PDE HW Notes for Problems 9, 11, and 7

November 19, 2024

1. Problem 9: Consider a circular cylinder of radius $R = 4.00$ cm and height $H = 20.0$ cm that obeys the steady-state heat equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0.$$

Find the temperature distribution, $u(r,z)$, given that $u(r,0) = 0°\text{C}$, $u(r,20) = 20°\text{C}$, and heat is lost through the sides due to Newton's Law of Cooling,

$$[u_r + hu]_{r=4} = 0,$$

for $h = 1.0\ \text{cm}^{-1}$.

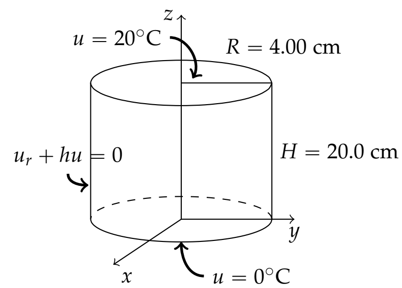

Figure 1: Geometry for the cylinder in Problem 9.

Show that separating variables gives the boundary value problems

$$Z'' - \lambda^2 Z = 0, \quad Z(0) = 0, \tag{1}$$

$$\phi'' + \frac{1}{r}\phi' + \lambda^2\phi = 0, \quad [\phi' + h\phi]_{r=R} = 0. \tag{2}$$

Equation (2) is a Bessel equation with solutions finite at the origin, $\phi(r) = J_0(\lambda r)$, satisfying

$$\lambda J_0'(\lambda R) + J_0(\lambda R) = 0. \tag{3}$$

The solution of (transcendental) Equation (3) will give the eigenvalues, λ_n. In Figure 2 we show the plot of

$$f(x) = xJ_0'(x) + J_0(x) = -xJ_1(x) + J_0(x),$$

where $x = \lambda R$. If we call the roots j_n, satisfying $-j_nJ_0'(j_n) + J_0(j_n) = 0$, then the eigenvalues are given in terms of these roots, $\lambda_n = j_n/R$. You need to numerically find several of these to obtain an approximate solution.

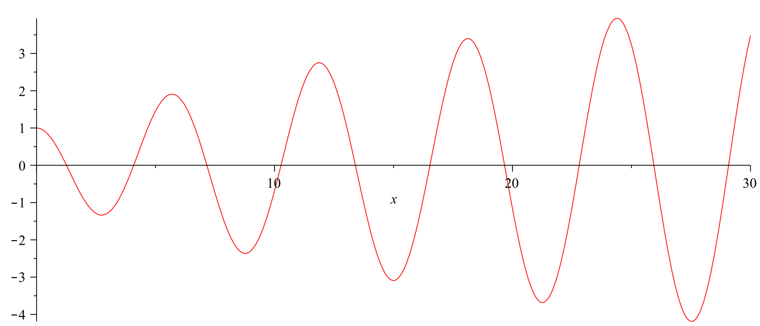

Figure 2: A plot of $f(x) = xJ_0'(x) + J_0(x) = -xJ_1(x) + J_0(x)$ in Problem 11.9 showing the location of the zeros.

Boundary value problem (1) leads to product solutions and eventually the general solution,

$$u(r,z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \sinh \lambda_n z.$$

The condition on the top of the cylinder, $u(r, H) = 20$, gives

$$20 = \sum_{n=1}^{\infty} (A_n \sinh \lambda_n H) J_0(\lambda_n r).$$

This is a Fourier-Bessel series. Finding the Fourier coefficients requires some manipulation of Green's Identity to enforce an orthogonality condition.

