A
Calculus Review

"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say." Bertrand Russell (1872-1970)

Before you begin our study of differential equations perhaps you should review some things from calculus. You definitely need to know something before taking this class. It is assumed that you have taken Calculus and are comfortable with differentiation and integration. Of course, you are not expected to know every detail from these courses. However, there are some topics and methods that will come up and it would be useful to have a handy reference to what it is you should know.

Most importantly, you should still have your calculus text to which you can refer throughout the course. Looking back on that old material, you will find that it appears easier than when you first encountered the material. That is the nature of learning mathematics and other subjects. Your understanding is continually evolving as you explore topics more in depth. It does not always sink in the first time you see it. In this chapter we will give a quick review of these topics. We will also mention a few new methods that might be interesting.

A.1 What Do I Need To Know From Calculus?

A.1.1 Introduction

There are two main topics in calculus: derivatives and integrals. You learned that derivatives are useful in providing rates of change in either time or space. Integrals provide areas under curves, but also are useful in providing other types of sums over continuous bodies, such as lengths, areas, volumes, moments of inertia, or flux integrals. In physics, one can look at graphs of position versus time and the slope (derivative) of such a function gives the velocity. (See Figure A.1.) By plotting velocity versus time you can either look at the derivative to obtain acceleration, or you could look

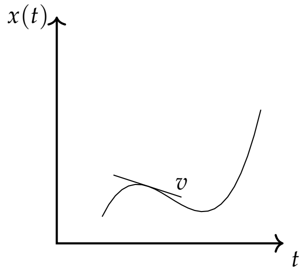

Figure A.1: Plot of position vs time.

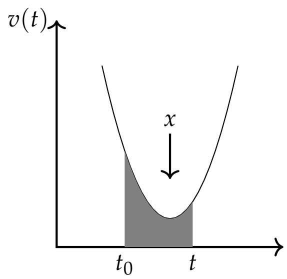

Figure A.2: Plot of velocity vs time.

at the area under the curve and get the displacement:

$$x = \int_{t_0}^{t} v\, dt. \tag{A.1}$$

This is shown in Figure A.2.

Of course, you need to know how to differentiate and integrate given functions. Even before getting into differentiation and integration, you need to have a bag of functions useful in physics. Common functions are the polynomial and rational functions. You should be fairly familiar with these. Polynomial functions take the general form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{A.2}$$

where $a_n \neq 0$. This is the form of a polynomial of degree n. Rational functions, $f(x) = \frac{g(x)}{h(x)}$, consist of ratios of polynomials. Their graphs can exhibit vertical and horizontal asymptotes.

Next are the exponential and logarithmic functions. The most common are the natural exponential and the natural logarithm. The natural exponential is given by $f(x) = e^x$, where $e \approx 2.718281828\ldots$. The natural logarithm is the inverse to the exponential, denoted by $\ln x$. (One needs to be careful, because some mathematics and physics books use log to mean natural exponential, whereas many of us were first trained to use this notation to mean the common logarithm, which is the 'log base 10'. Here we will use $\ln x$ for the natural logarithm.)

Exponential properties.

The properties of the exponential function follow from the basic properties for exponents. Namely, we have:

$$\begin{aligned} e^0 &= 1, & \text{(A.3)}\\ e^{-a} &= \frac{1}{e^a} & \text{(A.4)}\\ e^a e^b &= e^{a+b}, & \text{(A.5)}\\ (e^a)^b &= e^{ab}. & \text{(A.6)} \end{aligned}$$

The relation between the natural logarithm and natural exponential is given by

$$y = e^x \Leftrightarrow x = \ln y. \tag{A.7}$$

Logarithmic properties.

Some common logarithmic properties are

$$\begin{aligned} \ln 1 &= 0, & \text{(A.8)}\\ \ln \frac{1}{a} &= -\ln a, & \text{(A.9)}\\ \ln(ab) &= \ln a + \ln b, & \text{(A.10)}\\ \ln \frac{a}{b} &= \ln a - \ln b, & \text{(A.11)}\\ \ln \frac{1}{b} &= -\ln b. & \text{(A.12)} \end{aligned}$$

We will see applications of these relations as we progress through the course.

A.1.2 Trigonometric Functions

ANOTHER SET OF USEFUL FUNCTIONS are the trigonometric functions. These functions have probably plagued you since high school. They have their origins as far back as the building of the pyramids. Typical applications in your introductory math classes probably have included finding the heights of trees, flag poles, or buildings. It was recognized a long time ago that similar right triangles have fixed ratios of any pair of sides of the two similar triangles. These ratios only change when the non-right angles change.

Thus, the ratio of two sides of a right triangle only depends upon the angle. Since there are six possible ratios (think about it!), then there are six possible functions. These are designated as sine, cosine, tangent and their reciprocals (cosecant, secant and cotangent). In your introductory physics class, you really only needed the first three. You also learned that they are represented as the ratios of the opposite to hypotenuse, adjacent to hypotenuse, etc. Hopefully, you have this down by now.

You should also know the exact values of these basic trigonometric functions for the special angles $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{2}$, and their corresponding angles in the second, third and fourth quadrants. This becomes internalized after much use, but we provide these values in Table A.1 just in case you need a reminder.

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{2}$	0	1	undefined

Table A.1: Table of Trigonometric Values

The problems students often have using trigonometric functions in later courses stem from using, or recalling, identities. We will have many an occasion to do so in this class as well. What is an identity? It is a relation that holds true all of the time. For example, the most common identity for trigonometric functions is the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1. \tag{A.13}$$

This holds true for every angle θ! An even simpler identity is

$$\tan\theta = \frac{\sin\theta}{\cos\theta}. \tag{A.14}$$

Other simple identities can be derived from the Pythagorean identity. Dividing the identity by $\cos^2\theta$, or $\sin^2\theta$, yields

$$\tan^2\theta + 1 = \sec^2\theta, \tag{A.15}$$
$$1 + \cot^2\theta = \csc^2\theta. \tag{A.16}$$

Sum and difference identities.

Several other useful identities stem from the use of the sine and cosine of the sum and difference of two angles. Namely, we have that

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A, &\text{(A.17)} \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B. &\text{(A.18)} \end{aligned}$$

Note that the upper (lower) signs are taken together.

Example A.1. Evaluate $\sin \frac{\pi}{12}$.

$$\begin{aligned} \sin \frac{\pi}{12} &= \sin \left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \cos \frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} \\ &= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right). &\text{(A.19)} \end{aligned}$$

Double angle formulae.

The double angle formulae are found by setting $A = B$:

$$\begin{aligned} \sin(2A) &= 2 \sin A \cos B, &\text{(A.20)} \\ \cos(2A) &= \cos^2 A - \sin^2 A. &\text{(A.21)} \end{aligned}$$

Using Equation (A.13), we can rewrite (A.21) as

$$\begin{aligned} \cos(2A) &= 2\cos^2 A - 1, &\text{(A.22)} \\ &= 1 - 2\sin^2 A. &\text{(A.23)} \end{aligned}$$

Half angle formulae.

These, in turn, lead to the half angle formulae. Solving for $\cos^2 A$ and $\sin^2 A$, we find that

$$\begin{aligned} \sin^2 A = \frac{1 - \cos 2A}{2}, &&\text{(A.24)} \\ \cos^2 A = \frac{1 + \cos 2A}{2}. &&\text{(A.25)} \end{aligned}$$

Example A.2. Evaluate $\cos \frac{\pi}{12}$. In the last example, we used the sum/difference identities to evaluate a similar expression. We could have also used a half angle identity. In this example, we have

$$\begin{aligned} \cos^2 \frac{\pi}{12} &= \frac{1}{2}\left(1 + \cos \frac{\pi}{6}\right) \\ &= \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{4}\left(2 + \sqrt{3}\right) &\text{(A.26)} \end{aligned}$$

So, $\cos \frac{\pi}{12} = \frac{1}{2}\sqrt{2 + \sqrt{3}}$. This is not the simplest form and is called a nested radical. In fact, if we proceeded using the difference identity for cosines, then we would obtain

$$\cos\frac{\pi}{12} = \frac{\sqrt{2}}{4}(1+\sqrt{3}).$$

So, how does one show that these answers are the same?

Let's focus on the factor $\sqrt{2+\sqrt{3}}$. We seek to write this in the form $c + d\sqrt{3}$. Equating the two expressions and squaring, we have

$$\begin{aligned} 2+\sqrt{3} &= (c+d\sqrt{3})^2 \\ &= c^2 + 3d^2 + 2cd\sqrt{3}. \end{aligned} \tag{A.27}$$

It is useful at times to know when one can reduce square roots of such radicals, called denesting. More generally, one seeks to write $\sqrt{a+b\sqrt{q}} = c + d\sqrt{q}$. Following the procedure in this example, one has $d = \frac{b}{2c}$ and

$$c^2 = \frac{1}{2}\left(a \pm \sqrt{a^2 - qb^2}\right).$$

As long as $a^2 - qb^2$ is a perfect square, there is a chance to reduce the expression to a simpler form.

In order to solve for c and d, it would seem natural to equate the coefficients of $\sqrt{3}$ and the remaining terms. We obtain a system of two nonlinear algebraic equations,

$$c^2 + 3d^2 = 2 \tag{A.28}$$

$$2cd = 1. \tag{A.29}$$

Solving the second equation for $d = 1/2c$, and substituting the result into the first equation, we find

$$4c^4 - 8c^2 + 3 = 0.$$

This fourth order equation has four solutions,

$$c = \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{6}}{2}$$

and

$$b = \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{6}}{6}.$$

Thus,

$$\begin{aligned} \cos\frac{\pi}{12} &= \frac{1}{2}\sqrt{2+\sqrt{3}} \\ &= \pm\frac{1}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\sqrt{3}\right) \\ &= \pm\frac{\sqrt{2}}{4}(1+\sqrt{3}) \end{aligned} \tag{A.30}$$

and

$$\begin{aligned} \cos\frac{\pi}{12} &= \frac{1}{2}\sqrt{2+\sqrt{3}} \\ &= \pm\frac{1}{2}\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{6}\sqrt{3}\right) \\ &= \pm\frac{\sqrt{6}}{12}(3+\sqrt{3}). \end{aligned} \tag{A.31}$$

Of the four solutions, two are negative and we know the value of the cosine for this angle has to be positive. The remaining two solutions are actually equal! A quick computation will verify this:

$$\begin{aligned} \frac{\sqrt{6}}{12}(3+\sqrt{3}) &= \frac{\sqrt{3}\sqrt{2}}{12}(3+\sqrt{3}) \\ &= \frac{\sqrt{2}}{12}(3\sqrt{3}+3) \\ &= \frac{\sqrt{2}}{4}(\sqrt{3}+1). \end{aligned} \tag{A.32}$$

We could have bypassed this situation be requiring that the solutions for b and c were not simply proportional to $\sqrt{3}$ like they are in the second case.

Product Identities

Finally, another useful set of identities are the product identities. For example, if we add the identities for $\sin(A+B)$ and $\sin(A-B)$, the second terms cancel and we have

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B.$$

Thus, we have that

$$\boxed{\sin A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B)).} \tag{A.33}$$

Similarly, we have

$$\boxed{\cos A\cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B)).} \tag{A.34}$$

and

$$\boxed{\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B)).} \tag{A.35}$$

Know the above boxed identities!

These boxed equations are the most common trigonometric identities. They appear often and should just roll off of your tongue.

