6

Problems in Higher Dimensions

"Equations of such complexity as are the equations of the gravitational field can be found only through the discovery of a logically simple mathematical condition that determines the equations completely or at least almost completely."

"What I have to say about this book can be found inside this book." Albert Einstein (1879-1955)

In this chapter we will explore several examples of the solution of initial-boundary value problems involving higher spatial dimensions. These are described by higher dimensional partial differential equations, such as the ones presented in Table 1.1 in Chapter 1. The spatial domains of the problems span many different geometries, which will necessitate the use of rectangular, polar, cylindrical, or spherical coordinates.

We will solve many of these problems using the method of separation of variables, which we first saw in Chapter 1. Using separation of variables will result in a system of ordinary differential equations for each problem. Adding the boundary conditions, we will need to solve a variety of eigenvalue problems. The product solutions that result will involve trigonometric or some of the special functions that we had encountered in Chapter 5. These methods are used in solving the hydrogen atom and other problems in quantum mechanics and in electrostatic problems in electrodynamics. We will bring to this discussion many of the tools from earlier in this book showing how much of what we have seen can be used to solve some generic partial differential equations which describe oscillation and diffusion type problems.

As we proceed through the examples in this chapter, we will see some common features. For example, the two key equations that we have studied are the heat equation and the wave equation. For higher dimensional problems these take the form

$$u_t = k\nabla^2 u, \tag{6.1}$$

$$u_{tt} = c^2\nabla^2 u. \tag{6.2}$$

We can separate out the time dependence in each equation. Inserting a guess of $u(\mathbf{r},t) = \phi(\mathbf{r})T(t)$ into the heat and wave equations, we obtain

$$T'\phi = kT\nabla^2\phi, \tag{6.3}$$

$$T''\phi = c^2 T\nabla^2\phi. \tag{6.4}$$

Dividing each equation by $\phi(\mathbf{r})T(t)$, we can separate the time and space dependence just as we had in Chapter **??**. In each case we find that a function of time equals a function of the spatial variables. Thus, these functions must be constant functions. We set these equal to the constant $-\lambda$ and find the respective equations

$$\frac{1}{k}\frac{T'}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda, \tag{6.5}$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{\nabla^2\phi}{\phi} = -\lambda. \tag{6.6}$$

The sign of λ is chosen because we expect decaying solutions in time for the heat equation and oscillations in time for the wave equation and will pick $\lambda > 0$.

The respective equations for the temporal functions $T(t)$ are given by

$$T' = -\lambda k T, \tag{6.7}$$

$$T'' + c^2\lambda T = 0. \tag{6.8}$$

These are easily solved as we had seen in Chapter **??**. We have

$$T(t) = T(0)e^{-\lambda k t}, \tag{6.9}$$

$$T(t) = a\cos\omega t + b\sin\omega t, \quad \omega = c\sqrt{\lambda}, \tag{6.10}$$

where $T(0)$, a, and b are integration constants and ω is the angular frequency of vibration.

The Helmholtz equation.

In both cases the spatial equation is of the same form,

$$\nabla^2\phi + \lambda\phi = 0. \tag{6.11}$$

The Helmholtz equation is named after Hermann Ludwig Ferdinand von Helmholtz (1821-1894). He was both a physician and a physicist and made significant contributions in physiology, optics, acoustics, and electromagnetism.

This equation is called the Helmholtz equation. For one dimensional problems, which we have already solved, the Helmholtz equation takes the form $\phi'' + \lambda\phi = 0$. We had to impose the boundary conditions and found that there were a discrete set of eigenvalues, λ_n, and associated eigenfunctions, ϕ_n.

In higher dimensional problems we need to further separate out the spatial dependence. We will again use the boundary conditions to find the eigenvalues, λ, and eigenfunctions, $\phi(\mathbf{r})$, for the Helmholtz equation, though the eigenfunctions will be labeled with more than one index. The resulting boundary value problems are often second order ordinary differential equations, which can be set up as Sturm-Liouville problems. We know from Chapter 5 that such problems possess an orthogonal set of eigenfunctions. These can then be used to construct a general solution from the product solutions which may involve elementary, or special, functions, such as Legendre polynomials and Bessel functions.

We will begin our study of higher dimensional problems by considering the vibrations of two dimensional membranes. First we will solve the problem of a vibrating rectangular membrane and then we will turn our attention to a vibrating circular membrane. The rest of the chapter will be devoted to the study of other two and three dimensional problems possessing cylindrical or spherical symmetry.

6.1 *Vibrations of Rectangular Membranes*

Our first example will be the study of the vibrations of a rectangular membrane. You can think of this as a drumhead with a rectangular cross section as shown in Figure 6.1. We stretch the membrane over the drumhead and fasten the material to the boundary of the rectangle. The height of the vibrating membrane is described by its height from equilibrium, $u(x,y,t)$. This problem is a much simpler example of higher dimensional vibrations than that possessed by the oscillating electric and magnetic fields in the last chapter.

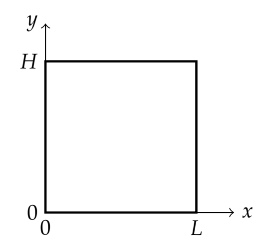

Figure 6.1: The rectangular membrane of length L and width H. There are fixed boundary conditions along the edges.

Example 6.1. The vibrating rectangular membrane.

The problem is given by the two dimensional wave equation in Cartesian coordinates,

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{6.12}$$

a set of boundary conditions,

$$\begin{aligned} &u(0,y,t) = 0, \quad u(L,y,t) = 0, \quad t > 0, \quad 0 < y < H, \\ &u(x,0,t) = 0, \quad u(x,H,t) = 0, \quad t > 0, \quad 0 < x < L, \end{aligned} \tag{6.13}$$

and a pair of initial conditions (since the equation is second order in time),

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y). \tag{6.14}$$

The first step is to separate the variables: $u(x,y,t) = X(x)Y(y)T(t)$. Inserting the guess, $u(x,y,t)$ into the wave equation, we have

$$X(x)Y(y)T''(t) = c^2\left(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)\right).$$

Dividing by both $u(x,y,t)$ and c^2, we obtain

$$\underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{Function of } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{Function of } x \text{ and } y} = -\lambda. \tag{6.15}$$

We see that we have a function of t equals a function of x and y. Thus, both expressions are constant. We expect oscillations in time, so we choose the constant λ to be positive, $\lambda > 0$. (Note: As usual, the primes mean differentiation with respect to the specific dependent variable. So, there should be no ambiguity.)

These lead to two equations:

$$T'' + c^2\lambda T = 0, \tag{6.16}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda. \tag{6.17}$$

We note that the spatial equation is just the separated form of Helmholtz's equation with $\phi(x, y) = X(x)Y(y)$.

The first equation is easily solved. We have

$$T(t) = a\cos\omega t + b\sin\omega t, \tag{6.18}$$

where

$$\omega = c\sqrt{\lambda}. \tag{6.19}$$

This is the angular frequency in terms of the separation constant, or eigenvalue. It leads to the frequency of oscillations for the various harmonics of the vibrating membrane as

$$\nu = \frac{\omega}{2\pi} = \frac{c}{2\pi}\sqrt{\lambda}. \tag{6.20}$$

Once we know λ, we can compute these frequencies.

Next we solve the spatial equation. We need carry out another separation of variables. Rearranging the spatial equation, we have

$$\underbrace{\frac{X''}{X}}_{\text{Function of } x} = \underbrace{-\frac{Y''}{Y} - \lambda}_{\text{Function of } y} = -\mu. \tag{6.21}$$

Here we have a function of x equal to a function of y. So, the two expressions are constant, which we indicate with a second separation constant, $-\mu < 0$. We pick the sign in this way because we expect oscillatory solutions for $X(x)$. This leads to two equations:

$$\begin{aligned} X'' + \mu X &= 0, \\ Y'' + (\lambda - \mu)Y &= 0. \end{aligned} \tag{6.22}$$

We now impose the boundary conditions. We have $u(0, y, t) = 0$ for all $t > 0$ and $0 < y < H$. This implies that $X(0)Y(y)T(t) = 0$ for all t and y in the domain. This is only true if $X(0) = 0$. Similarly, from the other boundary conditions we find that $X(L) = 0$, $Y(0) = 0$, and $Y(H) = 0$. We note that homogeneous boundary conditions are important in carrying out this process. Nonhomogeneous boundary conditions could be imposed just like we had in Section 1.7, but we still need the solutions for homogeneous boundary conditions before tackling the more general problems.

In summary, the boundary value problems we need to solve are:

$$\begin{aligned} X'' + \mu X = 0, &\quad X(0) = 0, X(L) = 0. \\ Y'' + (\lambda - \mu)Y = 0, &\quad Y(0) = 0, Y(H) = 0. \end{aligned} \tag{6.23}$$

We have seen boundary value problems of these forms in Chapter **??**. The solutions of the first eigenvalue problem are

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The second eigenvalue problem is solved in the same manner. The differences from the first problem are that the "eigenvalue" is $\lambda-\mu$, the independent variable is y, and the interval is $[0, H]$. Thus, we can quickly write down the solutions as

$$Y_m(y) = \sin\frac{m\pi x}{H}, \quad \lambda-\mu_m = \left(\frac{m\pi}{H}\right)^2, \quad m = 1,2,3,\ldots.$$

At this point we need to be careful about the indexing of the separation constants. So far, we have seen that μ depends on n and that the quantity $\kappa = \lambda-\mu$ depends on m. Solving for λ, we should write $\lambda_{nm} = \mu_n + \kappa_m$, or

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad n,m = 1,2,\ldots. \tag{6.24}$$

Since $\omega = c\sqrt{\lambda}$, we have that the discrete frequencies of the harmonics are given by

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}, \quad n,m = 1,2,\ldots. \tag{6.25}$$

The harmonics for the vibrating rectangular membrane are given by

$$\nu_{nm} = \frac{c}{2}\sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{H}\right)^2},$$

for $n,m = 1,2,\ldots$.

We have successfully carried out the separation of variables for the wave equation for the vibrating rectangular membrane. The product solutions can be written as

$$u_{nm} = (a\cos\omega_{nm}t + b\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H} \tag{6.26}$$

and the most general solution is written as a linear combination of the product solutions,

$$u(x,y,t) = \sum_{n,m}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}.$$

However, before we carry the general solution any further, we will first concentrate on the two dimensional harmonics of this membrane.

For the vibrating string the nth harmonic corresponds to the function $\sin\frac{n\pi x}{L}$ and several are shown in Figure 6.2. The various harmonics correspond to the pure tones supported by the string. These then lead to the corresponding frequencies that one would hear. The actual shapes of the harmonics are sketched by locating the nodes, or places on the string that do not move.

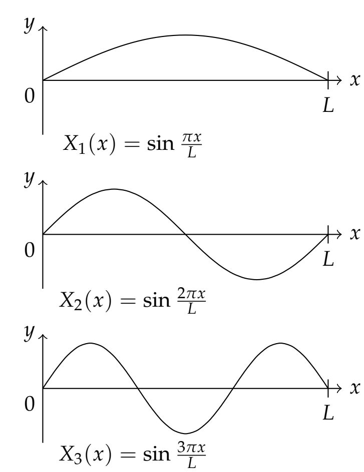

Figure 6.2: The first harmonics of the vibrating string

In the same way, we can explore the shapes of the harmonics of the vibrating membrane. These are given by the spatial functions

$$\phi_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{6.27}$$

Instead of nodes, we will look for the nodal curves, or nodal lines. These are the points (x,y) at which $\phi_{nm}(x,y) = 0$. Of course, these depend on the indices, n and m.

A discussion of the nodal lines.

For example, when $n = 1$ and $m = 1$, we have

$$\sin\frac{\pi x}{L}\sin\frac{\pi y}{H} = 0.$$

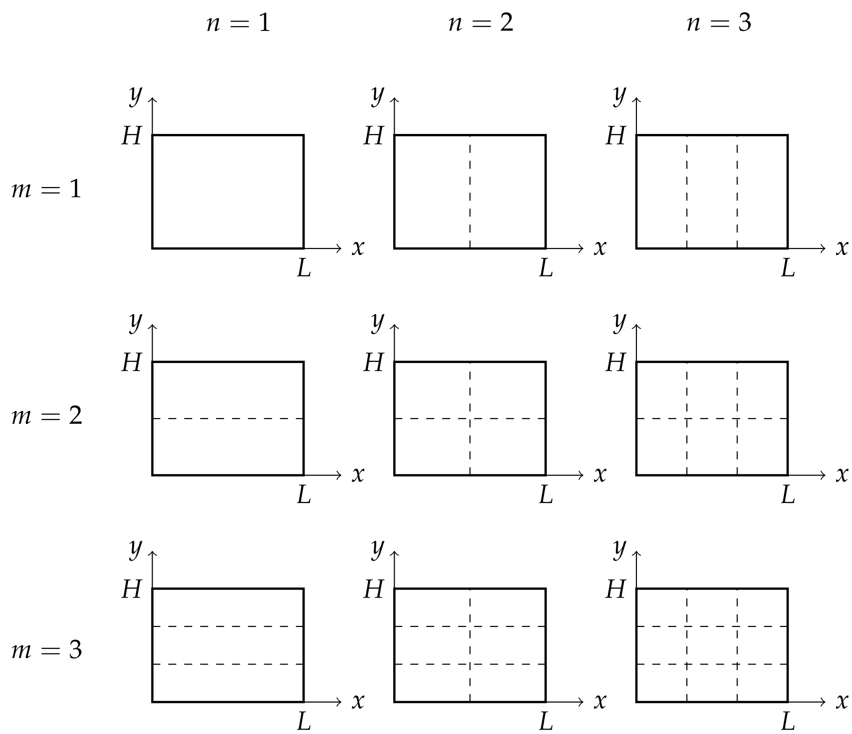

Figure 6.3: The first few modes of the vibrating rectangular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these the nodal lines to the 3D view in Figure 6.1

These are zero when either

$$\sin \frac{\pi x}{L} = 0, \quad \text{or} \quad \sin \frac{\pi y}{H} = 0.$$

Of course, this can only happen for $x = 0, L$ and $y = 0, H$. Thus, there are no interior nodal lines.

When $n = 2$ and $m = 1$, we have $y = 0, H$ and

$$\sin \frac{2\pi x}{L} = 0,$$

or, $x = 0, \frac{L}{2}, L$. Thus, there is one interior nodal line at $x = \frac{L}{2}$. These points stay fixed during the oscillation and all other points oscillate on either side of this line. A similar solution shape results for the (1,2)-mode; i.e., $n = 1$ and $m = 2$.

In Figure 6.3 we show the nodal lines for several modes for $n, m = 1, 2, 3$ with different columns corresponding to different n-values while the rows are labeled with different m-values. The blocked regions appear to vibrate independently. A better view is the three dimensional view depicted in Figure 6.1 . The frequencies of vibration are easily computed using the formula for ω_{nm}.

For completeness, we now return to the general solution and apply the initial conditions. The general solution is given by a linear superposition of the product solutions. There are two indices to sum over. Thus, the general solution is

The general solution for the vibrating rectangular membrane.

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \quad (6.28)$$

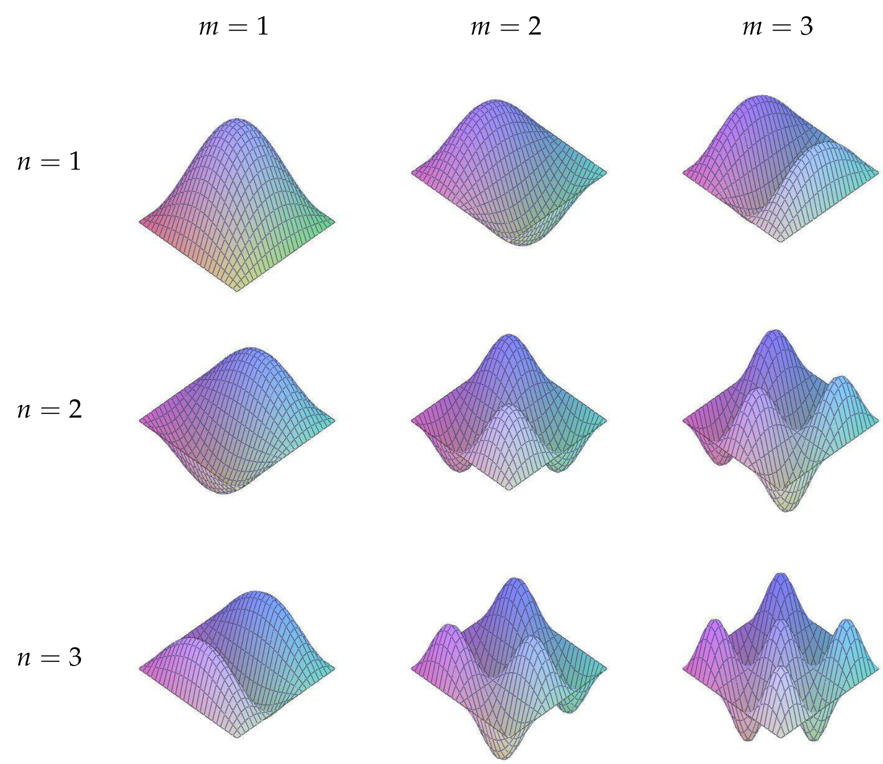

Table 6.1: A three dimensional view of the vibrating rectangular membrane for the lowest modes. Compare these images with the nodal lines in Figure 6.3

where

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{6.29}$$

The first initial condition is $u(x, y, 0) = f(x, y)$. Setting $t = 0$ in the general solution, we obtain

$$f(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}. \tag{6.30}$$

This is a double Fourier sine series. The goal is to find the unknown coefficients a_{nm}.

The coefficients a_{nm} can be found knowing what we already know about Fourier sine series. We can write the initial condition as the single sum

$$f(x, y) = \sum_{n=1}^{\infty} A_n(y) \sin\frac{n\pi x}{L}, \tag{6.31}$$

where

$$A_n(y) = \sum_{m=1}^{\infty} a_{nm} \sin\frac{m\pi y}{H}. \tag{6.32}$$

These are two Fourier sine series. Recalling from Chapter **??** that the coefficients of Fourier sine series can be computed as integrals, we have

$$\begin{aligned} A_n(y) &= \frac{2}{L}\int_0^L f(x, y) \sin\frac{n\pi x}{L}\, dx, \\ a_{nm} &= \frac{2}{H}\int_0^H A_n(y) \sin\frac{m\pi y}{H}\, dy. \end{aligned} \tag{6.33}$$

Inserting the integral for $A_n(y)$ into that for a_{nm}, we have an integral representation for the Fourier coefficients in the double Fourier sine series,

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy. \tag{6.34}$$

The Fourier coefficients for the double Fourier sine series.

We can carry out the same process for satisfying the second initial condition, $u_t(x,y,0) = g(x,y)$ for the initial velocity of each point. Inserting the general solution into this initial condition, we obtain

$$g(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm}\omega_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{6.35}$$

Again, we have a double Fourier sine series. But, now we can quickly determine the Fourier coefficients using the above expression for a_{nm} to find that

$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy. \tag{6.36}$$

This completes the full solution of the vibrating rectangular membrane problem. Namely, we have obtained the solution

The full solution of the vibrating rectangular membrane.

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \tag{6.37}$$

where

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy, \tag{6.38}$$

$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x,y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy, \tag{6.39}$$

and the angular frequencies are given by

$$\omega_{nm} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \tag{6.40}$$

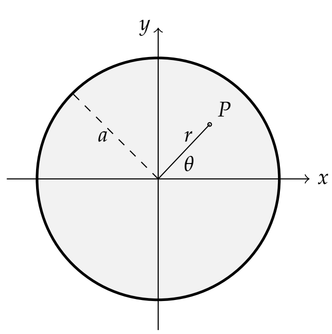

Figure 6.4: The circular membrane of radius a. A general point on the membrane is given by the distance from the center, r, and the angle, . There are fixed boundary conditions along the edge at $r = a$.

6.2 *Vibrations of a Kettle Drum*

In this section we consider the vibrations of a circular membrane of radius a as shown in Figure 6.4. Again we are looking for the harmonics of the vibrating membrane, but with the membrane fixed around the circular boundary given by $x^2 + y^2 = a^2$. However, expressing the boundary condition in Cartesian coordinates is awkward. Namely, we can only write $u(x,y,t) = 0$ for $x^2 + y^2 = a^2$. It is more natural to use polar coordinates as indicated in Figure 6.4. Let the height of the membrane be given by $u = u(r,\theta,t)$ at time t and position (r,θ). Now the boundary condition is given as $u(a,\theta,t) = 0$ for all $t > 0$ and $\theta \in [0,2\pi]$.