This is tricky for the mixed boundary conditions. So, I'll just state the following: In general, if

$$f(r) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) r\, dr$$

where $j_n J_0'(j_n) + J_0(j_n) = 0$ and $\lambda_n = j_n/R$, then the Fourier-Bessel coefficients are given by

$$a_n = \frac{2}{R^2 J_0^2(j_n)} \frac{j_n^2}{j_n^2 + R} \int_0^R f(r) J_0(\frac{j_n}{R} r) r\, dr.$$

If done correctly, you will need to integrate

$$\begin{aligned} \int_0^R J_0(\lambda_n r) r\, dr &= \frac{1}{\lambda_n^2} \int_0^{\lambda_n R} J_0(y) y\, dy \\ &= \frac{1}{\lambda_n^2} \int_0^{j_n} \frac{d}{dy} [y J_1(y)]\, dy \\ &= \frac{R^2}{j_n} J_1(j_n). \end{aligned}$$

2. Problem 11: Determine the steady-state temperature of a spherical ball maintained at the temperature

$$u(x, y, z) = x^2 + 2y^2 + 3z^2, \quad \rho = 1.$$

[Hint: Rewrite the problem in spherical coordinates and use the properties of spherical harmonics.]

From the text we have that solutions can be written as

$$u(\rho, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} \rho^{\ell} P_{\ell}^{m}(\cos\theta) e^{im\phi}.$$

[One can take real part to get a real valued solution.]

At $\rho = 1$, we have

$$x^2 + 2y^2 + 3z^2 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} P_{\ell}^{m}(\cos\theta) e^{im\phi}.$$

So, you need only write the Cartesian form of the initial condition in terms of associated Legendre functions. Write x, y, z in spherical

	$P_n^m(x)$	$P_n^m(\cos\theta)$
$P_0^0(x)$	1	1
$P_1^0(x)$	x	$\cos\theta$
$P_1^1(x)$	$-(1-x^2)^{\frac{1}{2}}$	$-\sin\theta$
$P_2^0(x)$	$\frac{1}{2}(3x^2-1)$	$\frac{1}{2}(3\cos^2\theta-1)$
$P_2^1(x)$	$-3x(1-x^2)^{\frac{1}{2}}$	$-3\cos\theta\sin\theta$
$P_2^2(x)$	$3(1-x^2)$	$3\sin^2\theta$

Table 1: Associated Legendre Functions, $P_n^m(x)$.

coordinates and write out a few terms of the right hand side in terms of trigonometric functions.

Using the the spherical coordinate transformation and Table 1 of associated Legendre functions, we have

$$\begin{aligned}
x^2+2y^2+3z^2 &= \sin^2\theta\cos^2\phi+2\sin^2\theta\sin^2\phi+3\cos^2\theta \\
&= \frac{1}{2}\sin^2\theta(1+\cos 2\phi)+\sin^2\theta(1-\cos 2\phi)+3\cos^2\theta \\
&= -\frac{1}{2}\sin^2\theta\cos 2\phi+\frac{3}{2}\sin^2\theta+3\cos^2\theta \\
&= -\frac{1}{2}\sin^2\theta\cos 2\phi+\frac{1}{2}(3\cos^2\theta-1)+2 \\
&= -\frac{1}{6}P_2^2(\cos\theta)\cos 2\phi+P_2^0(\cos\theta)+2P_0^0(\cos\theta).
\end{aligned}$$

Comparing this with the expansion for the boundary condition, we can identify a finite number of nonzero coefficients. The general solution is formed by inserting factors of ρ^ℓ in each term. Therefore, we have

$$\begin{aligned}
u(\rho,\theta,\phi) &= -\frac{1}{6}\rho^2P_2^2(\cos\theta)\cos 2\phi+\rho^2P_2^0(\cos\theta)+2P_0^0(\cos\theta) \\
&= -\frac{1}{2}\rho^2\sin^2\theta\cos 2\phi+\frac{1}{2}\rho^2(3\cos^2\theta-1)+2.
\end{aligned}$$

Another way to determine the nonzero coefficients is by computing the general Fourier coefficients. Denoting

$$u(1,\theta,\phi)=f(\theta,\phi)=\sin^2\theta\cos^2\phi+2\sin^2\theta\sin^2\phi+3\cos^2\theta,$$

the coefficients are given by

$$a_{\ell m}=\frac{\displaystyle\int_0^\pi\int_0^{2\pi} f(\theta,\phi)P_\ell^m(\cos\theta)e^{im\phi}\sin\theta\,d\phi d\theta}{\int_0^\pi\int_0^{2\pi}\left|P_\ell^m(\cos\theta)e^{im\phi}\right|^2\sin\theta\,d\phi d\theta}.$$