We will also need to understand the behaviors of trigonometric functions. In particular, we know that the sine and cosine functions are periodic. They are not the only periodic functions, as we shall see. [Just visualize the teeth on a carpenter's saw.] However, they are the most common periodic functions.

Periodic functions.

A periodic function $f(x)$ satisfies the relation

$$f(x+p) = f(x), \quad \text{for all } x$$

for some constant p. If p is the smallest such number, then p is called the period. Both the sine and cosine functions have period 2π. This means that the graph repeats its form every 2π units. Similarly, $\sin bx$ and $\cos bx$ have the common period $p = \frac{2\pi}{b}$. We will make use of this fact in later chapters.

In Feynman's *Surely You're Joking Mr. Feynman!*, Richard Feynman (1918-1988) talks about his invention of his own notation for both trigonometric and inverse trigonometric functions as the standard notation did not make sense to him.

Related to these are the inverse trigonometric functions. For example, $f(x) = \sin^{-1} x$, or $f(x) = \arcsin x$. Inverse functions give back angles, so you should think

$$\theta = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin\theta. \tag{A.36}$$

Also, you should recall that $y = \sin^{-1} x = \arcsin x$ is only a function if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Similar relations exist for $y = \cos^{-1} x = \arccos x$ and $\tan^{-1} x = \arctan x$.

Once you think about these functions as providing angles, then you can make sense out of more complicated looking expressions, like $\tan(\sin^{-1} x)$. Such expressions often pop up in evaluations of integrals. We can untangle this in order to produce a simpler form by referring to expression (A.36). $\theta = \sin^{-1} x$ is simple an angle whose sine is x. Knowing the sine is the opposite side of a right triangle divided by its hypotenuse, then one just draws a triangle in this proportion as shown in Figure A.3. Namely, the side opposite the angle has length x and the hypotenuse has length 1. Using the Pythagorean Theorem, the missing side (adjacent to the angle) is simply $\sqrt{1-x^2}$. Having obtained the lengths for all three sides, we can now produce the tangent of the angle as

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}.$$

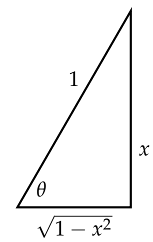

Figure A.3: $\theta = \sin^{-1} x \Rightarrow \tan\theta = \frac{x}{\sqrt{1-x^2}}$

A.1.3 Hyperbolic Functions

So, ARE THERE ANY OTHER FUNCTIONS that are useful in physics? Actually, there are many more. However, you have probably not see many of them to date. We will see by the end of the semester that there are many important functions that arise as solutions of some fairly generic, but important, physics problems. In your calculus classes you have also seen that some relations are represented in parametric form. However, there is at least one other set of elementary functions, which you should already know about. These are the hyperbolic functions. Such functions are useful in representing hanging cables, unbounded orbits, and special traveling waves called solitons. They also play a role in special and general relativity.

Solitons are special solutions to some generic nonlinear wave equations. They typically experience elastic collisions and play special roles in a variety of fields in physics, such as hydrodynamics and optics. A simple soliton solution is of the form

$$u(x,t) = 2\eta^2 \operatorname{sech}^2 \eta(x - 4\eta^2 t).$$

Hyperbolic functions are actually related to the trigonometric functions, as we shall see after a little bit of complex function theory. For now, we just want to recall a few definitions and identities. Just as all of the trigonometric functions can be built from the sine and the cosine, the hyperbolic functions can be defined in terms of the hyperbolic sine and hyperbolic cosine (shown in Figure A.4):

Hyperbolic functions; We will later see the connection between the hyperbolic and trigonometric functions in Chapter 9.

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \text{(A.37)}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}. \qquad \text{(A.38)}$$

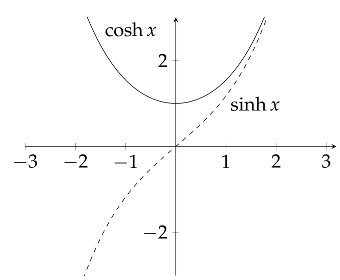

Figure A.4: Plots of $\cosh x$ and $\sinh x$. Note that $\sinh 0 = 0$, $\cosh 0 = 1$, and $\cosh x \geq 1$.

There are four other hyperbolic functions. These are defined in terms of the above functions similar to the relations between the trigonometric functions. We have

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \text{(A.39)}$$

$$\begin{aligned} \operatorname{sech} x &= \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, & \text{(A.40)} \\ \operatorname{csch} x &= \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, & \text{(A.41)} \\ \coth x &= \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. & \text{(A.42)} \end{aligned}$$

There are also a whole set of identities, similar to those for the trigonometric functions. For example, the Pythagorean identity for trigonometric functions, $\sin^2\theta + \cos^2\theta = 1$, is replaced by the identity

$$\cosh^2 x - \sinh^2 x = 1.$$

This is easily shown by simply using the definitions of these functions. This identity is also useful for providing a parametric set of equations describing hyperbolae. Letting $x = a\cosh t$ and $y = b\sinh t$, one has

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cosh^2 t - \sinh^2 t = 1.$$

Hyperbolic identities.

A list of commonly needed hyperbolic function identities are given by the following:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1, & \text{(A.43)} \\ \tanh^2 x + \operatorname{sech}^2 x &= 1, & \text{(A.44)} \\ \cosh(A \pm B) &= \cosh A \cosh B \pm \sinh A \sinh B, & \text{(A.45)} \\ \sinh(A \pm B) &= \sinh A \cosh B \pm \sinh B \cosh A, & \text{(A.46)} \\ \cosh 2x &= \cosh^2 x + \sinh^2 x, & \text{(A.47)} \\ \sinh 2x &= 2 \sinh x \cosh x, & \text{(A.48)} \\ \cosh^2 x &= \frac{1}{2}\left(1 + \cosh 2x\right), & \text{(A.49)} \\ \sinh^2 x &= \frac{1}{2}\left(\cosh 2x - 1\right). & \text{(A.50)} \end{aligned}$$

Note the similarity with the trigonometric identities. Other identities can be derived from these.

There also exist inverse hyperbolic functions and these can be written in terms of logarithms. As with the inverse trigonometric functions, we begin with the definition

$$y = \sinh^{-1} x \quad \Leftrightarrow \quad x = \sinh y. \tag{A.51}$$

The aim is to write y in terms of x without using the inverse function. First, we note that

$$x = \frac{1}{2}\left(e^y - e^{-y}\right). \tag{A.52}$$

Next we solve for e^y. This is done by noting that $e^{-y} = \frac{1}{e^y}$ and rewriting the previous equation as

$$0 = (e^y)^2 - 2xe^y - 1. \tag{A.53}$$

This equation is in quadratic form which we can solve using the quadratic formula as

$$e^y = x + \sqrt{1+x^2}.$$

(There is only one root as we expect the exponential to be positive.)

The final step is to solve for y,

$$y = \ln\left(x + \sqrt{1+x^2}\right). \tag{A.54}$$

The inverse hyperbolic functions care given by

$$\begin{aligned} \sinh^{-1} x &= \ln\left(x + \sqrt{1+x^2}\right), \\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right), \\ \tanh^{-1} x &= \frac{1}{2}\ln\frac{1+x}{1-x}. \end{aligned}$$

A.1.4 Derivatives

Now that we know some elementary functions, we seek their derivatives. We will not spend time exploring the appropriate limits in any rigorous way. We are only interested in the results. We provide these in Table A.2. We expect that you know the meaning of the derivative and all of the usual rules, such as the product and quotient rules.

Function	Derivative
a	0
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
$\csc ax$	$-a\csc ax\cot ax$
$\sec ax$	$a\sec ax\tan ax$
$\cot ax$	$-a\csc^2 ax$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\,\mathrm{sech}^2\, ax$
$\mathrm{csch}\, ax$	$-a\,\mathrm{csch}\, ax\coth ax$
$\mathrm{sech}\, ax$	$-a\,\mathrm{sech}\, ax\tanh ax$
$\coth ax$	$-a\,\mathrm{csch}^2\, ax$

Table A.2: Table of Common Derivatives (a is a constant).

Also, you should be familiar with the Chain Rule. Recall that this rule tells us that if we have a composition of functions, such as the elementary functions above, then we can compute the derivative of the composite function. Namely, if $h(x) = f(g(x))$, then

$$\frac{dh}{dx} = \frac{d}{dx}\left(f(g(x))\right) = \left.\frac{df}{dg}\right|_{g(x)}\frac{dg}{dx} = f'(g(x))g'(x). \tag{A.55}$$

Example A.3. Differentiate $H(x) = 5\cos\left(\pi\tanh 2x^2\right)$.

This is a composition of three functions, $H(x) = f(g(h(x)))$, where $f(x) = 5\cos x$, $g(x) = \pi\tanh x$, and $h(x) = 2x^2$. Then the derivative becomes

$$\begin{aligned} H'(x) &= 5\left(-\sin\left(\pi\tanh 2x^2\right)\right)\frac{d}{dx}\left(\left(\pi\tanh 2x^2\right)\right) \\ &= -5\pi\sin\left(\pi\tanh 2x^2\right)\operatorname{sech}^2 2x^2\frac{d}{dx}\left(2x^2\right) \\ &= -20\pi x\sin\left(\pi\tanh 2x^2\right)\operatorname{sech}^2 2x^2. \end{aligned} \tag{A.56}$$

A.1.5 Integrals

Integration is typically a bit harder. Imagine being given the last result in (A.56) and having to figure out what was differentiated in order to get the given function. As you may recall from the Fundamental Theorem of Calculus, the integral is the inverse operation to differentiation:

$$\int \frac{df}{dx}\,dx = f(x) + C. \tag{A.57}$$

It is not always easy to evaluate a given integral. In fact some integrals are not even doable! However, you learned in calculus that there are some methods that could yield an answer. While you might be happier using a computer algebra system, such as Maple or WolframAlpha.com, or a fancy calculator, you should know a few basic integrals and know how to use tables for some of the more complicated ones. In fact, it can be exhilarating when you can do a given integral without reference to a computer or a Table of Integrals. However, you should be prepared to do some integrals using what you have been taught in calculus. We will review a few of these methods and some of the standard integrals in this section.

First of all, there are some integrals you are expected to know without doing any work. These integrals appear often and are just an application of the Fundamental Theorem of Calculus to the previous Table A.2. The basic integrals that students should know off the top of their heads are given in Table A.3.

These are not the only integrals you should be able to do. We can expand the list by recalling a few of the techniques that you learned in calculus, the Method of Substitution, Integration by Parts, integration using partial fraction decomposition, and trigonometric integrals, and trigonometric substitution. There are also a few other techniques that you had not seen before. We will look at several examples.

Example A.4. Evaluate $\int \frac{x}{\sqrt{x^2+1}}\,dx$.

When confronted with an integral, you should first ask if a simple substitution would reduce the integral to one you know how to do.

The ugly part of this integral is the x^2+1 under the square root. So, we let $u = x^2 + 1$.

Noting that when $u = f(x)$, we have $du = f'(x)\,dx$. For our example, $du = 2x\,dx$.

Looking at the integral, part of the integrand can be written as $x\,dx = \frac{1}{2}u\,du$. Then, the integral becomes

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}}.$$

The substitution has converted our integral into an integral over u. Also, this integral is doable! It is one of the integrals we should know. Namely, we can write it as

$$\frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du.$$

This is now easily finished after integrating and using the substitution variable to give

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} + C = \sqrt{x^2+1} + C.$$

Note that we have added the required integration constant and that the derivative of the result easily gives the original integrand (after employing the Chain Rule).

Table A.3: Table of Common Integrals.

