta}$, we have

$$\begin{aligned} w &= \frac{1}{z_0 + re^{i\theta}} \\ &= \frac{\bar{z}_0 + re^{-i\theta}}{|z_0 + re^{i\theta}|^2} \\ &= w_0 + Re^{-i\theta}. \end{aligned} \tag{9.102}$$

Also, lines through the origin in the z-plane map into lines through the origin in the w-plane. Let $z = x + imx$. This corresponds to a line with slope m in the z-plane, $y = mx$. It maps to

$$\begin{aligned} f(z) &= \frac{1}{z} \\ &= \frac{1}{x+imx} \\ &= \frac{x-imx}{(1+m^2)x}. \end{aligned} \tag{9.103}$$

So, $u = \frac{x}{(1+m^2)x}$ and $v = -\frac{mx}{(1+m^2)x} = -mu$. This is a line through the origin in the w-plane with slope $-m$. This is shown in Figure 9.70 Note how the potion of the line $y = 2x$ that is inside the unit disk maps to the outside of the disk in the w-plane.

The bilinear transformation.

Another interesting class of transformation, of which the inversion is contained, is the bilinear transformation. The bilinear transformation is given by

$$w = f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

where a, b, c, and d are complex constants. These transformations were studied by mappings was studied by August Ferdinand Möbius (1790-1868) and are also called Möbius transformations, or linear fractional transformations. We further note that if $ad - bc = 0$, then the transformation reduces to the constat function.

We can seek to invert the transformation. Namely, solving for z, we have

$$z = f^{-1}(w) = \frac{-dw+b}{cw-a}, \quad w \neq \frac{a}{c}.$$

Since $f^{-1}(w)$ is not defined for $w \neq \frac{a}{c}$, we can say that $w \neq \frac{a}{c}$ maps to the point at infinity, or $f^{-1}\left(\frac{a}{c}\right) = \infty$. Similarly, we can let $z \to \infty$ to obtain

$$f(\infty) = \lim_{n\to\infty} f(z) = -\frac{d}{c}.$$

Thus, we have that the bilinear transformation is a one-to-one mapping of the extended complex z-plane to the extended complex w-plane.

The extended complex plane is the union of the complex plane plus the point at infinity. This is usually described in more detail using stereographic projection, which we will not review here.

If $c = 0$, $f(z)$ is easily seen to be a linear transformation. Linear transformations transform lines into lines and circles into circles.

When $c \neq 0$, we can write

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{c(az+b)}{c(cz+d)} \\ &= \frac{acz+ad-ad+bc}{c(cz+d)} \\ &= \frac{a(cz+d)-ad+bc}{c(cz+d)} \\ &= \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}. \end{aligned} \tag{9.104}$$

We note that if $bc - ad = 0$, then $f(z) = \frac{a}{c}$ is a constant, as noted above. The new form for $f(z)$ shows that it is the composition of a linear function $\zeta = cz + d$, an inversion, $g(\zeta) = \frac{1}{\zeta}$, and another linear transformation, $h(\zeta) = \frac{a}{c} + \frac{bc-ad}{c}\zeta$. Since linear transformations and inversions transform the set of circles and lines in the extended complex plane into circles and lines in the extended complex plane, then a bilinear does so as well.

What is important in out applications of complex analysis to the solution of Laplace's equation in the transformation of regions of the complex plane into other regions of the complex plane. Needed transformations can be found using the following property of bilinear transformations:

A given set of three points in the z-plane can be transformed into a given set of points in the w-plane using a bilinear transformation.

This statement is based on the following observation: There are three independent numbers that determine a bilinear transformation. If $a \neq 0$, then

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{z+\frac{b}{a}}{\frac{c}{a}z+\frac{d}{a}} \\ &\equiv \frac{z+\alpha}{\beta z+\gamma}. \end{aligned} \tag{9.105}$$

For $w = \frac{z+\alpha}{\beta z+\gamma}$, we have

$$\begin{aligned} w &= \frac{z+\alpha}{\beta z+\gamma} \\ w(\beta z+\gamma) &= z+\alpha \\ -\alpha + wz\beta + w\gamma &= z. \end{aligned} \tag{9.106}$$

Now, let $w_i = f(z_i)$, $i = 1,2,3$. This gives three equations for the three unknowns α, β, and γ. Namely,

$$\begin{aligned} -\alpha + w_1 z_1 \beta + w_1\gamma &= z_1, \\ -\alpha + w_2 z_2 \beta + w_2\gamma &= z_2, \\ -\alpha + w_3 z_3 \beta + w_3\gamma &= z_3. \end{aligned} \tag{9.107}$$

This systems of linear equation can be put into matrix form as

$$\begin{pmatrix} -1 & w_1 z_1 & w_1 \\ -1 & w_2 z_2 & w_2 \\ -1 & w_3 z_3 & w_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

It is only a matter of solving this system for $(\alpha, \beta, \gamma)^T$ in order to find the bilinear transformation.

