2

Trigonometric Fourier Series

"Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say." Bertrand Russell (1872-1970)

2.1 Introduction to Fourier Series

WE WILL NOW TURN TO THE STUDY of trigonometric series. You have seen that functions have series representations as expansions in powers of x, or $x-a$, in the form of Maclaurin and Taylor series. Recall that the Taylor series expansion is given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where the expansion coefficients are determined as

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

From the study of the heat equation and wave equation, we have found that there are infinite series expansions over other functions, such as sine functions. We now turn to such expansions and in the next chapter we will find out that expansions over special sets of functions are not uncommon in physics. But, first we turn to Fourier trigonometric series.

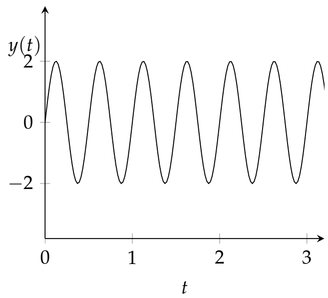
We will begin with the study of the Fourier trigonometric series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}.$$

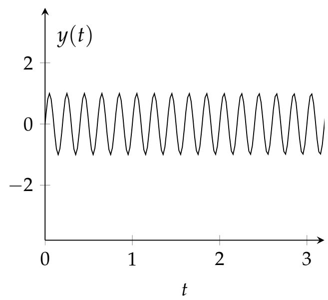
We will find expressions useful for determining the Fourier coefficients $\{a_n, b_n\}$ given a function $f(x)$ defined on $[-L, L]$. We will also see if the resulting infinite series reproduces $f(x)$. However, we first begin with some basic ideas involving simple sums of sinusoidal functions.

There is a natural appearance of such sums over sinusoidal functions in music. A pure note can be represented as

$$y(t) = A\sin(2\pi f t),$$



(a) $y(t) = 2\sin(4\pi f t)$



(b) $y(t) = \sin(10\pi f t)$

Figure 2.1: Plots of $y(t) = A\sin(2\pi f t)$ on $[0, 5]$ for $f = 2$ Hz and $f = 5$ Hz.

where A is the amplitude, f is the frequency in hertz (Hz), and t is time in seconds. The amplitude is related to the volume of the sound. The larger the amplitude, the louder the sound. In Figure 2.1 we show plots of two such tones with $f = 2$ Hz in the top plot and $f = 5$ Hz in the bottom one.

In these plots you should notice the difference due to the amplitudes and the frequencies. You can easily reproduce these plots and others in your favorite plotting utility.

As an aside, you should be cautious when plotting functions, or sampling data. The plots you get might not be what you expect, even for a simple sine function. In Figure 2.2 we show four plots of the function $y(t) = 2\sin(4\pi t)$. In the top left you see a proper rendering of this function. However, if you use a different number of points to plot this function, the results may be surprising. In this example we show what happens if you use $N = 200, 100, 101$ points instead of the 201 points used in the first plot. Such disparities are not only possible when plotting functions, but are also present when collecting data. Typically, when you sample a set of data, you only gather a finite amount of information at a fixed rate. This could happen when getting data on ocean wave heights, digitizing music and other audio to put on your computer, or any other process when you attempt to analyze a continuous signal.

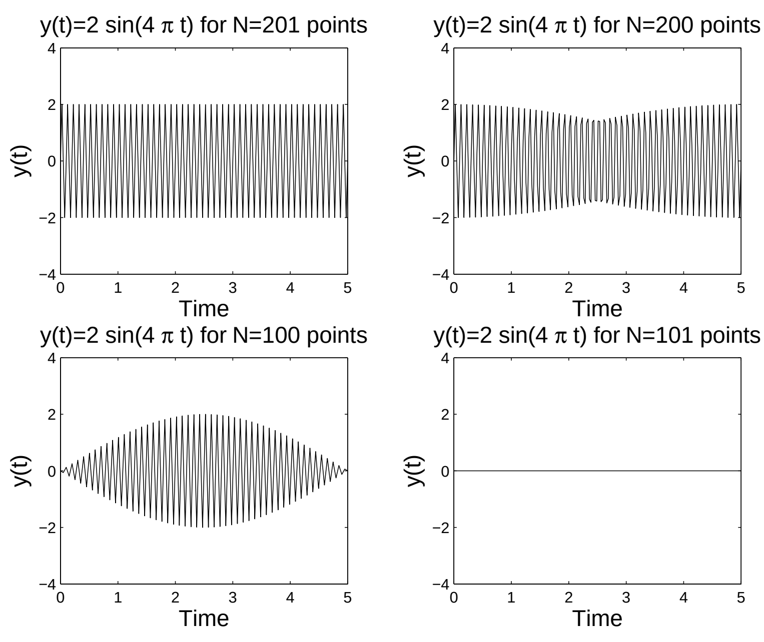


Figure 2.2: Problems can occur while plotting. Here we plot the function $y(t) = 2\sin 4\pi t$ using $N = 201, 200, 100, 101$ points.

Next, we consider what happens when we add several pure tones. After all, most of the sounds that we hear are in fact a combination of pure tones with different amplitudes and frequencies. In Figure 2.3 we see what happens when we add several sinusoids. Note that as one adds more and more tones with different characteristics, the resulting signal gets more complicated. However, we still have a function of time. In this chapter we will ask, "Given a function $f(t)$, can we find a set of sinusoidal functions whose sum converges to $f(t)$?"

Looking at the superpositions in Figure 2.3, we see that the sums yield functions that appear to be periodic. This is not to be unexpected. We recall that a periodic function is one in which the function values repeat over the domain of the function. The length of the smallest part of the domain which repeats is called the period. We can define this more precisely: A function is said to be periodic with period T if $f(t+T) = f(t)$ for all t and the smallest such positive number T is called the period.

For example, we consider the functions used in Figure 2.3. We began with $y(t) = 2\sin(4\pi t)$. Recall from your first studies of trigonometric functions that one can determine the period by dividing the coefficient of t into 2π to get the period. In this case we have

$$T = \frac{2\pi}{4\pi} = \frac{1}{2}.$$

Looking at the top plot in Figure 2.1 we can verify this result. (You can count the full number of cycles in the graph and divide this into the total time to get a more accurate value of the period.)

In general, if $y(t) = A\sin(2\pi f t)$, the period is found as

$$T = \frac{2\pi}{2\pi f} = \frac{1}{f}.$$

Of course, this result makes sense, as the unit of frequency, the hertz, is also defined as s^{-1}, or cycles per second.

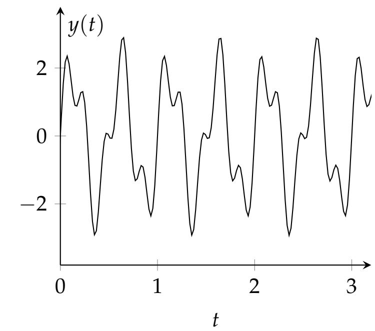
Returning to Figure 2.3, the functions $y(t) = 2\sin(4\pi t)$, $y(t) = \sin(10\pi t)$, and $y(t) = 0.5\sin(16\pi t)$ have periods of 0.5s, 0.2s, and 0.125s, respectively. Each superposition in Figure 2.3 retains a period that is the least common multiple of the periods of the signals added. For both plots, this is 1.0s $= 2(0.5)\text{s} = 5(.2)\text{s} = 8(.125)\text{s}$.

Our goal will be to start with a function and then determine the amplitudes of the simple sinusoids needed to sum to that function. We will see that this might involve an infinite number of such terms. Thus, we will be studying an infinite series of sinusoidal functions.

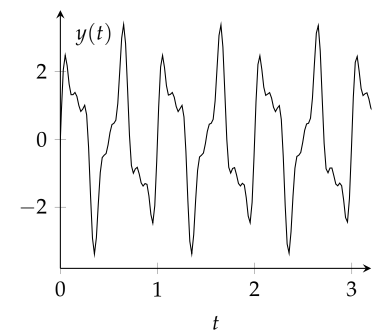
Secondly, we will find that using just sine functions will not be enough either. This is because we can add sinusoidal functions that do not necessarily peak at the same time. We will consider two signals that originate at different times. This is similar to when your music teacher would make sections of the class sing a song like "Row, Row, Row your Boat" starting at slightly different times.

We can easily add shifted sine functions. In Figure 2.4 we show the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum. Note that this shifted sine function can be written as $y(t) = 2\sin(4\pi(t + 7/32))$. Thus, this corresponds to a time shift of $-7/32$.

So, we should account for shifted sine functions in the general sum. Of course, we would then need to determine the unknown time shift as well as the amplitudes of the sinusoidal functions that make up the signal, $f(t)$.

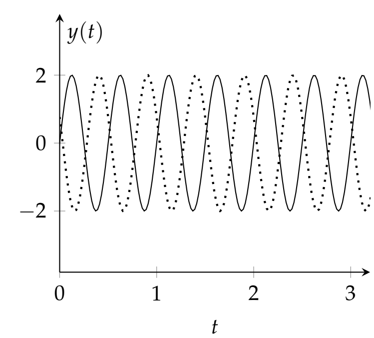


(a) Sum of signals with frequencies $f = 2$ Hz and $f = 5$ Hz.

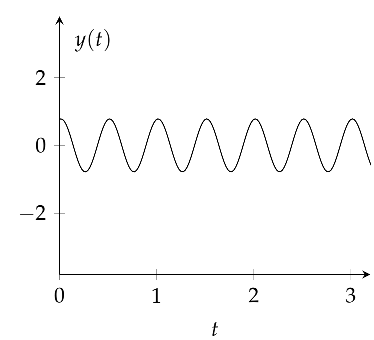


(b) Sum of signals with frequencies $f = 2$ Hz, $f = 5$ Hz, and $f = 8$ Hz.

Figure 2.3: Superposition of several sinusoids.



(a) Plot of each function.



(b) Plot of the sum of the functions.

Figure 2.4: Plot of the functions $y(t) = 2\sin(4\pi t)$ and $y(t) = 2\sin(4\pi t + 7\pi/8)$ and their sum.

While this is one approach that some researchers use to analyze signals, there is a more common approach. This results from another reworking of the shifted function.

Consider the general shifted function

$$y(t) = A\sin(2\pi f t + \phi). \tag{2.1}$$

Note that $2\pi f t + \phi$ is called the phase of the sine function and ϕ is called the phase shift. We can use the trigonometric identity (**??**) for the sine of the sum of two angles[1] to obtain

$$\begin{aligned} y(t) &= A\sin(2\pi f t + \phi) \\ &= A\sin(\phi)\cos(2\pi f t) + A\cos(\phi)\sin(2\pi f t). \end{aligned} \tag{2.2}$$

Defining $a = A\sin(\phi)$ and $b = A\cos(\phi)$, we can rewrite this as

$$y(t) = a\cos(2\pi f t) + b\sin(2\pi f t).$$

Thus, we see that the signal in Equation (2.1) is a sum of sine and cosine functions with the same frequency and different amplitudes. If we can find a and b, then we can easily determine A and ϕ:

$$A = \sqrt{a^2 + b^2}, \quad \tan\phi = \frac{b}{a}.$$

We are now in a position to state our goal.