Function	Indefinite Integral
a	ax
x^n	$\frac{x^{n+1}}{n+1}$
e^{ax}	$\frac{1}{a}e^{ax}$
$\frac{1}{x}$	$\ln x$
$\sin ax$	$-\frac{1}{a}\cos ax$
$\cos ax$	$\frac{1}{a}\sin ax$
$\sec^2 ax$	$\frac{1}{a}\tan ax$
$\sinh ax$	$\frac{1}{a}\cosh ax$
$\cosh ax$	$\frac{1}{a}\sinh ax$
$\operatorname{sech}^2 ax$	$\frac{1}{a}\tanh ax$
$\sec x$	$\ln\|\sec x + \tan x\|$
$\frac{1}{a+bx}$	$\frac{1}{b}\ln(a+bx)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a} = \ln\|\sqrt{x^2-a^2}+x\|$

Often we are faced with definite integrals, in which we integrate between two limits. There are several ways to use these limits. However, students often forget that a change of variables generally means that the limits have to change.

Example A.5. Evaluate $\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx$.

This is the last example but with integration limits added. We proceed as before. We let $u = x^2+1$. As x goes from 0 to 2, u takes values from 1 to 5. So, this substitution gives

$$\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int_1^5 \frac{du}{\sqrt{u}} = \sqrt{u}|_1^5 = \sqrt{5}-1.$$

When you becomes proficient at integration, you can bypass some of these steps. In the next example we try to demonstrate the thought process involved in using substitution without explicitly using the substitution variable.

Example A.6. Evaluate $\int_0^2 \frac{x}{\sqrt{9+4x^2}}\,dx$

As with the previous example, one sees that the derivative of $9+4x^2$ is proportional to x, which is in the numerator of the integrand. Thus a substitution would give an integrand of the form $u^{-1/2}$. So, we expect the answer to be proportional to $\sqrt{u} = \sqrt{9+4x^2}$. The starting point is therefore,

$$\int \frac{x}{\sqrt{9+4x^2}}\,dx = A\sqrt{9+4x^2},$$

where A is a constant to be determined.

We can determine A through differentiation since the derivative of the answer should be the integrand. Thus,

$$\begin{aligned} \frac{d}{dx}A(9+4x^2)^{\frac{1}{2}} &= A(9+4x^2)^{-\frac{1}{2}}\left(\frac{1}{2}\right)(8x) \\ &= 4xA(9+4x^2)^{-\frac{1}{2}}. \end{aligned} \tag{A.58}$$

Comparing this result with the integrand, we see that the integrand is obtained when $A = \frac{1}{4}$. Therefore,

$$\int \frac{x}{\sqrt{9+4x^2}}\,dx = \frac{1}{4}\sqrt{9+4x^2}.$$

We now complete the integral,

$$\int_0^2 \frac{x}{\sqrt{9+4x^2}}\,dx = \frac{1}{4}[5-3] = \frac{1}{2}.$$

The function

$$\mathrm{gd}(x) = \int_0^x \frac{dx}{\cosh x} = 2\tan^{-1}e^x - \frac{\pi}{2}$$

is called the Gudermannian and connects trigonometric and hyperbolic functions. This function was named after Christoph Gudermann (1798-1852), but introduced by Johann Heinrich Lambert (1728-1777), who was one of the first to introduce hyperbolic functions.

Example A.7. Evaluate $\int \frac{dx}{\cosh x}$.

This integral can be performed by first using the definition of $\cosh x$ followed by a simple substitution.

$$\begin{aligned} \int \frac{dx}{\cosh x} &= \int \frac{2}{e^x+e^{-x}}\,dx \\ &= \int \frac{2e^x}{e^{2x}+1}\,dx. \end{aligned} \tag{A.59}$$

Now, we let $u = e^x$ and $du = e^x dx$. Then,

$$\begin{aligned} \int \frac{dx}{\cosh x} &= \int \frac{2}{1+u^2}\,du \\ &= 2\tan^{-1}u + C \\ &= 2\tan^{-1}e^x + C. \end{aligned} \tag{A.60}$$

Integration by Parts

When the Method of Substitution fails, there are other methods you can try. One of the most used is the Method of Integration by Parts. Recall the Integration by Parts Formula:

Integration by Parts Formula.

$$\int u\,dv = uv - \int v\,du. \tag{A.61}$$

The idea is that you are given the integral on the left and you can relate it to an integral on the right. Hopefully, the new integral is one you can do, or at least it is an easier integral than the one you are trying to evaluate.

However, you are not usually given the functions u and v. You have to determine them. The integral form that you really have is a function of another variable, say x. Another form of the Integration by Parts Formula can be written as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx. \tag{A.62}$$

This form is a bit more complicated in appearance, though it is clearer than the u-v form as to what is happening. The derivative has been moved from one function to the other. Recall that this formula was derived by integrating the product rule for differentiation. (See your calculus text.)

Note: Often in physics one needs to move a derivative between functions inside an integrand. The key - use integration by parts to move the derivative from one function to the other under an integral.

These two formulae can be related by using the differential relations

$$\begin{aligned} u = f(x) \quad &\to \quad du = f'(x)\,dx, \\ v = g(x) \quad &\to \quad dv = g'(x)\,dx. \end{aligned} \tag{A.63}$$

This also gives a method for applying the Integration by Parts Formula.

Example A.8. Consider the integral $\int x \sin 2x\,dx$. We choose $u = x$ and $dv = \sin 2x\,dx$. This gives the correct left side of the Integration by Parts Formula. We next determine v and du:

$$du = \frac{du}{dx}dx = dx,$$

$$v = \int dv = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x.$$

We note that one usually does not need the integration constant. Inserting these expressions into the Integration by Parts Formula, we have

$$\int x \sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx.$$

We see that the new integral is easier to do than the original integral. Had we picked $u = \sin 2x$ and $dv = x\,dx$, then the formula still works, but the resulting integral is not easier.

For completeness, we finish the integration. The result is

$$\int x \sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C.$$

As always, you can check your answer by differentiating the result, a step students often forget to do. Namely,

$$\begin{aligned} \frac{d}{dx}\left(-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C\right) &= -\frac{1}{2}\cos 2x + x\sin 2x + \frac{1}{4}(2\cos 2x) \\ &= x\sin 2x. \end{aligned} \tag{A.64}$$

So, we do get back the integrand in the original integral.

We can also perform integration by parts on definite integrals. The general formula is written as

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx. \tag{A.65}$$

Integration by Parts for Definite Integrals.

Example A.9. Consider the integral

$$\int_0^\pi x^2\cos x\,dx.$$

This will require two integrations by parts. First, we let $u = x^2$ and $dv = \cos x$. Then,

$$du = 2x\,dx. \quad v = \sin x.$$

Inserting into the Integration by Parts Formula, we have

$$\begin{aligned} \int_0^\pi x^2\cos x\,dx &= x^2\sin x\Big|_0^\pi - 2\int_0^\pi x\sin x\,dx \\ &= -2\int_0^\pi x\sin x\,dx. \end{aligned} \tag{A.66}$$

We note that the resulting integral is easier that the given integral, but we still cannot do the integral off the top of our head (unless we look at Example 3!). So, we need to integrate by parts again. (Note: In your calculus class you may recall that there is a tabular method for carrying out multiple applications of the formula. We will show this method in the next example.)

We apply integration by parts by letting $U = x$ and $dV = \sin x\,dx$. This gives $dU = dx$ and $V = -\cos x$. Therefore, we have

$$\begin{aligned} \int_0^\pi x\sin x\,dx &= -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx \\ &= \pi + \sin x\Big|_0^\pi \\ &= \pi. \end{aligned} \tag{A.67}$$

The final result is

$$\int_0^\pi x^2\cos x\,dx = -2\pi.$$

There are other ways to compute integrals of this type. First of all, there is the Tabular Method to perform integration by parts. A second method is to use differentiation of parameters under the integral. We will demonstrate this using examples.

Example A.10. Compute the integral $\int_0^\pi x^2 \cos x \, dx$ using the Tabular Method.

First we identify the two functions under the integral, x^2 and $\cos x$. We then write the two functions and list the derivatives and integrals of each, respectively. This is shown in Table A.4. Note that we stopped when we reached zero in the left column.

Next, one draws diagonal arrows, as indicated, with alternating signs attached, starting with $+$. The indefinite integral is then obtained by summing the products of the functions at the ends of the arrows along with the signs on each arrow:

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

To find the definite integral, one evaluates the antiderivative at the given limits.

$$\begin{aligned} \int_0^\pi x^2 \cos x \, dx &= \left[x^2 \sin x + 2x \cos x - 2 \sin x \right]_0^\pi \\ &= (\pi^2 \sin \pi + 2\pi \cos \pi - 2 \sin \pi) - 0 \\ &= -2\pi. \end{aligned} \tag{A.68}$$

Actually, the Tabular Method works even if a zero does not appear in the left column. One can go as far as possible, and if a zero does not appear, then one needs only integrate, if possible, the product of the functions in the last row, adding the next sign in the alternating sign progression. The next example shows how this works.

Table A.4: Tabular Method

D		I
x^2	$+$ ↘	$\cos x$
$2x$	$-$ ↘	$\sin x$
2	$+$ ↘	$-\cos x$
0		$-\sin x$

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Example A.11. Use the Tabular Method to compute $\int e^{2x} \sin 3x \, dx$.

As before, we first set up the table as shown in Table A.5.

Putting together the pieces, noting that the derivatives in the left column will never vanish, we have

$$\int e^{2x} \sin 3x \, dx = (\frac{1}{2} \sin 3x - \frac{3}{4} \cos 3x) e^{2x} + \int (-9 \sin 3x) \left(\frac{1}{4} e^{2x} \right) dx.$$

The integral on the right is a multiple of the one on the left, so we can combine them,

$$\frac{13}{4} \int e^{2x} \sin 3x \, dx = (\frac{1}{2} \sin 3x - \frac{3}{4} \cos 3x) e^{2x},$$

Table A.5: Tabular Method, showing a nonterminating example.

D		I
$\sin 3x$	$+$	e^{2x}
$3\cos 3x$	$-$	$\frac{1}{2}e^{2x}$
$-9\sin 3x$		$\frac{1}{4}e^{2x}$

or

$$\int e^{2x}\sin 3x\,dx = (\frac{2}{13}\sin 3x - \frac{3}{13}\cos 3x)e^{2x}.$$

Differentiation Under the Integral

Differentiation Under the Integral Sign and Feynman's trick.

Another method that one can use to evaluate this integral is to differentiate under the integral sign. This is mentioned in the Richard Feynman's memoir *Surely You're Joking, Mr. Feynman!*. In the book Feynman recounts using this "trick" to be able to do integrals that his MIT classmates could not do. This is based on a theorem found in Advanced Calculus texts. Reader's unfamiliar with partial derivatives should be able to grasp their use in the following example.

Theorem A.1. *Let the functions $f(x,t)$ and $\frac{\partial f(x,t)}{\partial x}$ be continuous in both t, and x, in the region of the (t,x) plane which includes $a(x) \le t \le b(x)$, $x_0 \le x \le x_1$, where the functions $a(x)$ and $b(x)$ are continuous and have continuous derivatives for $x_0 \le x \le x_1$. Defining*

$$F(x) \equiv \int_{a(x)}^{b(x)} f(x,t)\,dt,$$

then

$$\begin{aligned} \frac{dF(x)}{dx} &= \left(\frac{\partial F}{\partial b}\right)\frac{db}{dx} + \left(\frac{\partial F}{\partial a}\right)\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt \\ &= f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt. \end{aligned} \tag{A.69}$$

for $x_0 \le x \le x_1$. This is a generalized version of the Fundamental Theorem of Calculus.