Before solving the initial-boundary value problem, we have to cast the full problem in polar coordinates. This means that we need to rewrite the

Laplacian in r and θ. To do so would require that we know how to transform derivatives in x and y into derivatives with respect to r and θ. Using the results from Section **??** on curvilinear coordinates, we know that the Laplacian can be written in polar coordinates. In fact, we could use the results from Problem **??** in Chapter **??** for cylindrical coordinates for functions which are z-independent, $f = f(r, \theta)$. Then, we would have

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

Derivation of Laplacian in polar coordinates.

We can obtain this result using a more direct approach, namely applying the Chain Rule in higher dimensions. First recall the transformations between polar and Cartesian coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}.$$

Now, consider a function $f = f(x(r,\theta), y(r,\theta)) = g(r,\theta)$. (Technically, once we transform a given function of Cartesian coordinates we obtain a new function g of the polar coordinates. Many texts do not rigorously distinguish between the two functions.) Thinking of $x = x(r,\theta)$ and $y = y(r,\theta)$, we have from the chain rule for functions of two variables:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x} \\
&= \frac{\partial g}{\partial r}\frac{x}{r} - \frac{\partial g}{\partial \theta}\frac{y}{r^2} \\
&= \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}. \qquad (6.41)
\end{aligned}$$

Here we have used

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r};$$

and

$$\frac{\partial \theta}{\partial x} = \frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right) = \frac{-y/x^2}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{r^2}.$$

Similarly,

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y} \\
&= \frac{\partial g}{\partial r}\frac{y}{r} + \frac{\partial g}{\partial \theta}\frac{x}{r^2} \\
&= \sin\theta\frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}. \qquad (6.42)
\end{aligned}$$

The 2D Laplacian can now be computed as

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = {} & \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) \\
& + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \cos\theta \frac{\partial}{\partial r}\left(\cos\theta \frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\
&\quad - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\
&\quad + \sin\theta \frac{\partial}{\partial r}\left(\sin\theta \frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\
&\quad + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial r} + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\
&= \cos\theta\left(\cos\theta \frac{\partial^2 g}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial g}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 g}{\partial r \partial \theta}\right) \\
&\quad - \frac{\sin\theta}{r}\left(\cos\theta \frac{\partial^2 g}{\partial \theta \partial r} - \frac{\sin\theta}{r}\frac{\partial^2 g}{\partial \theta^2} - \sin\theta \frac{\partial g}{\partial r} - \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}\right) \\
&\quad + \sin\theta\left(\sin\theta \frac{\partial^2 g}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2 g}{\partial r \partial \theta} - \frac{\cos\theta}{r^2}\frac{\partial g}{\partial \theta}\right) \\
&\quad + \frac{\cos\theta}{r}\left(\sin\theta \frac{\partial^2 g}{\partial \theta \partial r} + \frac{\cos\theta}{r}\frac{\partial^2 g}{\partial \theta^2} + \cos\theta \frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}\right) \\
&= \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}.
\end{aligned}
\tag{6.43}
$$

The last form often occurs in texts because it is in the form of a Sturm-Liouville operator. Also, it agrees with the result from using the Laplacian written in cylindrical coordinates as given in Problem **??** of Chapter **??**.

Now that we have written the Laplacian in polar coordinates we can pose the problem of a vibrating circular membrane.

Example 6.2. The vibrating circular membrane.

This problem is given by a partial differential equation,[1]

$$u_{tt} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right], \tag{6.44}$$

$$t > 0, \quad 0 < r < a, \quad -\pi < \theta < \pi,$$

the boundary condition,

$$u(a,\theta,t) = 0, \quad t > 0, \quad -\pi < \theta < \pi, \tag{6.45}$$

and the initial conditions,

$$
\begin{aligned}
u(r,\theta,0) &= f(r,\theta), \quad 0 < r < a, -\pi < \theta < \pi, \\
u_t(r,\theta,0) &= g(r,\theta),, \quad 0 < r < a, -\pi < \theta < \pi.
\end{aligned}
\tag{6.46}
$$

Now we are ready to solve this problem using separation of variables. As before, we can separate out the time dependence. Let $u(r,\theta,t) = T(t)\phi(r,\theta)$. As usual, $T(t)$ can be written in terms of sines and cosines. This leads to the Helmholtz equation,

$$\nabla^2\phi + \lambda\phi = 0.$$

[1] Here we state the problem of a vibrating circular membrane. We have chosen $-\pi < \theta < \pi$, but could have just as easily used $0 < \theta < 2\pi$. The symmetric interval about $\theta = 0$ will make the use of boundary conditions simpler.

We now separate the Helmholtz equation by letting $\phi(r,\theta) = R(r)\Theta(\theta)$. This gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\Theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 R\Theta}{\partial \theta^2} + \lambda R\Theta = 0. \tag{6.47}$$

Dividing by $u = R\Theta$, as usual, leads to

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \lambda = 0. \tag{6.48}$$

The last term is a constant. The first term is a function of r. However, the middle term involves both r and θ. This can be remedied by multiplying the equation by r^2. Rearranging the resulting equation, we can separate out the θ-dependence from the radial dependence. Letting μ be another separation constant, we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu. \tag{6.49}$$

This gives us two ordinary differential equations:

$$\begin{aligned} \frac{d^2\Theta}{d\theta^2} + \mu\Theta &= 0, \\ r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - \mu)R &= 0. \end{aligned} \tag{6.50}$$

Let's consider the first of these equations. It should look familiar by now. For $\mu > 0$, the general solution is

$$\Theta(\theta) = a\cos\sqrt{\mu}\theta + b\sin\sqrt{\mu}\theta.$$

The next step typically is to apply the boundary conditions in θ. However, when we look at the given boundary conditions in the problem, we do not see anything involving θ. This is a case for which the boundary conditions that are needed are implied and not stated outright.

We can determine the hidden boundary conditions by making some observations. Let's consider the solution corresponding to the endpoints $\theta = \pm\pi$. We note that at these θ-values we are at the same physical point for any $r < a$. So, we would expect the solution to have the same value at $\theta = -\pi$ as it has at $\theta = \pi$. Namely, the solution is continuous at these physical points. Similarly, we expect the slope of the solution to be the same at these points. This can be summarized using the boundary conditions

The boundary conditions in θ are periodic boundary conditions.

$$\Theta(\pi) = \Theta(-\pi), \quad \Theta'(\pi) = \Theta'(-\pi).$$

Such boundary conditions are called periodic boundary conditions.

Let's apply these conditions to the general solution for $\Theta(\theta)$. First, we set $\Theta(\pi) = \Theta(-\pi)$ and use the symmetries of the sine and cosine functions to obtain

$$a\cos\sqrt{\mu}\pi + b\sin\sqrt{\mu}\pi = a\cos\sqrt{\mu}\pi - b\sin\sqrt{\mu}\pi.$$

This implies that

$$\sin\sqrt{\mu}\pi = 0.$$

This can only be true for $\sqrt{\mu} = m$, for $m = 0, 1, 2, 3, \ldots$. Therefore, the eigenfunctions are given by

$$\Theta_m(\theta) = a\cos m\theta + b\sin m\theta, \quad m = 0, 1, 2, 3, \ldots.$$

For the other half of the periodic boundary conditions, $\Theta'(\pi) = \Theta'(-\pi)$, we have that

$$-am\sin m\pi + bm\cos m\pi = am\sin m\pi + bm\cos m\pi.$$

But, this gives no new information since this equation boils down to $bm = bm$..

To summarize what we know at this point, we have found the general solutions to the temporal and angular equations. The product solutions will have various products of $\{\cos\omega t, \sin\omega t\}$ and $\{\cos m\theta, \sin m\theta\}_{m=0}^{\infty}$. We also know that $\mu = m^2$ and $\omega = c\sqrt{\lambda}$.

We still need to solve the radial equation. Inserting $\mu = m^2$, the radial equation has the form

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (\lambda r^2 - m^2)R = 0. \tag{6.51}$$

Expanding the derivative term, we have

$$r^2R''(r) + rR'(r) + (\lambda r^2 - m^2)R(r) = 0. \tag{6.52}$$

The reader should recognize this differential equation from Equation (5.66). It is a Bessel equation with bounded solutions $R(r) = J_m(\sqrt{\lambda}r)$.

Recall there are two linearly independent solutions of this second order equation: $J_m(\sqrt{\lambda}r)$, the Bessel function of the first kind of order m, and $N_m(\sqrt{\lambda}r)$, the Bessel function of the second kind of order m, or Neumann functions. Plots of these functions are shown in Figures 5.8 and 5.9. So, we have the general solution of the radial equation is

$$R(r) = c_1J_m(\sqrt{\lambda}r) + c_2N_m(\sqrt{\lambda}r).$$

Now we are ready to apply the boundary conditions to the radial factor in the product solutions. Looking at the original problem we find only one condition: $u(a, \theta, t) = 0$ for $t > 0$ and $-\pi < \ < \pi$. This implies that $R(a) = 0$. But where is the second condition?

This is another unstated boundary condition. Look again at the plots of the Bessel functions. Notice that the Neumann functions are not well behaved at the origin. Do you expect that the solution will become infinite at the center of the drum? No, the solutions should be finite at the center. So, this observation leads to the second boundary condition. Namely, $|R(0)| < \infty$. This implies that $c_2 = 0$.

Now we are left with

$$R(r) = J_m(\sqrt{\lambda}r).$$

We have set $c_1 = 1$ for simplicity. We can apply the vanishing condition at $r = a$. This gives

$$J_m(\sqrt{\lambda}a) = 0.$$

Looking again at the plots of $J_m(x)$, we see that there are an infinite number of zeros, but they are not as easy as π! In Table 6.2 we list the nth zeros of J_m, which were first seen in Table 5.3.

n	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.324	14.796	16.223	17.616	18.980
5	14.931	16.471	17.960	19.409	20.827	22.218
6	18.071	19.616	21.117	22.583	24.019	25.430
7	21.212	22.760	24.270	25.748	27.199	28.627
8	24.352	25.904	27.421	28.908	30.371	31.812
9	27.493	29.047	30.569	32.065	33.537	34.989

Table 6.2: The zeros of Bessel Functions, $J_m(j_{mn}) = 0$.

Let's denote the nth zero of $J_m(x)$ by j_{mn}. Then, the boundary condition tells us that

$$\sqrt{\lambda}a = j_{mn}, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

This gives us the eigenvalues as

$$\lambda_{mn} = \left(\frac{j_{mn}}{a}\right)^2, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

Thus, the radial function satisfying the boundary conditions is

$$R_{mn}(r) = J_m\left(\frac{j_{mn}}{a} r\right).$$

We are finally ready to write out the product solutions for the vibrating circular membrane. They are given by

Product solutions for the vibrating circular membrane.

$$u(r, \theta, t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m\left(\frac{j_{mn}}{a} r\right). \tag{6.53}$$

Here we have indicated choices with the braces, leading to four different types of product solutions. Also, the angular frequency depends on the zeros of the Bessel functions,

$$\omega_{mn} = \frac{j_{mn}}{a} c, \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

As with the rectangular membrane, we are interested in the shapes of the harmonics. So, we consider the spatial solution ($t = 0$)

$$\phi(r, \theta) = (\cos m\theta) J_m\left(\frac{j_{mn}}{a} r\right).$$

Including the solutions involving $\sin m\theta$ will only rotate these modes. The nodal curves are given by $\phi(r, \theta) = 0$. This can be satisfied if $\cos m\theta = 0$, or $J_m(\frac{j_{mn}}{a} r) = 0$. The various nodal curves which result are shown in Figure 6.5.

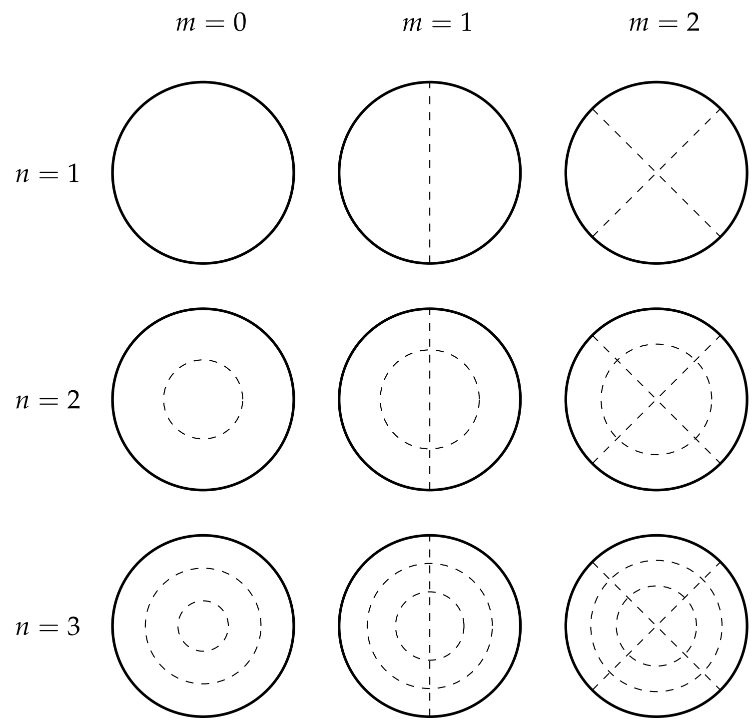

Figure 6.5: The first few modes of the vibrating circular membrane. The dashed lines show the nodal lines indicating the points that do not move for the particular mode. Compare these nodal lines with the three dimensional images in Figure 6.3.

For the angular part, we easily see that the nodal curves are radial lines, $\theta =$const. For $m = 0$, there are no solutions, since $\cos m\theta = 1$ for $m = 0$. in Figure 6.5 this is seen by the absence of radial lines in the first column.

For $m = 1$, we have $\cos\theta = 0$. This implies that $\theta = \pm\frac{\pi}{2}$. These values give the vertical line as shown in the second column in Figure 6.5. For $m = 2$, $\cos 2\theta = 0$ implies that $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. This results in the two lines shown in the last column of Figure 6.5.

We can also consider the nodal curves defined by the Bessel functions. We seek values of r for which $\frac{j_{mn}}{a}r$ is a zero of the Bessel function and lies in the interval $[0, a]$. Thus, we have

$$\frac{j_{mn}}{a}r = j_{mj}, \quad 1 \le j \le n,$$

or

$$r = \frac{j_{mj}}{j_{mn}}a, \quad 1 \le j \le n.$$

These will give circles of these radii with $j_{mj} \le j_{mn}$, or $j \le n$. For $m = 0$ and $n = 1$, there is only one zero and $r = a$. In fact, for all $n = 1$ modes, there is only one zero giving $r = a$. Thus, the first row in Figure 6.5 shows no interior nodal circles.

For a three dimensional view, one can look at Figure 6.3. Imagine that the various regions are oscillating independently and that the points on the nodal curves are not moving.

We should note that the nodal circles are not evenly spaced and that the radii can be computed relatively easily. For the $n = 2$ modes, we have two circles, $r = a$ and $r = \frac{j_{m1}}{j_{m2}}a$ as shown in the second row of Figure 6.5. For

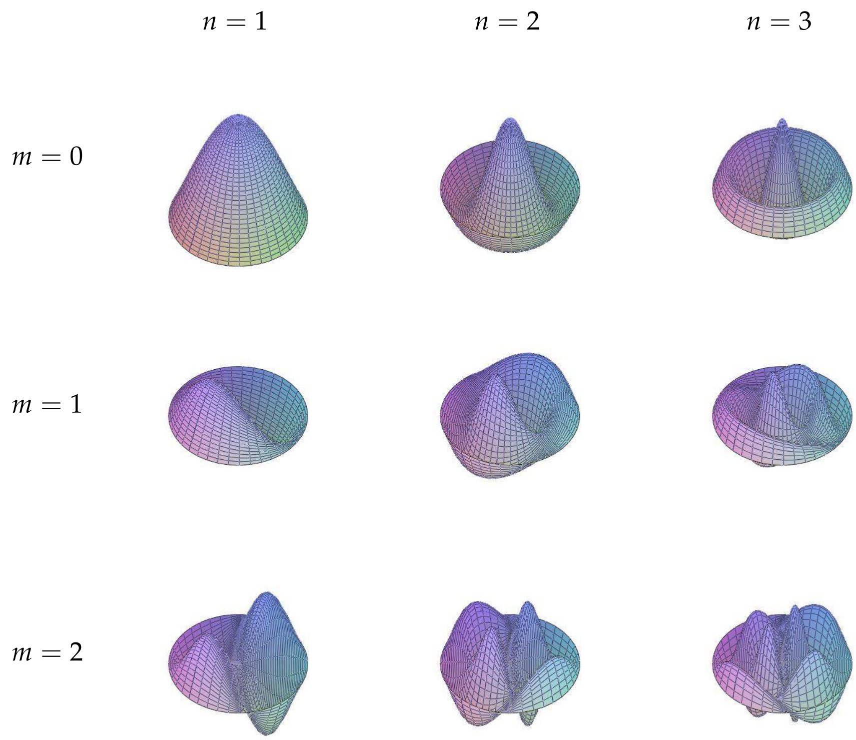

Table 6.3: A three dimensional view of the vibrating circular membrane for the lowest modes. Compare these images with the nodal line plots in Figure 6.5.

$m = 0$,

$$r = \frac{2.405}{5.520}a \approx 0.4357a$$

for the inner circle. For $m = 1$,

$$r = \frac{3.832}{7.016}a \approx 0.5462a,$$

and for $m = 2$,

$$r = \frac{5.136}{8.417}a \approx 0.6102a.$$

For $n = 3$ we obtain circles of radii

$$r = a, \quad r = \frac{j_{m1}}{j_{m3}}a, \text{ and } r = \frac{j_{m2}}{j_{m3}}a.$$

For $m = 0$,

$$r = a, \quad \frac{5.520}{8.654}a \approx 0.6379a, \quad \frac{2.405}{8.654}a \approx 0.2779a.$$

Similarly, for $m = 1$,

$$r = a, \quad \frac{3.832}{10.173}a \approx 0.3767a, \quad \frac{7.016}{10.173}a \approx 0.6897a,$$

and for $m = 2$,

$$r = a, \quad \frac{5.136}{11.620}a \approx 0.4420a, \quad \frac{8.417}{11.620}a \approx 0.7224a.$$

Example 6.3. Vibrating Annulus

More complicated vibrations can be dreamt up for this geometry. Consider an annulus in which the drum is formed from two concentric circular cylinders and the membrane is stretch between the two with an annular cross section as shown in Figure 6.6. The separation would follow as before except now the boundary conditions are that the membrane is fixed around the two circular boundaries. In this case we cannot toss out the Neumann functions because the origin is not part of the drum head.

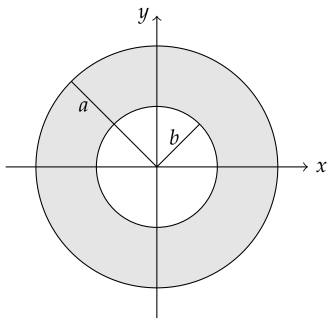

Figure 6.6: An annular membrane with radii a and $b > a$. There are fixed boundary conditions along the edges at $r = a$ and $r = b$.