Goal - Fourier Analysis

Given a signal $f(t)$, we would like to determine its frequency content by finding out what combinations of sines and cosines of varying frequencies and amplitudes will sum to the given function. This is called Fourier Analysis.

We should note that the form in the lower plot of Figure 2.4 looks like a simple sinusoidal function for a reason. Let

$$y_1(t) = 2\sin(4\pi t),$$

$$y_2(t) = 2\sin(4\pi t + 7\pi/8).$$

Then,

$$\begin{aligned} y_1 + y_2 &= 2\sin(4\pi t + 7\pi/8) + 2\sin(4\pi t) \\ &= 2[\sin(4\pi t + 7\pi/8) + \sin(4\pi t)] \\ &= 4\cos\frac{7\pi}{16}\sin\left(4\pi t + \frac{7\pi}{16}\right). \end{aligned}$$

[1] Recall the identities (**??**)-(**??**)

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \sin y\cos x, \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y. \end{aligned}$$

2.2 *Fourier Trigonometric Series*

As we have seen in the last section, we are interested in finding representations of functions in terms of sines and cosines. Given a function $f(x)$ we seek a representation in the form

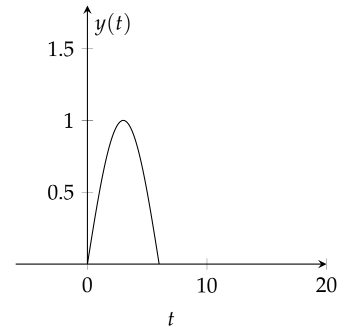
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right]. \tag{2.3}$$

Notice that we have opted to drop the references to the time-frequency form of the phase. This will lead to a simpler discussion for now and one can always make the transformation $nx = 2\pi f_n t$ when applying these ideas to applications.

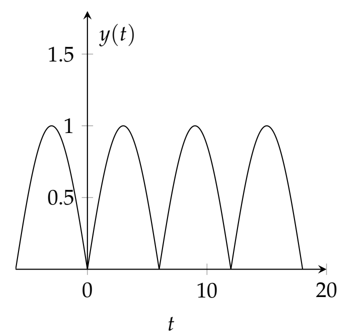
The series representation in Equation (2.3) is called a Fourier trigonometric series. We will simply refer to this as a Fourier series for now. The set of constants a_0, a_n, b_n, $n = 1, 2, \dots$ are called the Fourier coefficients. The constant term is chosen in this form to make later computations simpler, though some other authors choose to write the constant term as a_0. Our goal is to find the Fourier series representation given $f(x)$. Having found the Fourier series representation, we will be interested in determining when the Fourier series converges and to what function it converges.

From our discussion in the last section, we see that The Fourier series is periodic. The periods of $\cos nx$ and $\sin nx$ are $\frac{2\pi}{n}$. Thus, the largest period, $T = 2\pi$, comes from the $n = 1$ terms and the Fourier series has period 2π. This means that the series should be able to represent functions that are periodic of period 2π.

While this appears restrictive, we could also consider functions that are defined over one period. In Figure 2.5 we show a function defined on $[0, 2\pi]$. In the same figure, we show its periodic extension. These are just copies of the original function shifted by the period and glued together. The extension can now be represented by a Fourier series and restricting the Fourier series to $[0, 2\pi]$ will give a representation of the original function. Therefore, we will first consider Fourier series representations of functions defined on this interval. Note that we could just as easily considered functions defined on $[-\pi, \pi]$ or any interval of length 2π. We will consider more general intervals later in the chapter.



(a) Plot of function $f(t)$.



(b) Periodic extension of $f(t)$.

Figure 2.5: Plot of the function $f(t)$ defined on $[0, 2\pi]$ and its periodic extension.

Fourier Coefficients

Theorem 2.1. *The Fourier series representation of $f(x)$ defined on $[0, 2\pi]$, when it exists, is given by (2.3) with Fourier coefficients*

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \dots. \end{aligned} \tag{2.4}$$

These expressions for the Fourier coefficients are obtained by considering special integrations of the Fourier series. We will now derive the a_n integrals in (2.4).

We begin with the computation of a_0. Integrating the Fourier series term by term in Equation (2.3), we have

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} \frac{a_0}{2}\,dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]\,dx. \tag{2.5}$$

We will assume that we can integrate the infinite sum term by term. Then we will need to compute

Evaluating the integral of an infinite series by integrating term by term depends on the convergence properties of the series.

$$\begin{aligned} \int_0^{2\pi} \frac{a_0}{2}\,dx &= \frac{a_0}{2}(2\pi) = \pi a_0, \\ \int_0^{2\pi} \cos nx\,dx &= \left[\frac{\sin nx}{n}\right]_0^{2\pi} = 0, \\ \int_0^{2\pi} \sin nx\,dx &= \left[\frac{-\cos nx}{n}\right]_0^{2\pi} = 0. \end{aligned} \tag{2.6}$$

From these results we see that only one term in the integrated sum does not vanish leaving

$$\int_0^{2\pi} f(x)\,dx = \pi a_0.$$

This confirms the value for a_0.[2]

[2] Note that $\frac{a_0}{2}$ is the average of $f(x)$ over the interval $[0, 2\pi]$. Recall from the first semester of calculus, that the average of a function defined on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

For $f(x)$ defined on $[0, 2\pi]$, we have

$$f_{\text{ave}} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{a_0}{2}.$$

Next, we will find the expression for a_n. We multiply the Fourier series (2.3) by $\cos mx$ for some positive integer m. This is like multiplying by $\cos 2x$, $\cos 5x$, etc. We are multiplying by all possible $\cos mx$ functions for different integers m all at the same time. We will see that this will allow us to solve for the a_n's.

We find the integrated sum of the series times $\cos mx$ is given by

$$\begin{aligned}\int_0^{2\pi} f(x)\cos mx\,dx &= \int_0^{2\pi} \frac{a_0}{2}\cos mx\,dx \\ &+ \int_0^{2\pi} \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n \sin nx\right]\cos mx\,dx.\end{aligned} \tag{2.7}$$

Integrating term by term, the right side becomes

$$\int_0^{2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos mx\,dx + b_n\int_0^{2\pi}\sin nx\cos mx\,dx\right]. \tag{2.8}$$

We have already established that $\int_0^{2\pi}\cos mx\,dx = 0$, which implies that the first term vanishes.

Next we need to compute integrals of products of sines and cosines. This requires that we make use of some of the trigonometric identities listed in Chapter 1. For quick reference, we list these here.

Useful Trigonometric Identities

$$\sin(x \pm y) = \sin x\cos y \pm \sin y \cos x \tag{2.9}$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x \sin y \tag{2.10}$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \tag{2.11}$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) \tag{2.12}$$

$$\sin x\sin y = \frac{1}{2}(\cos(x-y)-\cos(x+y)) \tag{2.13}$$

$$\cos x\cos y = \frac{1}{2}(\cos(x+y)+\cos(x-y)) \tag{2.14}$$

$$\sin x\cos y = \frac{1}{2}(\sin(x+y)+\sin(x-y)) \tag{2.15}$$

We first want to evaluate $\int_0^{2\pi}\cos nx\cos mx\,dx$. We do this by using the product identity (2.14). We have

$$\begin{aligned}\int_0^{2\pi} \cos nx \cos mx \, dx &= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x] \, dx \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} \\ &= 0. \end{aligned} \tag{2.16}$$

There is one caveat when doing such integrals. What if one of the denominators $m \pm n$ vanishes? For this problem $m+n \neq 0$, since both m and n are positive integers. However, it is possible for $m = n$. This means that the vanishing of the integral can only happen when $m \neq n$. So, what can we do about the $m = n$ case? One way is to start from scratch with our integration. (Another way is to compute the limit as n approaches m in our result and use L'Hopital's Rule. Try it!)

For $n = m$ we have to compute $\int_0^{2\pi} \cos^2 mx \, dx$. This can also be handled using a trigonometric identity. Using the half angle formula, (2.12), with $\theta = mx$, we find

$$\begin{aligned}\int_0^{2\pi} \cos^2 mx \, dx &= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2mx) \, dx \\ &= \frac{1}{2}\left[x + \frac{1}{2m}\sin 2mx\right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi) = \pi. \end{aligned} \tag{2.17}$$

To summarize, we have shown that

$$\int_0^{2\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n. \end{cases} \tag{2.18}$$

This holds true for $m, n = 0, 1, \ldots$. [Why did we include $m, n = 0$?] When we have such a set of functions, they are said to be an orthogonal set over the integration interval. A set of (real) functions $\{\phi_n(x)\}$ is said to be orthogonal on $[a, b]$ if $\int_a^b \phi_n(x)\phi_m(x) \, dx = 0$ when $n \neq m$. Furthermore, if we also have that $\int_a^b \phi_n^2(x) \, dx = 1$, these functions are called orthonormal.

Definition of an orthogonal set of functions and orthonormal functions.

The set of functions $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal on $[0, 2\pi]$. Actually, they are orthogonal on any interval of length 2π. We can make them orthonormal by dividing each function by $\sqrt{\pi}$ as indicated by Equation (2.17). This is sometimes referred to normalization of the set of functions.

The notion of orthogonality is actually a generalization of the orthogonality of vectors in finite dimensional vector spaces. The integral $\int_a^b f(x)f(x) \, dx$ is the generalization of the dot product, and is called the scalar product of $f(x)$ and $g(x)$, which are thought of as vectors in an infinite dimensional vector space spanned by a set of orthogonal functions. We will return to these ideas in the next chapter.