However, from the form of $f(\theta,\phi)$, we only need a_{00}, a_{20}, and $a_{2,\pm 2}$.

we can use Table 1 to compute the needed integrals.

$$a_{00} = \frac{\displaystyle\int_0^\pi\int_0^{2\pi} f(\theta,\phi)\sin\theta\,d\phi d\theta}{\int_0^\pi\int_0^{2\pi}\sin\theta\,d\phi d\theta}$$

$$\begin{aligned}
&= \frac{8\pi}{4\pi} = 2. && (4)\\
a_{20} &= \frac{\displaystyle\int_0^{\pi}\int_0^{2\pi} f(\theta,\phi)P_2(\cos\theta)\sin\theta\, d\phi d\theta}{\int_0^{\pi}\int_0^{2\pi} |P_2(\cos\theta)|^2 \sin\theta\, d\phi d\theta}\\
&= \frac{5}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta,\phi)P_2(\cos\theta)\sin\theta\, d\phi d\theta = 1. && (5)\\
a_{2,\pm 2} &= \frac{\displaystyle\int_0^{\pi}\int_0^{2\pi} f(\theta,\phi)P_2^2(\cos\theta)e^{\pm 2i\phi}\sin\theta\, d\phi d\theta}{\int_0^{\pi}\int_0^{2\pi} \left|P_2^2(\cos\theta)\right|^2 \sin\theta\, d\phi d\theta}\\
&= \frac{3\displaystyle\int_0^{\pi}\int_0^{2\pi} f(\theta,\phi)e^{\pm 2i\phi}\sin^3\theta\, d\phi d\theta}{18\pi\int_0^{\pi}\sin^5\theta\, d\theta}\\
&= \frac{-8\pi/15}{6\pi(16/15)} = -\frac{1}{12}. && (6)
\end{aligned}$$

Writing,$u(\rho,\theta,\phi)$ as

$$a_{00}P_0^0(\cos\theta) + a_{20}\rho^2 P_2^0(\cos\theta) + a_{22}\rho^2 P_2^2(\cos\theta)e^{2i\phi} + a_{2,-2}\rho^2 P_2^{-2}(\cos\theta)e^{-2i\phi},$$

we obtain the same solution as before.

3. **Problem 7:** A copper cube 10.0 cm on a side is heated to 100° C. The block is placed on a surface that is kept at 0° C. The sides of the block are insulated, so the normal derivatives on the sides are zero. Heat flows from the top of the block to the air governed by the gradient $u_z = -10°\mathrm{C/m}$. Determine the temperature of the block at its center after 1.0 minute. Note that the thermal diffusivity is given by $k = \frac{K}{\rho c_p}$, where K is the thermal conductivity, ρ is the density, and c_p is the specific heat capacity.

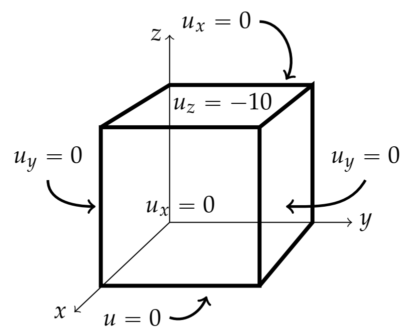

Figure 3: The cube for problem 10.7.

This is a heat conduction problem:

$$u_t = k\left(u_{xx} + u_{yy} + u_{zz}\right),$$

with nonhomogeneous boundary conditions given by ($s = 10.0$ cm)

$$\begin{aligned}
u(x,y,0,t) &= 0, \qquad & u_z(x,y,s,t) &= -0.1°\mathrm{C/cm}, \quad & 0 \leqslant x,y \leqslant s,\\
u_x(0,y,z,t) &= 0, & u_x(0,y,z,t) &= 0, & 0 \leqslant y,z \leqslant s,\\
u_y(x,0,z,t) &= 0, & u_y(x,0,z,t) &= 0, & 0 \leqslant x,z \leqslant s,
\end{aligned}$$

and the initial condition is

$$u(x,y,z,0) = 100°\mathrm{C}.$$

Convert to homogeneous boundary conditions.