The domain for this problem is shown in Figure 6.6 and the problem is given by the partial differential equation

$$u_{tt} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right], \tag{6.54}$$

$$t > 0, \quad b < r < a, \quad -\pi < \theta < \pi,$$

the boundary conditions,

$$u(b,\theta,t) = 0, \quad u(a,\theta,t) = 0, \quad t > 0, \quad -\pi < \theta < \pi, \tag{6.55}$$

and the initial conditions,

$$\begin{aligned} u(r,\theta,0) &= f(r,\theta), \quad b < r < a, -\pi < \theta < \pi, \\ u_t(r,\theta,0) &= g(r,\theta),, \quad b < r < a, -\pi < \theta < \pi. \end{aligned} \tag{6.56}$$

Since we cannot dispose of the Neumann functions, the product solutions take the form

$$u(r,\theta,t) = \left\{\begin{array}{c}\cos\omega t\\ \sin\omega t\end{array}\right\}\left\{\begin{array}{c}\cos m\theta\\ \sin m\theta\end{array}\right\}R_m(r), \tag{6.57}$$

where

$$R_m(r) = c_1 J_m(\sqrt{\lambda}r) + c_2 N_m(\sqrt{\lambda}r)$$

and $\omega = c\sqrt{\lambda}, \quad m = 0, 1, \ldots.$

For this problem the radial boundary conditions are that the membrane is fixed at $r = a$ and $r = b$. Taking $b < a$, we then have to satisfy the conditions

$$\begin{aligned} R(a) &= c_1 J_m(\sqrt{\lambda}a) + c_2 N_m(\sqrt{\lambda}a) = 0, \\ R(b) &= c_1 J_m(\sqrt{\lambda}b) + c_2 N_m(\sqrt{\lambda}b) = 0. \end{aligned} \tag{6.58}$$

This leads to two homogeneous equations for c_1 and c_2. The coefficient determinant of this system has to vanish if there are to be nontrivial solutions. This gives the eigenvalue equation for λ :

$$J_m(\sqrt{\lambda}a)N_m(\sqrt{\lambda}b) - J_m(\sqrt{\lambda}b)N_m(\sqrt{\lambda}a) = 0.$$

There are an infinite number of zeros of the function

$$F(\lambda) = \lambda : J_m(\sqrt{\lambda}a)N_m(\sqrt{\lambda}b) - J_m(\sqrt{\lambda}b)N_m(\sqrt{\lambda}a).$$

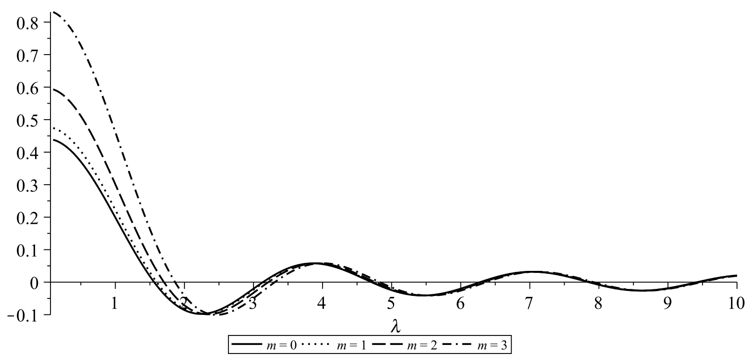

Figure 6.7: Plot of the function $F(\lambda) = J_m(\sqrt{\lambda}a)N_m(\sqrt{\lambda}b) - J_m(\sqrt{\lambda}b)N_m(\sqrt{\lambda}a)$ for $a = 4$ and $b = 2$ and $m = 0, 1, 2, 3$.

In Figure 6.7 we show a plot of $F(\lambda)$ for $a = 4$, $b = 2$ and $m = 0, 1, 2, 3$.

This eigenvalue equation needs to be solved numerically. Choosing $a = 2$ and $b = 4$, we have for the first few modes

$$\begin{aligned} \sqrt{\lambda_{mn}} &\approx 1.562, 3.137, 4.709, \quad m = 0 \\ &\approx 1.598, 3.156, 4.722, \quad m = 1 \\ &\approx 1.703, 3.214, 4.761, \quad m = 2. \end{aligned} \tag{6.59}$$

Note, since $\omega_{mn} = c\sqrt{\lambda_{mn}}$, these numbers essentially give us the frequencies of oscillation.

For these particular roots, we can solve for c_1 and c_2 up to a multiplicative constant. A simple solution is to set

$$c_1 = N_m(\sqrt{\lambda_{mn}}b), \quad c_2 = J_m(\sqrt{\lambda_{mn}}b).$$

This leads to the basic modes of vibration,

$$R_{mn}(r)\Theta_m(\theta) = \cos m\theta \left(N_m(\sqrt{\lambda_{mn}}b)J_m(\sqrt{\lambda_{mn}}r) - J_m(\sqrt{\lambda_{mn}}b)N_m(\sqrt{\lambda_{mn}}r) \right),$$

for $m = 0, 1, \ldots$, and $n = 1, 2, \ldots$. In Figure 6.4 we show various modes for the particular choice of annular membrane dimensions, $a = 2$ and $b = 4$.

6.3 *Laplace's Equation in 2D*

ANOTHER OF THE GENERIC PARTIAL DIFFERENTIAL EQUATIONS is Laplace's equation, $\nabla^2 u = 0$. This equation first appeared in the chapter on complex variables when we discussed harmonic functions. Another example is the electric potential for electrostatics. As we described Chapter **??**, for static electromagnetic fields,

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \quad \mathbf{E} = \nabla\phi.$$

In regions devoid of charge, these equations yield the Laplace equation $\nabla^2\phi = 0$.

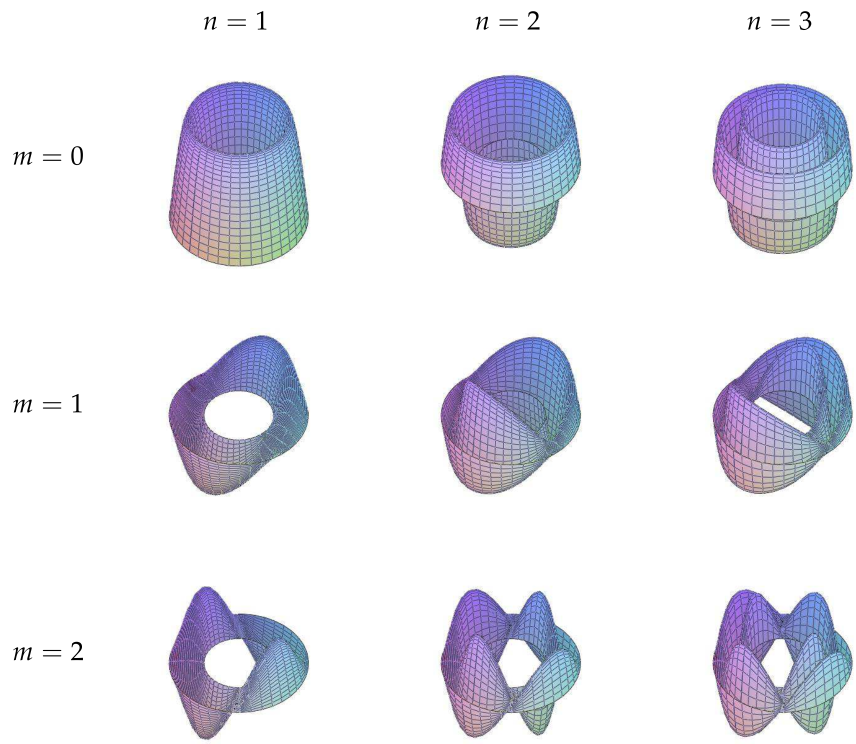

Table 6.4: A three dimensional view of the vibrating annular membrane for the lowest modes.

Another example comes from studying temperature distributions. Consider a thin rectangular plate with the boundaries set at fixed temperatures. Temperature changes of the plate are governed by the heat equation. The solution of the heat equation subject to these boundary conditions is time dependent. In fact, after a long period of time the plate will reach thermal equilibrium. If the boundary temperature is zero, then the plate temperature decays to zero across the plate. However, if the boundaries are maintained at a fixed nonzero temperature, which means energy is being put into the system to maintain the boundary conditions, the internal temperature may reach a nonzero equilibrium temperature. Reaching thermal equilibrium means that asymptotically in time the solution becomes time independent. Thus, the equilibrium state is a solution of the time independent heat equation, which is another Laplace equation, $\nabla^2 u = 0$.

Thermodynamic equilibrium, $\nabla^2 u = 0$.

Incompressible, irrotational fluid flow, $\nabla^2 \phi = 0$, for velocity $\mathbf{v} = \nabla \phi$.

As another example we could look at fluid flow. For an incompressible flow, $\nabla \cdot \mathbf{v} = 0$. If the flow is irrotational, then $\nabla \times \mathbf{v} = 0$. We can introduce a velocity potential, $\mathbf{v} = \nabla \phi$. Thus, $\nabla \times \mathbf{v}$ vanishes by a vector identity and $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2 \phi = 0$. So, once again we obtain Laplace's equation.

In this section we will look at examples of Laplace's equation in two dimensions. The solutions in these examples could be examples from any of the application in the above physical situations and the solutions can be applied appropriately.

Example 6.4. Equilibrium Temperature Distribution for a Rectangular Plate

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$

with the boundary conditions

$$u(0,y) = 0, \quad u(L,y) = 0, \quad u(x,0) = f(x), \quad u(x,H) = 0.$$

The boundary conditions are shown in Figure 6.8

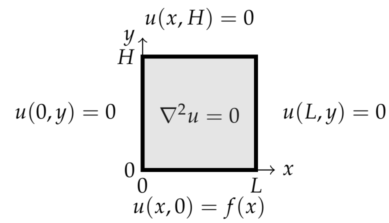

Figure 6.8: In this figure we show the domain and boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x,y) = X(x)Y(y)$. Then, Laplace's equation becomes

$$X''Y + XY'' = 0$$

and we can separate the x and y dependent functions and introduce a separation constant, λ,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Thus, we are led to two differential equations,

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned} \tag{6.60}$$

From the boundary condition $u(0,y) = 0, u(L,y) = 0$, we have $X(0) = 0, X(L) = 0$. So, we have the usual eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0.$$

The solutions to this problem are given by

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}.$$

The boundary condition $u(x,H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda}H} + c_2 e^{-\sqrt{\lambda}H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda}H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda}y} - c_1 e^{2\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \\ &= c_1 e^{\sqrt{\lambda}H}\left(e^{-\sqrt{\lambda}H} e^{\sqrt{\lambda}y} - e^{\sqrt{\lambda}H} e^{-\sqrt{\lambda}y}\right) \\ &= c_1 e^{\sqrt{\lambda}H}\left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)}\right) \\ &= -2c_1 e^{\sqrt{\lambda}H} \sinh\sqrt{\lambda}(H-y). \end{aligned} \tag{6.61}$$

Note: Having carried out this computation, we can now see that it would be better to guess this form in the future. So, for $Y(H) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}(H-y)$. For $Y(0) = 0$, one would guess a solution $Y(y) = \sinh\sqrt{\lambda}y$. Similarly, if $Y'(H) = 0$, one would guess a solution $Y(y) = \cosh\sqrt{\lambda}(H-y)$.

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that the y-dependence is given by

$$Y_n(y) = \sinh \frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x,y) = \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \quad n = 1, 2, \ldots.$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions and in the problem.

The remaining boundary condition, $u(x,0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}. \tag{6.62}$$

Now we apply the boundary condition, $u(x,0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi H}{L} \sin \frac{n\pi x}{L}. \tag{6.63}$$

Defining $b_n = a_n \sinh \frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{6.64}$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series. The Fourier coefficients are easily found as

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{6.65}$$

Since $a_n = b_n / \sinh \frac{n\pi H}{L}$, we can finish solving the problem. The solution is

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \tag{6.66}$$

where

$$a_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{6.67}$$

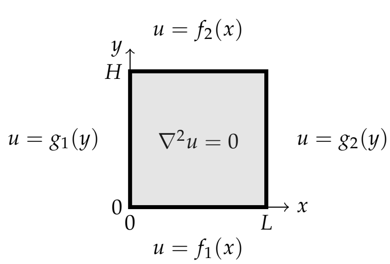

Figure 6.9: In this figure we show the domain and general boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

Example 6.5. Equilibrium Temperature Distribution for a Rectangular Plate for General Boundary Conditions

A more general problem is to seek solutions to Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, 0 < y < H$$

with non-zero boundary conditions on more than one side of the domain,

$$u(0,y) = g_1(y), \quad u(L,y) = g_2(y), \quad 0 < y < H,$$

$$u(x,0) = f_1(x), \quad u(x,H) = f_2(x), \quad 0 < x < L.$$

These boundary conditions are shown in Figure 6.9

The problem with this example is that none of the boundary conditions are homogeneous. This means that the corresponding eigenvalue problems will not have the homogeneous boundary conditions which Sturm-Liouville theory in Section 4 needs. However, we can express this problem in terms of four different problems with nonhomogeneous boundary conditions on only one side of the rectangle.

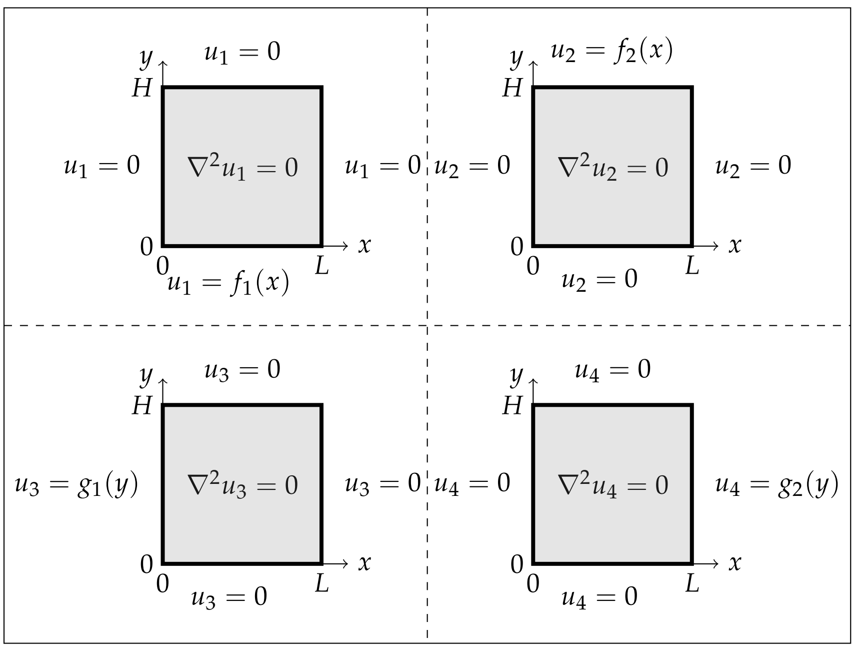

Figure 6.10: The general boundary value problem for a rectangular plate can be written as the sum of these four separate problems.

In Figure 6.10 we show how the problem can be broken up into four separate problems for functions $u_i(x,y)$, $i = 1, \ldots, 4$. Since the boundary conditions and Laplace's equation are linear, the solution to the general problem is simply the sum of the solutions to these four problems,

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

Then, this solution satisfies Laplace's equation,

$$\nabla^2 u(x,y) = \nabla^2 u_1(x,y) + \nabla^2 u_2(x,y) + \nabla^2 u_3(x,y) + \nabla^2 u_4(x,y) = 0,$$

and the boundary conditions. For example, using the boundary conditions defined in Figure 6.10, we have for $y = 0$,

$$u(x,0) = u_1(x,0) + u_2(x,0) + u_3(x,0) + u_4(x,0) = f_1(x).$$

The other boundary conditions can also be shown to hold.

We can solve each of the problems in Figure 6.10 quickly based on the solution we obtained in the last example. The solution for $u_1(x,y)$, which satisfies the boundary conditions

$$u_1(0,y) = 0, \quad u_1(L,y) = 0, \quad 0 < y < H,$$

$$u_1(x,0) = f_1(x), \quad u_1(x,H) = 0, \quad 0 < x < L,$$

is the easiest to write down. It is given by

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}. \tag{6.68}$$

where

$$a_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f_1(x) \sin \frac{n\pi x}{L}\, dx. \tag{6.69}$$

For the boundary conditions

$$u_2(0,y) = 0, \quad u_2(L,y) = 0, \quad 0 < y < H,$$

$$u_2(x,0) = 0, \quad u_2(x,H) = f_2(x), \quad 0 < x < L.$$

the boundary conditions for $X(x)$ are $X(0) = 0$ and $X(L) = 0$. So, we get the same form for the eigenvalues and eigenfunctions as before:

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, \ldots.$$

The remaining homogeneous boundary condition is now $Y(0) = 0$. Recalling that the equation satisfied by $Y(y)$ is

$$Y'' - \lambda Y = 0,$$

we can write the general solution as

$$Y(y) = c_1 \cosh \sqrt{\lambda} y + c_2 \sinh \sqrt{\lambda} y.$$

Requiring $Y(0) = 0$, we have $c_1 = 0$, or

$$Y(y) = c_2 \sinh \sqrt{\lambda} y.$$

Then, the general solution is

$$u_2(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}. \tag{6.70}$$

We now force the nonhomogeneous boundary condition, $u_2(x,H) = f_2(x)$,

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi H}{L}. \tag{6.71}$$

Once again we have a Fourier sine series. The Fourier coefficients are given by

$$b_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f_2(x) \sin \frac{n\pi x}{L}\, dx. \tag{6.72}$$

Next we turn to the problem with the boundary conditions

$$u_3(0,y) = g_1(y), \quad u_3(L,y) = 0, \quad 0 < y < H,$$
$$u_3(x,0) = 0, \quad u_3(x,H) = 0, \quad 0 < x < L.$$

In this case the pair of homogeneous boundary conditions $u_3(x,0) = 0, \quad u_3(x,H) = 0$ lead to solutions

$$Y_n(y) = \sin\frac{n\pi y}{H}, \quad \lambda_n = -\left(\frac{n\pi}{H}\right)^2, \quad n = 1, 2 \ldots.$$

The condition $u_3(L,0) = 0$ gives $X(x) = \sinh\frac{n\pi(L-x)}{H}$.

The general solution satisfying the homogeneous conditions is

$$u_3(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi(L-x)}{H}. \tag{6.73}$$

Applying the nonhomogeneous boundary condition, $u_3(0,y) = g_1(y)$, we obtain the Fourier sine series

$$g_1(y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi L}{H}. \tag{6.74}$$

The Fourier coefficients are found as

$$c_n = \frac{2}{H \sinh\frac{n\pi L}{H}} \int_0^H g_1(y) \sin\frac{n\pi y}{H}\, dy. \tag{6.75}$$

Finally, we can find the solution

$$u_4(0,y) = 0, \quad u_4(L,y) = g_2(y), \quad 0 < y < H,$$
$$u_4(x,0) = 0, \quad u_4(x,H) = 0, \quad 0 < x < L.$$

Following the above analysis, we find the general solution

$$u_4(x,y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi x}{H}. \tag{6.76}$$

The nonhomogeneous boundary condition, $u(L,y) = g_2(y)$, is satisfied if

$$g_2(y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi L}{H}. \tag{6.77}$$

The Fourier coefficients, d_n, are given by

$$d_n = \frac{2}{H \sinh\frac{n\pi L}{H}} \int_0^H g_1(y) \sin\frac{n\pi y}{H}\, dy. \tag{6.78}$$

The solution to the general problem is given by the sum of these four solutions.

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} \left[\left(a_n \sinh\frac{n\pi(H-y)}{L} + b_n \sinh\frac{n\pi y}{L} \right) \sin\frac{n\pi x}{L} \right. \\ &\quad \left. + \left(c_n \sinh\frac{n\pi(L-x)}{H} + d_n \sinh\frac{n\pi x}{H} \right) \sin\frac{n\pi y}{H} \right], \end{aligned} \tag{6.79}$$

where the coefficients are given by the above Fourier integrals.

Example 6.6. Laplace's Equation on a Disk

We now turn to solving Laplace's equation on a disk of radius a as shown in Figure 6.11. Laplace's equation in polar coordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0, \quad 0<r<a, \quad -\pi<\theta<\pi. \tag{6.80}$$

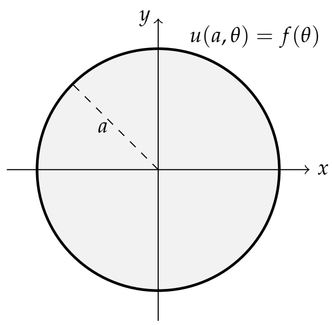

Figure 6.11: The disk of radius a with boundary condition along the edge at $r=a$.

The boundary conditions are given as

$$u(a,\theta)=f(\theta), \quad -\pi<\theta<\pi, \tag{6.81}$$

plus periodic boundary conditions in θ.

Separation of variable proceeds as usual. Let $u(r,\theta)=R(r)\Theta(\theta)$. Then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(R\Theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2(R\Theta)}{\partial \theta^2}=0, \tag{6.82}$$

or

$$\Theta\frac{1}{r}(rR')'+\frac{1}{r^2}R\Theta''=0. \tag{6.83}$$

Dividing by $u(r,\theta)=R(r)\Theta(\theta)$, multiplying by r^2, and rearranging, we have

$$\frac{r}{R}(rR')'=-\frac{\Theta''}{\Theta}=\lambda. \tag{6.84}$$

Since this equation gives a function of r equal to a function of θ, we set the equation equal to a constant. Thus, we have obtained two differential equations, which can be written as

$$r(rR')'-\lambda R = 0, \tag{6.85}$$
$$\Theta''+\lambda\Theta = 0. \tag{6.86}$$

We can solve the second equation subject to the periodic boundary conditions in the θ variable. The reader should be able to confirm that

$$\Theta(\theta)=a_n\cos n\theta+b_n\sin n\theta, \quad \lambda=n^2, n=0,1,2,\ldots$$

is the solution. Note that the $n=0$ case just leads to a constant solution.