Returning to the integrals in equation (2.8), we still have to evaluate $\int_0^{2\pi} \sin nx \cos mx \, dx$. We can use the trigonometric identity involving products of sines and cosines, (2.15). Setting $A = nx$ and $B = mx$, we find that

$$\begin{aligned}
\int_0^{2\pi} \sin nx \cos mx \, dx &= \frac{1}{2} \int_0^{2\pi} [\sin(n+m)x + \sin(n-m)x] \, dx \\
&= \frac{1}{2} \left[\frac{-\cos(n+m)x}{n+m} + \frac{-\cos(n-m)x}{n-m} \right]_0^{2\pi} \\
&= (-1+1) + (-1+1) = 0. \qquad (2.19)
\end{aligned}$$

So,

$$\int_0^{2\pi} \sin nx \cos mx \, dx = 0. \qquad (2.20)$$

For these integrals we also should be careful about setting $n = m$. In this special case, we have the integrals

$$\int_0^{2\pi} \sin mx \cos mx \, dx = \frac{1}{2} \int_0^{2\pi} \sin 2mx \, dx = \frac{1}{2} \left[\frac{-\cos 2mx}{2m} \right]_0^{2\pi} = 0.$$

Finally, we can finish evaluating the expression in Equation (2.8). We have determined that all but one integral vanishes. In that case, $n = m$. This leaves us with

$$\int_0^{2\pi} f(x) \cos mx \, dx = a_m \pi.$$

Solving for a_m gives

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx.$$

Since this is true for all $m = 1, 2, \ldots$, we have proven this part of the theorem. The only part left is finding the b_n's This will be left as an exercise for the reader.

We now consider examples of finding Fourier coefficients for given functions. In all of these cases we define $f(x)$ on $[0, 2\pi]$.

Example 2.1. $f(x) = 3\cos 2x$, $x \in [0, 2\pi]$.

We first compute the integrals for the Fourier coefficients.

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \, dx = 0. \\
a_n &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \cos nx \, dx = 0, \quad n \neq 2. \\
a_2 &= \frac{1}{\pi} \int_0^{2\pi} 3\cos^2 2x \, dx = 3, \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} 3\cos 2x \sin nx \, dx = 0, \forall n. \qquad (2.21)
\end{aligned}$$

The integrals for a_0, $a_n, n \neq 2$, and b_n are the result of orthogonality. For a_2, the integral can be computed as follows:

$$a_2 = \frac{1}{\pi} \int_0^{2\pi} 3\cos^2 2x \, dx$$

$$
\begin{aligned}
&= \frac{3}{2\pi}\int_0^{2\pi} [1+\cos 4x]\ dx \\
&= \frac{3}{2\pi}\left[x + \underbrace{\frac{1}{4}\sin 4x}_{\text{This term vanishes!}}\right]_0^{2\pi} = 3. \qquad (2.22)
\end{aligned}
$$

Therefore, we have that the only nonvanishing coefficient is $a_2 = 3$. So there is one term and $f(x) = 3\cos 2x$.

Well, we should have known the answer to the last example before doing all of those integrals. If we have a function expressed simply in terms of sums of simple sines and cosines, then it should be easy to write down the Fourier coefficients without much work. This is seen by writing out the Fourier series,

$$
\begin{aligned}
f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]. \\
&= \frac{a_0}{2} + a_1\cos x + b_1 \sin x + + a_2 \cos 2x + b_2 \sin 2x + \ldots. \qquad (2.23)
\end{aligned}
$$

For the last problem, $f(x) = 3\cos 2x$. Comparing this to the expanded Fourier series, one can immediately read off the Fourier coefficients without doing any integration. In the next example we emphasize this point.

Example 2.2. $f(x) = \sin^2 x$, $x \in [0, 2\pi]$.

We could determine the Fourier coefficients by integrating as in the last example. However, it is easier to use trigonometric identities. We know that

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

There are no sine terms, so $b_n = 0$, $n = 1, 2, \ldots$. There is a constant term, implying $a_0/2 = 1/2$. So, $a_0 = 1$. There is a $\cos 2x$ term, corresponding to $n = 2$, so $a_2 = -\frac{1}{2}$. That leaves $a_n = 0$ for $n \neq 0, 2$. So, $a_0 = 1$, $a_2 = -\frac{1}{2}$, and all other Fourier coefficients vanish.

Example 2.3. $f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$.

This example will take a little more work. We cannot bypass evaluating any integrals this time. As seen in Figure 2.6, this function is discontinuous. So, we will break up any integration into two integrals, one over $[0, \pi]$ and the other over $[\pi, 2\pi]$.

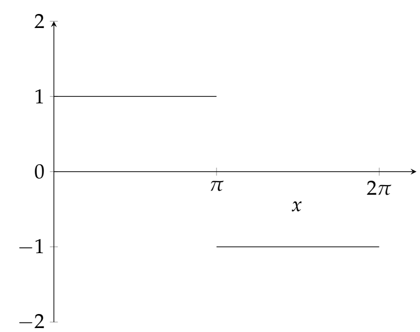


Figure 2.6: Plot of discontinuous function in Example 2.3.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(-1)\,dx \\
&= \frac{1}{\pi}(\pi) + \frac{1}{\pi}(-2\pi+\pi) = 0. \qquad (2.24)
\end{aligned}
$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$\begin{aligned}
&= \frac{1}{\pi}\left[\int_0^{\pi} \cos nx\,dx - \int_{\pi}^{2\pi} \cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left(\frac{1}{n}\sin nx\right)_0^{\pi} - \left(\frac{1}{n}\sin nx\right)_{\pi}^{2\pi}\right] \\
&= 0. \qquad (2.25)
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi} \sin nx\,dx - \int_{\pi}^{2\pi} \sin nx\,dx\right] \\
&= \frac{1}{\pi}\left[\left(-\frac{1}{n}\cos nx\right)_0^{\pi} + \left(\frac{1}{n}\cos nx\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n} + \frac{1}{n} - \frac{1}{n}\cos n\pi\right] \\
&= \frac{2}{n\pi}(1-\cos n\pi). \qquad (2.26)
\end{aligned}$$

Often we see expressions involving $\cos n\pi = (-1)^n$ and $1 \pm \cos n\pi = 1 \pm (-1)^n$. This is an example showing how to re-index series containing $\cos n\pi$.

We have found the Fourier coefficients for this function. Before inserting them into the Fourier series (2.3), we note that $\cos n\pi = (-1)^n$. Therefore,

$$1 - \cos n\pi = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases} \qquad (2.27)$$

So, half of the b_n's are zero. While we could write the Fourier series representation as

$$f(x) \sim \frac{4}{\pi}\sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{1}{n}\sin nx,$$

we could let $n = 2k-1$ in order to capture the odd numbers only. The answer can be written as

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1},$$

Having determined the Fourier representation of a given function, we would like to know if the infinite series can be summed; i.e., does the series converge? Does it converge to $f(x)$? We will discuss this question later in the chapter after we generalize the Fourier series to intervals other than for $x \in [0, 2\pi]$.

2.3 *Fourier Series Over Other Intervals*

In many applications we are interested in determining Fourier series representations of functions defined on intervals other than $[0, 2\pi]$. In this section we will determine the form of the series expansion and the Fourier coefficients in these cases.

The most general type of interval is given as $[a, b]$. However, this often is too general. More common intervals are of the form $[-\pi, \pi]$, $[0, L]$, or $[-L/2, L/2]$. The simplest generalization is to the interval $[0, L]$. Such intervals arise often in applications. For example, for the problem of a one dimensional string of length L we set up the axes with the left end at $x = 0$ and the right end at $x = L$. Similarly for the temperature distribution along a one dimensional rod of length L we set the interval to $x \in [0, 2\pi]$. Such problems naturally lead to the study of Fourier series on intervals of length L. We will see later that symmetric intervals, $[-a, a]$, are also useful.

Given an interval $[0, L]$, we could apply a transformation to an interval of length 2π by simply rescaling the interval. Then we could apply this transformation to the Fourier series representation to obtain an equivalent one useful for functions defined on $[0, L]$.

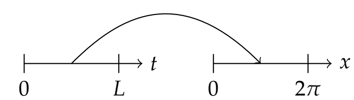


Figure 2.7: A sketch of the transformation between intervals $x \in [0, 2\pi]$ and $t \in [0, L]$.

We define $x \in [0, 2\pi]$ and $t \in [0, L]$. A linear transformation relating these intervals is simply $x = \frac{2\pi t}{L}$ as shown in Figure 2.7. So, $t = 0$ maps to $x = 0$ and $t = L$ maps to $x = 2\pi$. Furthermore, this transformation maps $f(x)$ to a new function $g(t) = f(x(t))$, which is defined on $[0, L]$. We will determine the Fourier series representation of this function using the representation for $f(x)$ from the last section.

Recall the form of the Fourier representation for $f(x)$ in Equation (2.3):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]. \tag{2.28}$$

Inserting the transformation relating x and t, we have

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi t}{L} + b_n \sin \frac{2n\pi t}{L}\right]. \tag{2.29}$$

This gives the form of the series expansion for $g(t)$ with $t \in [0, L]$. But, we still need to determine the Fourier coefficients.

Recall, that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

We need to make a substitution in the integral of $x = \frac{2\pi t}{L}$. We also will need to transform the differential, $dx = \frac{2\pi}{L} dt$. Thus, the resulting form for the Fourier coefficients is

$$a_n = \frac{2}{L} \int_0^L g(t) \cos \frac{2n\pi t}{L} \, dt. \tag{2.30}$$

Similarly, we find that

$$b_n = \frac{2}{L} \int_0^L g(t) \sin \frac{2n\pi t}{L} \, dt. \tag{2.31}$$

We note first that when $L = 2\pi$ we get back the series representation that we first studied. Also, the period of $\cos \frac{2n\pi t}{L}$ is L/n, which means that the representation for $g(t)$ has a period of L corresponding to $n = 1$.

Integration of even and odd functions over symmetric intervals, $[-a, a]$.

At the end of this section we present the derivation of the Fourier series representation for a general interval for the interested reader. In Table 2.1 we summarize some commonly used Fourier series representations.

At this point we need to remind the reader about the integration of even and odd functions on symmetric intervals.

Even Functions.

We first recall that $f(x)$ is an even function if $f(-x) = f(x)$ for all x. One can recognize even functions as they are symmetric with respect to the y-axis as shown in Figure 2.8.

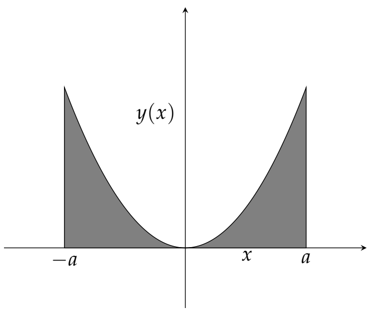


Figure 2.8: Area under an even function on a symmetric interval, $[-a, a]$.

If one integrates an even function over a symmetric interval, then one has that

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx. \tag{2.32}$$

One can prove this by splitting off the integration over negative values of x, using the substitution $x = -y$, and employing the evenness of $f(x)$. Thus,

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{a}^{0} f(-y)\,dy + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{a} f(y)\,dy + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx. \end{aligned} \tag{2.33}$$

This can be visually verified by looking at Figure 2.8.

Odd Functions.

