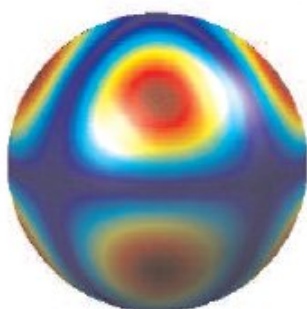
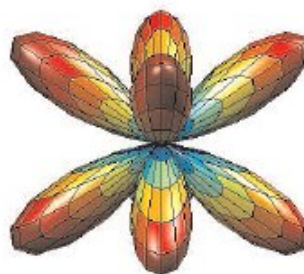
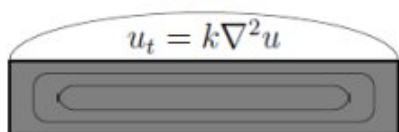
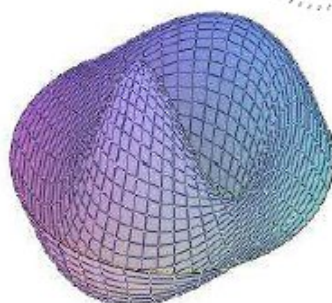
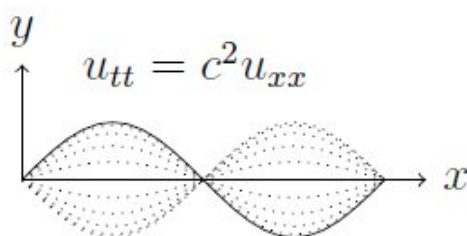


RUSSELL L. HERMAN

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS



$$\int_V (\varphi \nabla^2 \chi - \chi \nabla^2 \varphi) dV = \oint_S (\varphi \nabla \chi - \chi \nabla \varphi) \cdot \hat{\mathbf{n}} dS$$



$$T(r, z, t) = T_b + \frac{8(T_i - T_b)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{(2n-1)\pi z}{Z}}{(2n-1)} \frac{J_0\left(\frac{r}{a} j_{0m}\right) e^{-\lambda_{nm} k t}}{j_{0m} J_1(j_{0m})}$$

R. L. HERMAN - VERSION DATE: JANUARY 11, 2015

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*Dedicated to those students who have endured
previous versions of my notes.*