Inserting $\lambda=n^2$ into the radial equation, we find

$$r^2R''+rR'-n^2R=0.$$

This is a Cauchy-Euler type of ordinary differential equation. Recall that we solve such equations by guessing a solution of the form $R(r)=r^m$. This leads to the characteristic equation $m^2-n^2=0$. Therefore, $m=\pm n$. So,

$$R(r)=c_1r^n+c_2r^{-n}.$$

Since we expect finite solutions at the origin, $r=0$, we can set $c_2=0$. Thus, the general solution is

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\theta+b_n\sin n\theta\right)r^n. \tag{6.87}$$

Note that we have taken the constant term out of the sum and put it into a familiar form.

Now we can impose the remaining boundary condition, $u(a,\theta) = f(\theta)$, or

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) a^n. \tag{6.88}$$

This is a Fourier trigonometric series. The Fourier coefficients can be determined using the results from Chapter 4:

$$\begin{aligned} a_n &= \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta, \quad n = 0, 1, \ldots, & (6.89) \\ b_n &= \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \quad n = 1, 2 \ldots. & (6.90) \end{aligned}$$

6.3.1 *Poisson Integral Formula*

WE CAN PUT THE SOLUTION FROM THE LAST EXAMPLE in a more compact form by inserting the Fourier coefficients into the general solution. Doing this, we have

$$\begin{aligned} u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, d\phi \\ &\quad + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \left[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\right] \left(\frac{r}{a}\right)^n f(\phi)\, d\phi \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n\right] f(\phi)\, d\phi. \end{aligned} \tag{6.91}$$

The term in the brackets can be summed. We note that

$$\begin{aligned} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n &= \operatorname{Re}\left(e^{in(\theta-\phi)} \left(\frac{r}{a}\right)^n\right) \\ &= \operatorname{Re}\left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n. \end{aligned} \tag{6.92}$$

Therefore,

$$\sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n = \operatorname{Re}\left(\sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n\right).$$

The right hand side of this equation is a geometric series with common ratio of $\frac{r}{a} e^{i(\theta-\phi)}$, which is also the first term of the series. Since $\left|\frac{r}{a} e^{i(\theta-\phi)}\right| = \frac{r}{a} < 1$, the series converges. Summing the series, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n &= \frac{\frac{r}{a} e^{i(\theta-\phi)}}{1 - \frac{r}{a} e^{i(\theta-\phi)}} \\ &= \frac{r e^{i(\theta-\phi)}}{a - r e^{i(\theta-\phi)}} \end{aligned} \tag{6.93}$$

We need to rewrite this result so that we can easily take the real part. Thus, we multiply and divide by the complex conjugate of the denominator to obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n &= \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}\frac{a-re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}} \\ &= \frac{are^{-i(\theta-\phi)}-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}. \end{aligned} \tag{6.94}$$

The real part of the sum is given as

$$\operatorname{Re}\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n\right) = \frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}.$$

Therefore, the factor in the brackets under the integral in Equation (6.91) is

$$\begin{aligned}\frac{1}{2}+\sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n &= \frac{1}{2}+\frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)} \\ &= \frac{a^2-r^2}{2(a^2+r^2-2ar\cos(\theta-\phi))}. \end{aligned} \tag{6.95}$$

Poisson Integral Formula

Thus, we have shown that the solution of Laplace's equation on a disk of radius a with boundary condition $u(a,\theta) = f(\theta)$ can be written in the closed form

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}f(\phi)\,d\phi. \tag{6.96}$$

This result is called the Poisson Integral Formula and

$$K(\theta,\phi) = \frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}$$

is called the Poisson kernel.

Example 6.7. Evaluate the solution (6.96) at the center of the disk.

We insert $r = 0$ into the solution (6.96) to obtain

$$u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\,d\phi.$$

Recalling that the average of a function $g(x)$ on $[a,b]$ is given by

$$g_{ave} = \frac{1}{b-a}\int_a^b g(x)\,dx,$$

we see that the value of the solution u at the center of the disk is the average of the boundary values. This is sometimes referred to as the mean value theorem.

6.4 Three Dimensional Cake Baking

In the rest of the chapter we will extend our studies to three dimensional problems. In this section we will solve the heat equation as we look at examples of baking cakes.

We consider cake batter, which is at room temperature of $T_i = 80°$F. It is placed into an oven, also at a fixed temperature, $T_b = 350°$F. For simplicity, we will assume that the thermal conductivity and cake density are constant. Of course, this is not quite true. However, it is an approximation which simplifies the model. We will consider two cases, one in which the cake is a rectangular solid, such as baking it in a $13'' \times 9'' \times 2''$ baking pan. The other case will lead to a cylindrical cake, such as you would obtain from a round cake pan.

This discussion of cake baking is adapted from R. Wilkinson's thesis work. That in turn was inspired by work done by Dr. Olszewski,(2006) From baking a cake to solving the diffusion equation. *American Journal of Physics* 74(6).

Assuming that the heat constant k is indeed constant and the temperature is given by $T(\mathbf{r}, t)$, we begin with the heat equation in three dimensions,

$$\frac{\partial T}{\partial t} = k\nabla^2 T. \tag{6.97}$$

We will need to specify initial and boundary conditions. Let T_i be the initial batter temperature, $T(x, y, z, 0) = T_i$.

We choose the boundary conditions to be fixed at the oven temperature T_b. However, these boundary conditions are not homogeneous and would lead to problems when carrying out separation of variables. This is easily remedied by subtracting the oven temperature from all temperatures involved and defining $u(\mathbf{r}, t) = T(\mathbf{r}, t) - T_b$. The heat equation then becomes

$$\frac{\partial u}{\partial t} = k\nabla^2 u \tag{6.98}$$

with initial condition

$$u(\mathbf{r}, 0) = T_i - T_b.$$

The boundary conditions are now homogeneous. We cannot be any more specific than this until we specify the geometry.

Example 6.8. Temperature of a Rectangular Cake

We will consider a rectangular cake with dimensions $0 \le x \le W$, $0 \le y \le L$, and $0 \le z \le H$ as show in Figure 6.12. For this problem, we seek solutions of the heat equation plus the conditions

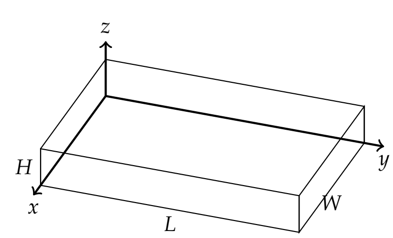

Figure 6.12: The dimensions of a rectangular cake.

$$\begin{aligned} u(x, y, z, 0) &= T_i - T_b, \\ u(0, y, z, t) = u(W, y, z, t) &= 0, \\ u(x, 0, z, t) = u(x, L, z, t) &= 0, \\ u(x, y, 0, t) = u(x, y, H, t) &= 0. \end{aligned}$$

Using the method of separation of variables, we seek solutions of the form

$$u(x, y, z, t) = X(x)Y(y)Z(z)G(t). \tag{6.99}$$

Substituting this form into the heat equation, we get

$$\frac{1}{k}\frac{G'}{G} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{6.100}$$

Setting these expressions equal to $-\lambda$, we get

$$\frac{1}{k}\frac{G'}{G} = -\lambda \quad \text{and} \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda. \tag{6.101}$$

Therefore, the equation for $G(t)$ is given by

$$G' + k\lambda G = 0.$$

We further have to separate out the functions of x, y, and z. We anticipate that the homogeneous boundary conditions will lead to oscillatory solutions in these variables. Therefore, we expect separation of variables will lead to the eigenvalue problems

$$\begin{aligned} X'' + \mu^2 X &= 0, \quad X(0) = X(W) = 0, \\ Y'' + \nu^2 Y &= 0, \quad Y(0) = Y(L) = 0, \\ Z'' + \kappa^2 Z &= 0, \quad Z(0) = Z(H) = 0. \end{aligned} \tag{6.102}$$

Noting that

$$\frac{X''}{X} = -\mu^2, \quad \frac{Y''}{Y} = -\nu^2, \quad \frac{Z''}{Z} = -\kappa^2,$$

we find from the heat equation that the separation constants are related,

$$\lambda^2 = \mu^2 + \nu^2 + \kappa^2.$$

We could have gotten to this point quicker by writing the first separated equation labeled with the separation constants as

$$\underbrace{\frac{1}{k}\frac{G'}{G}}_{-\lambda} = \underbrace{\frac{X''}{X}}_{-\mu} + \underbrace{\frac{Y''}{Y}}_{-\nu} + \underbrace{\frac{Z''}{Z}}_{-\kappa}.$$

Then, we can read off the eigenvalues problems and determine that $\lambda^2 = \mu^2 + \nu^2 + \kappa^2$.

From the boundary conditions, we get product solutions for $u(x, y, z, t)$ in the form

$$u_{mn\ell}(x, y, z, t) = \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z \, e^{-\lambda_{mn\ell} kt},$$

for

$$\lambda_{mnl} = \mu_m^2 + \nu_n^2 + \kappa_\ell^2 = \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{\ell\pi}{H}\right)^2, \quad m, n, \ell = 1, 2, \ldots.$$

The general solution is a linear combination of all of the product solutions, summed over three different indices,

$$u(x, y, z, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl} \sin \mu_m x \sin \nu_n y \sin \kappa_\ell z \, e^{-\lambda_{mn\ell} kt}, \tag{6.103}$$

where the $A_{mn\ell}$'s are arbitrary constants.

We can use the initial condition $u(x,y,z,0) = T_i - T_b$ to determine the $A_{mn\ell}$'s. We find

$$T_i - T_b = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl} \sin\mu_m x \sin\nu_n y \sin\kappa_\ell z. \qquad (6.104)$$

This is a triple Fourier sine series.

We can determine these coefficients in a manner similar to how we handled double Fourier sine series earlier in the chapter. Defining

$$b_m(y,z) = \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mnl} \sin\nu_n y \sin\kappa_\ell z,$$

we obtain a simple Fourier sine series:

$$T_i - T_b = \sum_{m=1}^{\infty} b_m(y,z) \sin\mu_m x. \qquad (6.105)$$

The Fourier coefficients can then be found as

$$b_m(y,z) = \frac{2}{W}\int_0^W (T_i - T_b) \sin\mu_m x\, dx.$$

Using the same technique for the remaining sine series and noting that $T_i - T_b$ is constant, we can determine the general coefficients A_{mnl} by carrying out the needed integrations:

$$\begin{aligned}
A_{mnl} &= \frac{8}{WLH}\int_0^H\int_0^L\int_0^W (T_i - T_b)\sin\mu_m x \sin\nu_n y \sin\kappa_\ell z\, dxdydz \\
&= (T_i - T_b)\frac{8}{\pi^3}\left[\frac{\cos\left(\frac{m\pi x}{W}\right)}{m}\right]_0^W \left[\frac{\cos\left(\frac{n\pi y}{L}\right)}{n}\right]_0^L \left[\frac{\cos\left(\frac{\ell\pi z}{H}\right)}{\ell}\right]_0^H \\
&= (T_i - T_b)\frac{8}{\pi^3}\left[\frac{\cos m\pi - 1}{m}\right]\left[\frac{\cos n\pi - 1}{n}\right]\left[\frac{\cos \ell\pi - 1}{\ell}\right] \\
&= (T_i - T_b)\frac{8}{\pi^3}\begin{cases} 0, & \text{for at least one } m,n,\ell \text{ even}, \\ \left[\frac{-2}{m}\right]\left[\frac{-2}{n}\right]\left[\frac{-2}{\ell}\right], & \text{for } m,n,\ell \text{ all odd}. \end{cases}
\end{aligned}$$

Since only the odd multiples yield non-zero $A_{mn\ell}$ we let $m = 2m' - 1$, $n = 2n' - 1$, and $\ell = 2\ell' - 1$ for $m', n', \ell' = 1, 2, \ldots$. The expansion coefficients can now be written in the simpler form

$$A_{mnl} = \frac{64(T_b - T_i)}{(2m'-1)(2n'-1)(2\ell'-1)\pi^3}.$$

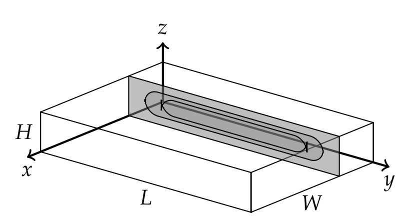

Figure 6.13: Rectangular cake showing a vertical slice.

Substituting this result into general solution and dropping the primes, we find

$$u(x,y,z,t) = \frac{64(T_b - T_i)}{\pi^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\sin\mu_m x \sin\nu_n y \sin\kappa_\ell z\, e^{-\lambda_{mn\ell}kt}}{(2m-1)(2n-1)(2\ell-1)},$$

where

$$\lambda_{mn\ell} = \left(\frac{(2m-1)\pi}{W}\right)^2 + \left(\frac{(2n-1)\pi}{L}\right)^2 + \left(\frac{(2\ell-1)\pi}{H}\right)^2$$

for $m, n, \ell = 1, 2, \ldots$.

Recalling that the solution to the physical problem is

$$T(x,y,z,t) = u(x,y,z,t) + T_b,$$

we have the final solution is given by

$$T(x,y,z,t) = T_b + \frac{64(T_b - T_i)}{\pi^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\sin \hat{\mu}_m x \sin \hat{\nu}_n y \sin \hat{\kappa}_\ell z \, e^{-\hat{\lambda}_{mn\ell} kt}}{(2m-1)(2n-1)(2\ell-1)}.$$

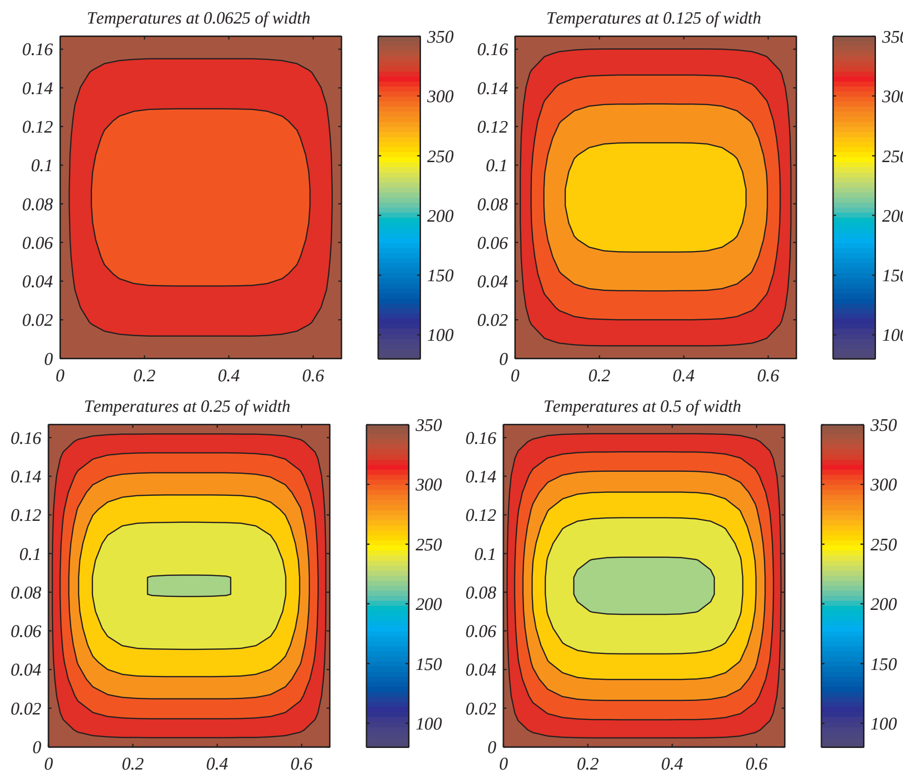

Figure 6.14: Temperature evolution for a $13'' \times 9'' \times 2''$ cake shown as vertical slices at the indicated length in feet.

We show some temperature distributions in Figure 6.14. Since we cannot capture the entire cake, we show vertical slices such as depicted in Figure 6.13. Vertical slices are taken at the positions and times indicated for a $13'' \times 9'' \times 2''$ cake. Obviously, this is not accurate because the cake consistency is changing and this will affect the parameter k. A more realistic model would be to allow $k = k(T(x,y,z,t))$. However, such problems are beyond the simple methods described in this book.

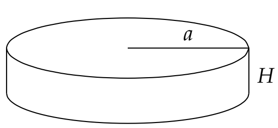

Figure 6.15: Geometry for a cylindrical cake.

Example 6.9. Circular Cakes

In this case the geometry of the cake is cylindrical as show in Figure 6.15. Therefore, we need to express the boundary conditions and heat equation in cylindrical coordinates. Also, we will assume that the solution, $u(r,z,t) = T(r,z,t) - T_b$, is independent of θ due to axial symmetry. This gives the heat equation in θ-independent cylindrical coordinates as

$$\frac{\partial u}{\partial t} = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right), \tag{6.106}$$

where $0 \leq r \leq a$ and $0 \leq z \leq Z$. The initial condition is

$$u(r,z,0) = T_i - T_b,$$

and the homogeneous boundary conditions on the side, top, and bottom of the cake are

$$u(a,z,t) = 0,$$
$$u(r,0,t) = u(r,Z,t) = 0.$$

Again, we seek solutions of the form $u(r,z,t) = R(r)H(z)G(t)$. Separation of variables leads to

$$\underbrace{\frac{1}{k}\frac{G'}{G}}_{-\lambda} = \underbrace{\frac{1}{rR}\frac{d}{dr}\left(rR'\right)}_{-\mu^2} + \underbrace{\frac{H''}{H}}_{-\nu^2}. \tag{6.107}$$

Here we have indicated the separation constants, which lead to three ordinary differential equations. These equations and the boundary conditions are

$$\begin{aligned}
G' + k\lambda G &= 0, \\
\frac{d}{dr}\left(rR'\right) + \mu^2 rR &= 0, \quad R(a) = 0, \quad R(0) \text{ is finite}, \\
H'' + \nu^2 H &= 0, \quad H(0) = H(Z) = 0.
\end{aligned} \tag{6.108}$$

We further note that the separation constants are related by $\lambda = \mu^2 + \nu^2$.

We can easily write down the solutions for $G(t)$ and $H(z)$,

$$G(t) = Ae^{-\lambda kt}$$

and

$$H_n(z) = \sin\frac{n\pi z}{Z}, \quad n = 1,2,3,\ldots,$$

where $\nu = \frac{n\pi}{Z}$. Recalling from the rectangular case that only odd terms arise in the Fourier sine series coefficients for the constant initial condition, we proceed by rewriting $H(z)$ as

$$H_n(z) = \sin\frac{(2n-1)\,\pi z}{Z}, \quad n = 1,2,3,\ldots \tag{6.109}$$

with $\nu = \frac{(2n-1)\pi}{Z}$.

The radial equation can be written in the form

$$r^2R'' + rR' + \mu^2 r^2 R = 0.$$

This is a Bessel equation of the first kind of order zero which we had seen in Section 5.5. Therefore, the general solution is a linear combination of Bessel functions of the first and second kind,

$$R(r) = c_1 J_0(\mu r) + c_2 N_0(\mu r). \tag{6.110}$$

Since $R(r)$ is bounded at $r = 0$ and $N_0(\mu r)$ is not well behaved at $r = 0$, we set $c_2 = 0$. Up to a constant factor, the solution becomes

$$R(r) = J_0(\mu r). \tag{6.111}$$

The boundary condition $R(a) = 0$ gives the eigenvalues as

$$\mu_m = \frac{j_{0m}}{a}, \quad m = 1, 2, 3, \ldots,$$

where j_{0m} is the m^{th} roots of the zeroth-order Bessel function, $J_0(j_{0m}) = 0$.

Therefore, we have found the product solutions

$$H_n(z)R_m(r)G(t) = \sin\frac{(2n-1)\,\pi z}{Z} J_0\left(\frac{r}{a} j_{0m}\right) e^{-\lambda_{nm} kt}, \tag{6.112}$$

where $m = 1, 2, 3, \ldots, n = 1, 2, \ldots$. Combining the product solutions, the general solution is found as

$$u(r, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{(2n-1)\,\pi z}{Z} J_0\left(\frac{r}{a} j_{0m}\right) e^{-\lambda_{nm} kt} \tag{6.113}$$

with

$$\lambda_{nm} = \left(\frac{(2n-1)\pi}{Z}\right)^2 + \left(\frac{j_{0m}}{a}\right)^2,$$

for $n, m = 1, 2, 3, \ldots$.