In the next examples we show how we can use this theorem to bypass integration by parts.

Example A.12. Use differentiation under the integral sign to evaluate $\int xe^x\,dx$. First, consider the integral

$$I(x,a) = \int e^{ax}\,dx = \frac{e^{ax}}{a}.$$

Then,

$$\frac{\partial I(x,a)}{\partial a} = \int xe^{ax}\,dx.$$

So,

$$\begin{aligned}
\int xe^{ax}\,dx &= \frac{\partial I(x,a)}{\partial a} \\
&= \frac{\partial}{\partial a}\left(\int e^{ax}\,dx\right) \\
&= \frac{\partial}{\partial a}\left(\frac{e^{ax}}{a}\right) \\
&= \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}. \qquad \text{(A.70)}
\end{aligned}$$

Evaluating this result at $a = 1$, we have

$$\int xe^{x}\,dx = (x-1)e^{x}.$$

The reader can verify this result by employing the previous methods or by just differentiating the result.

Example A.13. We will do the integral $\int_0^\pi x^2 \cos x\,dx$ once more. First, consider the integral

$$\begin{aligned}
I(a) &\equiv \int_0^\pi \cos ax\,dx \\
&= \left.\frac{\sin ax}{a}\right|_0^\pi \\
&= \frac{\sin a\pi}{a}. \qquad \text{(A.71)}
\end{aligned}$$

Differentiating the integral $I(a)$ with respect to a twice gives

$$\frac{d^2 I(a)}{da^2} = -\int_0^\pi x^2 \cos ax\,dx. \qquad \text{(A.72)}$$

Evaluation of this result at $a = 1$ leads to the desired result. Namely,

$$\begin{aligned}
\int_0^\pi x^2 \cos x\,dx &= -\left.\frac{d^2 I(a)}{da^2}\right|_{a=1} \\
&= -\frac{d^2}{da^2}\left(\frac{\sin a\pi}{a}\right)\Big|_{a=1} \\
&= -\frac{d}{da}\left(\frac{a\pi\cos a\pi - \sin a\pi}{a^2}\right)\Big|_{a=1} \\
&= -\left(\frac{a^2\pi^2 \sin a\pi + 2a\pi\cos a\pi - 2\sin a\pi}{a^3}\right)\Big|_{a=1} \\
&= -2\pi. \qquad \text{(A.73)}
\end{aligned}$$

Trigonometric Integrals

Other types of integrals that you will see often are trigonometric integrals. In particular, integrals involving powers of sines and cosines. For odd powers, a simple substitution will turn the integrals into simple powers.

Example A.14. For example, consider

$$\int \cos^3 x\, dx.$$

This can be rewritten as

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx.$$

Integration of odd powers of sine and cosine.

Let $u = \sin x$. Then, $du = \cos x\, dx$. Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\begin{aligned} \int \cos^3 x\, dx &= \int \cos^2 x \cos x\, dx \\ &= \int (1-u^2)\, du \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C. \end{aligned} \tag{A.74}$$

A quick check confirms the answer:

$$\begin{aligned} \frac{d}{dx}\left(\sin x - \frac{1}{3}\sin^3 x + C\right) &= \cos x - \sin^2 x \cos x \\ &= \cos x (1 - \sin^2 x) \\ &= \cos^3 x. \end{aligned} \tag{A.75}$$

Integration of even powers of sine and cosine.

Even powers of sines and cosines are a little more complicated, but doable. In these cases we need the half angle formulae (A.24)-(A.25).

Example A.15. As an example, we will compute

$$\int_0^{2\pi} \cos^2 x\, dx.$$

Substituting the half angle formula for $\cos^2 x$, we have

$$\begin{aligned} \int_0^{2\pi} \cos^2 x\, dx &= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2x)\, dx \\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)_0^{2\pi} \\ &= \pi. \end{aligned} \tag{A.76}$$

Recall that RMS averages refer to the root mean square average. This is computed by first computing the average, or mean, of the square of some quantity. Then one takes the square root. Typical examples are RMS voltage, RMS current, and the average energy in an electromagnetic wave. AC currents oscillate so fast that the measured value is the RMS voltage.

We note that this result appears often in physics. When looking at root mean square averages of sinusoidal waves, one needs the average of the square of sines and cosines. Recall that the average of a function on interval $[a, b]$ is given as

$$f\text{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx. \tag{A.77}$$

So, the average of $\cos^2 x$ over one period is

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^2 x\, dx = \frac{1}{2}. \tag{A.78}$$

The root mean square is then found by taking the square root, $\frac{1}{\sqrt{2}}$.

Trigonometric Function Substitution

Another class of integrals typically studied in calculus are those involving the forms $\sqrt{1-x^2}$, $\sqrt{1+x^2}$, or $\sqrt{x^2-1}$. These can be simplified through the use of trigonometric substitutions. The idea is to combine the two terms under the radical into one term using trigonometric identities. We will consider some typical examples.

Example A.16. Evaluate $\int \sqrt{1-x^2}\,dx$.

Since $1-\sin^2\theta = \cos^2\theta$, we perform the sine substitution

$$x = \sin\theta, \quad dx = \cos\theta\,d\theta.$$

Then,

$$\begin{aligned} \int \sqrt{1-x^2}\,dx &= \int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta \\ &= \int \cos^2\theta\,d\theta. \end{aligned} \tag{A.79}$$

In any of these computations careful attention has to be paid to simplifying the radical. This is because

$$\sqrt{x^2} = |x|.$$

For example, $\sqrt{(-5)^2} = \sqrt{25} = 5$. For $x = \sin\theta$, one typically specifies the domain $-\pi/2 \le \theta \le \pi/2$. In this domain we have $|\cos\theta| = \cos\theta$.

Using the last example, we have

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C.$$

However, we need to write the answer in terms of x. We do this by first using the double angle formula for $\sin 2\theta$ and $\cos\theta = \sqrt{1-x^2}$ to obtain

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\sin^{-1}x - x\sqrt{1-x^2}\right) + C.$$

Similar trigonometric substitutions result for integrands involving $\sqrt{1+x^2}$ and $\sqrt{x^2-1}$. The substitutions are summarized in Table A.6. The simplification of the given form is then obtained using trigonometric identities. This can also be accomplished by referring to the right triangles shown in Figure A.5.

Form	Substitution	Differential
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$dx = a\cos\theta\,d\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta\,d\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta\,d\theta$

Table A.6: Standard trigonometric substitutions.

Example A.17. Evaluate $\int_0^2 \sqrt{x^2+4}\,dx$.

Let $x = 2\tan\theta$. Then, $dx = 2\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = 2\sec\theta.$$

So, the integral becomes

$$\int_0^2 \sqrt{x^2+4}\,dx = 4\int_0^{\pi/4} \sec^3\theta\,d\theta.$$

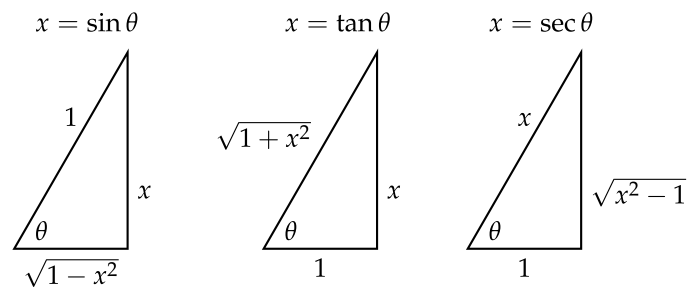

Figure A.5: Geometric relations used in trigonometric substitution.

One has to recall, or look up,

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\left(\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|\right) + C.$$

This gives

$$\begin{aligned} \int_0^2 \sqrt{x^2+4}\,dx &= 2\left[\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|\right]_0^{\pi/4} \\ &= 2\left(\sqrt{2} + \ln|\sqrt{2}+1| - (0+\ln(1))\right) \\ &= 2(\sqrt{2}+\ln(\sqrt{2}+1)). \end{aligned} \quad (A.80)$$

Example A.18. Evaluate $\int \frac{dx}{\sqrt{x^2-1}}$, $x \geq 1$.

In this case one needs the secant substitution. This yields

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{\sec^2\theta - 1}} \\ &= \int \frac{\sec\theta\tan\theta\, d\theta}{\tan\theta} \\ &= \int \sec\theta d\theta \\ &= \ln(\sec\theta + \tan\theta) + C \\ &= \ln(x + \sqrt{x^2-1}) + C. \end{aligned} \quad (A.81)$$

Example A.19. Evaluate $\int \frac{dx}{x\sqrt{x^2-1}}$, $x \geq 1$.

Again we can use a secant substitution. This yields

$$\begin{aligned} \int \frac{dx}{x\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\sqrt{\sec^2\theta-1}} \\ &= \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\, d\theta \\ &= \int d\theta = \theta + C = \sec^{-1} x + C. \end{aligned} \quad (A.82)$$

Hyperbolic Function Substitution

Even though trigonometric substitution plays a role in the calculus program, students often see hyperbolic function substitution used in physics courses. The reason might be because hyperbolic function substitution is sometimes simpler. The idea is the same as for trigonometric substitution. We use an identity to simplify the radical.

Example A.20. Evaluate $\int_0^2 \sqrt{x^2+4}\,dx$ using the substitution $x = 2\sinh u$.

Since $x = 2\sinh u$, we have $dx = 2\cosh u\,du$. Also, we can use the identity $\cosh^2 u - \sinh^2 u = 1$ to rewrite

$$\sqrt{x^2+4} = \sqrt{4\sinh^2 u + 4} = 2\cosh u.$$

The integral can be now be evaluated using these substitutions and some hyperbolic function identities,

$$\begin{aligned}
\int_0^2 \sqrt{x^2+4}\,dx &= 4\int_0^{\sinh^{-1} 1} \cosh^2 u\,du \\
&= 2\int_0^{\sinh^{-1} 1} (1+\cosh 2u)\,du \\
&= 2\left[u + \frac{1}{2}\sinh 2u\right]_0^{\sinh^{-1} 1} \\
&= 2\left[u + \sinh u \cosh u\right]_0^{\sinh^{-1} 1} \\
&= 2\left(\sinh^{-1} 1 + \sqrt{2}\right). \qquad \text{(A.83)}
\end{aligned}$$

In Example A.17 we used a trigonometric substitution and found

$$\int_0^2 \sqrt{x^2+4} = 2(\sqrt{2} + \ln(\sqrt{2}+1)).$$

This is the same result since $\sinh^{-1} 1 = \ln(1+\sqrt{2})$.

Example A.21. Evaluate $\int \frac{dx}{\sqrt{x^2-1}}$ for $x \geq 1$ using hyperbolic function substitution.

This integral was evaluated in Example A.19 using the trigonometric substitution $x = \sec\theta$ and the resulting integral of $\sec\theta$ had to be recalled. Here we will use the substitution

$$x = \cosh u, \quad dx = \sinh u\,du, \quad \sqrt{x^2-1} = \sqrt{\cosh^2 u - 1} = \sinh u.$$

Then,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2-1}} &= \int \frac{\sinh u\,du}{\sinh u} \\
&= \int du = u + C \\
&= \cosh^{-1} x + C \\
&= \frac{1}{2}\ln(x + \sqrt{x^2-1}) + C, \quad x \geq 1. \qquad \text{(A.84)}
\end{aligned}$$

This is the same result as we had obtained previously, but this derivation was a little cleaner.

