MAT 519 Final Exam Practice Problems

1. Find the Laplace transform:
   a. $\mathcal{L}\left[e^{-5t}\sin 2t\right] =$

   b. $\mathcal{L}\left[2e^{4t} + te^{-2t}\right] =$

2. Find the inverse Laplace transform:
   a. $\mathcal{L}^{-1}\left[\dfrac{s}{(s-2)(s+3)}\right] =$

   b. $\mathcal{L}^{-1}\left[\dfrac{2}{s^2 + 4s + 20}\right] =$

3. Find the inverse Laplace transforms in the last problem using Bromwich integrals.

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4. Solve $y''-2y'+3y=t,\ y(0)=1,\ y'(0)=2,$ using the Laplace transform.

5. Using contour integration, prove that $L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at.$

6. Consider the one-dimensional partial differential equation $u_t+u+u_{xxx}-u_{xx}=0.$
   a. Find the dispersion relation.

   b. Determine the group and phase velocities.

   c. Use the dispersion relation to write the general solution as a Fourier transforms integral.

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7. Consider the following problem: $$u_{xx} + u_{yy} = u,\ 0 \le y \le 1,\ x \ge 0,$$ $$u_x(0, y) = g(y),\ u_y(x,0) = 0,\ u_y(x,1) = 0.$$
   a. Pick an appropriate transform and convert the PDE to an ODE.

   b. What are the boundary conditions for the ODE?

   c. Describe how one would proceed to now obtain the solution to the original problem.

8. The vertical deflection of an infinite beam under a load of $f(x)$ satisfies an equation of the form $\frac{d^4u}{dx^4} + \alpha u = f\left(x\right)$. Assuming that u, u', u'', u''' approach zero as $|x| \to \infty$, use Fourier transforms to obtain the solution in the form $u(x) = \int_{-\infty}^{\infty} f(\xi)G(\xi, x)\, d\xi.$

9. Find a bilinear transformation which takes the unit circle into the real axis.

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10. Find the Fourier cosine transform, $\hat{f}(\omega) = \int_0^\infty f(x)\cos\omega x\, dx$, of the following:

a. $f(x) = \dfrac{1}{x^2 + 4}$.

b. $\dfrac{d^2u}{dx^2}$ [State any needed assumptions.]

11. Let $F(z) = i \ln \dfrac{z+i}{z-i}$.

a. Find the equipotential lines and streamlines.

b. Describe these curves and the type of flow associated with them.

12. Develop a finite difference approximation to the beam problem for a finite length beam.

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13. Consider the finite difference scheme, $A\mathbf{u}^{(m+1)} = B\mathbf{u}^{(m)}$, which produces the vector $\mathbf{u}^{(m+1)}$ at time step $m+1$ from $\mathbf{u}^{(m)} = (u_1^m, \ldots, u_N^m)$ where A and B are tridiagonal matrices with diagonal entries $(-r, 1+2r, -r)$ and $(r, 1-2r, r)$, respectively. Here $r = \dfrac{2\Delta t}{\Delta x^2}$.

    a. What PDE is being solved?
    b. What is the truncation error?
    c. When is this scheme stable?

14. Do the following integrals:

    a. $\displaystyle\oint_{|z+1|=1} \frac{e^{\pi z}}{2z^2 + z - 1}\, dz$

    b. $\displaystyle\int_C (\overline{z} + z)\, dz$ for the semicircle $z = 2e^{i\theta}$, $0 \le \theta \le \pi$.

    c. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1}\, dx$

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15. Do the following integral: $\oint\limits_{|z|=3} \frac{z^2+1}{z^2(z-2)}\,dz$

16. Consider the function $u(x, y) = x^2 - x - y^2$. Find the harmonic conjugate $v(x, y)$.

17. Compute $\int\limits_C \frac{dz}{z^2}$ for C a positively oriented arc of radius 2 from $z = 2i$ to $z = -2$.

18. Consider $f(z) = \cos z$

   a. Find the real and imaginary parts in the simplest form; i.e., find functions *u(x,y)* and *v(x,y)* such that $f(x+iy) = u(x, y) + iv(x, y)$.

   b. Show that $u(x, y)$ satisfies Laplace's equation.

   c. Is $f(z) = \cos z$ differentiable. If so, find the derivative.

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19. Let C be the ellipse $9x^2+4y^2=36$ traversed once in the counterclockwise direction. Define $g(z)=\int_C \frac{s^2+s+1}{s-z}\,ds$. Find $g(i)$ and $g(4i)$.

20. Find the Laurent series expansion for $f(z)=\frac{\cosh z}{z^3}$ about $z=0$.

21. Find a bilinear transformation which takes the unit circle into the real axis.

22. Find series representations for all indicated regions: $f(z)=\frac{z}{z-1}$, $|z|<1, |z|>1$.

23. Find series representations for $f(z)=\frac{1}{(2z-1)(z+4)}$ convergent in different regions of the complex plane not including the points $z=\frac{1}{2},-4$.