Inserting the solution into the constant initial condition, we have

$$T_i - T_b = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{(2n-1)\pi z}{Z} J_0\left(\frac{r}{a} j_{0m}\right).$$

This is a double Fourier series but it involves a Fourier-Bessel expansion. Writing

$$b_n(r) = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right),$$

the condition becomes

$$T_i - T_b = \sum_{n=1}^{\infty} b_n(r) \sin\frac{(2n-1)\pi z}{Z}.$$

As seen previously, this is a Fourier sine series and the Fourier coefficients are given by

$$\begin{aligned} b_n(r) &= \frac{2}{Z}\int_0^Z (T_i - T_b) \sin\frac{(2n-1)\,\pi z}{Z} dz \\ &= \frac{2(T_i - T_b)}{Z}\left[-\frac{Z}{(2n-1)\,\pi}\cos\frac{(2n-1)\,\pi z}{Z}\right]_0^Z \\ &= \frac{4(T_i - T_b)}{(2n-1)\,\pi}. \end{aligned}$$

We insert this result into the Fourier-Bessel series,

$$\frac{4(T_i - T_b)}{(2n-1)\,\pi} = \sum_{m=1}^{\infty} A_{nm} J_0\left(\frac{r}{a} j_{0m}\right),$$

and recall from Section 5.5 that we can determine the Fourier coefficients A_{nm} using the Fourier-Bessel series,

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(j_{pn} \frac{x}{a}), \tag{6.114}$$

where the Fourier-Bessel coefficients are found as

$$c_n = \frac{2}{a^2 \left[J_{p+1}(j_{pn}) \right]^2} \int_0^a x f(x) J_p(j_{pn} \frac{x}{a})\, dx. \tag{6.115}$$

Comparing these series expansions, we have

$$A_{nm} = \frac{2}{a^2 J_1^2(j_{0m})} \frac{4(T_i - T_b)}{(2n-1)\,\pi} \int_0^a J_0(\mu_m r) r\, dr. \tag{6.116}$$

In order to evaluate $\int_0^a J_0(\mu_m r) r\, dr$, we let $y = \mu_m r$ and get

$$\begin{aligned}
\int_0^a J_0(\mu_m r) r dr &= \int_0^{\mu_m a} J_0(y) \frac{y}{\mu_m} \frac{dy}{\mu_m} \\
&= \frac{1}{\mu_m^2} \int_0^{\mu_m a} J_0(y) y\, dy \\
&= \frac{1}{\mu_m^2} \int_0^{\mu_m a} \frac{d}{dy} \left(y J_1(y) \right)\, dy \\
&= \frac{1}{\mu_m^2} (\mu_m a) J_1(\mu_m a) = \frac{a^2}{j_{0m}} J_1(j_{0m}).
\end{aligned} \tag{6.117}$$

Here we have made use of the identity $\frac{d}{dx}\left(x J_1(x) \right) = J_0(x)$ from Section 5.5.

Substituting the result of this integral computation into the expression for A_{nm}, we find

$$A_{nm} = \frac{8(T_i - T_b)}{(2n-1)\pi} \frac{1}{j_{0m} J_1(j_{0m})}.$$

Substituting this result into the original expression for $u(r, z, t)$, gives

$$u(r, z, t) = \frac{8(T_i - T_b)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{(2n-1)\pi z}{Z}}{(2n-1)} \frac{J_0(\frac{r}{a} j_{0m}) e^{-\lambda_{nm} k t}}{j_{0m} J_1(j_{0m})}.$$

Therefore, $T(r, z, t)$ is found as

$$T(r, z, t) = T_b + \frac{8(T_i - T_b)}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{(2n-1)\pi z}{Z}}{(2n-1)} \frac{J_0(\frac{r}{a} j_{0m}) e^{-\lambda_{nm} k t}}{j_{0m} J_1(j_{0m})},$$

where

$$\lambda_{nm} = \left(\frac{(2n-1)\pi}{Z} \right)^2 + \left(\frac{j_{0m}}{a} \right)^2, \quad n, m = 1, 2, 3, \ldots.$$

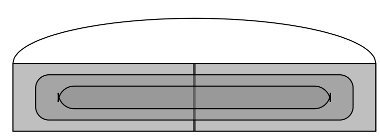

Figure 6.16: Depiction of a sideview of a vertical slice of a circular cake.

We have therefore found the general solution for the three-dimensional heat equation in cylindrical coordinates with constant diffusivity. Similar to the solutions shown in Figure 6.14 of the previous section, we show in Figure 6.17 the temperature evolution throughout a standard 9″ round cake pan. These are vertical slices similar to what is depicted in Figure 6.16.

Again, one could generalize this example to considerations of other types of cakes with cylindrical symmetry. For example, there are muffins, Boston steamed bread which is steamed in tall cylindrical cans. One could also consider an annular pan, such as a bundt cake pan. In fact, such problems extend beyond baking cakes to possible heating molds in manufacturing.

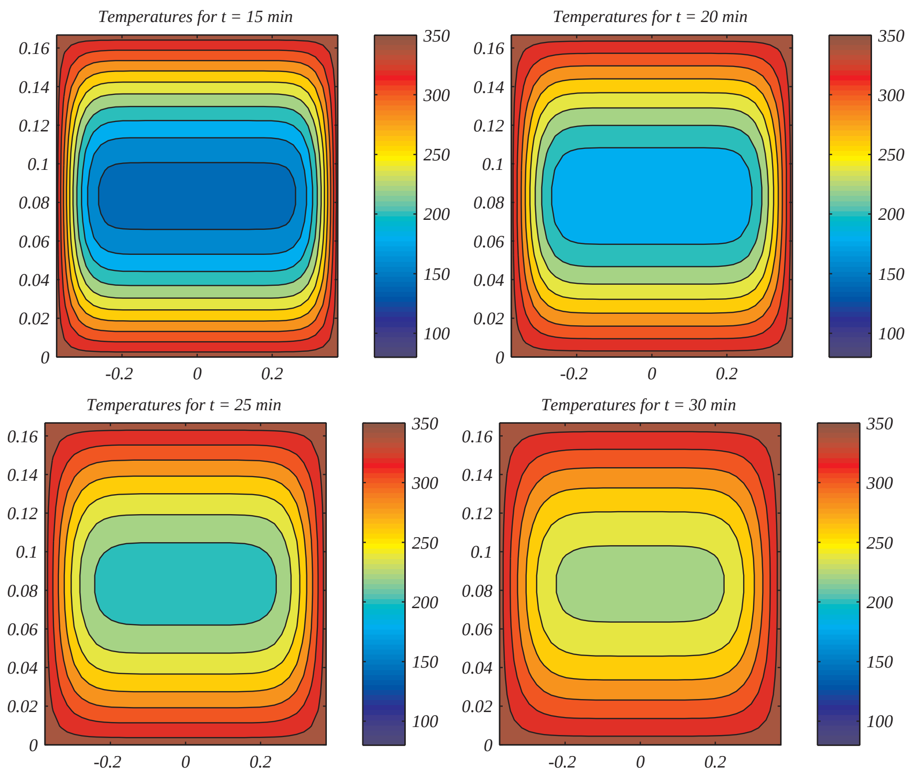

Figure 6.17: Temperature evolution for a standard 9″ cake shown as vertical slices through the center.

6.5 Laplace's Equation and Spherical Symmetry

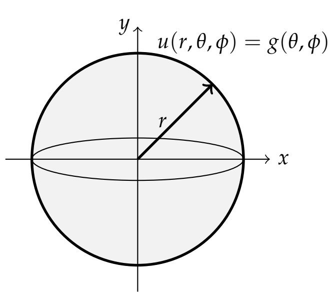

Figure 6.18: A sphere of radius r with the boundary condition $u(r,\theta,\phi) = g(\theta,\phi)$.

We have seen that Laplace's equation, $\nabla^2 u = 0$, arises in electrostatics as an equation for electric potential outside a charge distribution and it occurs as the equation governing equilibrium temperature distributions. As we had seen in the last chapter, Laplace's equation generally occurs in the study of potential theory, which also includes the study of gravitational and fluid potentials. The equation is named after Pierre-Simon Laplace (1749-1827) who had studied the properties of this equation. Solutions of Laplace's equation are called harmonic functions.

Example 6.10. Solve Laplace's equation in spherical coordinates.

We seek solutions of this equation inside a sphere of radius r subject to the boundary condition as shown in Figure 6.18. The problem is given by Laplace's equation Laplace's equation in spherical coordinates[2]

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0, \qquad (6.118)$$

[2] The Laplacian in spherical coordinates is given in Problem **??** in Chapter 8.

where $u = u(\rho, \theta, \phi)$.

The boundary conditions are given by

$$u(r, \theta, \phi) = g(\theta, \phi), \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi,$$

and the periodic boundary conditions

$$u(\rho, \theta, 0) = u(\rho, \theta, 2\pi), \quad u_\phi(\rho, \theta, 0) = u_\phi(\rho, \theta, 2\pi),$$

where $0 < \rho < \infty$, and $\quad 0 < \theta < \pi$.

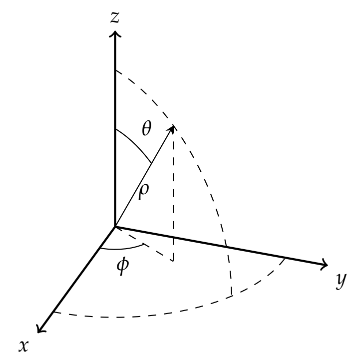

Figure 6.19: Definition of spherical coordinates (ρ, θ, ϕ). Note that there are different conventions for labeling spherical coordinates. This labeling is used often in physics.

As before, we perform a separation of variables by seeking product solutions of the form $u(\rho, \theta, \phi) = R(\rho)\Theta(\theta)\Phi(\phi)$. Inserting this form into the Laplace equation, we obtain

$$\frac{\Theta\Phi}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{R\Phi}{\rho^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{\rho^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0. \tag{6.119}$$

Multiplying this equation by ρ^2 and dividing by $R\Theta\Phi$, yields

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \tag{6.120}$$

Note that the first term is the only term depending upon ρ. Thus, we can separate out the radial part. However, there is still more work to do on the other two terms, which give the angular dependence. Thus, we have

$$-\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) = \frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2} = -\lambda, \tag{6.121}$$

where we have introduced the first separation constant. This leads to two equations:

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \lambda R = 0 \tag{6.122}$$

and

$$\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2} = -\lambda. \tag{6.123}$$

The final separation can be performed by multiplying the last equation by $\sin^2\theta$, rearranging the terms, and introducing a second separation constant:

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \mu. \tag{6.124}$$

Equation (6.123) is a key equation which occurs when studying problems possessing spherical symmetry. It is an eigenvalue problem for $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$, $LY = -\lambda Y$, where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The eigenfunctions of this operator are referred to as spherical harmonics.

From this expression we can determine the differential equations satisfied by $\Theta(\theta)$ and $\Phi(\phi)$:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + (\lambda\sin^2\theta - \mu)\Theta = 0, \tag{6.125}$$

and

$$\frac{d^2\Phi}{d\phi^2} + \mu\Phi = 0. \tag{6.126}$$

We now have three ordinary differential equations to solve. These are the radial equation (6.122) and the two angular equations (6.125)-(6.126). We note that all three are in Sturm-Liouville form. We will solve each eigenvalue problem subject to appropriate boundary conditions.

The simplest of these differential equations is Equation (6.126) for $\Phi(\phi)$. We have seen equations of this form many times and the general solution is a linear combination of sines and cosines. Furthermore, in this problem $u(\rho,\theta,\phi)$ is periodic in ϕ,

$$u(\rho,\theta,0) = u(\rho,\theta,2\pi), \quad u_\phi(\rho,\theta,0) = u_\phi(\rho,\theta,2\pi).$$

Since these conditions hold for all ρ and θ, we must require that $\Phi(\phi)$ satisfy the periodic boundary conditions

$$\Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi).$$

The eigenfunctions and eigenvalues for Equation (6.126) are then found as

$$\Phi(\phi) = \{\cos m\phi, \sin m\phi\}, \quad \mu = m^2, \quad m = 0, 1, \ldots. \tag{6.127}$$

Next we turn to solving equation, (6.125). We first transform this equation in order to identify the solutions. Let $x = \cos\theta$. Then the derivatives with respect to θ transform as

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}.$$

Letting $y(x) = \Theta(\theta)$ and noting that $\sin^2\theta = 1 - x^2$, Equation (6.125) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0. \tag{6.128}$$

We further note that $x \in [-1, 1]$, as can be easily confirmed by the reader.

This is a Sturm-Liouville eigenvalue problem. The solutions consist of a set of orthogonal eigenfunctions. For the special case that $m = 0$ Equation (6.128) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \lambda y = 0. \tag{6.129}$$

In a course in differential equations one learns to seek solutions of this equation in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

This leads to the recursion relation

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n.$$

Setting $n = 0$ and seeking a series solution, one finds that the resulting series does not converge for $x = \pm 1$. This is remedied by choosing $\lambda = \ell(\ell+1)$ for $\ell = 0, 1, \ldots$, leading to the differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \ell(\ell+1)y = 0. \tag{6.130}$$

We saw this equation in Chapter 5 in the form

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0.$$

The solutions of this differential equation are Legendre polynomials, denoted by $P_\ell(x)$.

For the more general case, $m \neq 0$, the differential equation (6.128) with $\lambda = \ell(\ell+1)$ becomes

associated Legendre functions

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)y = 0. \tag{6.131}$$

The solutions of this equation are called the associated Legendre functions. The two linearly independent solutions are denoted by $P_\ell^m(x)$ and $Q_\ell^m(x)$. The latter functions are not well behaved at $x = \pm 1$, corresponding to the north and south poles of the original problem. So, we can throw out these solutions in many physical cases, leaving

$$\Theta(\theta) = P_\ell^m(\cos\theta)$$

as the needed solutions. In Table 6.5 we list a few of these.

	$P_n^m(x)$	$P_n^m(\cos\theta)$
$P_0^0(x)$	1	1
$P_1^0(x)$	x	$\cos\theta$
$P_1^1(x)$	$-(1-x^2)^{\frac{1}{2}}$	$-\sin\theta$
$P_2^0(x)$	$\frac{1}{2}(3x^2-1)$	$\frac{1}{2}(3\cos^2\theta - 1)$
$P_2^1(x)$	$-3x(1-x^2)^{\frac{1}{2}}$	$-3\cos\theta\sin\theta$
$P_2^2(x)$	$3(1-x^2)$	$3\sin^2\theta$
$P_3^0(x)$	$\frac{1}{2}(5x^3-3x)$	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$
$P_3^1(x)$	$-\frac{3}{2}(5x^2-1)(1-x^2)^{\frac{1}{2}}$	$-\frac{3}{2}(5\cos^2\theta - 1)\sin\theta$
$P_3^2(x)$	$15x(1-x^2)$	$15\cos\theta\sin^2\theta$
$P_3^3(x)$	$-15(1-x^2)^{\frac{3}{2}}$	$-15\sin^3\theta$

Table 6.5: Associated Legendre Functions, $P_n^m(x)$.

The associated Legendre functions are related to the Legendre polynomials by[3]

$$P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x), \tag{6.132}$$

for $\ell = 0, 1, 2, , \ldots$ and $m = 0, 1, \ldots, \ell$. We further note that $P_\ell^0(x) = P_\ell(x)$, as one can see in the table. Since $P_\ell(x)$ is a polynomial of degree ℓ, then for $m > \ell$, $\frac{d^m}{dx^m}P_\ell(x) = 0$ and $P_\ell^m(x) = 0$.

Furthermore, since the differential equation only depends on m^2, $P_\ell^{-m}(x)$ is proportional to $P_\ell^m(x)$. One normalization is given by

$$P_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x).$$

The associated Legendre functions also satisfy the orthogonality condition

Orthogonality relation.

[3] The factor of $(-1)^m$ is known as the Condon-Shortley phase and is useful in quantum mechanics in the treatment of agular momentum. It is sometimes omitted by some

$$\int_{-1}^{1} P_\ell^m(x) P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}. \quad (6.133)$$

The last differential equation we need to solve is the radial equation. With $\lambda = \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$, the radial equation (6.122) can be written as

$$\rho^2 R'' + 2\rho R' - \ell(\ell+1)R = 0. \quad (6.134)$$

The radial equation is a Cauchy-Euler type of equation. So, we can guess the form of the solution to be $R(\rho) = \rho^s$, where s is a yet to be determined constant. Inserting this guess into the radial equation, we obtain the characteristic equation

$$s(s+1) = \ell(\ell+1).$$

Solving for s, we have

$$s = \ell, -(\ell+1).$$

Thus, the general solution of the radial equation is

$$R(\rho) = a\rho^\ell + b\rho^{-(\ell+1)}. \quad (6.135)$$

We would normally apply boundary conditions at this point. The boundary condition $u(r, \theta, \phi) = g(\theta, \phi)$ is not a homogeneous boundary condition, so we will need to hold off using it until we have the general solution to the three dimensional problem. However, we do have a hidden condition. Since we are interested in solutions inside the sphere, we need to consider what happens at $\rho = 0$. Note that $\rho^{-(\ell+1)}$ is not defined at the origin. Since the solution is expected to be bounded at the origin, we can set $b = 0$. So, in the current problem we have established that

$$R(\rho) = a\rho^\ell.$$

When seeking solutions outside the sphere, one considers the boundary condition $R(\rho) \to 0$ as $\rho \to \infty$. In this case, $R(\rho) = \rho^{-(\ell+1)}$.

We have carried out the full separation of Laplace's equation in spherical coordinates. The product solutions consist of the forms

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta)\cos m\phi$$

and

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta)\sin m\phi$$

for $\ell = 0, 1, 2, \ldots$ and $m = 0, \pm 1, , \ldots, \pm \ell$. These solutions can be combined to give a complex representation of the product solutions as

$$u(\rho, \theta, \phi) = \rho^\ell P_\ell^m(\cos\theta) e^{im\phi}.$$

The general solution is then given as a linear combination of these product solutions. As there are two indices, we have a double sum:[4]

$$u(\rho, \theta, \phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}\rho^\ell P_\ell^m(\cos\theta)e^{im\phi}. \quad (6.136)$$

[4] While this appears to be a complex-valued solution, it can be rewritten as a sum over real functions. The inner sum contains terms for both $m = k$ and $m = -k$. Adding these contributions, we have that

$$a_{\ell k}\rho^\ell P_\ell^k(\cos\theta)e^{ik\phi} + a_{\ell(-k)}\rho^\ell P_\ell^{-k}(\cos\theta)e^{-ik\phi}$$

can be rewritten as

$$(A_{\ell k}\cos k\phi + B_{\ell k}\sin k\phi)\rho^\ell P_\ell^k(\cos\theta).$$

Example 6.11. Laplace's Equation with Azimuthal Symmetry

As a simple example we consider the solution of Laplace's equation in which there is azimuthal symmetry. Let

$$u(r,\theta,\phi) = g(\theta) = 1 - \cos 2\theta.$$

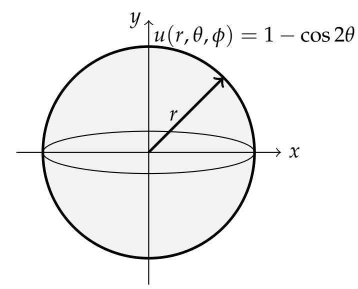

Figure 6.20: A sphere of radius r with the boundary condition

$$u(r,\theta,\phi) = 1 - \cos 2\theta.$$

This function is zero at the poles and has a maximum at the equator. So, this could be a crude model of the temperature distribution of the Earth with zero temperature at the poles and a maximum near the equator.

In problems in which there is no ϕ-dependence, only the $m = 0$ terms of the general solution survives. Thus, we have that

$$u(\rho,\theta,\phi) = \sum_{\ell=0}^{\infty} a_\ell \rho^\ell P_\ell(\cos\theta). \tag{6.137}$$

Here we have used the fact that $P_\ell^0(x) = P_\ell(x)$. We just need to determine the unknown expansion coefficients, a_ℓ. Imposing the boundary condition at $\rho = r$, we are lead to

$$g(\theta) = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta). \tag{6.138}$$

This is a Fourier-Legendre series representation of $g(\theta)$. Since the Legendre polynomials are an orthogonal set of eigenfunctions, we can extract the coefficients.