A similar computation could be done for odd functions. $f(x)$ is an odd function if $f(-x) = -f(x)$ for all x. The graphs of such functions are symmetric with respect to the origin as shown in Figure 2.9. If one integrates an odd function over a symmetric interval, then one has that

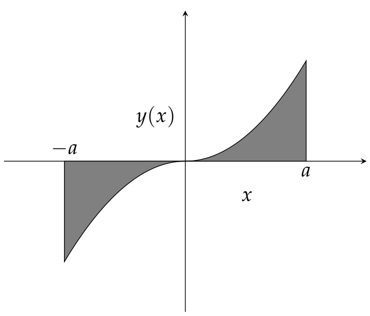


Figure 2.9: Area under an odd function on a symmetric interval, $[-a, a]$.

$$\int_{-a}^{a} f(x)\,dx = 0. \tag{2.34}$$

Example 2.4. Let $f(x) = |x|$ on $[-\pi, \pi]$ We compute the coefficients, beginning as usual with a_0. We have, using the fact that $|x|$ is an even function,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \pi \end{aligned} \tag{2.35}$$

We continue with the computation of the general Fourier coefficients for $f(x) = |x|$ on $[-\pi, \pi]$. We have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx. \tag{2.36}$$

Here we have made use of the fact that $|x|\cos nx$ is an even function.

In order to compute the resulting integral, we need to use integration by parts ,

$$\int_{a}^{b} u\,dv = uv\Big|_{a}^{b} - \int_{a}^{b} v\,du,$$

by letting $u = x$ and $dv = \cos nx\,dx$. Thus, $du = dx$ and $v = \int dv = \frac{1}{n}\sin nx$.

Table 2.1: Special Fourier Series Representations on Different Intervals

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right]. \tag{2.37}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{2.38}$$

Fourier Series on $[-\frac{L}{2}, \frac{L}{2}]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right]. \tag{2.39}$$

$$\begin{aligned} a_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2n\pi x}{L}\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2n\pi x}{L}\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{2.40}$$

Fourier Series on $[-\pi, \pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]. \tag{2.41}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{2.42}$$

Continuing with the computation, we have

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx. \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right] \\ &= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} \\ &= -\frac{2}{\pi n^2}(1 - (-1)^n). \end{aligned} \tag{2.43}$$

Here we have used the fact that $\cos n\pi = (-1)^n$ for any integer n. This leads to a factor $(1 - (-1)^n)$. This factor can be simplified as

$$1 - (-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \tag{2.44}$$

So, $a_n = 0$ for n even and $a_n = -\frac{4}{\pi n^2}$ for n odd.

Computing the b_n's is simpler. We note that we have to integrate $|x| \sin nx$ from $x = -\pi$ to π. The integrand is an odd function and this is a symmetric interval. So, the result is that $b_n = 0$ for all n.

Putting this all together, the Fourier series representation of $f(x) = |x|$ on $[-\pi, \pi]$ is given as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos nx}{n^2}. \tag{2.45}$$

While this is correct, we can rewrite the sum over only odd n by reindexing. We let $n = 2k - 1$ for $k = 1, 2, 3, \ldots$. Then we only get the odd integers. The series can then be written as

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}. \tag{2.46}$$

Throughout our discussion we have referred to such results as Fourier representations. We have not looked at the convergence of these series. Here is an example of an infinite series of functions. What does this series sum to? We show in Figure 2.10 the first few partial sums. They appear to be converging to $f(x) = |x|$ fairly quickly.

Even though $f(x)$ was defined on $[-\pi, \pi]$ we can still evaluate the Fourier series at values of x outside this interval. In Figure 2.11, we see that the representation agrees with $f(x)$ on the interval $[-\pi, \pi]$. Outside this interval we have a periodic extension of $f(x)$ with period 2π.

Another example is the Fourier series representation of $f(x) = x$ on $[-\pi, \pi]$ as left for Problem 7. This is determined to be

$$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{2.47}$$

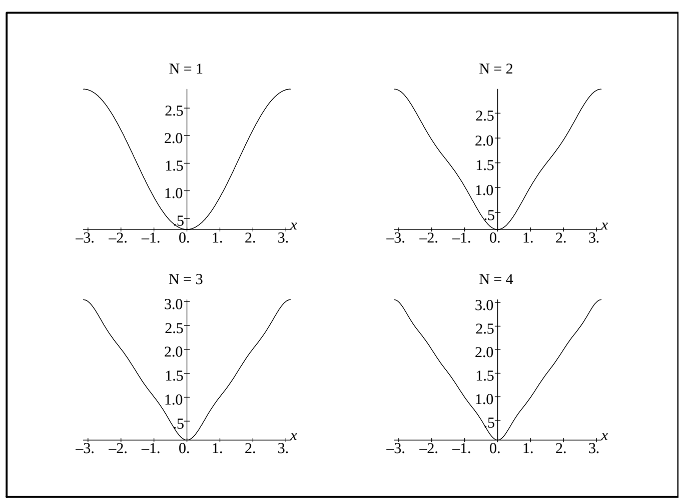


Figure 2.10: Plot of the first partial sums of the Fourier series representation for $f(x) = |x|$.

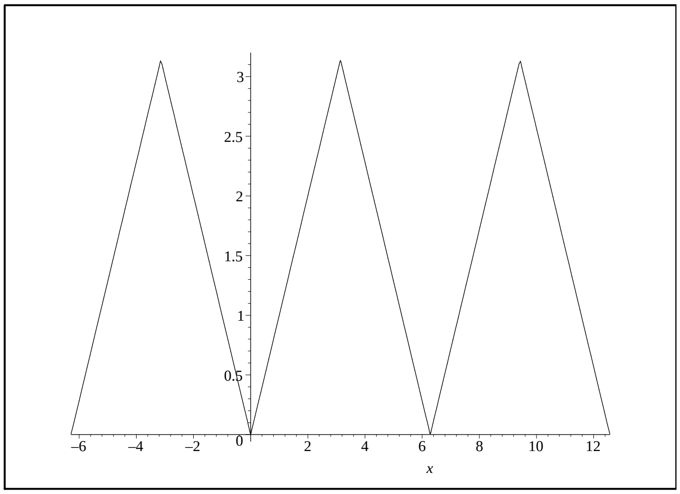


Figure 2.11: Plot of the first 10 terms of the Fourier series representation for $f(x) = |x|$ on the interval $[-2\pi, 4\pi]$.

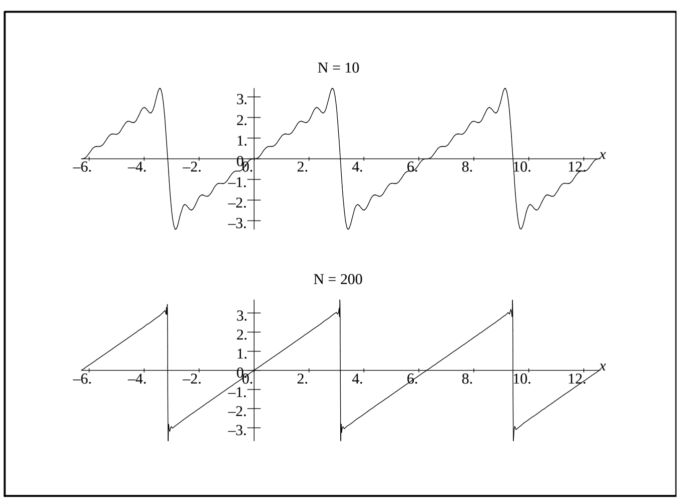


Figure 2.12: Plot of the first 10 terms and 200 terms of the Fourier series representation for $f(x) = x$ on the interval $[-2\pi, 4\pi]$.

As seen in Figure 2.12 we again obtain the periodic extension of the function. In this case we needed many more terms. Also, the vertical parts of the first plot are nonexistent. In the second plot we only plot the points and not the typical connected points that most software packages plot as the default style.

Example 2.5. It is interesting to note that one can use Fourier series to obtain sums of some infinite series. For example, in the last example we found that

$$x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Now, what if we chose $x = \frac{\pi}{2}$? Then, we have

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right].$$

This gives a well known expression for π:

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right].$$

2.3.1 *Fourier Series on* $[a, b]$

This section can be skipped on first reading. It is here for completeness and the end result, Theorem 2.2 provides the result of the section.

A Fourier series representation is also possible for a general interval, $t \in [a, b]$. As before, we just need to transform this interval to $[0, 2\pi]$. Let

$$x = 2\pi \frac{t-a}{b-a}.$$

Inserting this into the Fourier series (2.3) representation for $f(x)$ we obtain

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a}\right]. \tag{2.48}$$

Well, this expansion is ugly. It is not like the last example, where the transformation was straightforward. If one were to apply the theory to applications, it might seem to make sense to just shift the data so that $a = 0$ and be done with any complicated expressions. However, some students enjoy the challenge of developing such generalized expressions. So, let's see what is involved.

First, we apply the addition identities for trigonometric functions and rearrange the terms.

$$\begin{aligned} g(t) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos \frac{2n\pi(t-a)}{b-a} + b_n \sin \frac{2n\pi(t-a)}{b-a}\right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\cos \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} + \sin \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a}\right)\right. \\ &\quad \left. + b_n\left(\sin \frac{2n\pi t}{b-a} \cos \frac{2n\pi a}{b-a} - \cos \frac{2n\pi t}{b-a} \sin \frac{2n\pi a}{b-a}\right)\right] \end{aligned}$$

$$\begin{aligned} &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\cos \frac{2n\pi t}{b-a} \left(a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \right) \right. \\ &\quad \left. + \sin \frac{2n\pi t}{b-a} \left(a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a} \right) \right]. \end{aligned} \tag{2.49}$$

Defining $A_0 = a_0$ and

$$\begin{aligned} A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\ B_n &\equiv a_n \sin \frac{2n\pi a}{b-a} + b_n \cos \frac{2n\pi a}{b-a}, \end{aligned} \tag{2.50}$$

we arrive at the more desirable form for the Fourier series representation of a function defined on the interval $[a, b]$.

$$g(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos \frac{2n\pi t}{b-a} + B_n \sin \frac{2n\pi t}{b-a} \right]. \tag{2.51}$$

We next need to find expressions for the Fourier coefficients. We insert the known expressions for a_n and b_n and rearrange. First, we note that under the transformation $x = 2\pi \frac{t-a}{b-a}$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi(t-a)}{b-a} \, dt, \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx \\ &= \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi(t-a)}{b-a} \, dt. \end{aligned} \tag{2.53}$$

Then, inserting these integrals in A_n, combining integrals and making use of the addition formula for the cosine of the sum of two angles, we obtain

$$\begin{aligned} A_n &\equiv a_n \cos \frac{2n\pi a}{b-a} - b_n \sin \frac{2n\pi a}{b-a} \\ &= \frac{2}{b-a} \int_a^b g(t) \left[\cos \frac{2n\pi(t-a)}{b-a} \cos \frac{2n\pi a}{b-a} - \sin \frac{2n\pi(t-a)}{b-a} \sin \frac{2n\pi a}{b-a} \right] dt \\ &= \frac{2}{b-a} \int_a^b g(t) \cos \frac{2n\pi t}{b-a} \, dt. \end{aligned} \tag{2.54}$$

A similar computation gives

$$B_n = \frac{2}{b-a} \int_a^b g(t) \sin \frac{2n\pi t}{b-a} \, dt. \tag{2.55}$$

Summarizing, we have shown that:

Theorem 2.2. *The Fourier series representation of $f(x)$ defined on $[a,b]$ when it exists, is given by*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a} \right]. \tag{2.56}$$

with Fourier coefficients

$$\begin{aligned} a_n &= \frac{2}{b-a} \int_a^b f(x) \cos \frac{2n\pi x}{b-a}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi x}{b-a}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{2.57}$$

2.4 *Sine and Cosine Series*

In the last two examples ($f(x) = |x|$ and $f(x) = x$ on $[-\pi, \pi]$) we have seen Fourier series representations that contain only sine or cosine terms. As we know, the sine functions are odd functions and thus sum to odd functions. Similarly, cosine functions sum to even functions. Such occurrences happen often in practice. Fourier representations involving just sines are called sine series and those involving just cosines (and the constant term) are called cosine series.