Also, we can extend this result to values $x \leq -1$ by letting $x = -\cosh u$. This gives

$$\int \frac{dx}{\sqrt{x^2-1}} = \frac{1}{2}\ln(x + \sqrt{x^2-1}) + C, \quad x \leq -1.$$

Combining these results, we have shown

$$\int \frac{dx}{\sqrt{x^2-1}} = \frac{1}{2}\ln(|x| + \sqrt{x^2-1}) + C, \quad x^2 \geq 1.$$

We have seen in the last example that the use of hyperbolic function substitution allows us to bypass integrating the secant function in Example A.19 when using trigonometric substitutions. In fact, we can use hyperbolic substitutions to evaluate integrals of powers of secants. Comparing Examples A.19 and A.21, we consider the transformation $\sec\theta = \cosh u$. The relation between differentials is found by differentiation, giving

$$\sec\theta\tan\theta\, d\theta = \sinh u\, du.$$

Since

$$\tanh^2\theta = \sec^2\theta - 1,$$

we have $\tan\theta = \sinh u$, therefore

$$d\theta = \frac{du}{\cosh u}.$$

Evaluation of $\int \sec\theta\, d\theta$.

In the next example we show how useful this transformation is.

Example A.22. Evaluate $\int \sec\theta\, d\theta$ using hyperbolic function substitution.

From the discussion in the last paragraph, we have

$$\begin{aligned} \int \sec\theta\, d\theta &= \int du \\ &= u + C \\ &= \cosh^{-1}(\sec\theta) + C \end{aligned} \tag{A.85}$$

We can express this result in the usual form by using the logarithmic form of the inverse hyperbolic cosine,

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}).$$

The result is

$$\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta).$$

This example was fairly simple using the transformation $\sec\theta = \cosh u$. Another common integral that arises often is integrations of $\sec^3\theta$. In a typical calculus class this integral is evaluated using integration by parts. However. that leads to a tricky manipulation that is a bit scary the first time it is encountered (and probably upon several more encounters.) In the next example, we will show how hyperbolic function substitution is simpler.

Evaluation of $\int \sec^3\theta\, d\theta$.

Example A.23. Evaluate $\int \sec^3\theta\, d\theta$ using hyperbolic function substitution.

First, we consider the transformation $\sec\theta = \cosh u$ with $d\theta = \frac{du}{\cosh u}$. Then,

$$\int \sec^3\theta\, d\theta = \int \frac{du}{\cosh u}.$$

This integral was done in Example A.7, leading to

$$\int \sec^3\theta\, d\theta = 2\tan^{-1} e^u + C.$$

While correct, this is not the form usually encountered. Instead, we make the slightly different transformation $\tan\theta = \sinh u$. Since $\sec^2\theta = 1 + \tan^2\theta$, we find $\sec\theta = \cosh u$. As before, we find

$$d\theta = \frac{du}{\cosh u}.$$

Using this transformation and several identities, the integral becomes

$$\begin{aligned}
\int \sec^3\theta\, d\theta &= \int \cosh^2 u\, du \\
&= \frac{1}{2}\int (1+\cosh 2u)\, du \\
&= \frac{1}{2}\left(u + \frac{1}{2}\sinh 2u\right) \\
&= \frac{1}{2}\left(u + \sinh u \cosh u\right) \\
&= \frac{1}{2}\left(\cosh^{-1}(\sec\theta) + \tan\theta\sec\theta\right) \\
&= \frac{1}{2}\left(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\right). \qquad \text{(A.86)}
\end{aligned}$$

There are many other integration methods, some of which we will visit in other parts of the book, such as partial fraction decomposition and numerical integration. Another topic which we will revisit is power series.

A.1.6 Geometric Series

Geometric series are fairly common and will be used throughout the book. You should learn to recognize them and work with them.

INFINITE SERIES OCCUR OFTEN in mathematics and physics. Two series which occur often are the geometric series and the binomial series. we will discuss these next.

A geometric series is of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots. \qquad \text{(A.87)}$$

Here a is the first term and r is called the ratio. It is called the ratio because the ratio of two consecutive terms in the sum is r.

Example A.24. For example,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

is an example of a geometric series. We can write this using summation notation,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} 1\left(\frac{1}{2}\right)^n.$$

Thus, $a = 1$ is the first term and $r = \frac{1}{2}$ is the common ratio of successive terms. Next, we seek the sum of this infinite series, if it exists.

The sum of a geometric series, when it exists, can easily be determined. We consider the nth partial sum:

$$s_n = a + ar + \ldots + ar^{n-2} + ar^{n-1}. \tag{A.88}$$

Now, multiply this equation by r.

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n. \tag{A.89}$$

Subtracting these two equations, while noting the many cancelations, we have

$$\begin{aligned} (1-r)s_n &= (a + ar + \ldots + ar^{n-2} + ar^{n-1}) \\ &\quad -(ar + ar^2 + \ldots + ar^{n-1} + ar^n) \\ &= a - ar^n \\ &= a(1-r^n). \end{aligned} \tag{A.90}$$

Thus, the nth partial sums can be written in the compact form

$$s_n = \frac{a(1-r^n)}{1-r}. \tag{A.91}$$

The sum, if it exists, is given by $S = \lim_{n\to\infty} s_n$. Letting n get large in the partial sum (A.91), we need only evaluate $\lim_{n\to\infty} r^n$. From the special limits in the Appendix we know that this limit is zero for $|r| < 1$. Thus, we have

Geometric Series

The sum of the geometric series exists for $|r| < 1$ and is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad |r| < 1. \tag{A.92}$$

The reader should verify that the geometric series diverges for all other values of r. Namely, consider what happens for the separate cases $|r| > 1$, $r = 1$ and $r = -1$.

Next, we present a few typical examples of geometric series.

Example A.25. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

In this case we have that $a = 1$ and $r = \frac{1}{2}$. Therefore, this infinite series converges and the sum is

$$S = \frac{1}{1-\frac{1}{2}} = 2.$$

Example A.26. $\sum_{k=2}^{\infty} \frac{4}{3^k}$

In this example we first note that the first term occurs for $k = 2$. It sometimes helps to write out the terms of the series,

$$\sum_{k=2}^{\infty} \frac{4}{3^k} = \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \frac{4}{3^5} + \ldots.$$

Looking at the series, we see that $a = \frac{4}{9}$ and $r = \frac{1}{3}$. Since $|r|<1$, the geometric series converges. So, the sum of the series is given by

$$S = \frac{\frac{4}{9}}{1 - \frac{1}{3}} = \frac{2}{3}.$$

Example A.27. $\sum_{n=1}^{\infty}(\frac{3}{2^n} - \frac{2}{5^n})$

Finally, in this case we do not have a geometric series, but we do have the difference of two geometric series. Of course, we need to be careful whenever rearranging infinite series. In this case it is allowed [1]. Thus, we have

[1] A rearrangement of terms in an infinite series is allowed when the series is absolutely convergent. (See the Appendix.)

$$\sum_{n=1}^{\infty}\left(\frac{3}{2^n} - \frac{2}{5^n}\right) = \sum_{n=1}^{\infty}\frac{3}{2^n} - \sum_{n=1}^{\infty}\frac{2}{5^n}.$$

Now we can add both geometric series to obtain

$$\sum_{n=1}^{\infty}\left(\frac{3}{2^n} - \frac{2}{5^n}\right) = \frac{\frac{3}{2}}{1-\frac{1}{2}} - \frac{\frac{2}{5}}{1-\frac{1}{5}} = 3 - \frac{1}{2} = \frac{5}{2}.$$

Geometric series are important because they are easily recognized and summed. Other series which can be summed include special cases of Taylor series and *telescoping series.* Next, we show an example of a telescoping series.

Example A.28. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ The first few terms of this series are

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots.$$

It does not appear that we can sum this infinite series. However, if we used the partial fraction expansion

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

then we find the kth partial sum can be written as

$$\begin{aligned} s_k &= \sum_{n=1}^{k}\frac{1}{n(n+1)} \\ &= \sum_{n=1}^{k}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{k} - \frac{1}{k+1}\right). \end{aligned} \quad \text{(A.93)}$$

We see that there are many cancelations of neighboring terms, leading to the series collapsing (like a retractable telescope) to something manageable:

$$s_k = 1 - \frac{1}{k+1}.$$

Taking the limit as $k \to \infty$, we find $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$.

A.1.7 Power Series

Actually, what are now known as Taylor and Maclaurin series were known long before they were named. James Gregory (1638-1675) has been recognized for discovering Taylor series, which were later named after Brook Taylor (1685-1731) . Similarly, Colin Maclaurin (1698-1746) did not actually discover Maclaurin series, but the name was adopted because of his particular use of series.

Another example of an infinite series that the student has encountered in previous courses is the power series. Examples of such series are provided by Taylor and Maclaurin series.

A power series expansion about $x = a$ with coefficient sequence c_n is given by $\sum_{n=0}^{\infty} c_n(x-a)^n$. For now we will consider all constants to be real numbers with x in some subset of the set of real numbers.

Consider the following expansion about $x = 0$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots. \tag{A.94}$$

We would like to make sense of such expansions. For what values of x will this infinite series converge? Until now we did not pay much attention to which infinite series might converge. However, this particular series is already familiar to us. It is a geometric series. Note that each term is gotten from the previous one through multiplication by $r = x$. The first term is $a = 1$. So, from Equation (A.92), we have that the sum of the series is given by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

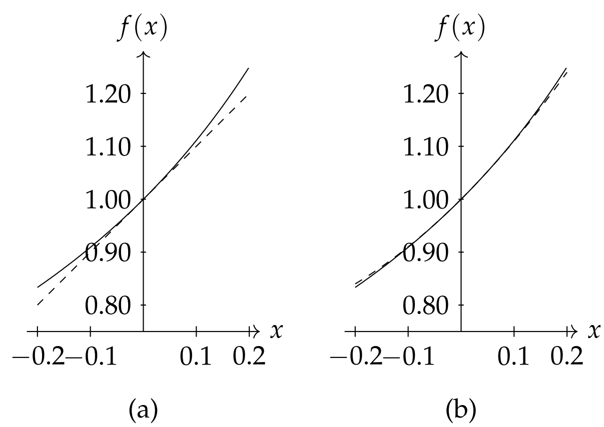

Figure A.6: (a) Comparison of $\frac{1}{1-x}$ (solid) to $1 + x$ (dashed) for $x \in [-0.2, 0.2]$. (b) Comparison of $\frac{1}{1-x}$ (solid) to $1 + x + x^2$ (dashed) for $x \in [-0.2, 0.2]$.

In this case we see that the sum, when it exists, is a simple function. In fact, when x is small, we can use this infinite series to provide approximations to the function $(1-x)^{-1}$. If x is small enough, we can write

$$(1-x)^{-1} \approx 1 + x.$$

In Figure A.6a we see that for small values of x these functions do agree.

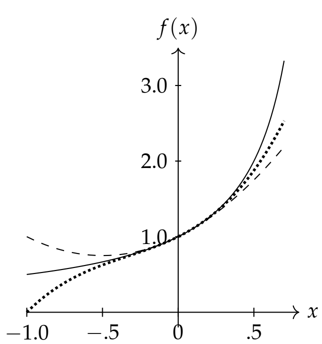

Figure A.7: Comparison of $\frac{1}{1-x}$ (solid) to $1 + x + x^2$ (dashed) and $1 + x + x^2 + x^3$ (dotted) for $x \in [-1.0, 0.7]$.