A quicker method is to use the implicit form of the transformation,

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}.$$

Note that this implicit relation works upon insertion of the values w_i, z_i, for $i = 1,2,3$.

Example 9.58. Find the bilinear transformation that maps the points -1, i, 1 to the points -1, 0, 1.

The implicit form of the transformation becomes

$$\begin{aligned} \frac{(z+1)(i-1)}{(z-1)(i+1)} &= \frac{(w+1)(0-1)}{(w-1)(0+1)} \\ \frac{z+1}{z-1}\frac{i-1}{i+1} &= -\frac{w+1}{w-1}. \end{aligned} \tag{9.108}$$

Solving for w, we have

$$w = f(z) = \frac{(i-1)z+1+i}{(1+i)z-1+i}.$$

We can use the transformation in the last example to map the unit disk containing the points -1, i, and 1 to the half plane $w > 0$. We see that the unit circle gets mapped to the real axis with $z = -i$ mapped to the point at infinity. The point $z = 0$ gets mapped to

$$w = \frac{1+i}{-1+i} = \frac{1+i}{-1+i}\frac{-1-i}{-1-i} = \frac{2}{2} = 1.$$

Thus, interior points of the unit disk get mapped to the upper half plane. This is shown in Figure 9.71.

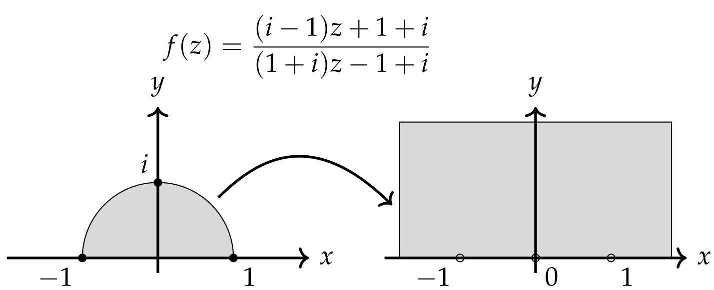


Figure 9.71: The bilinear transformation $f(z) = \frac{(i-1)z+1+i}{(1+i)z-1+i}$ maps the unit disk to the upper half plane.

Problems

1. Write the following in standard form.

a. $(4+5i)(2-3i)$.

b. $(1+i)^3$.

c. $\frac{5+3i}{1-i}$.

2. Write the following in polar form, $z = re^{i\theta}$.

a. $i-1$.

b. $-2i$.

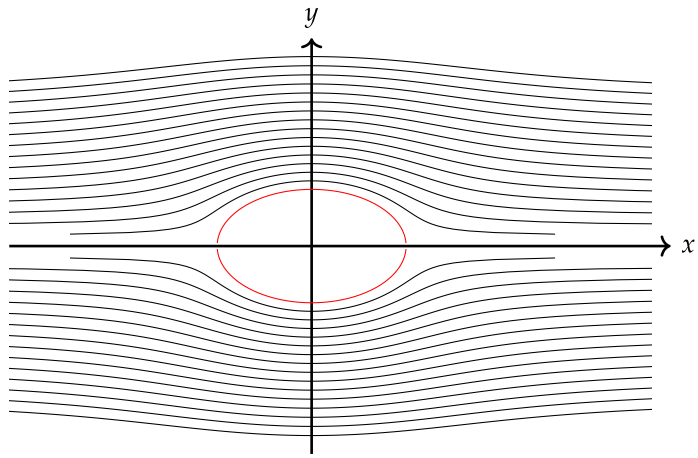


Figure 9.72: Flow about an elliptical cylinder.

c. $\sqrt{3}+3i$.

3. Write the following in rectangular form, $z = a + ib$.

a. $4e^{i\pi/6}$.

b. $\sqrt{2}e^{5i\pi/4}$.

c. $(1-i)^{100}$.