Another interesting result, based upon these examples, is that the original functions, $|x|$ and x agree on the interval $[0, \pi]$. Note from Figures 2.10-2.12 that their Fourier series representations do as well. Thus, more than one series can be used to represent functions defined on finite intervals. All they need to do is to agree with the function over that particular interval. Sometimes one of these series is more useful because it has additional properties needed in the given application.

We have made the following observations from the previous examples:

1. There are several trigonometric series representations for a function defined on a finite interval.
2. Odd functions on a symmetric interval are represented by sine series and even functions on a symmetric interval are represented by cosine series.

These two observations are related and are the subject of this section. We begin by defining a function $f(x)$ on interval $[0, L]$. We have seen that the Fourier series representation of this function appears to converge to a periodic extension of the function.

In Figure 2.13 we show a function defined on $[0, 1]$. To the right is its periodic extension to the whole real axis. This representation has a period of $L = 1$. The bottom left plot is obtained by first reflecting f about the y-axis to make it an even function and then graphing the periodic extension of this new function. Its period will be $2L = 2$. Finally, in the last plot we flip the function about each axis and graph the periodic extension of the new odd function. It will also have a period of $2L = 2$.

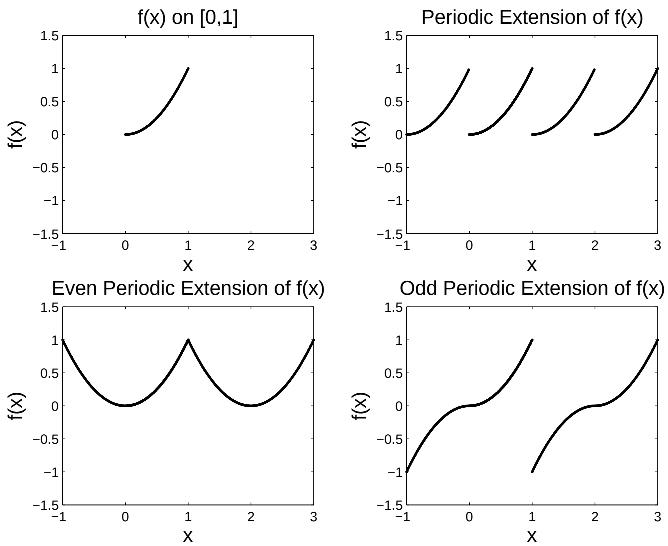


Figure 2.13: This is a sketch of a function and its various extensions. The original function $f(x)$ is defined on $[0,1]$ and graphed in the upper left corner. To its right is the periodic extension, obtained by adding replicas. The two lower plots are obtained by first making the original function even or odd and then creating the periodic extensions of the new function.

In general, we obtain three different periodic representations. In order to distinguish these we will refer to them simply as the periodic, even and odd extensions. Now, starting with $f(x)$ defined on $[0,L]$, we would like to determine the Fourier series representations leading to these extensions. [For easy reference, the results are summarized in Table 2.2]

We have already seen from Table 2.1 that the periodic extension of $f(x)$, defined on $[0,L]$, is obtained through the Fourier series representation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right], \tag{2.58}$$

where

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0,1,2,\ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1,2,\ldots. \end{aligned} \tag{2.59}$$

Even periodic extension.

Given $f(x)$ defined on $[0,L]$, the even periodic extension is obtained by simply computing the Fourier series representation for the even function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ f(-x) & -L < x < 0. \end{cases} \tag{2.60}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L,L]$, we expect that the resulting Fourier series will not contain sine terms. Therefore, the series expansion will be given by [Use the general case in (2.56) with $a = -L$ and $b = L$.]:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{2.67}$$

Table 2.2: Fourier Cosine and Sine Series Representations on $[0, L]$

Fourier Series on $[0, L]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right]. \tag{2.61}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \end{aligned} \tag{2.62}$$

Fourier Cosine Series on $[0, L]$

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{2.63}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \quad n = 0, 1, 2, \ldots. \tag{2.64}$$

Fourier Sine Series on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{2.65}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \quad n = 1, 2, \ldots. \tag{2.66}$$

with Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\frac{n\pi x}{L}\,dx. \quad n = 0,1,2,\ldots. \tag{2.68}$$

However, we can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the Cosine Series Representation of $f(x)$ for $x \in [0, L]$ is given as

Fourier Cosine Series.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \tag{2.69}$$

where

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx. \quad n = 0,1,2,\ldots. \tag{2.70}$$

Similarly, given $f(x)$ defined on $[0, L]$, the odd periodic extension is obtained by simply computing the Fourier series representation for the odd function

Odd periodic extension.

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L, \\ -f(-x) & -L < x < 0. \end{cases} \tag{2.71}$$

The resulting series expansion leads to defining the Sine Series Representation of $f(x)$ for $x \in [0, L]$ as

Fourier Sine Series Representation.

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{2.72}$$

where

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \quad n = 1,2,\ldots. \tag{2.73}$$

Example 2.6. In Figure 2.13 we actually provided plots of the various extensions of the function $f(x) = x^2$ for $x \in [0, 1]$. Let's determine the representations of the periodic, even and odd extensions of this function.

For a change, we will use a CAS (Computer Algebra System) package to do the integrals. In this case we can use Maple. A general code for doing this for the periodic extension is shown in Table 2.3.

Example 2.7. Periodic Extension - Trigonometric Fourier Series Using the code in Table 2.3, we have that $a_0 = \frac{2}{3}$, $a_n = \frac{1}{n^2\pi^2}$, and $b_n = -\frac{1}{n\pi}$. Thus, the resulting series is given as

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}\cos 2n\pi x - \frac{1}{n\pi}\sin 2n\pi x\right].$$

In Figure 2.14 we see the sum of the first 50 terms of this series. Generally, we see that the series seems to be converging to the periodic extension of f. There appear to be some problems with the convergence around integer values of x. We will later see that this is because of the discontinuities in the periodic extension and the resulting overshoot is referred to as the Gibbs phenomenon which is discussed in the last section of this chapter.

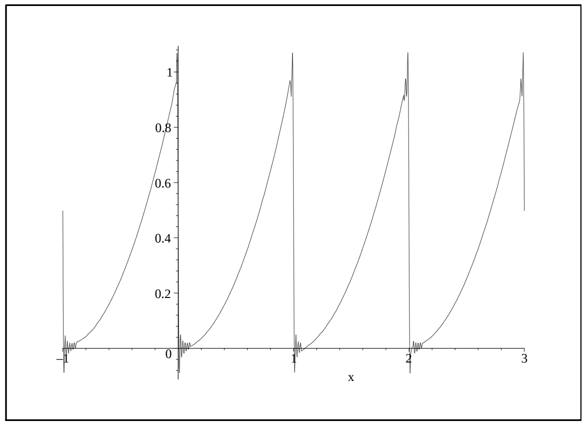


Figure 2.14: The periodic extension of $f(x) = x^2$ on $[0,1]$.

Table 2.3: Maple code for computing Fourier coefficients and plotting partial sums of the Fourier series.

```
> restart:
> L:=1:
> f:=x^2:
> assume(n,integer):
> a0:=2/L*int(f,x=0..L);
                              a0 := 2/3
> an:=2/L*int(f*cos(2*n*Pi*x/L),x=0..L);
                                     1
                            an := -------
                                   2   2
                                 n~  Pi
> bn:=2/L*int(f*sin(2*n*Pi*x/L),x=0..L);
                                      1
                            bn := - -----
                                    n~ Pi
> F:=a0/2+sum((1/(k*Pi)^2)*cos(2*k*Pi*x/L)
     -1/(k*Pi)*sin(2*k*Pi*x/L),k=1..50):
> plot(F,x=-1..3,title=‘Periodic Extension‘,
      titlefont=[TIMES,ROMAN,14],font=[TIMES,ROMAN,14]);
```

Example 2.8. Even Periodic Extension - Cosine Series

In this case we compute $a_0 = \frac{2}{3}$ and $a_n = \frac{4(-1)^n}{n^2\pi^2}$. Therefore, we have

$$f(x) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x.$$

In Figure 2.15 we see the sum of the first 50 terms of this series. In this case the convergence seems to be much better than in the periodic extension case. We also see that it is converging to the even extension.

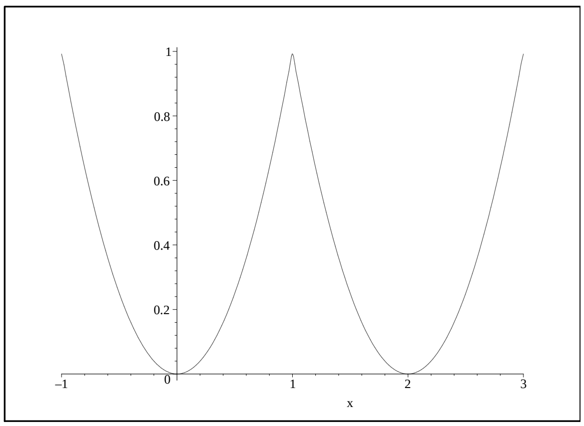


Figure 2.15: The even periodic extension of $f(x) = x^2$ on $[0, 1]$.

