PDE HW Notes for Problems 9 and 11

April 16, 2024

1. Problem 9: Consider a circular cylinder of radius $R = 4.00$ cm and height $H = 20.0$ cm that obeys the steady-state heat equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0.$$

Find the temperature distribution, $u(r,z)$, given that $u(r,0) = 0°\text{C}$, $u(r,20) = 20°\text{C}$, and heat is lost through the sides due to Newton's Law of Cooling,

$$[u_r + hu]_{r=4} = 0,$$

for $h = 1.0\ \text{cm}^{-1}$.

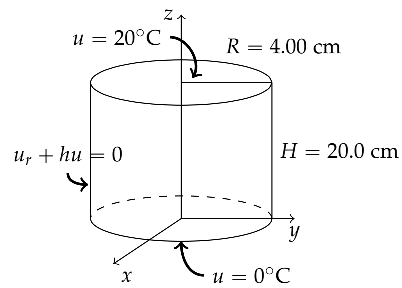

Figure 1: Geometry for the cylinder in Problem 9.

Show that separating variables gives the boundary value problems

$$Z'' - \lambda^2 Z = 0, \quad Z(0) = 0, \tag{1}$$
$$\phi'' + \frac{1}{r}\phi' + \lambda^2\phi = 0, \quad [\phi' + h\phi]_{r=R} = 0. \tag{2}$$

Equation (2) is a Bessel equation with solutions finite at the origin, $\phi(r) = J_0(\lambda r)$, satisfying

$$\lambda J_0'(\lambda R) + J_0(\lambda R) = 0. \tag{3}$$

The solution of (transcendental) Equation (3) will give the eigenvalues, λ_n. In Figure 2 we show the plot of

$$f(x) = xJ_0'(x) + J_0(x) = -xJ_1(x) + J_0(x),$$

where $x = \lambda R$. If we call the roots j_n, satisfying $-j_n J_0'(j_n) + J_0(j_n) = 0$, then the eigenvalues are given in terms of these roots, $\lambda_n = j_n/R$. You need to numerically find several of these to obtain an approximate solution.

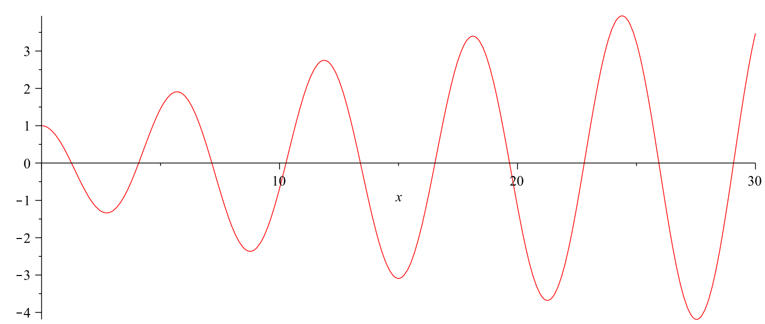

Figure 2: A plot of $f(x) = xJ_0'(x) + J_0(x) = -xJ_1(x) + J_0(x)$ in Problem 11.9 showing the location of the zeros.

Boundary value problem (1) leads to product solutions and eventually the general solution,

$$u(r,z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \sinh \lambda_n z.$$

The condition on the top of the cylinder, $u(r, H) = 20$, gives

$$20 = \sum_{n=1}^{\infty} (A_n \sinh \lambda_n H) J_0(\lambda_n r).$$

This is a Fourier-Bessel series. Finding the Fourier coefficients requires some manipulation of Green's Identity to enforce an orthogonality condition.

This is tricky for the mixed boundary conditions. So, I'll just state the following: In general, if

$$f(r) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) r\, dr$$

where $j_n J_0'(j_n) + J_0(j_n) = 0$ and $\lambda_n = j_n/R$, then the Fourier-Bessel coefficients are given by

$$a_n = \frac{2}{R^2 J_0^2(j_n)} \frac{j_n^2}{j_n^2 + R} \int_0^R f(r) J_0(\frac{j_n}{R} r) r\, dr.$$

If done correctly, you will need to integrate

$$\begin{aligned} \int_0^R J_0(\lambda_n r) r\, dr &= \frac{1}{\lambda_n^2} \int_0^{\lambda_n R} J_0(y) y\, dy \\ &= \frac{1}{\lambda_n^2} \int_0^{j_n} \frac{d}{dy} [y J_1(y)]\, dy \\ &= \frac{R^2}{j_n} J_1(j_n). \end{aligned}$$

2. Problem 11: Determine the steady-state temperature of a spherical ball maintained at the temperature

$$u(x, y, z) = x^2 + 2y^2 + 3z^2, \quad \rho = 1.$$

[Hint: Rewrite the problem in spherical coordinates and use the properties of spherical harmonics.]

From the text we have that solutions can be written as

$$u(\rho, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \rho^{\ell} P_{\ell}^{m}(\cos\theta) e^{im\phi}.$$

[One can take real part to get a real valued solution.]

At $\rho = 1$, we have

$$x^2 + 2y^2 + 3z^2 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} P_{\ell}^{m}(\cos\theta) e^{im\phi}.$$

So, you need only write the Cartesian form of the initial condition in terms of associated Legendre functions. Write x, y, z in spherical coordinates and write out a few terms of the right hand side in terms of trigonometric functions.