Green's Function Cheat Sheet

PDE II Spring 2021, Dr. R. L. Herman

ODEs

Linear ODE:

$a(t)y''(t)+b(t)y'(t)+c(t)y(t) = f(t),$

Sturm-Liouville Form:

$$\mathcal{L}[y] = \frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)y(x) = f(x),$$

$$p(x) = \exp\left[\int^x \frac{b(\xi)}{a(\xi)}\,d\xi\right]$$

Initial Value Green's Function

$y(0) = y_0, y'(0) = v_0,$

$$y(t) = y_h(t)+\int_0^t G(t,\tau)f(\tau)\,d\tau,$$

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}$$

$y_1, y_2, y_h =$ solutions of

$a(t)y''(t)+b(t)y'(t)+c(t)y(t) = 0,$

$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0,$

$y_h(0) = y_0, y_h'(0) = v_0.$

Boundary Value Green's Function

$\mathcal{L}[y] = f, a < x < b, \quad y(a) = 0, y(b) = 0.$

$$y(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi,$$

$$G(x,\xi) = \begin{cases} \dfrac{y_1(\xi)y_2(x)}{pW}, & a \leq \xi \leq x, \\ \dfrac{y_1(x)y_2(\xi)}{pW}, & x \leq \xi \leq b, \end{cases}$$

$\mathcal{L}[y_1] = 0,\ \mathcal{L}[y_2] = 0.$
$y_1(a) = 0,\ y_2(b) = 0$
$y_1(b) \neq 0,\ y_2(a) \neq 0.$

$$\begin{aligned} W(y_1,y_2) &= \begin{vmatrix} y1(x) & y_2(x) \\ y1'(x) & y_2'(x) \end{vmatrix} \\ &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \end{aligned}$$

BVP Green's Function Properties

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right)+q(x)G(x,\xi) = 0, x \neq \xi$$

$G(x,\xi)$ satisfies homogeneous BCs.
$G(x,\xi) = G(\xi,x)$
$G(\xi^+,\xi) = G(\xi^-,\xi)$

$$\left.\frac{\partial G(x,\xi)}{\partial x}\right|_{\xi^+} - \left.\frac{\partial G(x,\xi)}{\partial x}\right|_{\xi^-} = \frac{1}{p(\xi)}$$

$$\begin{aligned} y(\xi) &= \int_a^b f(x)G(x,\xi)\,dx \\ &\quad - \left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi) - G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b}. \end{aligned}$$

Series Representation

Eigenfunctions:

$\mathcal{L}[\phi_n] = -\lambda_n\sigma\phi_n, \quad n = 1,2,\ldots$

$$G(x,\xi) = \sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n}.$$

$N_n = \int_a^b \phi_n^2(x)\,dx$
if $\lambda_k = 0$ need a Generalized Green's Function

PDEs

Heat Equation - Fixed BCs

$$\begin{aligned} u_t &= ku_{xx}, \quad 0 < t, \quad 0 \leq x \leq L, \\ u(x,0) &= f(x), \quad 0 < x < L, \\ u(0,t) &= 0, \quad t > 0, \\ u(L,t) &= 0, \quad t > 0. \end{aligned}$$

$$u(x,t) = \int_0^L G(x,\xi;t,0)f(\xi)\,d\xi.$$

$$G(x,\xi;t,0) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{\lambda_n kt}.$$

General GF: next page.

Wave Equation - Fixed BCs

$$\begin{aligned} u_{tt} &= c^2u_{xx}, \quad 0 < t, \quad 0 \leq x \leq L, \\ u(0,t) &= 0, \quad u(L,0) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L, \end{aligned}$$

$$u(x,t) = \int_0^L G_c(x,\xi,t,0)f(\xi)\,d\xi+\int_0^L G_s(x,\xi,t,0)g(\xi)\,d\xi.$$

$$\begin{aligned} G_c(x,\xi,t,0) &= \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\cos\frac{n\pi ct}{L}, \\ G_s(x,\xi,t,0) &= \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\frac{\sin\frac{n\pi ct}{L}}{n\pi c/L}. \end{aligned}$$