Of course, if we want better agreement, we select more terms. In Figure A.6b we see what happens when we do so. The agreement is much better. But extending the interval, we see in Figure A.7 that keeping only quadratic terms may not be good enough. Keeping the cubic terms gives better agreement over the interval.

Finally, in Figure A.8 we show the sum of the first 21 terms over the entire interval $[-1, 1]$. Note that there are problems with approximations near the endpoints of the interval, $x = \pm 1$.

Such polynomial approximations are called Taylor polynomials. Thus, $T_3(x) = 1 + x + x^2 + x^3$ is the third order Taylor polynomial approximation of $f(x) = \frac{1}{1-x}$.

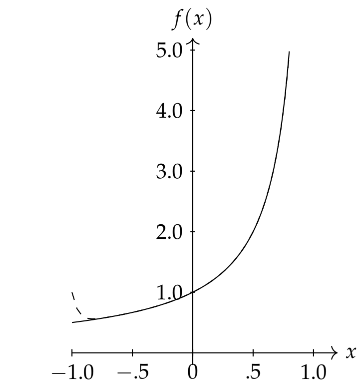

Figure A.8: Comparison of $\frac{1}{1-x}$ (solid) to $\sum_{n=0}^{20} x^n$ for $x \in [-1, 1]$.

With this example we have seen how useful a series representation might be for a given function. However, the series representation was a simple geometric series, which we already knew how to sum. Is there a way to begin with a function and then find its series representation? Once we have such a representation, will the series converge to the function with which we started? For what values of x will it converge? These questions can be answered by recalling the definitions of Taylor and Maclaurin series.

Taylor series expansion.

A Taylor series expansion of $f(x)$ about $x = a$ is the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{A.95}$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{A.96}$$

Note that we use $\sim$ to indicate that we have yet to determine when the series may converge to the given function. A special class of series are those Taylor series for which the expansion is about $x = 0$. These are called Maclaurin series.

Maclaurin series expansion.

A Maclaurin series expansion of $f(x)$ is a Taylor series expansion of $f(x)$ about $x = 0$, or

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n, \tag{A.97}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{A.98}$$

Example A.29. Expand $f(x) = e^x$ about $x = 0$.

We begin by creating a table. In order to compute the expansion coefficients, c_n, we will need to perform repeated differentiations of $f(x)$. So, we provide a table for these derivatives. Then, we only need to evaluate the second column at $x = 0$ and divide by $n!$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	e^x	$e^0 = 1$	$\frac{1}{0!} = 1$
1	e^x	$e^0 = 1$	$\frac{1}{1!} = 1$
2	e^x	$e^0 = 1$	$\frac{1}{2!}$
3	e^x	$e^0 = 1$	$\frac{1}{3!}$

Next, we look at the last column and try to determine a pattern so that we can write down the general term of the series. If there is only a need to get a polynomial approximation, then the first few terms may be sufficient. In this case, the pattern is obvious: $c_n = \frac{1}{n!}$. So,

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Example A.30. Expand $f(x) = e^x$ about $x = 1$.

Here we seek an expansion of the form $e^x \sim \sum_{n=0}^{\infty} c_n (x-1)^n$. We could create a table like the last example. In fact, the last column would have values of the form $\frac{e}{n!}$. (You should confirm this.) However, we will make use of the Maclaurin series expansion for e^x and get the result quicker. Note that $e^x = e^{x-1+1} = ee^{x-1}$. Now, apply the known expansion for e^x :

$$e^x \sim e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \dots\right) = \sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}.$$

Example A.31. Expand $f(x) = \frac{1}{1-x}$ about $x = 0$.

This is the example with which we started our discussion. We can set up a table in order to find the Maclaurin series coefficients. We see from the last column of the table that we get back the geometric series (A.94).

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	$\frac{1}{1-x}$	1	$\frac{1}{0!} = 1$
1	$\frac{1}{(1-x)^2}$	1	$\frac{1}{1!} = 1$
2	$\frac{2(1)}{(1-x)^3}$	$2(1)$	$\frac{2!}{2!} = 1$
3	$\frac{3(2)(1)}{(1-x)^4}$	$3(2)(1)$	$\frac{3!}{3!} = 1$

So, we have found

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n.$$

We can replace $\sim$ by equality if we can determine the range of x-values for which the resulting infinite series converges. We will investigate such convergence shortly.

Series expansions for many elementary functions arise in a variety of applications. Some common expansions are provided in Table A.7.

We still need to determine the values of x for which a given power series converges. The first five of the above expansions converge for all reals, but the others only converge for $|x| < 1$.

Series Expansions You Should Know				
e^x	=	$1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$	=	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$
$\cos x$	=	$1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots$	=	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$
$\sin x$	=	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$	=	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$
$\cosh x$	=	$1+\frac{x^2}{2}+\frac{x^4}{4!}+\dots$	=	$\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$
$\sinh x$	=	$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots$	=	$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$
$\frac{1}{1-x}$	=	$1+x+x^2+x^3+\dots$	=	$\sum_{n=0}^{\infty}x^n$
$\frac{1}{1+x}$	=	$1-x+x^2-x^3+\dots$	=	$\sum_{n=0}^{\infty}(-x)^n$
$\tan^{-1}x$	=	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$	=	$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$
$\ln(1+x)$	=	$x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$	=	$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$

Table A.7: Common Mclaurin Series Expansions

We consider the convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$. For $x=a$ the series obviously converges. Will it converge for other points? One can prove

Theorem A.2. *If $\sum_{n=0}^{\infty} c_n(b-a)^n$ converges for $b \neq a$, then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely for all x satisfying $|x-a|<|b-a|$.*

This leads to three possibilities

1. $\sum_{n=0}^{\infty} c_n(x-a)^n$ may only converge at $x=a$.
2. $\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge for all real numbers.
3. $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a|<R$ and diverges for $|x-a|>R$.

Interval and radius of convergence.

The number R is called the radius of convergence of the power series and $(a-R, a+R)$ is called the interval of convergence. Convergence at the endpoints of this interval has to be tested for each power series.

In order to determine the interval of convergence, one needs only note that when a power series converges, it does so absolutely. So, we need only test the convergence of $\sum_{n=0}^{\infty}|c_n(x-a)^n| = \sum_{n=0}^{\infty}|c_n||x-a|^n$. This is easily done using either the ratio test or the nth root test. We first identify the nonnegative terms $a_n=|c_n||x-a|^n$. Then, we apply one of the convergence tests from Calculus II.

For example, the nth Root Test gives the convergence condition for $a_n = |c_n||x-a|^n$,

$$\rho = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{|c_n|}|x-a| < 1.$$

Since $|x-a|$ is independent of n,, we can factor it out of the limit and divide the value of the limit to obtain

$$|x-a| < \left(\lim_{n\to\infty} \sqrt[n]{|c_n|}\right)^{-1} \equiv R.$$

Thus, we have found the radius of convergence, R.

Similarly, we can apply the Ratio Test.

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}|x-a| < 1.$$

Again, we rewrite this result to determine the radius of convergence:

$$|x-a| < \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)^{-1} \equiv R.$$

Example A.32. Find the radius of convergence of the series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Since there is a factorial, we will use the Ratio Test.

$$\rho = \lim_{n\to\infty} \frac{|n!|}{|(n+1)!|}|x| = \lim_{n\to\infty} \frac{1}{n+1}|x| = 0.$$

Since $\rho = 0$, it is independent of $|x|$ and thus the series converges for all x. We also can say that the radius of convergence is infinite.

Example A.33. Find the radius of convergence of the series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

In this example we will use the nth Root Test.

$$\rho = \lim_{n\to\infty} \sqrt[n]{1}|x| = |x| < 1.$$

Thus, we find that we have absolute convergence for $|x| < 1$. Setting $x = 1$ or $x = -1$, we find that the resulting series do not converge. So, the endpoints are not included in the complete interval of convergence.

In this example we could have also used the Ratio Test. Thus,

$$\rho = \lim_{n\to\infty} \frac{1}{1}|x| = |x| < 1.$$

We have obtained the same result as when we used the nth Root Test.

Example A.34. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n}$.

In this example, we have an expansion about $x = 2$. Using the nth Root Test we find that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n}}|x-2| = 3|x-2| < 1.$$

Solving for $|x-2|$ in this inequality, we find $|x-2| < \frac{1}{3}$. Thus, the radius of convergence is $R = \frac{1}{3}$ and the interval of convergence is $\left(2-\frac{1}{3}, 2+\frac{1}{3}\right) = \left(\frac{5}{3}, \frac{7}{3}\right)$.

As for the endpoints, we first test the point $x = \frac{7}{3}$. The resulting series is $\sum_{n=1}^{\infty} \frac{3^n(\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series, and thus it does not converge. Inserting $x = \frac{5}{3}$, we get the alternating harmonic series. This series does converge. So, we have convergence on $[\frac{5}{3}, \frac{7}{3})$. However, it is only conditionally convergent at the left endpoint, $x = \frac{5}{3}$.

Example A.35. Find an expansion of $f(x) = \frac{1}{x+2}$ about $x = 1$.

Instead of explicitly computing the Taylor series expansion for this function, we can make use of an already known function. We first write $f(x)$ as a function of $x - 1$, since we are expanding about $x = 1$; i.e., we are seeking a series whose terms are powers of $x - 1$.

This expansion is easily done by noting that $\frac{1}{x+2} = \frac{1}{(x-1)+3}$. Factoring out a 3, we can rewrite this expression as a sum of a geometric series. Namely, we use the expansion for

$$\begin{aligned} g(z) &= \frac{1}{1+z} \\ &= 1 - z + z^2 - z^3 + \dots . \end{aligned} \tag{A.99}$$

and then we rewrite $f(x)$ as

$$\begin{aligned} f(x) &= \frac{1}{x+2} \\ &= \frac{1}{(x-1)+3} \\ &= \frac{1}{3[1+\frac{1}{3}(x-1)]} \\ &= \frac{1}{3}\frac{1}{1+\frac{1}{3}(x-1)}. \end{aligned} \tag{A.100}$$

Note that $f(x) = \frac{1}{3}g(\frac{1}{3}(x-1))$ for $g(z) = \frac{1}{1+z}$. So, the expansion becomes

$$f(x) = \frac{1}{3}\left[1 - \frac{1}{3}(x-1) + \left(\frac{1}{3}(x-1)\right)^2 - \left(\frac{1}{3}(x-1)\right)^3 + \dots\right].$$

This can further be simplified as

$$f(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \dots .$$

Convergence is easily established. The expansion for $g(z)$ converges for $|z| < 1$. So, the expansion for $f(x)$ converges for $|-\frac{1}{3}(x-1)| < 1$. This implies that $|x-1| < 3$. Putting this inequality in interval notation, we have that the power series converges absolutely for $x \in (-2, 4)$. Inserting the endpoints, one can show that the series diverges for both $x = -2$ and $x = 4$. You should verify this!

Example A.36. Prove Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

As a final application, we can derive Euler's Formula ,

Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, is an important formula and is used throughout the text.

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $i = \sqrt{-1}$. We naively use the expansion for e^x with $x = i\theta$. This leads us to

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots .$$

Next we note that each term has a power of i. The sequence of powers of i is given as $\{1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \dots\}$. See the pattern? We conclude that

$$i^n = i^r, \text{ where } r = \text{ remainder after dividing } n \text{ by } 4.$$

This gives

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right).$$

We recognize the expansions in the parentheses as those for the cosine and sine functions. Thus, we end with Euler's Formula.

