Solution of the Nonhomogeneous Heat Equation

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Homogeneous Problem

Consider the problem[a]

$$\begin{aligned} u_t &= ku_{xx}, \quad 0 < x < L, \quad t > 0 \\ u(0,t) &= 0, \quad u(0,t) = 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{1}$$

We use the Method of Separation of Variables.

Assuming $u(x,t) = X(x)T(t)$, we need to solve

$$X'' + \kappa^2 X = 0, \quad X(0) = X(L) = 0.$$

The solution of this eigenvalue problem is

$$X(x) = \sin\frac{n\pi x}{L}, \quad \kappa = \frac{n\pi}{L}, n = 1, 2, \ldots.$$

The general solution of (1) is then

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2}. \tag{2}$$

The Fourier coefficients, b_n, are found as

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

[a] Note - Other boundary conditions, such as insulating, mixed, or periodic, boundary conditions will lead to other solutions.

Time-Independent BCs

Assume nonhomogeneous fixed conditions:

$$\begin{aligned} u_t &= ku_{xx}, \quad 0 < x < L, \quad t > 0 \\ u(0,t) &= a, \quad u(L,t) = b, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{3}$$

We seek **steady-state** ($u_t = 0$) solutions satisfying the boundary conditions.

$$\begin{aligned} w''(x) &= 0, \quad 0 < x < L, \\ w(0) &= a, \quad w(L) = b. \end{aligned} \tag{4}$$

Therefore, $w(x) = cx + d$, and the BCs give

$$w(x) = \frac{(b-a)x}{L} + a. \tag{5}$$

If $u(x,t) = w(x) + v(x,t)$, then $v(x,t)$ solves

$$\begin{aligned} v_t &= kv_{xx}, \quad 0 < x < L, \quad t > 0 \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0 \\ v(x,0) &= f(x) - w(x), \quad 0 \le x \le L. \end{aligned} \tag{6}$$

The general solution of (3) is found as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2 kt/L^2} + \frac{(b-a)x}{L} + a, \tag{7}$$

where b_n is determined using the modified initial condition, $v(x,0) = f(x) - w(x)$.

General Problem with Time-Dependent Boundary Conditions

The nonhomogeneous heat equation with time-dependent boundary conditions is given by

$$\begin{aligned} u_t - ku_{xx} &= F(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= a(t), \quad u(L,t) = b(t), \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{8}$$

We seek solutions of the form

$$u(x,t) = v(x,t) + w(x,t),$$

where $w(x,t)$ satisfies

$$w(x,t) = [b(t) - a(t)]\frac{x}{L} + a(t) \tag{9}$$

and $v(x,t)$ satisfies a nonhomogeneous problem with homogeneous boundary conditions,

$$\begin{aligned} v_t - kv_{xx} &= F(x,t) - [b'(t) - a'(t)]\frac{x}{L} - a'(t), \\ v(0,t) &= 0, \quad v(L,t) = 0, \\ v(x,0) &= f(x) - [b(0) - a(0)]\frac{x}{L} - a(0). \end{aligned}$$

This is a nonhomogeneous heat equation with homogeneous boundary conditions.

Nonhomogeneous Heat Equation with Homogeneous BCs

The equation for $v(x,t)$ can be written in the general form

$$\begin{aligned} v_t - kv_{xx} &= h(x,t), \quad 0 < x < L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= g(x), \quad 0 \le x \le L. \end{aligned} \tag{10}$$

Once again, we split Problem (10) into two problems. Let

$$v(x,t) = u_1(x,t) + u_2(x,t),$$

where u_1 and u_2 satisfy the following two problems.

Problem for $u_1(x,t)$

$$\begin{aligned} u_{1t} - ku_{1xx} &= 0, \quad 0 < x < L, \quad t > 0, \\ u_1(0,t) &= 0, \quad u_1(L,t) = 0, \quad t > 0, \\ u_1(x,0) &= g(x), \quad 0 \le x \le L. \end{aligned}$$

This is the familiar homogeneous heat equation with homogeneous boundary conditions. The solutions are found using the Method of Separation of Variables.

Problem for $u_2(x,t)$

$$\begin{aligned} u_{2t} - ku_{2xx} &= h(x,t), \quad 0 < x < L, \quad t > 0, \\ u_2(0,t) &= 0, \quad u_2(L,t) = 0, \quad t > 0, \\ u_2(x,0) &= 0, \quad 0 \le x \le L. \end{aligned}$$

This is a nonhomogeneous heat equation with homogeneous boundary and initial conditions. We use **Duhamel's Principle** to convert this problem with a source to an initial value problem.

Solution to General Problem

From these simpler problems we form the general solution:

$$u(x,t) = u_1(x,t) + u_2(x,t) + [b(t) - a(t)]\frac{x}{L} + a(t) \tag{11}$$

Duhamel's Principle

The solution of the heat equation with a source and homogeneous boundary and initial conditions may be found by solving a homogeneous heat equation with nonhomogeneous initial conditions.

ODE Version

Let $\mathbf{X} : \mathbb{R} \to \mathbb{R}$ and $\mathbf{X}(t) = U(t)\mathbf{X}_0$ be the solution of $\dot{\mathbf{X}} = A\mathbf{X}$, $\mathbf{X}(0) = \mathbf{X}_0$.

Consider

$$\mathbf{X}(t) = \int_0^t U(t-s)\mathbf{Y}(s)\,ds.$$

$\mathbf{X}(t)$ satisfies the inhomogeneous problem

$$\left(\frac{d}{dt} - A\right)\mathbf{X} = \mathbf{Y}(s), \quad \mathbf{X}(0) = \mathbf{0}.$$

Solution for $u_2(x,t)$

Solve for $\tilde{v}(x,t;s)$ in the problem

$$\begin{aligned} \tilde{v}_t - k\tilde{v}_{xx} &= 0, \quad 0 < x < L, \quad t > 0, \\ \tilde{v}(0,t;s) &= 0, \quad \tilde{v}(L,t;s) = 0, \\ \tilde{v}(x,0;s) &= h(x,s). \end{aligned} \tag{12}$$

Then, $v(x,t;s) = \tilde{v}(x,t-s;s)$ satisfies

$$\begin{aligned} v_t - kv_{xx} &= 0, \quad 0 < x < L, \quad t \ge s, \\ v(0,t;s) &= 0, \quad v(L,t;s) = 0, \\ v(x,s;s) &= h(x,s). \end{aligned} \tag{13}$$

$v(x,t;s)$ is the solution when the source is turned on at time $t = s - \Delta s$ and turned off at $t = s$. A superposition of these incremental sources gives the solution

$$\begin{aligned} u_2(x,t) &= \int_0^t v(x,t;s)\,ds \\ &= \int_0^t \tilde{v}(x,t-s;s)\,ds. \end{aligned} \tag{14}$$

Green's Function, $G(x,y)$

The **steady state solution**, satisfying

$$\begin{aligned} -kw_{xx} &= h(x), \quad 0 < x < L, \\ w(0) &= a, \quad w(L) = b, \end{aligned} \tag{15}$$

can be found by direct integration as

$$w(x) = -\int_0^L G(x,y)\left(-\frac{1}{k}h(y)\right)dy + (b-a)\frac{x}{L} + a.$$