In Chapter 5 we had proven that

$$\int_0^\pi P_n(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = \int_{-1}^1 P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}.$$

So, multiplying the expression for $g(\theta)$ by $P_m(\cos\theta)\sin\theta$ and integrating, we obtain the expansion coefficients:

$$a_\ell = \frac{2\ell+1}{2r^\ell}\int_0^\pi g(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta. \tag{6.139}$$

Sometimes it is easier to rewrite $g(\theta)$ as a polynomial in $\cos\theta$ and avoid the integration. For this example we see that

$$\begin{aligned} g(\theta) &= 1 - \cos 2\theta \\ &= 2\sin^2\theta \\ &= 2 - 2\cos^2\theta. \end{aligned} \tag{6.140}$$

Thus, setting $x = \cos\theta$ and $G(x) = g(\theta(x))$, we have $G(x) = 2 - 2x^2$.

We seek the form

$$G(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x),$$

where $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. Since $G(x) = 2 - 2x^2$ does not have any x terms, we know that $c_1 = 0$. So,

$$2 - 2x^2 = c_0(1) + c_2\frac{1}{2}(3x^2 - 1) = c_0 - \frac{1}{2}c_2 + \frac{3}{2}c_2 x^2.$$

By observation we have $c_2 = -\frac{4}{3}$ and thus, $c_0 = 2 + \frac{1}{2}c_2 = \frac{4}{3}$. Therefore, $G(x) = \frac{4}{3}P_0(x) - \frac{4}{3}P_2(x)$.

We have found the expansion of $g(\theta)$ in terms of Legendre polynomials,

$$g(\theta) = \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}P_2(\cos\theta). \tag{6.141}$$

Therefore, the nonzero coefficients in the general solution become

$$a_0 = \frac{4}{3}, \quad a_2 = \frac{4}{3}\frac{1}{r^2},$$

and the rest of the coefficients are zero. Inserting these into the general solution, we have the final solution

$$\begin{aligned} u(\rho,\theta,\phi) &= \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}\left(\frac{\rho}{r}\right)^2 P_2(\cos\theta) \\ &= \frac{4}{3} - \frac{2}{3}\left(\frac{\rho}{r}\right)^2 (3\cos^2\theta - 1). \end{aligned} \tag{6.142}$$

6.5.1 *Spherical Harmonics*

$Y_{\ell m}(\theta,\phi)$, are the spherical harmonics. Spherical harmonics are important in applications from atomic electron configurations to gravitational fields, planetary magnetic fields, and the cosmic microwave background radiation.

THE SOLUTIONS OF THE ANGULAR PARTS OF THE PROBLEM are often combined into one function of two variables, as problems with spherical symmetry arise often, leaving the main differences between such problems confined to the radial equation. These functions are referred to as spherical harmonics, $Y_{\ell m}(\theta,\phi)$, which are defined with a special normalization as

$$Y_{\ell m}(\theta,\phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}. \tag{6.143}$$

These satisfy the simple orthogonality relation

$$\int_0^\pi \int_0^{2\pi} Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)\sin\theta\, d\phi\, d\theta = \delta_{\ell\ell'}\delta_{mm'}.$$

As seen earlier in the chapter, the spherical harmonics are eigenfunctions of the eigenvalue problem $LY = -\lambda Y$, where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

This operator appears in many problems in which there is spherical symmetry, such as obtaining the solution of Schrödinger's equation for the hydrogen atom as we will see later. Therefore, it is customary to plot spherical harmonics. Because the $Y_{\ell m}$'s are complex functions, one typically plots either the real part or the modulus squared. One rendition of $|Y_{\ell m}(\theta,\phi)|^2$ is shown in Figure 6.6 for $\ell, m = 0,1,2,3$.

We could also look for the nodal curves of the spherical harmonics like we had for vibrating membranes. Such surface plots on a sphere are shown in Figure 6.7. The colors provide for the amplitude of the $|Y_{\ell m}(\theta,\phi)|^2$. We can match these with the shapes in Figure 6.6 by coloring the plots with some of the same colors as shown in Figure 6.7. However, by plotting just the sign of the spherical harmonics, as in Figure 6.8, we can pick out the nodal curves much easier.

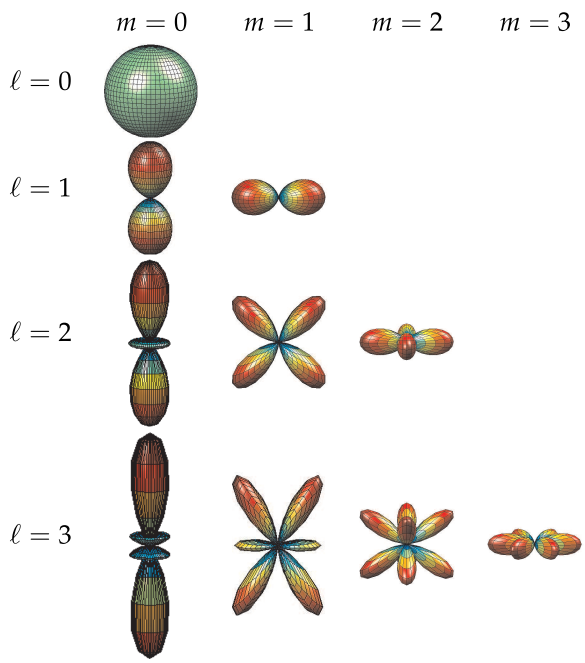

Table 6.6: The first few spherical harmonics, $|Y_{\ell m}(\theta, \phi)|^2$

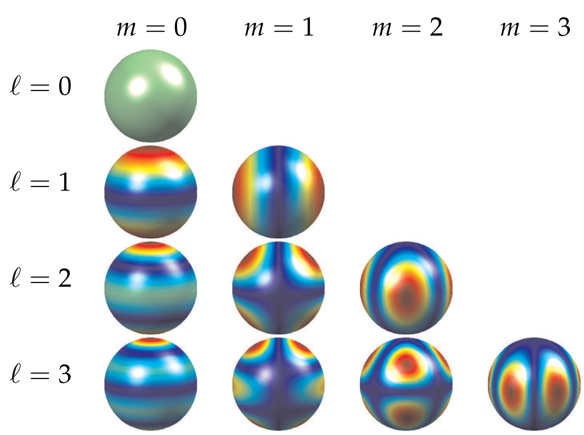

Table 6.7: Spherical harmonic contours for $|Y_{\ell m}(\theta, \phi)|^2$.

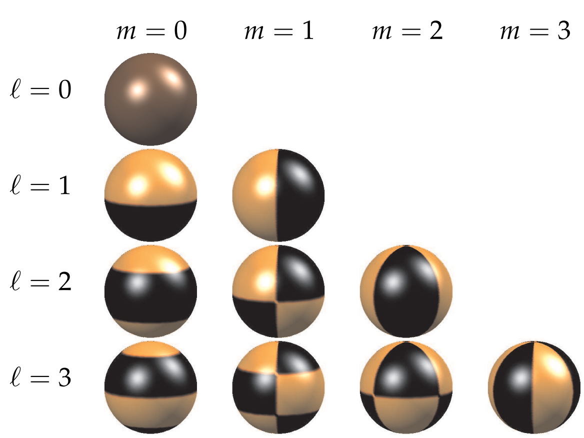

Table 6.8: In these figures we show the nodal curves of $|Y_{\ell m}(\theta, \phi)|^2$ Along the first column ($m = 0$) are the zonal harmonics seen as ℓ horizontal circles. Along the top diagonal ($m = \ell$) are the sectional harmonics. These look like orange sections formed from m vertical circles. The remaining harmonics are tesseral harmonics. They look like a checkerboard pattern formed from intersections of $\ell - m$ horizontal circles and m vertical circles.

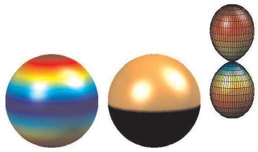

Figure 6.21: Zonal harmonics, $\ell = 1$, $m = 0$.

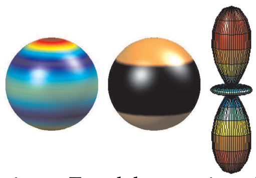

Figure 6.22: Zonal harmonics, $\ell = 2$, $m = 0$.

Figure 6.23: Sectoral harmonics, $\ell = 2$, $m = 2$.

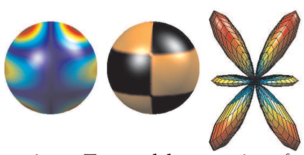

Figure 6.24: Tesseral harmonics, $\ell = 3$, $m = 1$.

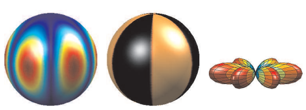

Figure 6.25: Sectoral harmonics, $\ell = 3$, $m = 3$.

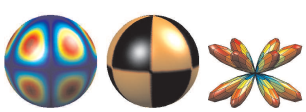

Figure 6.26: Tesseral harmonics, $\ell = 4$, $m = 3$.

Spherical, or surface, harmonics can be further grouped into zonal, sectoral, and tesseral harmonics. Zonal harmonics correspond to the $m = 0$ modes. In this case, one seeks nodal curves for which $P_\ell(\cos\theta) = 0$. Solutions of this equation lead to constant θ values such that $\cos\theta$ is a zero of the Legendre polynomial, $P_\ell(x)$. The zonal harmonics correspond to the first column in Figure 6.8. Since $P_\ell(x)$ is a polynomial of degree ℓ, the zonal harmonics consist of ℓ latitudinal circles.

Sectoral, or meridional, harmonics result for the case that $m = \pm\ell$. For this case, we note that $P_\ell^{\pm\ell}(x) \propto (1 - x^2)^{m/2}$. This function vanishes for $x = \pm 1$, or $\theta = 0, \pi$. Therefore, the spherical harmonics can only produce nodal curves for $e^{im\phi} = 0$. Thus, one obtains the meridians satisfying the condition $A\cos m\phi + B\sin m\phi = 0$. Solutions of this equation are of the form $\phi =$ constant. These modes can be seen in Figure 6.8 in the top diagonal and can be described as m circles passing through the poles, or longitudinal circles.

Tesseral harmonics consist of the rest of the modes, which typically look like a checker board glued to the surface of a sphere. Examples can be seen in the pictures of nodal curves, such as Figure 6.8. Looking in Figure 6.8 along the diagonals going downward from left to right, one can see the same number of latitudinal circles. In fact, there are $\ell - m$ latitudinal nodal curves in these figures

In summary, the spherical harmonics have several representations, as show in Figures 6.7-6.8. Note that there are ℓ nodal lines, m meridional curves, and $\ell - m$ horizontal curves in these figures. The plots in Figure 6.6 are the typical plots shown in physics for discussion of the wavefunctions of the hydrogen atom. Those in 6.7 are useful for describing gravitational or electric potential functions, temperature distributions, or wave modes on a spherical surface. The relationships between these pictures and the nodal curves can be better understood by comparing respective plots. Several modes were separated out in Figures 6.21-6.26 to make this comparison easier.

6.6 *Spherically Symmetric Vibrations*

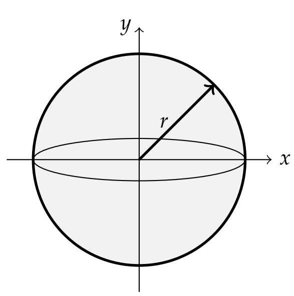

Figure 6.27: A vibrating sphere of radius r with the initial conditions

$$u(\theta,\phi,0) = f(\theta,\phi),$$
$$u_t(\theta,\phi,0) = g(\theta,\phi).$$

ANOTHER APPLICATION OF SPHERICAL HARMONICS IS A VIBRATING SPHERICAL MEMBRANE, such as a balloon. Just as for the two-dimensional membranes encountered earlier, we let $u(\theta,\phi,t)$ represent the vibrations of the surface about a fixed radius obeying the wave equation, $u_{tt} = c^2\nabla^2 u$, and satisfying the initial conditions

$$u(\theta,\phi,0) = f(\theta,\phi), \quad u_t(\theta,\phi,0) = g(\theta,\phi).$$

In spherical coordinates, we have (for $\rho = r =$ constant.)

$$u_{tt} = \frac{c^2}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right), \tag{6.144}$$

where $u = u(\theta, \phi, t)$.

The boundary conditions are given by the periodic boundary conditions

$$u(\theta, 0, t) = u(\theta, 2\pi, t), \quad u_\phi(\theta, 0, t) = u_\phi(\theta, 2\pi, t),$$

where $0 < t$, and $0 < \theta < \pi$, and that $u = u(\theta, \phi, t)$ should remain bounded.

Noting that the wave equation takes the form

$$u_{tt} = \frac{c^2}{r^2} Lu, \quad \text{where} \quad LY_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta, \phi) = P_\ell^m(\cos\theta) e^{im\phi}$, then we can seek product solutions of the form

$$u_{\ell m}(\theta, \phi, t) = T(t) Y_{\ell m}(\theta, \phi).$$

Inserting this form into the wave equation in spherical coordinates, we find

$$T'' Y_{\ell m} = -\frac{c^2}{r^2} T(t) \ell(\ell+1) Y_{\ell m},$$

or

$$T'' + \ell(\ell+1) \frac{c^2}{r^2} T(t).$$

The solutions of this equation are easily found as

$$T(t) = A \cos \omega_\ell t + B \sin \omega_\ell t, \quad \omega_\ell = \sqrt{\ell(\ell+1)} \frac{c}{r}.$$

Therefore, the product solutions are given by

$$u_{\ell m}(\theta, \phi, t) = \left[A \cos \omega_\ell t + B \sin \omega_\ell t\right] Y_{\ell m}(\theta, \phi)$$

for $\ell = 0, 1, \ldots, m = -\ell, -\ell+1, \ldots, \ell$.

In Figure 6.28 we show several solutions for $r = c = 1$ at $t = 10$.

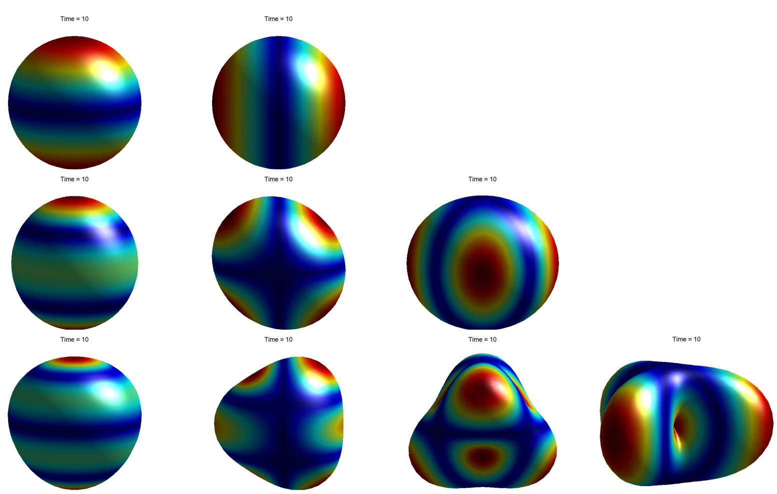

Figure 6.28: Modes for a vibrating spherical membrane:
Row 1: $(1,0), (1,1)$;
Row 2: $(2,0), (2,1), (2,2)$;
Row 3 $(3,0), (3,1), (3,2), (3,3)$.

The general solution is found as

$$u(\theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} \cos \omega_\ell t + B_{\ell m} \sin \omega_\ell t\right] Y_{\ell m}(\theta, \phi).$$

An interesting problem is to consider hitting the balloon with a velocity impulse while at rest. An example of such a solution is shown in Figure 6.29. In this images several modes are excited after the impulse.

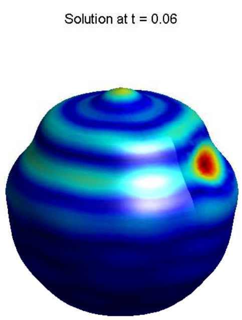

Figure 6.29: A moment captured from a simulation of a spherical membrane after hit with a velocity impulse.

6.7 Baking a Spherical Turkey

During one year as this course was being taught, an instructor returned from the American holiday of Thanksgiving, where it is customary to cook a turkey. Such a turkey is shown in Figure 6.30. This reminded the instructor of a typical problem, such as in Weinberger, (1995, p. 92.), where one is given a roast of a certain volume and one is asked to find the time it takes to cook one double the size. In this section, we explore a similar problem for cooking a turkey.

Figure 6.30: A 12-lb turkey leaving the oven.

Often during this time of the year, November, articles appear with some scientific evidence as to how to gauge how long it takes to cook a turkey of a given weight. Inevitably it refers to the story, as told in `http://today.slac.stanford.edu/a/2008/11-26.htm`http://today.slac.stanford.edu/a/2008/11-26.htm that Pief Panofsky, a former SLAC Director, was determined to find a nonlinear equation for determining cooking times instead of using the rule of thumb of 30 minutes per pound of turkey. He had arrived at the form,

$$t = \frac{W^{2/3}}{1.5},$$

where t is the cooking time and W is the weight of the turkey in pounds. Nowadays, one can go to Wolframalpha.com and enter the question "how long should you cook a turkey" and get results based on a similar formula.

Before turning to the solution of the heat equation for a turkey, let's consider a simpler problem.

Example 6.12. If it takes 4 hours to cook a 10 pound turkey in a 350^0 F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

In all of our analysis, we will consider a spherical turkey. While the turkey in Figure 6.30 is not quite spherical, we are free to approximate the turkey as such. If you prefer, we could imagine a spherical turkey like the one shown in Figure 6.31.

This problem is one of scaling. Thinking of the turkey as being spherically symmetric, then the baking follows the heat equation in the form

$$u_t = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

We can rescale the variables from coordinates (r, t) to (ρ, τ) as $r = \beta\rho$, and $t = \alpha\tau$. Then the derivatives transform as

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = \frac{1}{\beta}\frac{\partial}{\partial \rho}, \\ \frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{1}{\alpha}\frac{\partial}{\partial \tau}. \end{aligned} \tag{6.145}$$

Figure 6.31: The depiction of a spherical turkey.

Inserting these transformations into the heat equation, we have

$$u_\tau = \frac{\alpha}{\beta^2}\frac{k}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right).$$

To keep conditions the same, then we need $\alpha = \beta^2$. So, the transformation that keeps the form of the heat equation the same, or makes it invariant, is $r = \beta\rho$, and $t = \beta^2\tau$. This is also known as a self-similarity transformation.

So, if the radius increases by a factor of β, then the time to cook the turkey (reaching a given temperature, u), would increase by β^2. Returning to the problem, if the weight of the doubles, then the volume doubles, assuming that the density is held constant. However, the volume is proportional to r^3. So, r increases by a factor of $2^{1/3}$. Therefore, the time increases by a factor of $2^{2/3} \approx 1.587$. This give the time for cooking a 20 lb turkey as $t = 4(2^{2/3}) = 2^{8/3} \approx 6.35$ hours.

The previous example shows the power of using similarity transformations to get general information about solutions of differential equations. However, we have focussed on using the method of separation of variables for most of the book so far. We should be able to find a solution to the spherical turkey model using these methods as well. This will be shown in the next example.

Example 6.13. Find the temperature, $T(\rho, t)$ inside a spherical turkey, initially at 40°, which is F placed in a 350° F. Assume that the turkey is of constant density and that the surface of the turkey is maintained at the oven temperature. [We will also neglect convection and radaition processes inside the oven.]

The problem can be formulated as a heat equation problem for $T(\rho, t)$:

$$\begin{aligned} T_t &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right), \quad 0 < \rho < a, t > 0, \\ T(a,t) &= 350, \quad T(\rho,t) \quad \text{finite at } \rho = 0, \quad t > 0, \\ T(\rho,0) &= 40. \end{aligned} \tag{6.146}$$

We note that the boundary condition is not homogeneous. However, we can fix that by introducing the auxiliary function (the difference between the turkey and oven temperatures) $u(\rho,t) = T(\rho,t) - T_a$, where $T_a = 350$. Then, the problem to be solved becomes

$$\begin{aligned} u_t &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \quad 0 < \rho < a, t > 0, \\ u(a,t) &= 0, \quad u(\rho,t) \quad \text{finite at } \rho = 0, \quad t > 0, \\ u(\rho,0) &= T_a i - T_a = -310, \end{aligned} \tag{6.147}$$

where $T_i = 40$.

