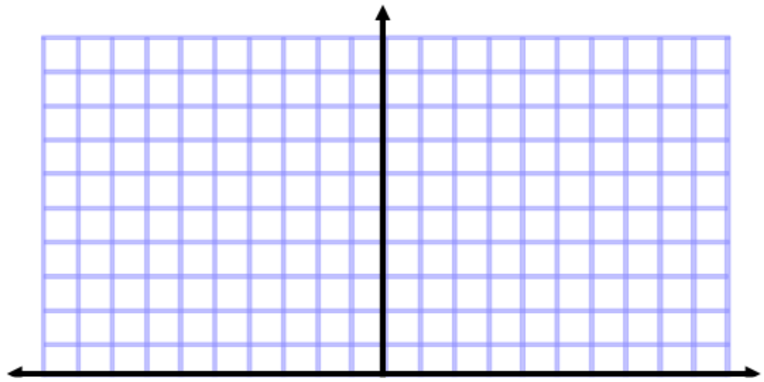
MAT 519 Practice Problems for Midterm Exam

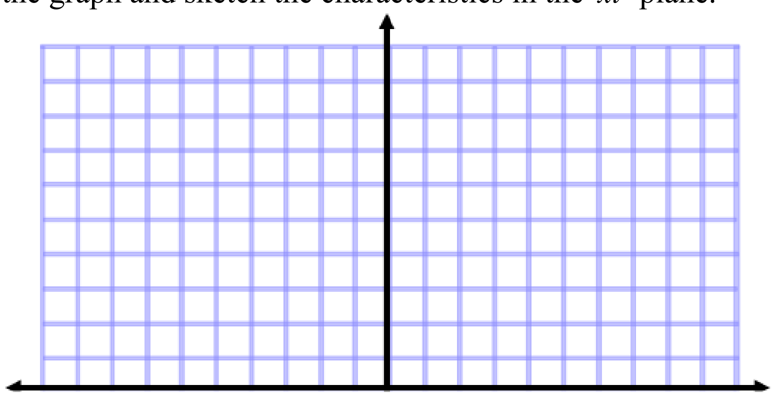
1. Recall the Green's function $G(x,\xi)=\begin{cases} C\xi(1-x), & 0\le\xi\le x \\ Cx(1-\xi), & x\le\xi\le 1\end{cases}$ is used to solve the problem $y''=f(x)$ for some specific boundary conditions.
   a. What are the boundary conditions?
   b. Prove that the Green's function is symmetric.
   c. Determine C.
2. Consider the boundary value problem $\frac{d^2y}{dx^2}=6\sin\frac{3x}{2},\ y(0)=0,\ y'(\pi)=0.$
   a. Solve this problem by direct integration.
   b. Determine the Green's function for this problem, $\frac{\partial^2 G(x,\xi)}{\partial x^2}=\delta(x-\xi),$

      $G(0,\xi)=0,\ \frac{\partial}{\partial x}G(\pi,\xi)=0.$
   c. Use the Green's function to solve the nonhomogeneous problem.
   d. Find the eigenfunctions of the corresponding eignenvalue problem, $\phi''(x)+\lambda\phi(x)=0,\quad \phi(0)=0,\ \phi'(\pi)=0.$
   e. Use these eigenfunctions to solve the nonhomogeneous problem; i.e., assume that $y(x)=\sum c_n\phi_n(x)$ and find the expansion coefficients.
3. Consider the boundary value problem $x^2y''-2xy+2y=0,\quad y(1)=0,\ y(2)=1.$
   a. Solve the given problem.
   b. Using what you have learned in this course, describe how you would solve the nonhomogeneous problem: $x^2y''-2xy'+2y=f(x),\quad y(1)=0,\ y(2)=1.$
4. Use the Method of Variation of Parameters to solve: $y''+4y=\sec 2x$
5. Reduce $u_t=ku_{xx}+Q(x,t),\ u(x,0)=f(x)$ to homogeneous boundary conditions if $u(0,t)=A(t),\ u_x(L,t)=B(t).$
6. Solve $u_t=ku_{xx}+Q(x,t),\ u(x,0)=f(x),\ u(0,t)=A(t),\ u_x(L,t)=B(t),$ using an eigenfunction expansion.
7. Solve the problem: $\nabla^2u=e^{2y}\sin x,\ u(0,y)=0,\ u(\pi,y)=0,\ u(x,0)=0,$ $u(x,1)=f(x).$
8. Find the Green's function for $\nabla^2u=F(x,y),\ u(0,y)=0,\ u(\pi,y)=0,\ u(x,0)=0,$ $u(x,1)=f(x).$ From Green's identity, derive the representation of the solution to this problem.
9. Consider the problem $u_t+(1+u)u_x=0,\ u(x,0)=\begin{cases} 1, & x\le 0, \\ 1-x, & 0<x<1, \\ 0, & x\ge 1.\end{cases}$
   a. Use the Method of Characteristics to obtain the general solution

b. Label the graph and sketch the initial condition $u(x,0)$.



c. Label the graph and sketch the characteristics in the xt-plane.



d. At what time does the shock form? $t =$ ____________

e. Determine the shock path, $x_s(t)$.

10. A general PDE for $u = u(x,y)$ takes the form $F(x,y,u,p,q) = 0$, where $p = u_x$ and $q = u_y$. The Charpit equations for solving this equation are given by

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}.$$

For the special case that $F = F(p,q)$, find the most general form for the solution, $u = u(x,y)$.

11. Solve the following equations for $u(x,y)$:

a. $u_x - 2u_y = u,\ u(0,y) = y.$

b. $xu_x + yu_y = u.$