Example 2.9. Odd Periodic Extension - Sine Series

Finally, we look at the sine series for this function. We find that

$$b_n = -\frac{2}{n^3\pi^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2).$$

Therefore,

$$f(x) \sim -\frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3}(n^2\pi^2(-1)^n - 2(-1)^n + 2) \sin n\pi x.$$

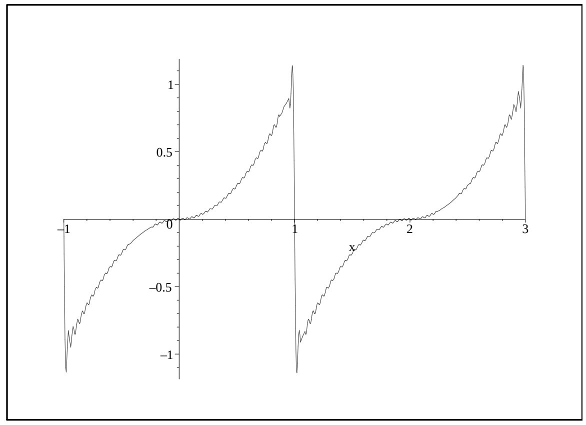
Once again we see discontinuities in the extension as seen in Figure 2.16. However, we have verified that our sine series appears to be converging to the odd extension as we first sketched in Figure 2.13.

2.5 *Solution of the Heat Equation*

We started this chapter seeking solutions of initial-boundary value problems involving the heat equation and the wave equation. In particular, we found the general solution for the problem of heat flow in a one dimensional rod of length L with fixed zero temperature ends. The problem was given by

$$\begin{array}{llll} \text{PDE} & u_t = ku_{xx}, & 0 < t, & 0 \le x \le L, \\ \text{IC} & u(x,0) = f(x), & & 0 < x < L, \\ \text{BC} & u(0,t) = 0, & & t > 0, \\ & u(L,t) = 0, & & t > 0. \end{array} \tag{2.74}$$

Figure 2.16: The odd periodic extension of $f(x) = x^2$ on $[0, 1]$.



We found the solution using separation of variables. This resulted in a sum over various product solutions:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \frac{n\pi x}{L},$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2.$$

This equation satisfies the boundary conditions. However, we had only gotten to state initial condition using this solution. Namely,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

We were left with having to determine the constants b_n. Once we know them, we have the solution.

Now we can get the Fourier coefficients when we are given the initial condition, $f(x)$. They are given by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots.$$

We consider a couple of examples with different initial conditions.

Example 2.10. Consider the solution of the heat equation with $f(x) = \sin x$ and $L = \pi$.

In this case the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin nx.$$

However, the initial condition takes the form of the first term in the expansion; i.e., the $n = 1$ term. So, we need not carry out the integral because we can immediately write $b_1 = 1$ and $b_n = 0$, $n = 2, 3, \ldots$. Therefore, the solution consists of just one term,

$$u(x,t) = e^{-kt} \sin x.$$

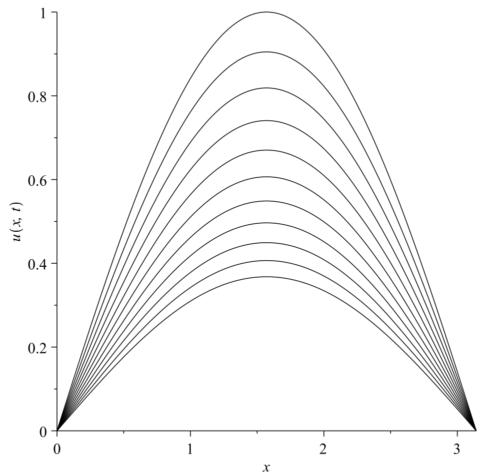


Figure 2.17: The evolution of the initial condition $f(x) = \sin x$ for $L = \pi$ and $k = 1$.

In Figure 2.17 we see that how this solution behaves for $k = 1$ and $t \in [0, 1]$.

Example 2.11. Consider solutions of the heat equation with $f(x) = x(1-x)$ and $L = 1$.

This example requires a bit more work. The solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt} \sin n\pi x,$$

where

$$b_n = 2\int_0^1 f(x) \sin n\pi x \, dx.$$

This integral is easily computed using integration by parts

$$\begin{aligned}
b_n &= 2\int_0^1 x(1-x) \sin n\pi x \, dx \\
&= \left[2x(1-x)\left(-\frac{1}{n\pi}\cos n\pi x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 (1-2x)\cos n\pi x \, dx \\
&= -\frac{2}{n^2\pi^2}\left\{[(1-2x)\sin n\pi x]_0^1 + 2\int_0^1 \sin n\pi x \, dx\right\} \\
&= \frac{4}{n^3\pi^3}\left[\cos n\pi x\right]_0^1 \\
&= \frac{4}{n^3\pi^3}\left(\cos n\pi - 1\right) \\
&= \begin{cases} 0, & n \text{ even} \\ -\frac{8}{n^3\pi^3}, & n \text{ odd} \end{cases}. \qquad (2.75)
\end{aligned}$$

So, we have that the solution can be written as

$$u(x,t) = \frac{8}{\pi^3}\sum_{\ell=1}^{\infty} \frac{1}{(2\ell-1)^3} e^{-(2\ell-1)^2\pi^2 kt} \sin(2\ell-1)\pi x.$$

In Figure 2.18 we see that how this solution behaves for $k = 1$ and $t \in [0, 1]$. Twenty terms were used. We see that this solution diffuses much faster than in the last example. Most of the terms damp out quickly as the solution asymptotically approaches the first term.

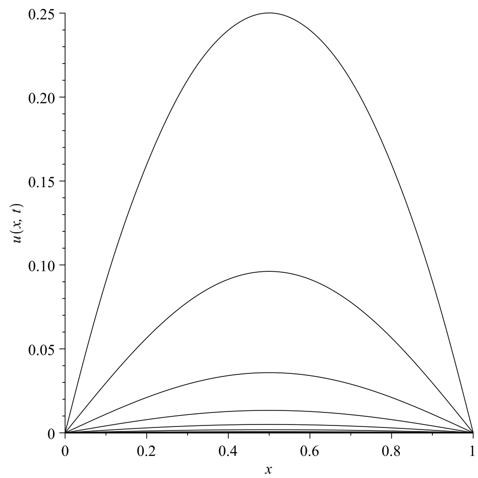


Figure 2.18: The evolution of the initial condition $f(x) = x(1-x)$ for $L = 1$ and $k = 1$.

2.6 Finite Length Strings

We now return to the physical example of wave propagation in a string. We found that the general solution can be represented as a sum over product solutions. We will restrict our discussion to the special case that the initial velocity is zero and the original profile is given by $u(x,0) = f(x)$. The solution is then

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} \tag{2.76}$$

satisfying

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \tag{2.77}$$

We have seen that the Fourier sine series coefficients are given by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{2.78}$$

We can rewrite this solution in a more compact form. First, we define the wave numbers,

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots,$$

and the angular frequencies,

$$\omega_n = ck_n = \frac{n\pi c}{L}.$$

Then, the product solutions take the form

$$\sin k_n x \cos \omega_n t.$$

Using trigonometric identities, these products can be written as

$$\sin k_n x \cos \omega_n t = \frac{1}{2}\left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t)\right].$$

Inserting this expression in the solution, we have

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t)\right]. \tag{2.79}$$

Since $\omega_n = ck_n$, we can put this into a more suggestive form:

$$u(x,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} A_n \sin k_n(x+ct) + \sum_{n=1}^{\infty} A_n \sin k_n(x-ct)\right]. \tag{2.80}$$

We see that each sum is simply the sine series for $f(x)$ but evaluated at either $x+ct$ or $x-ct$. Thus, the solution takes the form

The solution of the wave equation can be written as the sum of right and left traveling waves.

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]. \tag{2.81}$$

If $t = 0$, then we have $u(x,0) = \frac{1}{2}\left[f(x) + f(x)\right] = f(x)$. So, the solution satisfies the initial condition. At $t = 1$, the sum has a term $f(x-c)$.

Recall from your mathematics classes that this is simply a shifted version of $f(x)$. Namely, it is shifted to the right. For general times, the function is shifted by ct to the right. For larger values of t, this shift is further to the right. The function (wave) shifts to the right with velocity c. Similarly, $f(x+ct)$ is a wave traveling to the left with velocity $-c$.

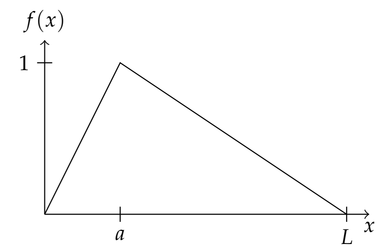


Figure 2.19: The initial profile for a string of length one plucked at $x = a$.

Thus, the waves on the string consist of waves traveling to the right and to the left. However, the story does not stop here. We have a problem when needing to shift $f(x)$ across the boundaries. The original problem only defines $f(x)$ on $[0, L]$. If we are not careful, we would think that the function leaves the interval leaving nothing left inside. However, we have to recall that our sine series representation for $f(x)$ has a period of $2L$. So, before we apply this shifting, we need to account for its periodicity. In fact, being a sine series, we really have the odd periodic extension of $f(x)$ being shifted. The details of such analysis would take us too far from our current goal. However, we can illustrate this with a few figures.

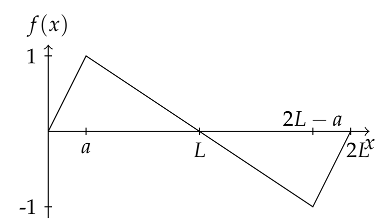


Figure 2.20: Odd extension about the right end of a plucked string.

We begin by plucking a string of length L. This can be represented by the function

$$f(x) = \begin{cases} \frac{x}{a} & 0 \le x \le a \\ \frac{L-x}{L-a} & a \le x \le L \end{cases} \tag{2.82}$$

where the string is pulled up one unit at $x = a$. This is shown in Figure 2.19.

Next, we create an odd function by extending the function to a period of $2L$. This is shown in Figure 2.20.

Finally, we construct the periodic extension of this to the entire line. In Figure 2.21 we show in the lower part of the figure copies of the periodic extension, one moving to the right and the other moving to the left. (Actually, the copies are $\frac{1}{2}f(x \pm ct)$.) The top plot is the sum of these solutions. The physical string lies in the interval [0,1]. Of course, this is better seen when the solution is animated.

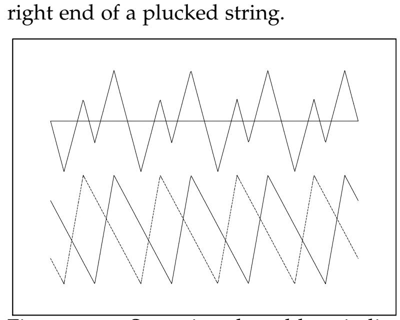


Figure 2.21: Summing the odd periodic extensions. The lower plot shows copies of the periodic extension, one moving to the right and the other moving to the left. The upper plot is the sum.