We can now employ the method of separation of variables. Let $u(\rho,t) = R(\rho)G(t)$. Inserting into the heat equation for u, we have

$$\frac{1}{k}\frac{G'}{G} = \frac{1}{R}\left(R'' + \frac{2}{\rho}R'\right) = -\lambda.$$

This give the two ordinary differential equations, the temporal equation,

$$G' = -k\lambda G, \tag{6.148}$$

and the radial equation,

$$\rho R'' + 2R' + \lambda\rho R = 0. \tag{6.149}$$

The temporal equation is easy to solve,

$$G(t) = G_0 e^{-\lambda k t}.$$

However, the radial equation is slightly more difficult. But, making the substitution $R(\rho) = y(\rho)/\rho$, it is readily transformed into a simpler form:[5]

$$y'' + \lambda y = 0.$$

The boundary conditions on $u(\rho,t) = R(\rho)G(t)$ transfer to $R(a) = 0$ and $R(\rho)$ finite at the origin. In turn, this means that $y(a) = 0$ and $y(\rho)$ has to vanish near the origin. If $y(\rho)$ does not vanish near the origin, then $R(\rho)$ is not finite as $\rho \to 0$.

[5] The radial equation almost looks familiar when it is multiplied by ρ :

$$\rho^2 R'' + 2\rho R' + \lambda\rho^2 R = 0.$$

If it were not for the '2', it would be the zeroth order Bessel equation. This is actually the zeroth order spherical Bessel equation. In general, the spherical Bessel functions, $j_n(x)$ and $y_n(x)$, satisfy

$$x^2 y'' + 2xy' + [x^2 - n(n+1)]y = 0.$$

So, the radial solution of the turkey problem is

$$R(\rho) = j_n(\sqrt{\lambda}\rho) = \frac{sin\sqrt{\lambda}\rho}{\sqrt{\lambda}\rho}.$$

We further note that

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$$

So, we need to solve the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(a) = 0.$$

This gives the well-known set of eigenfunctions

$$y(\rho) = \sin \frac{n\pi\rho}{a}, \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots.$$

Therefore, we have found

$$R(\rho) = \frac{\sin \frac{n\pi\rho}{a}}{\rho}, \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The general solution to the auxiliary problem is

$$u(\rho, t) = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

This gives the general solution for the temperature as

$$T(\rho, t) = T_a + \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

All that remains is to find the solution satisfying the initial condition, $T(\rho, 0) = 40$. Inserting $t = 0$, we have

$$T_i - T_a = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi\rho}{a}}{\rho}.$$

This is almost a Fourier sine series. Multiplying by ρ, we have

$$(T_i - T_a)\rho = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi\rho}{a}.$$

Now, we can solve for the coefficients,

$$\begin{aligned} A_n &= \frac{2}{a} \int_0^a (T_i - T_a)\rho \sin \frac{n\pi\rho}{a}\, d\rho \\ &= \frac{2a}{n\pi}(T_i - T_a)(-1)^{n+1}. \end{aligned} \tag{6.150}$$

This gives the final solution,

$$T(\rho, t) = T_a + \frac{2a(T_i - T_a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

For generality, the ambient and initial temperature were left in terms of T_a and T_i, respectively.

It is interesting to use the above solution to compare roasting different turkeys. We take the same conditions as above. Let the radius of the spherical turkey be six inches. We will assume that such a turkey takes four hours to cook, i.e., reach a temperature of 180° F. Plotting the solution with 400 terms, one finds that $k \approx 0.000089$. This gives a "baking time" of $t1 = 239.63$.

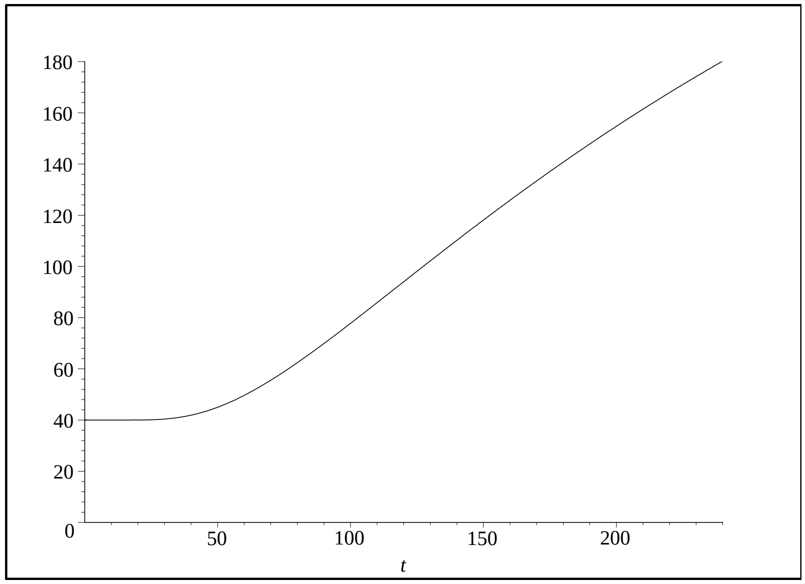

Figure 6.32: The temperature at the center of a turkey with radius $a = 0.5$ ft and $k \approx 0.000089$.

A plot of the temperature at the center point ($\rho = a/2$) of the bird is in Figure 6.32.

Using the same constants, but increasing the radius of a turkey to $a = 0.5(2^{1/3})$ ft, we obtain the temperature plot in Figure 6.33. This radius corresponds to doubling the volume of the turkey. Solving for the time at which the center temperature (at $\rho = a/2$) reaches 180° F, we obtained $t2 = 380.38$. Comparing the two temperatures, we find the ratio (using the full computation of the solution in Maple)

$$\frac{t2}{t1} = \frac{380.3813709}{239.6252478} \approx 1.587401054.$$

The compares well to

$$2^{2/3} \approx 1.587401052.$$

Of course, the temperature is not quite the center of the spherical turkey. The reader can work out the details for other locations. Perhaps other interesting models would be a spherical shell of turkey with a corse of bread stuffing. Or, one might consider an ellipsoidal geometry.

6.8 Schrödinger Equation in Spherical Coordinates

Another important eigenvalue problem in physics is the Schrödinger equation. The time-dependent Schrödinger equation is given by

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \tag{6.151}$$

Here $\Psi(\mathbf{r}, t)$ is the wave function, which determines the quantum state of a particle of mass m subject to a (time independent) potential, $V(\mathbf{r})$. From

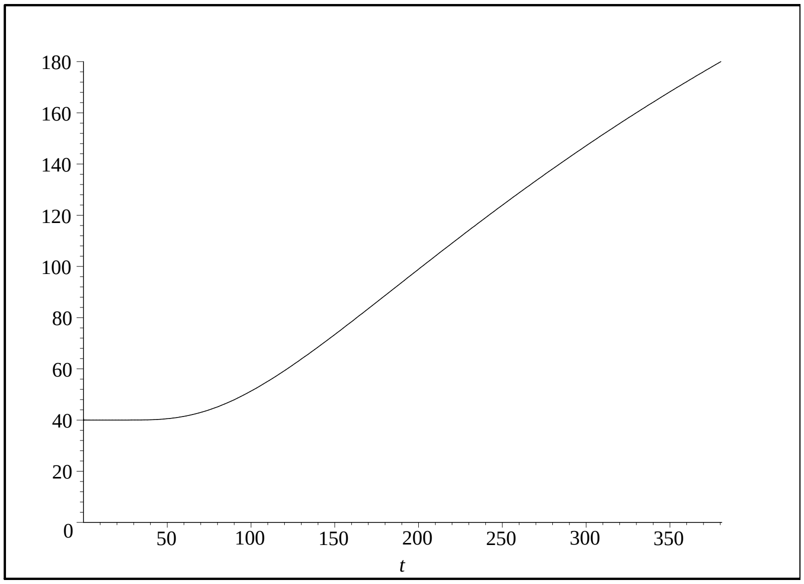

Figure 6.33: The temperature at the center of a turkey with radius $a = 0.5(2^{1/3})$ ft and $k \approx 0.000089$.

Planck's constant, h, one defines $\hbar = \frac{h}{2\pi}$. The probability of finding the particle in an infinitesimal volume, dV, is given by $|\Psi(\mathbf{r}, t)|^2 \, dV$, assuming the wave function is normalized,

$$\int_{\text{all space}} |\Psi(\mathbf{r}, t)|^2 \, dV = 1.$$

One can separate out the time dependence by assuming a special form, $\Psi(\mathbf{r}, t) = \psi(\mathbf{r}) e^{-iEt/\hbar}$, where E is the energy of the particular stationary state solution, or product solution. Inserting this form into the time-dependent equation, one finds that $\psi(\mathbf{r})$ satisfies the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi. \tag{6.152}$$

Assuming that the potential depends only on the distance from the origin, $V = V(\rho)$, we can further separate out the radial part of this solution using spherical coordinates. Recall that the Laplacian in spherical coordinates is given by

$$\nabla^2 = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}. \tag{6.153}$$

Then, the time-independent Schrödinger equation can be written as

$$\begin{aligned} - & \frac{\hbar^2}{2m} \left[\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] \\ = & \; [E - V(\rho)]\psi. \end{aligned} \tag{6.154}$$

Let's continue with the separation of variables. Assuming that the wave function takes the form $\psi(\rho, \theta, \phi) = R(\rho) Y(\theta, \phi)$, we obtain

$$- \frac{\hbar^2}{2m} \left[\frac{Y}{\rho^2} \frac{d}{d\rho} \left(\rho^2 \frac{dR}{d\rho} \right) + \frac{R}{\rho^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial Y}{\partial \theta} \right) + \frac{R}{\rho^2 \sin^2\theta} \frac{\partial^2 Y}{\partial \phi^2} \right]$$

$$= \quad RY[E - V(\rho)]\psi. \tag{6.155}$$

Dividing by $\psi = RY$, multiplying by $-\frac{2m\rho^2}{\hbar^2}$, and rearranging, we have

$$\frac{1}{R}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \frac{2m\rho^2}{\hbar^2}\left[V(\rho) - E\right] = -\frac{1}{Y}LY,$$

where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

We have a function of ρ equal to a function of the angular variables. So, we set each side equal to a constant. We will judiciously write the separation constant as $\ell(\ell+1)$. The resulting equations are then

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \frac{2m\rho^2}{\hbar^2}\left[V(\rho) - E\right]R = \ell(\ell+1)R, \tag{6.156}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)Y. \tag{6.157}$$

The second of these equations should look familiar from the last section. This is the equation for spherical harmonics,

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{2}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m e^{im\phi}. \tag{6.158}$$

So, any further analysis of the problem depends upon the choice of potential, $V(\rho)$, and the solution of the radial equation. For this, we turn to the determination of the wave function for an electron in orbit about a proton.

Example 6.14. The Hydrogen Atom - $\ell = 0$ States

Historically, the first test of the Schrödinger equation was the determination of the energy levels in a hydrogen atom. This is modeled by an electron orbiting a proton. The potential energy is provided by the Coulomb potential,

$$V(\rho) = -\frac{e^2}{4\pi\epsilon_0\rho}.$$

Solution of the hydrogen problem.

Thus, the radial equation becomes

$$\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{2m\rho^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0\rho} + E\right]R = \ell(\ell+1)R. \tag{6.159}$$

Before looking for solutions, we need to simplify the equation by absorbing some of the constants. One way to do this is to make an appropriate change of variables. Let $\rho = ar$. Then, by the Chain Rule we have

$$\frac{d}{d\rho} = \frac{dr}{d\rho}\frac{d}{dr} = \frac{1}{a}\frac{d}{dr}.$$

Under this transformation, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \frac{2ma^2r^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 ar} + E\right]u = \ell(\ell+1)u, \tag{6.160}$$

where $u(r) = R(\rho)$. Expanding the second term,

$$\frac{2ma^2r^2}{\hbar^2}\left[\frac{e^2}{4\pi\epsilon_0 ar}+E\right]u = \left[\frac{mae^2}{2\pi\epsilon_0\hbar^2}r+\frac{2mEa^2}{\hbar^2}r^2\right]u,$$

we see that we can define

$$\begin{aligned} a &= \frac{2\pi\epsilon_0\hbar^2}{me^2}, && (6.161)\\ \epsilon &= -\frac{2mEa^2}{\hbar^2} \\ &= -\frac{2(2\pi\epsilon_0)^2\hbar^2}{me^4}E. && (6.162)\end{aligned}$$

Using these constants, the radial equation becomes

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right)+ru-\ell(\ell+1)u = \epsilon r^2 u. \tag{6.163}$$

Expanding the derivative and dividing by r^2,

$$u''+\frac{2}{r}u'+\frac{1}{r}u-\frac{\ell(\ell+1)}{r^2}u = \epsilon u. \tag{6.164}$$

The first two terms in this differential equation came from the Laplacian. The third term came from the Coulomb potential. The fourth term can be thought to contribute to the potential and is attributed to angular momentum. Thus, ℓ is called the angular momentum quantum number. This is an eigenvalue problem for the radial eigenfunctions $u(r)$ and energy eigenvalues ϵ.

The solutions of this equation are determined in a quantum mechanics course. In order to get a feeling for the solutions, we will consider the zero angular momentum case, $\ell = 0$:

$$u''+\frac{2}{r}u'+\frac{1}{r}u = \epsilon u. \tag{6.165}$$

Even this equation is one we have not encountered in this book. Let's see if we can find some of the solutions.

First, we consider the behavior of the solutions for large r. For large r the second and third terms on the left hand side of the equation are negligible. So, we have the approximate equation

$$u''-\epsilon u = 0. \tag{6.166}$$

Therefore, the solutions behave like $u(r) = e^{\pm\sqrt{\epsilon}r}$ for large r. For bounded solutions, we choose the decaying solution.

This suggests that solutions take the form $u(r) = v(r)e^{-\sqrt{\epsilon}r}$ for some unknown function, $v(r)$. Inserting this guess into Equation (6.165), gives an equation for $v(r)$:

$$rv''+2\left(1-\sqrt{\epsilon}r\right)v'+(1-2\sqrt{\epsilon})v = 0. \tag{6.167}$$

Next we seek a series solution to this equation. Let

$$v(r)=\sum_{k=0}^{\infty} c_k r^k.$$

Inserting this series into Equation (6.167), we have

$$\sum_{k=1}^{\infty}[k(k-1)+2k]c_k r^{k-1}+\sum_{k=1}^{\infty}[1-2\sqrt{\epsilon}(k+1)]c_k r^k=0.$$

We can re-index the dummy variable in each sum. Let $k=m$ in the first sum and $k=m-1$ in the second sum. We then find that

$$\sum_{k=1}^{\infty}\left[m(m+1)c_m+(1-2m\sqrt{\epsilon})c_{m-1}\right]r^{m-1}=0.$$

Since this has to hold for all $m\geq 1$,

$$c_m=\frac{2m\sqrt{\epsilon}-1}{m(m+1)}c_{m-1}.$$

Further analysis indicates that the resulting series leads to unbounded solutions unless the series terminates. This is only possible if the numerator, $2m\sqrt{\epsilon}-1$, vanishes for $m=n$, $n=1,2\ldots$. Thus,

$$\epsilon=\frac{1}{4n^2}.$$

Since ϵ is related to the energy eigenvalue, E, we have

$$E_n=-\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2 n^2}.$$

Inserting the values for the constants, this gives

$$E_n=-\frac{13.6 \text{ eV}}{n^2}.$$

Energy levels for the hydrogen atom.

This is the well known set of energy levels for the hydrogen atom.

The corresponding eigenfunctions are polynomials, since the infinite series was forced to terminate. We could obtain these polynomials by iterating the recursion equation for the c_m's. However, we will instead rewrite the radial equation (6.167).

Let $x=2\sqrt{\epsilon}r$ and define $y(x)=v(r)$. Then

$$\frac{d}{dr}=2\sqrt{\epsilon}\frac{d}{dx}.$$

This gives

$$2\sqrt{\epsilon}xy''+(2-x)2\sqrt{\epsilon}y'+(1-2\sqrt{\epsilon})y=0.$$

Rearranging, we have

$$xy''+(2-x)y'+\frac{1}{2\sqrt{\epsilon}}(1-2\sqrt{\epsilon})y=0.$$

Noting that $2\sqrt{\epsilon} = \frac{1}{n}$, this equation becomes

$$xy'' + (2 - x)y' + (n - 1)y = 0. \tag{6.168}$$

The resulting equation is well known. It takes the form

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{6.169}$$

Solutions of this equation are the associated Laguerre polynomials. The solutions are denoted by $L_n^\alpha(x)$. They can be defined in terms of the Laguerre polynomials,

$$L_n(x) = e^x \left(\frac{d}{dx}\right)^n (e^{-x}x^n).$$

The associated Laguerre polynomials are defined as

$$L_{n-m}^m(x) = (-1)^m \left(\frac{d}{dx}\right)^m L_n(x).$$

Note: The Laguerre polynomials were first encountered in Problem 2 in Chapter 5 as an example of a classical orthogonal polynomial defined on $[0, \infty)$ with weight $w(x) = e^{-x}$. Some of these polynomials are listed in Table 6.9 and several Laguerre polynomials are shown in Figure 6.34.

The associated Laguerre polynomials are named after the French mathematician Edmond Laguerre (1834-1886).

Comparing Equation (6.168) with Equation (6.169), we find that $y(x) = L_{n-1}^1(x)$.

	$L_n^m(x)$
$L_0^0(x)$	1
$L_1^0(x)$	$1 - x$
$L_2^0(x)$	$\frac{1}{2}(x^2 - 4x + 2)$
$L_3^0(x)$	$\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$
$L_0^1(x)$	1
$L_1^1(x)$	$2 - x$
$L_2^1(x)$	$\frac{1}{2}(x^2 - 6x + 6)$
$L_3^1(x)$	$\frac{1}{6}(-x^3 + 3x^2 - 36x + 24)$
$L_0^2(x)$	1
$L_1^2(x)$	$3 - x$
$L_2^2(x)$	$\frac{1}{2}(x^2 - 8x + 12)$
$L_3^2(x)$	$\frac{1}{12}(-2x^3 + 30x^2 - 120x + 120)$

Table 6.9: Associated Laguerre Functions, $L_n^m(x)$. Note that $L_n^0(x) = L_n(x)$.

In summary, we have made the following transformations:

1. $R(\rho) = u(r)$, $\rho = ar$.
2. $u(r) = v(r)e^{-\sqrt{\epsilon}r}$.
3. $v(r) = y(x) = L_{n-1}^1(x)$, $x = 2\sqrt{\epsilon}r$.

In most derivation in quantum mechanics $a = \frac{a_0}{2}$. where $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ is the Bohr radius and $a_0 = 5.2917 \times 10^{-11}$m.

Therefore,

$$R(\rho) = e^{-\sqrt{\epsilon}\rho/a} L_{n-1}^1(2\sqrt{\epsilon}\rho/a).$$

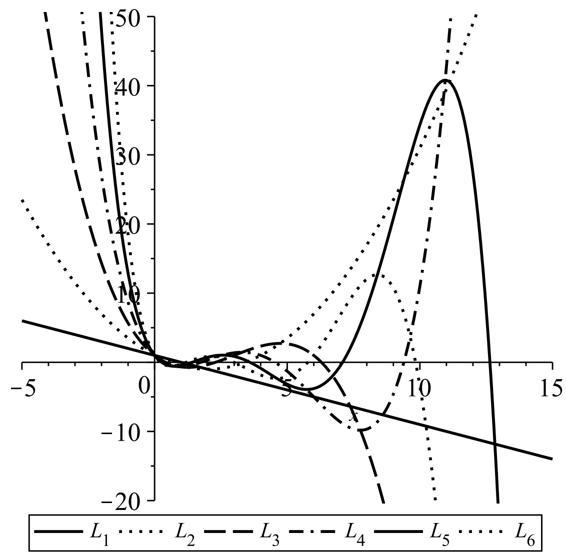

Figure 6.34: Plots of the first few Laguerre polynomials.

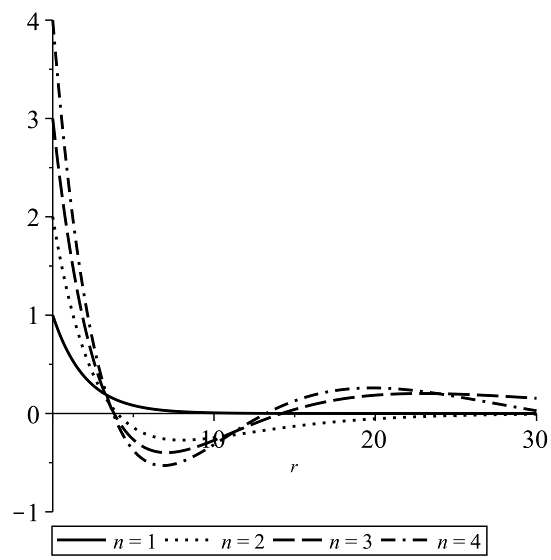

Figure 6.35: Plots of $R(\rho)$ for $a = 1$ and $n = 1, 2, 3, 4$ for the $\ell = 0$ states.