4. Find all z such that $z^4 = 16i$. Write the solutions in rectangular form, $z = a + ib$, with no decimal approximation or trig functions.

5. Show that $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$ using trigonometric identities and the exponential forms of these functions.

6. Find all z such that $\cos z = 2$, or explain why there are none. You will need to consider $\cos(x+iy)$ and equate real and imaginary parts of the resulting expression similar to problem 5.

7. Find the principal value of i^i. Rewrite the base, i, as an exponential first.

8. Consider the circle $|z-1| = 1$.

a. Rewrite the equation in rectangular coordinates by setting $z = x + iy$.

b. Sketch the resulting circle using part a.

c. Consider the image of the circle under the mapping $f(z) = z^2$, given by $|z^2 - 1| = 1$.

i. By inserting $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, find the equation of the image curve in polar coordinates.

ii. Sketch the image curve. You may need to refer to your Calculus II text for polar plots. [Maple might help.]

9. Find the real and imaginary parts of the functions:

a. $f(z) = z^3$.

b. $f(z) = \sinh(z)$.

c. $f(z) = \cos \bar{z}$.

10. Find the derivative of each function in Problem 9 when the derivative exists. Otherwise, show that the derivative does not exist.

11. Let $f(z) = u + iv$ be differentiable. Consider the vector field given by $\mathbf{F} = v\mathbf{i} + u\mathbf{j}$. Show that the equations $\nabla \cdot \mathbf{F} = \mathbf{0}$ and $\nabla \times \mathbf{F} = \mathbf{0}$ are equivalent to the Cauchy-Riemann equations. [You will need to recall from multivariable calculus the del operator, $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$.]

12. What parametric curve is described by the function

$$\gamma(t) = (t - 3) + i(2t + 1),$$

$0 \leq t \leq 2$? [Hint: What would you do if you were instead considering the parametric equations $x = t - 3$ and $y = 2t + 1$?]

13. Write the equation that describes the circle of radius 3 which is centered at $z = 2 - i$ in a) Cartesian form (in terms of x and y); b) polar form (in terms of θ and r); c) complex form (in terms of z, r, and $e^{i\theta}$).

14. Consider the function $u(x, y) = x^3 - 3xy^2$.

a. Show that $u(x, y)$ is harmonic; i.e., $\nabla^2 u = 0$.

b. Find its harmonic conjugate, $v(x, y)$.

c. Find a differentiable function, $f(z)$, for which $u(x, y)$ is the real part.

d. Determine $f'(z)$ for the function in part c. [Use $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and rewrite your answer as a function of z.]

15. Evaluate the following integrals:

a. $\int_C \bar{z}\, dz$, where C is the parabola $y = x^2$ from $z = 0$ to $z = 1 + i$.

b. $\int_C f(z)\, dz$, where $f(z) = 2z - \bar{z}$ and C is the path from $z = 0$ to $z = 2 + i$ consisting of two line segments from $z = 0$ to $z = 2$ and then $z = 2$ to $z = 2 + i$.

c. $\int_C \frac{1}{z^2+4}\, dz$ for C the positively oriented circle, $|z| = 2$. [Hint: Parametrize the circle as $z = 2e^{i\theta}$, multiply numerator and denominator by $e^{-i\theta}$, and put in trigonometric form.]

16. Let C be the positively oriented ellipse $3x^2 + y^2 = 9$. Define

$$F(z_0) = \int_C \frac{z^2 + 2z}{z - z_0}\, dz.$$

Find $F(2i)$ and $F(2)$. [Hint: Sketch the ellipse in the complex plane. Use the Cauchy Integral Theorem with an appropriate $f(z)$, or Cauchy's Theorem if z_0 is outside the contour.]

17. Show that

$$\int_C \frac{dz}{(z-1-i)^{n+1}} = \begin{cases} 0, & n \neq 0, \\ 2\pi i, & n = 0, \end{cases}$$

for C the boundary of the square $0 \leq x \leq 2$, $0 \leq y \leq 2$ taken counterclockwise. [Hint: Use the fact that contours can be deformed into simpler shapes (like a circle) as long as the integrand is analytic in the region between them. After picking a simpler contour, integrate using parametrization.]