The time evolution for this plucked string is shown for several times in Figure 2.22. This results in a wave that appears to reflect from the ends as time increases.

The relation between the angular frequency and the wave number, $\omega = ck$, is called a dispersion relation. In this case ω depends on k linearly. If one knows the dispersion relation, then one can find the wave speed as $c = \frac{\omega}{k}$. In this case, all of the harmonics travel at the same speed. In cases where they do not, we have nonlinear dispersion, which we will discuss later.

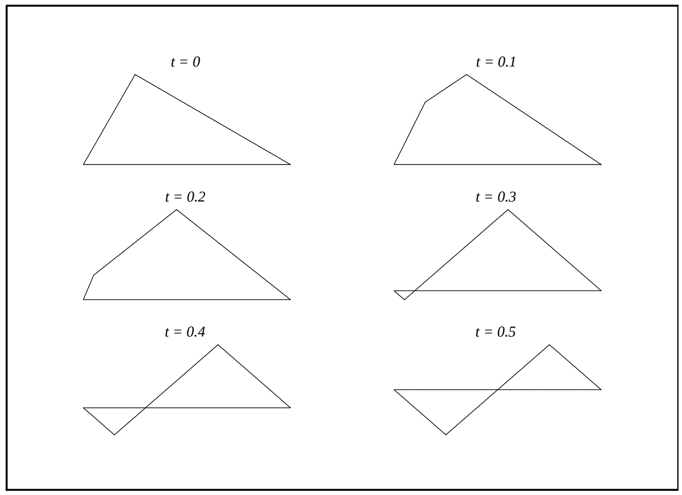


Figure 2.22: This Figure shows the plucked string at six successive times.

2.7 *The Gibbs Phenomenon*

WE HAVE SEEN THE GIBBS PHENOMENON when there is a jump discontinuity in the periodic extension of a function, whether the function originally had a discontinuity or developed one due to a mismatch in the values of the endpoints. This can be seen in Figures 2.12, 2.14 and 2.16. The Fourier series has a difficult time converging at the point of discontinuity and these graphs of the Fourier series show a distinct overshoot which does not go away. This is called the Gibbs phenomenon[3] and the amount of overshoot can be computed.

[3] The Gibbs phenomenon was named after Josiah Willard Gibbs (1839-1903) even though it was discovered earlier by the Englishman Henry Wilbraham (1825-1883). Wilbraham published a soon forgotten paper about the effect in 1848. In 1889 Albert Abraham Michelson (1852-1931), an American physicist,observed an overshoot in his mechanical graphing machine. Shortly afterwards J. Willard Gibbs published papers describing this phenomenon, which was later to be called the Gibbs phenomena. Gibbs was a mathematical physicist and chemist and is considered the father of physical chemistry.

In one of our first examples, Example 2.3, we found the Fourier series representation of the piecewise defined function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

to be

$$f(x) \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

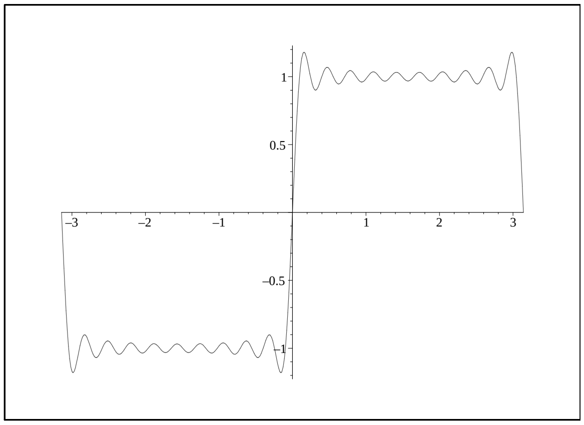


Figure 2.23: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$.

In Figure 2.23 we display the sum of the first ten terms. Note the wiggles, overshoots and under shoots. These are seen more when we plot the representation for $x \in [-3\pi, 3\pi]$, as shown in Figure 2.24.

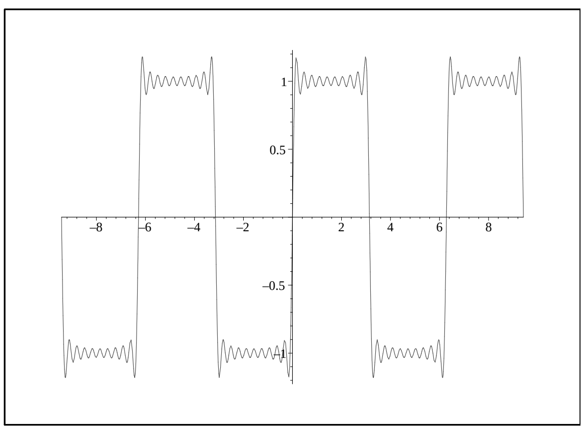


Figure 2.24: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 10$ plotted on $[-3\pi, 3\pi]$ displaying the periodicity.

We note that the overshoots and undershoots occur at discontinuities in the periodic extension of $f(x)$. These occur whenever $f(x)$ has a discontinuity or if the values of $f(x)$ at the endpoints of the domain do not agree.

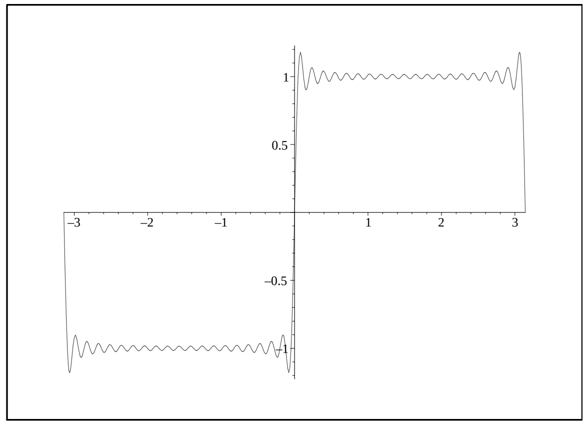


Figure 2.25: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 20$.

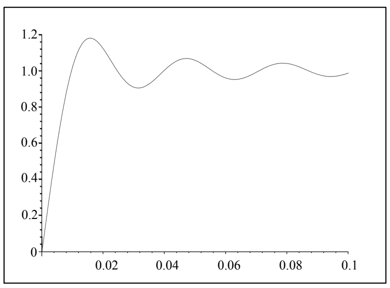


Figure 2.26: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 100$.

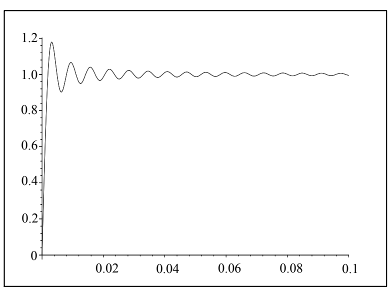


Figure 2.27: The Fourier series representation of a step function on $[-\pi, \pi]$ for $N = 500$.

One might expect that we only need to add more terms. In Figure 2.25 we show the sum for twenty terms. Note the sum appears to converge better for points far from the discontinuities. But, the overshoots and undershoots are still present. In Figures 2.26 and 2.27 show magnified plots of the overshoot at $x = 0$ for $N = 100$ and $N = 500$, respectively. We see that the overshoot persists. The peak is at about the same height, but its location seems to be getting closer to the origin. We will show how one can estimate the size of the overshoot.

We can study the Gibbs phenomenon by looking at the partial sums of general Fourier trigonometric series for functions $f(x)$ defined on the interval $[-L, L]$. Writing out the partial sums, inserting the Fourier coefficients and rearranging, we have

$$\begin{aligned}
S_N(x) &= a_0 + \sum_{n=1}^{N} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \\
&= \frac{1}{2L} \int_{-L}^{L} f(y)\, dy + \sum_{n=1}^{N} \left[\left(\frac{1}{L} \int_{-L}^{L} f(y) \cos \frac{n\pi y}{L}\, dy \right) \cos \frac{n\pi x}{L} \right. \\
&\qquad \left. + \left(\frac{1}{L} \int_{-L}^{L} f(y) \sin \frac{n\pi y}{L}\, dy. \right) \sin \frac{n\pi x}{L} \right] \\
&= \frac{1}{L} \int\limits_{-L}^{L} \left\{ \frac{1}{2} \right. \\
&\qquad \left. + \sum_{n=1}^{N} \left(\cos \frac{n\pi y}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi y}{L} \sin \frac{n\pi x}{L} \right) \right\} f(y)\, dy \\
&= \frac{1}{L} \int\limits_{-L}^{L} \left\{ \frac{1}{2} + \sum_{n=1}^{N} \cos \frac{n\pi (y-x)}{L} \right\} f(y)\, dy \\
&\equiv \frac{1}{L} \int\limits_{-L}^{L} D_N(y-x) f(y)\, dy
\end{aligned}$$

We have defined

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos\frac{n\pi x}{L},$$

which is called the N-th Dirichlet kernel .

We now prove

Lemma 2.1. *The N-th Dirichlet kernel is given by*

$$D_N(x) = \begin{cases} \frac{\sin((N+\frac{1}{2})\frac{\pi x}{L})}{2\sin\frac{\pi x}{2L}}, & \sin\frac{\pi x}{2L} \neq 0, \\ N+\frac{1}{2}, & \sin\frac{\pi x}{2L} = 0. \end{cases}$$

Proof. Let $\theta = \frac{\pi x}{L}$ and multiply $D_N(x)$ by $2\sin\frac{\theta}{2}$to obtain:

$$\begin{aligned} 2\sin\frac{\theta}{2}D_N(x) &= 2\sin\frac{\theta}{2}\left[\frac{1}{2}+\cos\theta+\cdots+\cos N\theta\right] \\ &= \sin\frac{\theta}{2}+2\cos\theta\sin\frac{\theta}{2}+2\cos 2\theta\sin\frac{\theta}{2}+\cdots+2\cos N\theta\sin\frac{\theta}{2} \\ &= \sin\frac{\theta}{2}+\left(\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}\right)+\left(\sin\frac{5\theta}{2}-\sin\frac{3\theta}{2}\right)+\cdots \\ &\quad +\left[\sin\left(N+\frac{1}{2}\right)\theta-\sin\left(N-\frac{1}{2}\right)\theta\right] \\ &= \sin\left(N+\frac{1}{2}\right)\theta. \end{aligned} \tag{2.83}$$

Thus,

$$2\sin\frac{\theta}{2}D_N(x) = \sin\left(N+\frac{1}{2}\right)\theta.$$

If $\sin\frac{\theta}{2} \neq 0$, then

$$D_N(x) = \frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}, \qquad \theta = \frac{\pi x}{L}.$$

If $\sin\frac{\theta}{2} = 0$, then one needs to apply L'Hospital's Rule as $\theta \to 2m\pi$:

$$\begin{aligned} \lim_{\theta\to 2m\pi}\frac{\sin\left(N+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} &= \lim_{\theta\to 2m\pi}\frac{(N+\frac{1}{2})\cos\left(N+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} \\ &= \frac{(N+\frac{1}{2})\cos\left(2m\pi N+m\pi\right)}{\cos m\pi} \\ &= \frac{(N+\frac{1}{2})(\cos 2m\pi N\cos m\pi-\sin 2m\pi N\sin m\pi)}{\cos m\pi} \\ &= N+\frac{1}{2}. \end{aligned} \tag{2.84}$$

□

We further note that $D_N(x)$ is periodic with period $2L$ and is an even function.