We assume the solution takes the form

$$u(x,y,z,t) = v(x,y,z,t) + f(z).$$

Then, $v(x,y,z,t)$ satisfies the new boundary value problem

$$v_t = k\left[v_{xx} + v_{yy} + v_{zz} + f''(z)\right],$$

$$\begin{aligned} v(x,y,0,t) &= -f(0), & v_z(x,y,s,t) &= -f'(s) - 0.1, \\ v_x(0,y,z,t) &= 0, & v_x(0,y,z,t) &= 0, \\ v_y(x,0,z,t) &= 0, & v_y(x,0,z,t) &= 0, \end{aligned}$$

and the initial condition is given by

$$v(x,y,z,0) = 100 - f(z).$$

To make the boundary conditions homogeneous, let $f(0) = 0$ and $f'(s) = -0.1$. The heat equation will be homogeneous if $f''(z) = 0$. So, we assume that $f(z) = Az + B$. The boundary conditions give $B = 0$ and $A = -0.1$. This yields $f(z) = -0.1z$.

Now, we solve for $v(x,y,z,t)$ using separation of variables. Let $v(x,y,z,t) = X(x)Y(y)Z(z)T(t)$. Then,

$$\frac{1}{k}\frac{T'}{T} = \underbrace{\frac{X''}{X}}_{-\lambda^2} + \underbrace{\frac{Y''}{Y}}_{\mu^2} + \underbrace{\frac{Z''}{Z}}_{-\nu^2}.$$

Setting the individual spatial terms equal to the given constants, we have the set of boundary value problems

$$\begin{aligned} X'' + \lambda^2 X &= 0, \quad X'(0) = X'(s) = 0, \\ Y'' + \mu^2 Y &= 0, \quad Y'(0) = Y'(s) = 0, \\ Z'' + \nu^2 Z &= 0, \quad Z(0) = Z'(s) = 0. \end{aligned}$$

The eigenfunctions for these are

$$\begin{aligned} X_n(x) &= \cos\frac{n\pi x}{s}, \quad Y_m(y) = \cos\frac{m\pi y}{s}, \quad n, m, = 0, 1, \dots, \\ Z(z) &= \sin\frac{(2\ell+1)\pi z}{2s}, \quad \ell = 0, 1, \dots, \end{aligned}$$

and eigenvalues $\lambda_n = \frac{n\pi}{s}$, $\mu_m = \frac{m\pi}{s}$, $\nu_\ell = \frac{(2\ell+1)\pi}{2s}$. The time dependence is given by

$$\begin{aligned} T_{nm\ell}(t) &= e^{-k(\lambda_n^2 + \mu_m^2 + \nu_\ell^2)t} \\ &= e^{-(n^2+m^2+(2\ell+1)^2/4)\pi^2 kt/s^2}, \quad n, m, \ell = 0, 1, \dots. \end{aligned}$$

So, the general solution of the homogeneous problem is

$$v(x,y,z,t) = \sum_{n,m,\ell=0}^{\infty} A_{nm\ell} \cos\frac{n\pi x}{s} \cos\frac{m\pi y}{s} \sin\frac{(2\ell+1)\pi z}{2s} T_{nm\ell}(t).$$

The initial condition, $v(x,y,z,t) = 100 + 0.1z$, can be used to find the expansion coefficients,

$$100 + 0.1z = \sum_{n,m,\ell=0}^{\infty} A_{nm\ell} \cos\frac{n\pi x}{s} \cos\frac{m\pi y}{s} \sin\frac{(2\ell+1)\pi z}{2s}.$$

Since the left hand side is independent of x and y, only the $A_{00\ell}$ terms can be nonzero and the triple sum reduces to

$$100 + 0.1z = \sum_{\ell=0}^{\infty} A_{00\ell} \sin\frac{(2\ell+1)\pi z}{2s}.$$