18. Show that for g and h analytic functions at z_0, with $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$,

$$\operatorname{Res}\left[\frac{g(z)}{h(z)}; z_0\right] = \frac{g(z_0)}{h'(z_0)}.$$

19. For the following determine if the given point is a removable singularity, an essential singularity, or a pole (indicate its order).

a. $\frac{1-\cos z}{z^2}$, $\quad z = 0$.
b. $\frac{\sin z}{z^2}$, $\quad z = 0$.
c. $\frac{z^2-1}{(z-1)^2}$, $\quad z = 1$.
d. $ze^{1/z}$, $\quad z = 0$.
e. $\cos \frac{\pi}{z-\pi}$, $\quad z = \pi$.

20. Find the Laurent series expansion for $f(z) = \frac{\sinh z}{z^3}$ about $z = 0$. [Hint: You need to first do a MacLaurin series expansion for the hyperbolic sine.]

21. Find series representations for all indicated regions.

a. $f(z) = \frac{z}{z-1}$, $|z| < 1$, $|z| > 1$.
b. $f(z) = \frac{1}{(z-i)(z+2)}$, $|z| < 1$, $1 < |z| < 2$, $|z| > 2$. [Hint: Use partial fractions to write this as a sum of two functions first.]

22. Find the residues at the given points:

a. $\frac{2z^2+3z}{z-1}$ at $z = 1$.
b. $\frac{\ln(1+2z)}{z}$ at $z = 0$.
c. $\frac{\cos z}{(2z-\pi)^3}$ at $z = \frac{\pi}{2}$.

23. Consider the integral $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$.

a. Evaluate this integral by making the substitution $2\cos\theta = z + \frac{1}{z}$, $z = e^{i\theta}$ and using complex integration methods.
b. In the 1800's Weierstrass introduced a method for computing integrals involving rational functions of sine and cosine. One makes the substitution $t = \tan\frac{\theta}{2}$ and converts the integrand into a rational function of t. Note that the integration around the unit circle corresponds to $t \in (-\infty, \infty)$.

   i. Show that

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$

ii. Show that

$$d\theta = \frac{2dt}{1+t^2}$$

iii. Use the Weierstrass substitution to compute the above integral.

24. Do the following integrals.

a.

$$\oint_{|z-i|=3} \frac{e^z}{z^2+\pi^2}\, dz.$$

b.

$$\oint_{|z-i|=3} \frac{z^2-3z+4}{z^2-4z+3}\, dz.$$

c.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4}\, dx.$$

[Hint: This is $\operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4}\, dx$.]

25. Evaluate the integral $\int_0^\infty \frac{(\ln x)^2}{1+x^2}\, dx$.
[Hint: Replace x with $z = e^t$ and use the rectangular contour in Figure 9.73 with $R \to \infty$.]

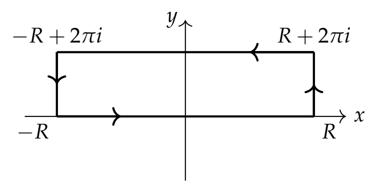


Figure 9.73: Rectangular contour for Problem 25.

26. Do the following integrals for fun!

a. For C the boundary of the square $|x| \le 2$, $|y| \le 2$,

$$\oint_C \frac{dz}{z(z-1)(z-3)^2}.$$

b.

$$\int_0^\pi \frac{\sin^2\theta}{13-12\cos\theta}\, d\theta.$$

c.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+5x+6}.$$

d.

$$\int_0^\infty \frac{\cos \pi x}{1-9x^2}\, dx.$$

e.

$$\int_0^\infty \frac{dx}{(x^2+9)(1-x)^2}.$$

f.

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\, dx.$$

g.

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\, dx.$$

27. Let $f(z) = u(x,y) + iv(x,y)$ be analytic in domain D. Prove that the Jacobian of the transformation from the xy-plane to the uv-plave is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = \left|f'(z)\right|^2.$$

28. Find the bilinear transformation which maps the points $1+i$, i, $2-i$ of the z-plane into the points $0, 1, i$ of the w-plane. Sketch the region in the w-plane to which the triangle formed by the points in the z-plane is mapped.

29. Find stream functions and potential functions for the following fluid motions. Describe the fluid flow from each.

a. $F(z) = z^2 + z$.

b. $F(z) = \cos z$.

c. $F(z) = U_0 \left(z + \frac{1}{z^2} \right)$.

d. $F(z) = \ln \left(1 + \frac{1}{z^2} \right)$.

30. Create a flow function which has two sources at $z = \pm a$ and one sink at $z = 0$ of equal magnitude strength. Verify your choice by finding and plotting the families of steam functions and potential functions.