We further derive relations from this result, which will be important for our next studies. From Euler's formula we have that for integer n:

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

We also have

$$e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n .$$

Equating these two expressions, we are led to de Moivre's Formula, named after Abraham de Moivre (1667-1754),

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \quad \text{(A.101)}$$

de Moivre's Formula.

This formula is useful for deriving identities relating powers of sines or cosines to simple functions. For example, if we take $n = 2$ in Equation (A.101), we find

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.$$

Looking at the real and imaginary parts of this result leads to the well known double angle identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

Here we see elegant proofs of well known trigonometric identities.

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta, \quad \text{(A.102)} \\ \sin 2\theta &= 2\sin\theta\cos\theta, \\ \cos^2\theta &= \frac{1}{2}(1 + \cos 2\theta), \\ \sin^2\theta &= \frac{1}{2}(1 - \cos 2\theta). \end{aligned}$$

Replacing $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$ leads to the half angle formulae:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

We can also use Euler's Formula to write sines and cosines in terms of complex exponentials. We first note that due to the fact that the cosine is an even function and the sine is an odd function, we have

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Trigonometric functions can be written in terms of complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Combining this with Euler's Formula, we have that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

We finally note that there is a simple relationship between hyperbolic functions and trigonometric functions. Recall that

$$\cosh x = \frac{e^{x}+e^{-x}}{2}.$$

If we let $x = i\theta$, then we have that $\cosh(i\theta) = \cos\theta$ and $\cos(ix) = \cosh x$. Similarly, we can show that $\sinh(i\theta) = i\sin\theta$ and $\sin(ix) = -i\sinh x$.

Hyperbolic functions and trigonometric functions are intimately related.

$$\cos(ix) = \cosh x,$$

$$\sin(ix) = -i\sinh x.$$

A.1.8 The Binomial Expansion

Another series expansion which occurs often in examples and applications is the binomial expansion. This is simply the expansion of the expression $(a+b)^p$ in powers of a and b. We will investigate this expansion first for nonnegative integer powers p and then derive the expansion for other values of p. While the binomial expansion can be obtained using Taylor series, we will provide a more intuitive derivation to show that

The binomial expansion is a special series expansion used to approximate expressions of the form $(a+b)^p$ for $b \ll a$, or $(1+x)^p$ for $|x| \ll 1$.

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r, \tag{A.103}$$

where the C_r^n are called the *binomial coefficients*.

Lets list some of the common expansions for nonnegative integer powers.

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
&\cdots
\end{aligned} \tag{A.104}$$

We now look at the patterns of the terms in the expansions. First, we note that each term consists of a product of a power of a and a power of b. The powers of a are decreasing from n to 0 in the expansion of $(a+b)^n$. Similarly, the powers of b increase from 0 to n. The sums of the exponents in each term is n. So, we can write the $(k+1)$st term in the expansion as $a^{n-k}b^k$. For example, in the expansion of $(a+b)^{51}$ the 6th term is $a^{51-5}b^5 = a^{46}b^5$. However, we do not yet know the numerical coefficients in the expansion.

Let's list the coefficients for the above expansions.

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array} \tag{A.105}$$

This pattern is the famous Pascal's triangle. There are many interesting features of this triangle. But we will first ask how each row can be generated.

Pascal's triangle is named after Blaise Pascal (1623-1662). While such configurations of numbers were known earlier in history, Pascal published them and applied them to probability theory.

Pascal's triangle has many unusual properties and a variety of uses:

- Horizontal rows add to powers of 2.
- The horizontal rows are powers of 11 (1, 11, 121, 1331, etc.).
- Adding any two successive numbers in the diagonal 1-3-6-10-15-21-28... results in a perfect square.
- When the first number to the right of the 1 in any row is a prime number, all numbers in that row are divisible by that prime number. The reader can readily check this for the $n=5$ and $n=7$ rows.
- Sums along certain diagonals leads to the Fibonacci sequence. These diagonals are parallel to the line connecting the first 1 for $n=3$ row and the 2 in the $n=2$ row.

We see that each row begins and ends with a one. The second term and next to last term have a coefficient of n. Next we note that consecutive pairs in each row can be added to obtain entries in the next row. For example, we have for rows $n=2$ and $n=3$ that $1+2=3$ and $2+1=3:$

$$\begin{array}{lccccccccc} n=2: & & 1 & & & 2 & & & 1 & \\ & & & \searrow & \swarrow & & \searrow & \swarrow & & \\ n=3: & 1 & & & 3 & & & 3 & & 1 \end{array} \tag{A.106}$$

With this in mind, we can generate the next several rows of our triangle.

$$\begin{array}{lccccccccccccc} n=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\ n=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ n=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ n=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array} \tag{A.107}$$

So, we use the numbers in row $n=4$ to generate entries in row $n=5:$ $1+4=5$, $4+6=10$. We then use row $n=5$ to get row $n=6$, etc.

Of course, it would take a while to compute each row up to the desired n. Fortunately, there is a simple expression for computing a specific coefficient. Consider the kth term in the expansion of $(a+b)^n$. Let $r=k-1$, for $k=1,\ldots,n+1$. Then this term is of the form $C_r^n a^{n-r}b^r$. We have seen that the coefficients satisfy

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}.$$

Actually, the binomial coefficients, C_r^n, have been found to take a simple form,

$$C_r^n = \frac{n!}{(n-r)!r!} \equiv \left(\begin{array}{c} n \\ r \end{array}\right).$$

This is nothing other than the combinatoric symbol for determining how to choose n objects r at a time. In the binomial expansions this makes sense. We have to count the number of ways that we can arrange r products of b with $n-r$ products of a. There are n slots to place the b's. For example, the $r=2$ case for $n=4$ involves the six products: $aabb$, $abab$, $abba$, $baab$, $baba$, and $bbaa$. Thus, it is natural to use this notation.

Andreas Freiherr von Ettingshausen (1796-1878) was a German mathematician and physicist who in 1826 introduced the notation $\left(\begin{array}{c} n \\ r \end{array}\right)$. However, the binomial coefficients were known by the Hindus centuries beforehand.

So, we have found that

$$(a+b)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) a^{n-r}b^r. \tag{A.108}$$

Now consider the geometric series $1+x+x^2+\ldots$. We have seen that such this geometric series converges for $|x|<1$, giving

$$1+x+x^2+\ldots = \frac{1}{1-x}.$$

But, $\frac{1}{1-x} = (1-x)^{-1}$. This is a binomial to a power, but the power is not an integer.

It turns out that the coefficients of such a binomial expansion can be written similar to the form in Equation(A.108). This example suggests that our sum may no longer be finite. So, for p a real number, $a = 1$ and $b = x$, we generalize Equation(A.108) as

$$(1+x)^p = \sum_{r=0}^{\infty} \begin{pmatrix} p \\ r \end{pmatrix} x^r \tag{A.109}$$

and see if the resulting series makes sense. However, we quickly run into problems with the coefficients in the series.

Consider the coefficient for $r = 1$ in an expansion of $(1+x)^{-1}$. This is given by

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{(-1)!}{(-1-1)!1!} = \frac{(-1)!}{(-2)!1!}.$$

But what is $(-1)!$? By definition, it is

$$(-1)! = (-1)(-2)(-3)\cdots.$$

This product does not seem to exist! But with a little care, we note that

$$\frac{(-1)!}{(-2)!} = \frac{(-1)(-2)!}{(-2)!} = -1.$$

So, we need to be careful not to interpret the combinatorial coefficient literally. There are better ways to write the general binomial expansion. We can write the general coefficient as

$$\begin{aligned} \begin{pmatrix} p \\ r \end{pmatrix} &= \frac{p!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)(p-r)!}{(p-r)!r!} \\ &= \frac{p(p-1)\cdots(p-r+1)}{r!}. \end{aligned} \tag{A.110}$$

With this in mind we now state the theorem:

General Binomial Expansion

The general binomial expansion for $(1+x)^p$ is a simple generalization of Equation (A.108). For p real, we have the following binomial series:

$$(1+x)^p = \sum_{r=0}^{\infty} \frac{p(p-1)\cdots(p-r+1)}{r!} x^r, \quad |x| < 1. \tag{A.111}$$

Often in physics we only need the first few terms for the case that $x \ll 1$:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + O(x^3). \tag{A.112}$$

Example A.37. Approximate $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ for $v \ll c$.

For $v \ll c$ the first approximation is found inserting $v/c = 0$. Thus, one obtains $\gamma = 1$. This is the Newtonian approximation and does not provide enough of an approximation for terrestrial speeds. Thus, we need to expand γ in powers of v/c.

The factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ is important in special relativity. Namely, this is the factor relating differences in time and length measurements by observers moving relative inertial frames. For terrestrial speeds, this gives an appropriate approximation.

First, we rewrite γ as

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}.$$

Using the binomial expansion for $p = -1/2$, we have

$$\gamma \approx 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) = 1 + \frac{v^2}{2c^2}.$$

Example A.38. Time Dilation Example

The average speed of a large commercial jet airliner is about 500 mph. If you flew for an hour (measured from the ground), then how much younger would you be than if you had not taken the flight, assuming these reference frames obeyed the postulates of special relativity?

This is the problem of time dilation. Let Δt be the elapsed time in a stationary reference frame on the ground and $\Delta\tau$ be that in the frame of the moving plane. Then from the Theory of Special Relativity these are related by

$$\Delta t = \gamma \Delta\tau.$$

The time differences would then be

$$\begin{aligned} \Delta t - \Delta\tau &= (1 - \gamma^{-1})\Delta t \\ &= \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)\Delta t. \end{aligned} \tag{A.113}$$

The plane speed, 500 mph, is roughly 225 m/s and $c = 3.00 \times 10^8$ m/s. Since $V \ll c$, we would need to use the binomial approximation to get a nonzero result.

$$\begin{aligned} \Delta t - \Delta\tau &= \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)\Delta t \\ &= \left(1 - \left(1 - \frac{v^2}{2c^2} + \ldots\right)\right)\Delta t \\ &\approx \frac{v^2}{2c^2}\Delta t \\ &= \frac{(225)^2}{2(3.00 \times 10^8)^2}(1 \text{ h}) = 1.01 \text{ ns}. \end{aligned} \tag{A.114}$$

Thus, you have aged one nanosecond less than if you did not take the flight.

Example A.39. Small differences in large numbers: Compute $f(R,h) = \sqrt{R^2+h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m.

Inserting these values into a scientific calculator, one finds that

$$f(6378164, 1) = \sqrt{6378164^2+1} - 6378164 = 1 \times 10^{-7} \text{ m}.$$

In some calculators one might obtain 0, in other calculators, or computer algebra systems like Maple, one might obtain other answers. What answer do you get and how accurate is your answer?

The problem with this computation is that $R \gg h$. Therefore, the computation of $f(R,h)$ depends on how many digits the computing device can handle. The best way to get an answer is to use the binomial approximation. Writing $h = Rx$, or $x = \frac{h}{R}$, we have

$$\begin{aligned} f(R,h) &= \sqrt{R^2+h^2} - R \\ &= R\sqrt{1+x^2} - R \\ &\simeq R\left[1+\frac{1}{2}x^2\right] - R \\ &= \frac{1}{2}Rx^2 \\ &= \frac{1}{2}\frac{h}{R^2} = 7.83926 \times 10^{-8} \text{ m}. \end{aligned} \quad \text{(A.115)}$$

Of course, you should verify how many digits should be kept in reporting the result.

In the next examples, we generalize this example. Such general computations appear in proofs involving general expansions without specific numerical values given.