Since

$$\int_0^s \sin^2\frac{(2\ell+1)\pi z}{2s}\,dz = \frac{s}{2},$$

we can obtain the Fourier coefficients as

$$\begin{aligned} A_{00\ell} &= \frac{2}{s}\int_0^s (100+0.1z)\sin\frac{(2\ell+1)\pi z}{2s}\,dz, \quad \ell = 0,1,\ldots \\ &= \frac{2}{s}\left[-\frac{2s}{(2\ell+1)\pi}(100+0.1z)\cos\frac{(2\ell+1)\pi z}{2s}\right. \\ &\quad \left. + \frac{0.4s^2}{(2\ell+1)^2\pi^2}\sin\frac{(2\ell+1)\pi z}{2s}\right]_0^s \\ &= \frac{2}{s}\left[\frac{200s}{(2\ell+1)\pi} + \frac{0.4s^2}{(2\ell+1)^2\pi^2}\sin\frac{(2\ell+1)\pi}{2}\right] \end{aligned}$$

So, the solution of the original problem can be written as

$$u(x,y,z,t) = -0.1z + \frac{2}{s}\sum_{\ell=0}^{\infty} b_\ell \sin\frac{(2\ell+1))\pi z}{2s} e^{-(2\ell+1)^2)\pi^2 kt/4s^2},$$

where

$$b_\ell = \frac{200s}{(2\ell+1)\pi} + (-1)^\ell \frac{0.4s^2}{(2\ell+1)^2\pi^2}.$$

In order to evaluate the temperature of the block at its center after 1.0 minute, we need the thermal diffusivity. We can look it up, to find it is around 1.15 cm^2/s, or we can use $k = \frac{K}{\rho c_p}$, where $K = 400$ W/m°C is the thermal conductivity, $\rho = 8.96$ g/cm^3 is the density, and $c_p = 0.385$ J/g°C is the specific heat capacity. In this case, we find essentially the same value for temperatures around 25°C to 100°C. Thus, we can use

$$k = 1.15\mathrm{cm}^2/\mathrm{s} = 1.15 \times 10^{-4}\mathrm{m}^2/\mathrm{s}.$$

As most of the units are in centimeters and seconds, we will stick with those. Thus, we want to find $u(5,5,5,60)$.

$$u(5,5,5,60) = -0.5 + 0.2\sum_{\ell=0}^{\infty} b_\ell \sin\frac{(2\ell+1))\pi}{4} e^{-1.70(2\ell+1)^2)\pi^2},$$

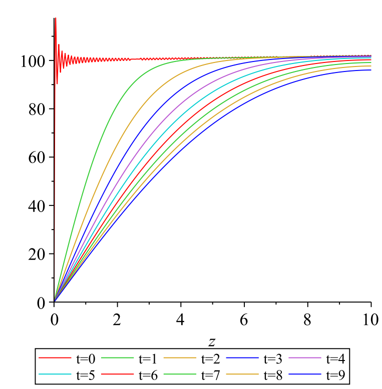

Figure 4: A plot of the temperature at the center of the cube in Problem 11.7 for $t = 0, 1, \ldots, 9$ s.

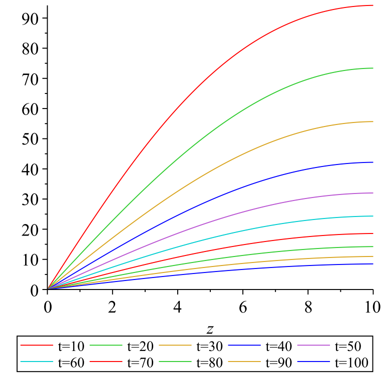

Figure 5: A plot of the temperature at the center of the cube in Problem 11.7 for $t = 10, \ldots, 100$ s.

where

$$b_\ell = \frac{2000}{(2\ell+1)\pi} + (-1)^\ell \frac{40}{(2\ell+1)^2\pi^2}.$$

In Figure 4 is shown the temperature at the center of the cube in Problem 11.7 for $t = 0, 1, \ldots, 9$ s for 201 terms. The initial temperature is seen to be $100°$C, though the convergence is slow. However, convergence for $t > 0$ seems to be quick. In Figure 5 is shown the temperature at the center of the cube in Problem 11.7 for $t = 10, 20 \ldots, 100$ s for 201 terms. One can find the value of the temperature at the center after one minute using only one or two terms. It is $u(5, 5, 5, 60) = 17.0°$C.