MAT 519 Final Topics

I. Complex Analysis – Starting from Laurent Series
   a. Series Expansions
      i. Power series, Laurent series, Taylor series
      ii. Circle of convergence
      iii. Using geometric series
         1. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z|<1,$
         2. $\frac{1}{1-z} = \frac{-1}{z} \times \frac{1}{1-\frac{1}{z}} = -\sum_{n=1}^{\infty} z^{-n}, |z|>1$
   b. Singularities – removable, poles, essential
   c. Integration
      i. Cauchy's Theorem
      ii. Cauchy Integral Theorem
      iii. When can one deform contours?
      iv. Computing Residues
         1. $\text{Res}[f(z); z=z_0] = \lim_{z\to z_0}(z-z_0)f(z)$ - simple poles
         2. $\text{Res}[f(z); z=z_0] = \lim_{z\to z_0} \frac{1}{k!}\frac{d^{k-1}}{dz^{k-1}}[(z-z_0)^k f(z)]$ - poles of order k
      v. Residue Theorem $\int_C f(z)\,dz = 2p\,i \sum_{\text{Poles inside C}}$ Residues
      vi. Integrands of the form $f(\cos q, \sin q)$:

         Use $\cos q = \frac{z+z^{-1}}{2}$, $\sin q = \frac{z-z^{-1}}{2i}$
      vii. Infinite (Principal Value) Integrals
      viii. Jordan's Lemma

II. Fourier Transforms
   a. Definition $F(w) = \frac{1}{2p}\int_{-\infty}^{\infty} f(x)e^{iwx}dx$, $f(x) = \int_{-\infty}^{\infty} F(w)e^{-iwx}dw$
   b. Properties – Transforms of derivatives,
   c. Gaussian Integrals, Transform of a Gaussian
   d. Convolution Theorem $F^{-1}[F(w)G(w)] = (f*g)(x) \equiv \int_{-\infty}^{\infty} f(x)g(x-x)\,dx$
   e. Parseval's Identity $\int_{-\infty}^{\infty} |F(w)|^2\,dw = \frac{1}{2p}\int_{-\infty}^{\infty} |f(x)|^2\,dx$
   f. Fourier Sine and Cosine Transforms and Semi-infinite intervals
   g. Heat Equation Examples

III. First Order PDEs
   a. Quasilinear Equations $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$
   b. Method of Characteristics $\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} = dt$ or $\frac{dx}{dt} = a, \frac{dy}{dt} = b, \frac{du}{dt} = c,$

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c. Conservation Laws

   i. Integral form: $\frac{dQ}{dt} = f(a,t) - f(b,t) + \int_a^b f(x,t)\,dx$

   ii. Local Form: $\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = f.$

   iii. Constitutive Equation: $f = f(u) \Rightarrow u_t + \frac{df}{du}u_x = f$

d. Solutions of $u_t + c(u)u_x = 0.$

   i. Sketching Characteristic Curves

   ii. Sketching Solutions based on Characteristics

   iii. Breaking Time $t_B = \left\{-\frac{1}{F'(x)}\right\}_{\min}$ for $F(x) = c(f(x))$ and $f(x) = u(x,0)$

   iv. Rarefactions - $u(x,t) = g(\frac{x}{t})$

   v. Shock Waves – Rankine-Hugonoit Condition $\frac{dx_s}{dt} = \frac{[f]}{[u]}$ where $[u] = u^+ - u^-$

e. General Nonlinear First Order PDEs

   i. $F(x,y,u,p,q) = 0$ for $p = u_x$, $q = u_y$

   ii. Charpit Equations $\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{-dp}{F_x + pF_u} = \frac{-dq}{F_y + qF_u} = dt$