Poisson's Equation - Dirichlet BCs

$$\begin{aligned} \nabla^2 u &= f, \quad \text{in } D, \\ u &= g, \quad \text{on } C, \end{aligned}$$

$$\begin{aligned} \nabla^2 G &= \delta(\xi - x, \eta - y), \quad \text{in } D, \\ G &\equiv 0, \quad \text{on } C. \end{aligned}$$

$$\begin{aligned} u(x,y) &= \int_D G(x,y;\xi,\eta)f(\xi,\eta)\,d\xi d\eta \\ &\quad + \int_C (u\nabla_{r'}G - G\nabla_{r'}u)\cdot d\mathbf{s}' \end{aligned}$$

Infinite Plane

$$G(x,y,\xi,\eta) = \frac{1}{4\pi}\ln((\xi-x)^2+(\eta-y)^2).$$

Half Plane

$$G(x,y,\xi,\eta) = \frac{1}{4\pi}\ln((\xi-x)^2+(\eta-y)^2)-\frac{1}{4\pi}\ln((\xi-x)^2+(\eta+y)^2).$$

Poisson Integral Formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)}f(\phi)\,d\phi$$

General GF:

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\frac{r}{r'\rho} = \frac{1}{4\pi}\ln\frac{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\theta' - \theta)}{1 + \rho^2\rho'^2 - 2\rho\rho'\cos(\theta' - \theta)}$$

$\rho = \sqrt{x^2 + y^2}, \qquad \rho' = \sqrt{\xi^2 + \eta^2}$

Identities

Lagrange's Identity

$u\mathcal{L}[v]-v\mathcal{L}[u]=p[uv'-u'v]$

Green's Identity

$$\int_a^b [u\mathcal{L}[v]-v\mathcal{L}[u]]\;dx=p[uv'-u'v]_a^b$$

Green's 1st Theorem

$$\oint_S \varphi\nabla\chi\cdot\hat{\mathbf{n}}\,dS=\int_V(\nabla\varphi\cdot\nabla\chi+\varphi\nabla^2\chi)\,dV.$$

Green's 2nd Theorem

$$\int_V(\varphi\nabla^2\chi-\chi\nabla^2\varphi)\,dV=\oint_S(\varphi\nabla\chi-\chi\nabla\varphi)\cdot\hat{\mathbf{n}}\,dS.$$

Laplace's Equation

$\nabla^2\psi=0.$
Boundary Conditions
Dirichlet: ψ given on S.
Neumann: $\frac{\partial\psi}{\partial n}$ given S.
If Neumann on all S, need

$$\int\frac{\partial\psi}{\partial n}\,dS=0.$$

Dirichlet conditions

$$\nabla^2 g_D(\vec{\mathbf{r}},\vec{\mathbf{r}}')=0,$$

$$g_D(\vec{\mathbf{r}}_s,\vec{\mathbf{r}}'_s)=\delta^{(2)}(\vec{\mathbf{r}}_s-\vec{\mathbf{r}}'_s),$$

$$\psi(\vec{\mathbf{r}})=\int g_D(\vec{\mathbf{r}},\vec{\mathbf{r}}'_{\mathbf{s}})\psi(\vec{\mathbf{r}}'_s)\,dS'.$$

Neumann conditions

$$\nabla^2 g_N(\vec{\mathbf{r}},\vec{\mathbf{r}}')=0,$$

$$\frac{\partial g_N}{\partial n}(\vec{\mathbf{r}}_s,\vec{\mathbf{r}}'_s)=\delta^{(2)}(\vec{\mathbf{r}}_s-\vec{\mathbf{r}}'_s),$$

$$\psi(\vec{\mathbf{r}})=\int g_N(\vec{\mathbf{r}},\vec{\mathbf{r}}'_s)\frac{\partial\psi}{\partial n}(\vec{\mathbf{r}}'_s)\,dS'.$$