Example A.40. Obtain an approximation to $(a+b)^p$ when a is much larger than b, denoted by $a \gg b$.

If we neglect b then $(a+b)^p \simeq a^p$. How good of an approximation is this? This is where it would be nice to know the order of the next term in the expansion. Namely, what is the power of b/a of the first neglected term in this expansion?

In order to do this we first divide out a as

$$(a+b)^p = a^p\left(1+\frac{b}{a}\right)^p.$$

Now we have a small parameter, $\frac{b}{a}$. According to what we have seen earlier, we can use the binomial expansion to write

$$\left(1+\frac{b}{a}\right)^n = \sum_{r=0}^{\infty}\begin{pmatrix} p \\ r \end{pmatrix}\left(\frac{b}{a}\right)^r. \quad \text{(A.116)}$$

Thus, we have a sum of terms involving powers of $\frac{b}{a}$. Since $a \gg b$, most of these terms can be neglected. So, we can write

$$\left(1+\frac{b}{a}\right)^p = 1 + p\frac{b}{a} + O\left(\left(\frac{b}{a}\right)^2\right).$$

Here we used $O()$, *big-Oh* notation, to indicate the size of the first neglected term.

Summarizing, we have

$$\begin{aligned}(a+b)^p &= a^p\left(1+\frac{b}{a}\right)^p \\ &= a^p\left(1+p\frac{b}{a}+O\left(\left(\frac{b}{a}\right)^2\right)\right) \\ &= a^p+pa^p\frac{b}{a}+a^pO\left(\left(\frac{b}{a}\right)^2\right). \qquad \text{(A.117)}\end{aligned}$$

Therefore, we can approximate $(a+b)^p \simeq a^p + pba^{p-1}$, with an error on the order of b^2a^{p-2}. Note that the order of the error does not include the constant factor from the expansion. We could also use the approximation that $(a+b)^p \simeq a^p$, but it is not typically good enough in applications because the error in this case is of the order ba^{p-1}.

Example A.41. Approximate $f(x) = (a+x)^p - a^p$ for $x \ll a$.

In an earlier example we computed $f(R,h) = \sqrt{R^2+h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m. We can make use of the binomial expansion to determine the behavior of similar functions in the form $f(x) = (a+x)^p - a^p$. Inserting the binomial expression into $f(x)$, we have as $\frac{x}{a} \to 0$ that

$$\begin{aligned}f(x) &= (a+x)^p - a^p \\ &= a^p\left[\left(1+\frac{x}{a}\right)^p - 1\right] \\ &= a^p\left[\frac{px}{a}+O\left(\left(\frac{x}{a}\right)^2\right)\right] \\ &= O\left(\frac{x}{a}\right) \quad \text{as } \frac{x}{a}\to 0. \qquad \text{(A.118)}\end{aligned}$$

This result might not be the approximation that we desire. So, we could back up one step in the derivation to write a better approximation as

$$(a+x)^p - a^p = a^{p-1}px + O\left(\left(\frac{x}{a}\right)^2\right) \quad \text{as } \frac{x}{a}\to 0.$$

We now use this approximation to compute $f(R,h) = \sqrt{R^2+h^2} - R$ for $R = 6378.164$ km and $h = 1.0$ m in the earlier example. We let $a = R^2$, $x = 1$ and $p = \frac{1}{2}$. Then, the leading order approximation would be of order

$$O\left(\left(\frac{x}{a}\right)^2\right) = O\left(\left(\frac{1}{6378164^2}\right)^2\right) \sim 2.4\times 10^{-14}.$$

Thus, we have

$$\sqrt{6378164^2+1} - 6378164 \approx a^{p-1}px$$

where

$$a^{p-1}px = (6378164^2)^{-1/2}(0.5)1 = 7.83926 \times 10^{-8}.$$

This is the same result we had obtained before. However, we have also an estimate of the size of the error and this might be useful in indicating how many digits we should trust in the answer.

Problems

1. Prove the following identities using only the definitions of the trigonometric functions, the Pythagorean identity, or the identities for sines and cosines of sums of angles.

a. $\cos 2x = 2\cos^2 x - 1$.

b. $\sin 3x = A\sin^3 x + B\sin x$, for what values of A and B?

c. $\sec\theta + \tan\theta = \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)$.

2. Determine the exact values of

a. $\sin\frac{\pi}{8}$.

b. $\tan 15^o$.

c. $\cos 105^o$.

3. Denest the following if possible.

a. $\sqrt{3 - 2\sqrt{2}}$.

b. $\sqrt{1 + \sqrt{2}}$.

c. $\sqrt{5 + 2\sqrt{6}}$.

d. $\sqrt[3]{\sqrt{5} + 2} - \sqrt[3]{\sqrt{5} - 2}$.

e. Find the roots of $x^2 + 6x - 4\sqrt{5} = 0$ in simplified form.

4. Determine the exact values of

a. $\sin\left(\cos^{-1}\frac{3}{5}\right)$.

b. $\tan\left(\sin^{-1}\frac{x}{7}\right)$.

c. $\sin^{-1}\left(\sin\frac{3\pi}{2}\right)$.

5. Do the following.

a. Write $(\cosh x - \sinh x)^6$ in terms of exponentials.

b. Prove $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$ using the exponential forms of the hyperbolic functions.

c. Prove $\cosh 2x = \cosh^2 x + \sinh^2 x$.

d. If $\cosh x = \frac{13}{12}$ and $x < 0$, find $\sinh x$ and $\tanh x$.

e. Find the exact value of $\sinh(\operatorname{arccosh} 3)$.

6. Prove that the inverse hyperbolic functions are the following logarithms:

a. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$.

b. $\tanh^{-1} x = \dfrac{1}{2} \ln \dfrac{1+x}{1-x}$.

7. Write the following in terms of logarithms:

a. $\cosh^{-1} \frac{4}{3}$.

b. $\tanh^{-1} \frac{1}{2}$.

c. $\sinh^{-1} 2$.

8. Solve the following equations for x.

a. $\cosh(x + \ln 3) = 3$.

b. $2 \tanh^{-1} \frac{x-2}{x-1} = \ln 2$.

c. $\sinh^2 x - 7 \cosh x + 13 = 0$.

9. Compute the following integrals.

a. $\int x e^{2x^2}\, dx$.

b. $\int_0^3 \dfrac{5x}{\sqrt{x^2+16}} dx$.

c. $\int x^3 \sin 3x\, dx$. (Do this using integration by parts, the Tabular Method, and differentiation under the integral sign.)

d. $\int \cos^4 3x\, dx$.

e. $\int_0^{\pi/4} \sec^3 x\, dx$.

f. $\int e^x \sinh x\, dx$.

g. $\int \sqrt{9 - x^2}\, dx$

h. $\int \dfrac{dx}{(4-x^2)^2}$, using the substitution $x = 2\tanh u$.

i. $\int_0^4 \dfrac{dx}{\sqrt{9+x^2}}$, using a hyperbolic function substitution.

j. $\int \dfrac{dx}{1-x^2}$, using the substitution $x = \tanh u$.

k. $\int \dfrac{dx}{(x^2+4)^{3/2}}$, using the substitutions $x = 2\tan\theta$ and $x = 2\sinh u$.

l. $\int \dfrac{dx}{\sqrt{3x^2 - 6x + 4}}$.

10. Find the sum for each of the series:

a. $5 + \frac{25}{7} + \frac{125}{49} + \frac{625}{343} + \cdots$.

b. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 3}{4^n}$.

c. $\sum_{n=2}^{\infty} \dfrac{2}{5^n}$.

d. $\sum_{n=-1}^{\infty}(-1)^{n+1}\left(\frac{e}{\pi}\right)^n$.

e. $\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$.

f. $\sum_{n=1}^{\infty}\frac{3}{n(n+3)}$.

g. What is $0.56\bar{9}$?

11. A superball is dropped from a 2.00 m height. After it rebounds, it reaches a new height of 1.65 m. Assuming a constant coefficient of restitution, find the (ideal) total distance the ball will travel as it keeps bouncing.

12. Here are some telescoping series problems.

a. Verify that

$$\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)}=\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}-\frac{n}{n+1}\right).$$

b. Find the nth partial sum of the series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}-\frac{n}{n+1}\right)$ and use it to determine the sum of the resulting *telescoping* series.

c. Sum the series $\sum_{n=1}^{\infty}\left[\tan^{-1}n-\tan^{-1}(n+1)\right]$ by first writing the Nth partial sum and then computing $\lim_{N\to\infty}s_N$.

13. Determine the radius and interval of convergence of the following infinite series:

a. $\sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^n}{n}$.

b. $\sum_{n=1}^{\infty}\frac{x^n}{2^n n!}$.

c. $\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x}{5}\right)^n$.

d. $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{\sqrt{n}}$.

14. Find the Taylor series centered at $x=a$ and its corresponding radius of convergence for the given function. In most cases, you need not employ the direct method of computation of the Taylor coefficients.

a. $f(x)=\sinh x$, $a=0$.

b. $f(x)=\sqrt{1+x}$, $a=0$.

c. $f(x)=\ln\frac{1+x}{1-x}$, $a=0$.

d. $f(x)=xe^x$, $a=1$.

e. $f(x)=\frac{1}{\sqrt{x}}$, $a=1$.

f. $f(x)=x^4+x-2$, $a=2$.

g. $f(x)=\frac{x-1}{2+x}$, $a=1$.

15. Consider Gregory's expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

a. Derive Gregory's expansion by using the definition

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2},$$

expanding the integrand in a Maclaurin series, and integrating the resulting series term by term.

b. From this result, derive Gregory's series for π by inserting an appropriate value for x in the series expansion for $\tan^{-1} x$.

16. In the event that a series converges uniformly, one can consider the derivative of the series to arrive at the summation of other infinite series.

a. Differentiate the series representation for $f(x) = \frac{1}{1-x}$ to sum the series $\sum_{n=1}^{\infty} nx^n$, $|x| < 1$.

b. Use the result from part a to sum the series $\sum_{n=1}^{\infty} \frac{n}{5^n}$.

c. Sum the series $\sum_{n=2}^{\infty} n(n-1)x^n$, $|x| < 1$.

d. Use the result from part c to sum the series $\sum_{n=2}^{\infty} \frac{n^2 - n}{5^n}$.

e. Use the results from this problem to sum the series $\sum_{n=4}^{\infty} \frac{n^2}{5^n}$.

17. Evaluate the integral $\int_0^{\pi/6} \sin^2 x \, dx$ by doing the following:

a. Compute the integral exactly.

b. Integrate the first three terms of the Maclaurin series expansion of the integrand and compare with the exact result.

18. Determine the next term in the time dilation example, A.38. That is, find the $\frac{v^4}{c^2}$ term and determine a better approximation to the time difference of 1 ns.

19. Evaluate the following expressions at the given point. Use your calculator or your computer (such as Maple). Then use series expansions to find an approximation to the value of the expression to as many places as you trust.

a. $\frac{1}{\sqrt{1+x^3}} - \cos x^2$ at $x = 0.015$.

b. $\ln \sqrt{\frac{1+x}{1-x}} - \tan x$ at $x = 0.0015$.

c. $f(x) = \frac{1}{\sqrt{1+2x^2}} - 1 + x^2$ at $x = 5.00 \times 10^{-3}$.

d. $f(R,h) = R - \sqrt{R^2 + h^2}$ for $R = 1.374 \times 10^3$ km and $h = 1.00$ m.

e. $f(x) = 1 - \frac{1}{\sqrt{1-x}}$ for $x = 2.5 \times 10^{-13}$.