So far, we have found that the Nth partial sum is given by

$$S_N(x) = \frac{1}{L}\int_{-L}^{L} D_N(y-x) f(y)\, dy. \tag{2.85}$$

Making the substitution $\xi = y - x$, we have

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L-x}^{L-x} D_N(\xi) f(\xi + x)\, d\xi \\ &= \frac{1}{L}\int_{-L}^{L} D_N(\xi) f(\xi + x)\, d\xi. \end{aligned} \tag{2.86}$$

In the second integral we have made use of the fact that $f(x)$ and $D_N(x)$ are periodic with period $2L$ and shifted the interval back to $[-L, L]$.

We now write the integral as the sum of two integrals over positive and negative values of ξ and use the fact that $D_N(x)$ is an even function. Then,

$$\begin{aligned} S_N(x) &= \frac{1}{L}\int_{-L}^{0} D_N(\xi) f(\xi + x)\, d\xi + \frac{1}{L}\int_{0}^{L} D_N(\xi) f(\xi + x)\, d\xi \\ &= \frac{1}{L}\int_{0}^{L} \left[f(x-\xi) + f(\xi + x)\right] D_N(\xi)\, d\xi. \end{aligned} \tag{2.87}$$

We can use this result to study the Gibbs phenomenon whenever it occurs. In particular, we will only concentrate on the earlier example. For this case, we have

$$S_N(x) = \frac{1}{\pi}\int_{0}^{\pi} \left[f(x-\xi) + f(\xi + x)\right] D_N(\xi)\, d\xi \tag{2.88}$$

for

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos nx.$$

Also, one can show that

$$f(x-\xi) + f(\xi + x) = \begin{cases} 2, & 0 \le \xi < x, \\ 0, & x \le \xi < \pi - x, \\ -2, & \pi - x \le \xi < \pi. \end{cases}$$

Thus, we have

$$\begin{aligned} S_N(x) &= \frac{2}{\pi}\int_{0}^{x} D_N(\xi)\, d\xi - \frac{2}{\pi}\int_{\pi - x}^{\pi} D_N(\xi)\, d\xi \\ &= \frac{2}{\pi}\int_{0}^{x} D_N(z)\, dz + \frac{2}{\pi}\int_{0}^{x} D_N(\pi - z)\, dz. \end{aligned} \tag{2.89}$$

Here we made the substitution $z = \pi - \xi$ in the second integral.

The Dirichlet kernel for $L = \pi$ is given by

$$D_N(x) = \frac{\sin(N + \frac{1}{2})x}{2\sin\frac{x}{2}}.$$

For N large, we have $N + \frac{1}{2} \approx N$, and for small x, we have $\sin\frac{x}{2} \approx \frac{x}{2}$. So, under these assumptions,

$$D_N(x) \approx \frac{\sin Nx}{x}.$$

Therefore,

$$S_N(x) \to \frac{2}{\pi}\int_0^x \frac{\sin N\xi}{\xi}\,d\xi \qquad \text{for large } N, \text{ and small } x.$$

If we want to determine the locations of the minima and maxima, where the undershoot and overshoot occur, then we apply the first derivative test for extrema to $S_N(x)$. Thus,

$$\frac{d}{dx}S_N(x) = \frac{2}{\pi}\frac{\sin Nx}{x} = 0.$$

The extrema occur for $Nx = m\pi$, $m = \pm 1, \pm 2, \ldots$. One can show that there is a maximum at $x = \pi/N$ and a minimum for $x = 2\pi/N$. The value for the overshoot can be computed as

$$\begin{aligned} S_N(\pi/N) &= \frac{2}{\pi}\int_0^{\pi/N} \frac{\sin N\xi}{\xi}\,d\xi \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt \\ &= \frac{2}{\pi}\text{Si}(\pi) \\ &= 1.178979744\ldots. \end{aligned} \tag{2.90}$$

Note that this value is independent of N and is given in terms of the sine integral,

$$\text{Si}(x) \equiv \int_0^x \frac{\sin t}{t}\,dt.$$

Problems

1. Write $y(t) = 3\cos 2t - 4\sin 2t$ in the form $y(t) = A\cos(2\pi f t + \phi)$.

2. Derive the coefficients b_n in Equation(2.4).

3. Let $f(x)$ be defined for $x \in [-L, L]$. Parseval's identity is given by

$$\frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Assuming the the Fourier series of $f(x)$ converges uniformly in $(-L, L)$, prove Parseval's identity by multiplying the Fourier series representation by $f(x)$ and integrating from $x = -L$ to $x = L$. [In section **??** we will encounter Parseval's equality for Fourier transforms which is a continuous version of this identity.]

4. Consider the square wave function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi. \end{cases}$$

a. Find the Fourier series representation of this function and plot the first 50 terms.

b. Apply Parseval's identity in Problem 3 to the result in part a.

c. Use the result of part b to show $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

5. For the following sets of functions: i) show that each is orthogonal on the given interval, and ii) determine the corresponding orthonormal set. [See page 53]

a. $\{\sin 2nx\}, \quad n = 1, 2, 3, \ldots, \quad 0 \le x \le \pi$.

b. $\{\cos n\pi x\}, \quad n = 0, 1, 2, \ldots, \quad 0 \le x \le 2$.

c. $\{\sin \frac{n\pi x}{L}\}, \quad n = 1, 2, 3, \ldots, \quad x \in [-L, L]$.

6. Consider $f(x) = 4\sin^3 2x$.

a. Derive the trigonometric identity giving $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$ using DeMoivre's Formula.

b. Find the Fourier series of $f(x) = 4\sin^3 2x$ on $[0, 2\pi]$ without computing any integrals.

7. Find the Fourier series of the following:

a. $f(x) = x$, $x \in [0, 2\pi]$.

b. $f(x) = \frac{x^2}{4}$, $|x| < \pi$.

c. $f(x) = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi, \\ -\frac{\pi}{2}, & \pi < x < 2\pi. \end{cases}$

8. Find the Fourier Series of each function $f(x)$ of period 2π. For each series, plot the Nth partial sum,

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos nx + b_n \sin nx\right],$$

for $N = 5, 10, 50$ and describe the convergence (is it fast? what is it converging to, etc.) [Some simple Maple code for computing partial sums is shown in the notes.]

a. $f(x) = x$, $|x| < \pi$.

b. $f(x) = |x|$, $|x| < \pi$.

c. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

9. Find the Fourier series of $f(x) = x$ on the given interval. Plot the Nth partial sums and describe what you see.

a. $0 < x < 2$.

b. $-2 < x < 2$.

c. $1 < x < 2$.

10. The result in problem 7b above gives a Fourier series representation of $\frac{x^2}{4}$. By picking the right value for x and a little arrangement of the series, show that [See Example 2.5.]

a.
$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots.$$

b.
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

Hint: Consider how the series in part a. can be used to do this.

11. Sketch (by hand) the graphs of each of the following functions over four periods. Then sketch the extensions each of the functions as both an even and odd periodic function. Determine the corresponding Fourier sine and cosine series and verify the convergence to the desired function using Maple.

a. $f(x) = x^2$, $0 < x < 1$.

b. $f(x) = x(2 - x)$, $0 < x < 2$.

c. $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$

d. $f(x) = \begin{cases} \pi, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$

12. Consider the function $f(x) = x$, $-\pi < x < \pi$.

a. Show that $x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$.

b. Integrate the series in part a and show that

$$x^2 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}.$$

c. Find the Fourier cosine series of $f(x) = x^2$ on $[0, \pi]$ and compare it to the result in part b.

13. Consider the function $f(x) = x$, $0 < x < 2$.

a. Find the Fourier sine series representation of this function and plot the first 50 terms.

b. Find the Fourier cosine series representation of this function and plot the first 50 terms.

c. Apply Parseval's identity in Problem 3 to the result in part b.

d. Use the result of part c to find the sum $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

14. Differentiate the Fourier sine series term by term in Problem 18. Show that the result is not the derivative of $f(x) = x$.

15. Find the general solution to the heat equation, $u_t - u_{xx} = 0$, on $[0, \pi]$ satisfying the boundary conditions $u_x(0,t) = 0$ and $u(\pi, t) = 0$. Determine the solution satisfying the initial condition,

$$u(x,0) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \le x \le \pi, \end{cases}$$

16. Find the general solution to the wave equation $u_{tt} = 2u_{xx}$, on $[0, 2\pi]$ satisfying the boundary conditions $u(0,t) = 0$ and $u_x(2\pi, t) = 0$. Determine the solution satisfying the initial conditions, $u(x,0) = x(4\pi - x)$, and $u_t(x,0) = 0$.

17. Recall the plucked string initial profile example in the last chapter given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\ell}{2}, \\ \ell - x, & \frac{\ell}{2} \le x \le \ell, \end{cases}$$

satisfying fixed boundary conditions at $x = 0$ and $x = \ell$. Find and plot the solutions at $t = 0, .2, ..., 1.0$, of $u_{tt} = u_{xx}$, for $u(x,0) = f(x)$, $u_t(x,0) = 0$, with $x \in [0,1]$.

18. Find and plot the solutions at $t = 0, .2, ..., 1.0$, of the problem

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 \le x \le 1, t > 0 \\ u(x,0) &= \begin{cases} 0, & 0 \le x < \frac{1}{4}, \\ 1, & \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} < x \le 1, \end{cases} \\ u_t(x,0) &= 0, \\ u(0,t) &= 0, \quad t > 0, \\ u(1,t) &= 0, \quad t > 0. \end{aligned}$$

19. Find the solution to Laplace's equation, $u_{xx} + u_{yy} = 0$, on the unit square, $[0,1] \times [0,1]$ satisfying the boundary conditions $u(0,y) = 0$, $u(1,y) = y(1-y)$, $u(x,0) = 0$, and $u(x,1) = 0$.