Eigenfunction Expansions

$$\begin{aligned} u_t &= ku_{xx}+Q(x,t),\quad 0<x<L,\quad t>0,\\ u(0,t) &= 0,\quad u(L,t)=0,\quad t>0,\\ u(x,0) &= f(x),\quad 0<x<L.\end{aligned}$$

$$\begin{aligned} u(x,t) &= \sum_{n=1}^\infty a_n(t)\phi_n(x)\\ f(x) &= \sum_{n=1}^\infty a_n(0)\phi_n(x),\\ Q(x,t) &= \sum_{n=1}^\infty q_n(t)\phi_n(x).\end{aligned}$$

$$\begin{aligned} u(x,t) &= \sum_{n=1}^\infty a_n(t)\phi_n(x)\\ &= \sum_{n=1}^\infty\left[a_n(0)e^{-k\lambda_n t}+\int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau\right]\phi_n(x).\end{aligned}$$

Forced Vibrating Membrane

$$\begin{aligned} u_{tt} &= c^2\nabla^2u+Q(\mathbf{r},t),\quad \mathbf{r}\in D,\quad t>0,\\ u(\mathbf{r},t) &= 0,\quad \mathbf{r}\in\partial D,\quad t>0,\\ u(\mathbf{r},0) &= f(\mathbf{r}),\quad u_t(\mathbf{r},0)=g(\mathbf{r}),\quad \mathbf{r}\in D.\end{aligned}$$

$$\begin{aligned} u(\mathbf{r},t) &= \sum_{\alpha\in J} a_\alpha(t)\phi_\alpha(\mathbf{r}),\\ Q(\mathbf{r},t) &= \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}).\\ f(\mathbf{r}) &= \sum_{\alpha\in J} a_\alpha(0)\phi_\alpha(\mathbf{r}),\\ g(\mathbf{r}) &= \sum_{\alpha\in J} \dot{a}_\alpha(0)\phi_\alpha(\mathbf{r}).\end{aligned}$$

$\ddot{a}_\alpha(t)+c^2\lambda_\alpha a_\alpha(t)=q_\alpha(t),$
given $a_\alpha(0),\dot{a}_\alpha(0)$
May lead to resonance.

General Green's Function

Nonhomogeneous Heat Equation

$$\begin{aligned} u(x,t) &= \int_0^L G(x,t;\xi,0)f(\xi)\,d\xi\\ &\quad+\int_0^t\int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\,d\xi\,d\tau.\end{aligned}$$

$$G(x,t;\xi,\tau)=\sum_{n=1}^\infty\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2},$$

Plus Nonhomogeneous Boundary Conditions

$u(0,t)=\alpha(t),\quad u(L,t)=\beta(t).$

$$\begin{aligned} u(x,t) &= \int_0^t\int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\,d\xi\,d\tau\\ &\quad+\int_0^L G(x,t;\xi,0)f(\xi)\,d\xi\\ &\quad+k\int_0^t\left[\alpha(\tau)\frac{\partial G}{\partial\xi}(x,0;t,\tau)-\beta(\tau)\frac{\partial G}{\partial\xi}(x,L;t,\tau)\right]\,d\tau.\end{aligned}$$

ODE Examples

$y''=f(x),\qquad y(0)=0=y(1)$

$$G(x,\xi)=\begin{cases}-\xi(1-x), & 0\le\xi\le x,\\ -x(1-\xi), & x\le\xi\le 1.\end{cases}$$

$y''+\omega^2y=f(x),\qquad y(0)=0=y(1)$

$$G(x,\xi)=\begin{cases}-\dfrac{\sin\omega(1-\xi)\sin\omega x}{\omega\sin\omega}, & 0\le x\le\xi,\\ -\dfrac{\sin\omega(1-x)\sin\omega\xi}{\omega\sin\omega}, & \xi\le x\le 1.\end{cases}$$

$y''+\omega^2y=f(x),\qquad y(0)=0=y'(1)$

$$G(x,\xi)=\begin{cases}-\dfrac{\cos\omega(1-\xi)\sin\omega x}{\omega\cos\omega}, & 0\le x\le\xi,\\ -\dfrac{\sin\omega(1-x)\cos\omega\xi}{\omega\cos\omega}, & \xi\le x\le 1.\end{cases}$$