However, we also found that $2\sqrt{\epsilon} = 1/n$. So,

$$R(\rho) = e^{-\rho/2na} L^1_{n-1}(\rho/na).$$

In Figure 6.35 we show a few of these solutions.

Example 6.15. Find the $\ell \geq 0$ solutions of the radial equation.

For the general case, for all $\ell \geq 0$, we need to solve the differential equation

$$u'' + \frac{2}{r}u' + \frac{1}{r}u - \frac{\ell(\ell+1)}{r^2}u = \epsilon u. \tag{6.170}$$

Instead of letting $u(r) = v(r)e^{-\sqrt{\epsilon}r}$, we let

$$u(r) = v(r) r^\ell e^{-\sqrt{\epsilon}r}.$$

This lead to the differential equation

$$rv'' + 2(\ell + 1 - \sqrt{\epsilon}r)v' + (1 - 2(\ell+1)\sqrt{\epsilon})v = 0. \tag{6.171}$$

as before, we let $x = 2\sqrt{\epsilon}r$ to obtain

$$xy'' + 2\left[\ell + 1 - \frac{x}{2}\right] v' + \left[\frac{1}{2\sqrt{\epsilon}} - \ell(\ell+1)\right] v = 0.$$

Noting that $2\sqrt{\epsilon} = 1/n$, we have

$$xy'' + 2\left[2(\ell+1) - x\right] v' + (n - \ell(\ell+1))v = 0.$$

We see that this is once again in the form of the associate Laguerre equation and the solutions are

$$y(x) = L^{2\ell+1}_{n-\ell-1}(x).$$

So, the solution to the radial equation for the hydrogen atom is given by

$$\begin{aligned} R(\rho) &= r^\ell e^{-\sqrt{\epsilon}r} L^{2\ell+1}_{n-\ell-1}(2\sqrt{\epsilon}r) \\ &= \left(\frac{\rho}{2na}\right)^\ell e^{-\rho/2na} L^{2\ell+1}_{n-\ell-1}\left(\frac{\rho}{na}\right). \end{aligned} \tag{6.172}$$

Interpretations of these solutions will be left for your quantum mechanics course.

6.9 Curvilinear Coordinates

In order to study solutions of the wave equation, the heat equation, or even Schrödinger's equation in different geometries, we need to see how differential operators, such as the Laplacian, appear in these geometries. The most common coordinate systems arising in physics are polar coordinates, cylindrical coordinates, and spherical coordinates. These reflect the common geometrical symmetries often encountered in physics.

In such systems it is easier to describe boundary conditions and to make use of these symmetries. For example, specifying that the electric potential is 10.0 V on a spherical surface of radius one, we would say $\phi(x,y,z) = 10$ for $x^2+y^2+z^2 = 1$. However, if we use spherical coordinates, (r,θ,ϕ), then we would say $\phi(r,\theta,\phi) = 10$ for $r = 1$, or $\phi(1,\theta,\phi) = 10$. This is a much simpler representation of the boundary condition.

However, this simplicity in boundary conditions leads to a more complicated looking partial differential equation in spherical coordinates. In this section we will consider general coordinate systems and how the differential operators are written in the new coordinate systems. This is a more general approach than that taken earlier in the chapter. For a more modern and elegant approach, one can use differential forms.

We begin by introducing the general coordinate transformations between Cartesian coordinates and the more general curvilinear coordinates. Let the Cartesian coordinates be designated by (x_1,x_2,x_3) and the new coordinates by (u_1,u_2,u_3). We will assume that these are related through the transformations

$$\begin{aligned} x_1 &= x_1(u_1,u_2,u_3), \\ x_2 &= x_2(u_1,u_2,u_3), \\ x_3 &= x_3(u_1,u_2,u_3). \end{aligned} \tag{6.173}$$

Thus, given the curvilinear coordinates (u_1,u_2,u_3) for a specific point in space, we can determine the Cartesian coordinates, (x_1,x_2,x_3), of that point. We will assume that we can invert this transformation: Given the Cartesian coordinates, one can determine the corresponding curvilinear coordinates.

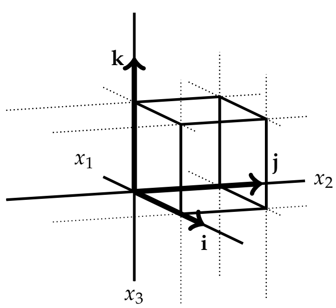

Figure 6.36: Plots of x_i-curves forming an orthogonal Cartesian grid.

In the Cartesian system we can assign an orthogonal basis, $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$. As a particle traces out a path in space, one locates its position by the coordinates (x_1,x_2,x_3). Picking x_2 and x_3 constant, the particle lies on the curve $x_1 =$ value of the x_1 coordinate. This line lies in the direction of the basis vector $\mathbf{i}$. We can do the same with the other coordinates and essentially map out a grid in three dimensional space as sown in Figure 6.36. All of the x_i-curves intersect at each point orthogonally and the basis vectors $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ lie along the grid lines and are mutually orthogonal. We would like to mimic this construction for general curvilinear coordinates. Requiring the orthogonality of the resulting basis vectors leads to orthogonal curvilinear coordinates.

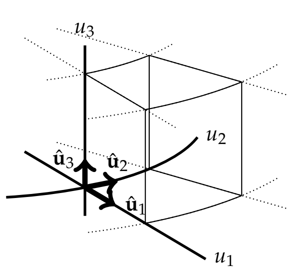

Figure 6.37: Plots of general u_i-curves forming an orthogonal grid.

As for the Cartesian case, we consider u_2 and u_3 constant. This leads to a curve parametrized by $u_1 : \mathbf{r} = x_1(u_1)\mathbf{i} + x_2(u_1)\mathbf{j} + x_3(u_1)\mathbf{k}$. We call this the u_1-curve. Similarly, when u_1 and u_3 are constant we obtain a u_2-curve and for u_1 and u_2 constant we obtain a u_3-curve. We will assume that these curves intersect such that each pair of curves intersect orthogonally as seen in Figure 6.37. Furthermore, we will assume that the unit tangent vectors to these curves form a right handed system similar to the $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ systems for Cartesian coordinates. We will denote these as $\{\hat{\mathbf{u}}_1,\hat{\mathbf{u}}_2,\hat{\mathbf{u}}_3\}$.

We can determine these tangent vectors from the coordinate transformations. Consider the position vector as a function of the new coordinates,

$$\mathbf{r}(u_1, u_2, u_3) = x_1(u_1, u_2, u_3)\mathbf{i} + x_2(u_1, u_2, u_3)\mathbf{j} + x_3(u_1, u_2, u_3)\mathbf{k}.$$

Then, the infinitesimal change in position is given by

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u_i} du_i.$$

We note that the vectors $\frac{\partial \mathbf{r}}{\partial u_i}$ are tangent to the u_i-curves. Thus, we define the unit tangent vectors

$$\hat{\mathbf{u}}_i = \frac{\frac{\partial \mathbf{r}}{\partial u_i}}{\left|\frac{\partial \mathbf{r}}{\partial u_i}\right|}.$$

Solving for the original tangent vector, we have

$$\frac{\partial \mathbf{r}}{\partial u_i} = h_i \hat{\mathbf{u}}_i,$$

where

$$h_i \equiv \left|\frac{\partial \mathbf{r}}{\partial u_i}\right|.$$

The scale factors, $h_i \equiv \left|\frac{\partial \mathbf{r}}{\partial u_i}\right|$.

The h_i's are called the scale factors for the transformation. The infinitesimal change in position in the new basis is then given by

$$d\mathbf{r} = \sum_{i=1}^{3} h_i u_i \hat{\mathbf{u}}_i.$$

Example 6.16. Determine the scale factors for the polar coordinate transformation.

The transformation for polar coordinates is

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Here we note that $x_1 = x$, $x_2 = y$, $u_1 = r$, and $u_2 = \theta$. The u_1-curves are curves with $\theta =$ const. Thus, these curves are radial lines. Similarly, the u_2-curves have $r =$ const. These curves are concentric circles about the origin as shown in Figure 6.38.

The unit vectors are easily found. We will denote them by $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\theta$. We can determine these unit vectors by first computing $\frac{\partial \mathbf{r}}{\partial u_i}$. Let

$$\mathbf{r} = x(r,\theta)\mathbf{i} + y(r,\theta)\mathbf{j} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}.$$

Then,

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial \theta} &= -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}. \end{aligned} \tag{6.174}$$

The first vector already is a unit vector. So,

$$\hat{\mathbf{u}}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}.$$

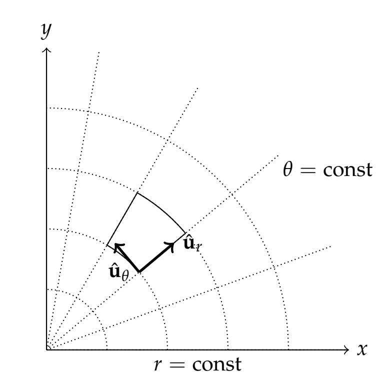

Figure 6.38: Plots an orthogonal polar grid.

The second vector has length r since $|-r\sin\theta\mathbf{i}+r\cos\theta\mathbf{j}| = r$. Dividing $\frac{\partial\mathbf{r}}{\partial\theta}$ by r, we have

$$\hat{\mathbf{u}}_\theta = -\sin\theta\mathbf{i}+\cos\theta\mathbf{j}.$$

We can see these vectors are orthogonal ($\hat{\mathbf{u}}_r\cdot\hat{\mathbf{u}}_\theta = 0$) and form a right hand system. That they form a right hand system can be seen by either drawing the vectors, or computing the cross product,

$$\begin{aligned}(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})\times(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j}) &= \cos^2\theta\mathbf{i}\times\mathbf{j}-\sin^2\theta\mathbf{j}\times\mathbf{i}\\ &= \mathbf{k}. \qquad (6.175)\end{aligned}$$

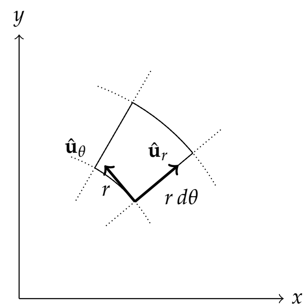

Figure 6.39: Infinitesimal area in polar coordinates.

Since

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial r} &= \hat{\mathbf{u}}_r,\\ \frac{\partial\mathbf{r}}{\partial\theta} &= r\hat{\mathbf{u}}_\theta,\end{aligned}$$

The scale factors are $h_r = 1$ and $h_\theta = r$.

Once we know the scale factors, we have that

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i.$$

The infinitesimal arclength is then given by the Euclidean line element

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = \sum_{i=1}^{3} h_i^2 du_i^2$$

when the system is orthogonal. The h_i^2 are referred to as the metric coefficients.

Example 6.17. Verify that $d\mathbf{r} = dr\hat{\mathbf{u}}_r + r\,d\theta\hat{\mathbf{u}}_\theta$ directly from $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$ and obtain the Euclidean line element for polar coordinates.

We begin by computing

$$\begin{aligned}d\mathbf{r} &= d(r\cos\theta\mathbf{i}+r\sin\theta\mathbf{j})\\ &= (\cos\theta\mathbf{i}+\sin\theta\mathbf{j})\,dr + r(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\,d\theta\\ &= dr\hat{\mathbf{u}}_r + r\,d\theta\hat{\mathbf{u}}_\theta. \qquad (6.176)\end{aligned}$$

This agrees with the form $d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i$ when the scale factors for polar coordinates are inserted.

The line element is found as

$$\begin{aligned}ds^2 &= d\mathbf{r}\cdot d\mathbf{r}\\ &= (dr\hat{\mathbf{u}}_r + r\,d\theta\hat{\mathbf{u}}_\theta)\cdot(dr\hat{\mathbf{u}}_r + r\,d\theta\hat{\mathbf{u}}_\theta)\\ &= dr^2 + r^2\,d\theta^2. \qquad (6.177)\end{aligned}$$

This is the Euclidean line element in polar coordinates.

Also, along the u_i-curves,

$$d\mathbf{r} = h_i du_i \hat{\mathbf{u}}_i, \quad \text{(no summation)}.$$

This can be seen in Figure 6.40 by focusing on the u_1 curve. Along this curve, u_2 and u_3 are constant. So, $du_2 = 0$ and $du_3 = 0$. This leaves $d\mathbf{r} = h_1 du_1 \hat{\mathbf{u}}_1$ along the u_1-curve. Similar expressions hold along the other two curves.

We can use this result to investigate infinitesimal volume elements for general coordinate systems as shown in Figure 6.40. At a given point (u_1, u_2, u_3) we can construct an infinitesimal parallelepiped of sides $h_i du_i$, $i = 1, 2, 3$. This infinitesimal parallelepiped has a volume of size

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| du_1 du_2 du_3.$$

The triple scalar product can be computed using determinants and the resulting determinant is call the Jacobian, and is given by

$$\begin{aligned} J &= \left| \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} \right| \\ &= \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| \\ &= \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}. \end{aligned} \tag{6.178}$$

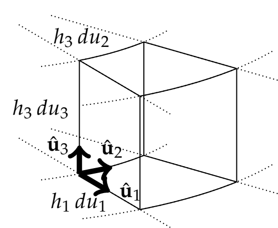

Figure 6.40: Infinitesimal volume element with sides of length $h_i\, du_i$.

Therefore, the volume element can be written as

$$dV = J\, du_1 du_2 du_3 = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(u_1, u_2, u_3)} \right| du_1 du_2 du_3.$$

Example 6.18. Determine the volume element for cylindrical coordinates (r, θ, z), given by

$$x = r \cos \theta, \tag{6.179}$$
$$y = r \sin \theta, \tag{6.180}$$
$$z = z. \tag{6.181}$$

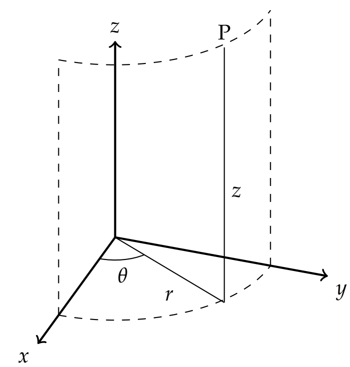

Figure 6.41: Cylindrical coordinate system.

Here, we have $(u_1, u_2, u_3) = (r, \theta, z)$ as displayed in Figure 6.41. Then, the Jacobian is given by

$$\begin{aligned} J &= \left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| \\ &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= r \end{aligned} \tag{6.182}$$

Thus, the volume element is given as

$$dV = r dr d\theta dz.$$

This result should be familiar from multivariate calculus.

Another approach is to consider the geometry of the infinitesimal volume element. The directed edge lengths are given by $d\mathbf{s}_i = h_i du_i \hat{\mathbf{u}}_i$ as seen in Figure 6.37. The infinitesimal area element of for the face in direction $\hat{\mathbf{u}}_k$ is found from a simple cross product,

$$d\mathbf{A}_k = d\mathbf{s}_i \times d\mathbf{s}_j = h_i h_j du_i du_j \hat{\mathbf{u}}_i \times \hat{\mathbf{u}}_j.$$

Since these are unit vectors, the areas of the faces of the infinitesimal volumes are $dA_k = h_i h_j du_i du_j$.

The infinitesimal volume is then obtained as

$$dV = |d\mathbf{s}_k \cdot d\mathbf{A}_k| = h_i h_j h_k du_i du_j du_k |\hat{\mathbf{u}}_i \cdot (\hat{\mathbf{u}}_k \times \hat{\mathbf{u}}_j)|.$$

Thus, $dV = h_1 h_2 h_3 du_1 du_1 du_3$. Of course, this should not be a surprise since

$$J = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| = |h_1 \hat{\mathbf{u}}_1 \cdot h_2 \hat{\mathbf{u}}_2 \times h_3 \hat{\mathbf{u}}_3| = h_1 h_2 h_3.$$

Example 6.19. For polar coordinates, determine the infinitesimal area element.

In an earlier example, we found the scale factors for polar coordinates as $h_r = 1$ and $h_\theta = r$. Thus, $dA = h_r h_\theta \, dr d\theta = r \, dr d\theta$. Also, the last example for cylindrical coordinates will yield similar results if we already know the scales factors without having to compute the Jacobian directly. Furthermore, the area element perpendicular to the z-coordinate gives the polar coordinate system result.

Next we will derive the forms of the gradient, divergence, and curl in curvilinear coordinates using several of the identities in section **??**. The results are given here for quick reference.

Gradient, divergence and curl in orthogonal curvilinear coordinates.

$$\begin{aligned}
\nabla\phi &= \sum_{i=1}^{3} \frac{\hat{\mathbf{u}}_i}{h_i} \frac{\partial \phi}{\partial u_i} \\
&= \frac{\hat{\mathbf{u}}_1}{h_1} \frac{\partial \phi}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2} \frac{\partial \phi}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3} \frac{\partial \phi}{\partial u_3}. \qquad (6.183)\\
\nabla \cdot \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_1 h_3 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right). \qquad (6.184)\\
\nabla \times \mathbf{F} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}. \qquad (6.185)\\
\nabla^2 \phi &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial u_2} \right) \right. \\
&\quad \left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right) \qquad (6.186)
\end{aligned}$$

Derivation of the gradient form.

We begin the derivations of these formulae by looking at the gradient, $\nabla\phi$, of the scalar function $\phi(u_1, u_2, u_3)$. We recall that the gradient operator appears in the differential change of a scalar function,

$$d\phi = \nabla\phi \cdot d\mathbf{r} = \sum_{i=1}^{3} \frac{\partial \phi}{\partial u_i} du_i.$$

Since

$$d\mathbf{r} = \sum_{i=1}^{3} h_i du_i \hat{\mathbf{u}}_i, \tag{6.187}$$

we also have that

$$d\phi = \nabla\phi \cdot d\mathbf{r} = \sum_{i=1}^{3} (\nabla\phi)_i \, h_i du_i.$$

Comparing these two expressions for $d\phi$, we determine that the components of the del operator can be written as

$$(\nabla\phi)_i = \frac{1}{h_i}\frac{\partial \phi}{\partial u_i}$$

and thus the gradient is given by

$$\nabla\phi = \frac{\hat{\mathbf{u}}_1}{h_1}\frac{\partial \phi}{\partial u_1} + \frac{\hat{\mathbf{u}}_2}{h_2}\frac{\partial \phi}{\partial u_2} + \frac{\hat{\mathbf{u}}_3}{h_3}\frac{\partial \phi}{\partial u_3}. \tag{6.188}$$

Next we compute the divergence,

Derivation of the divergence form.

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{3} \nabla \cdot (F_i \hat{\mathbf{u}}_i).$$

We can do this by computing the individual terms in the sum. We will compute $\nabla \cdot (F_1 \hat{\mathbf{u}}_1)$.

Using Equation (6.188), we have that

$$\nabla u_i = \frac{\hat{\mathbf{u}}_i}{h_i}.$$

Then

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_3}{h_2 h_3} = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}.$$

Solving for $\hat{\mathbf{u}}_1$, gives

$$\hat{\mathbf{u}}_1 = h_2 h_3 \nabla u_2 \times \nabla u_3.$$

Inserting this result into $\nabla \cdot (F_1 \hat{\mathbf{u}}_1)$ and using the vector identity 2c from section **??**,

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f,$$

we have

$$\begin{aligned} \nabla \cdot (F_1 \hat{\mathbf{u}}_1) &= \nabla \cdot (F_1 h_2 h_3 \nabla u_2 \times \nabla u_3) \\ &= \nabla (F_1 h_2 h_3) \cdot \nabla u_2 \times \nabla u_3 + F_1 h_2 h_2 \nabla \cdot (\nabla u_2 \times \nabla u_3). \end{aligned} \tag{6.189}$$

The second term of this result vanishes by vector identity 3c,

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

Since $\nabla u_2 \times \nabla u_3 = \frac{\hat{\mathbf{u}}_1}{h_2 h_3}$, the first term can be evaluated as

$$\nabla \cdot (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_2 h_3) \cdot \frac{\hat{\mathbf{u}}_1}{h_2 h_3} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (F_1 h_2 h_3).$$

Similar computations can be carried out for the remaining components, leading to the sought expression for the divergence in curvilinear coordinates:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_1 h_3 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right). \quad (6.190)$$

Example 6.20. Write the divergence operator in cylindrical coordinates.

In this case we have

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{1}{h_r h_\theta h_z} \left(\frac{\partial}{\partial r} (h_\theta h_z F_r) + \frac{\partial}{\partial \theta} (h_r h_z F_\theta) + \frac{\partial}{\partial \theta} (h_r h_\theta F_z) \right) \\ &= \frac{1}{r} \left(\frac{\partial}{\partial r} (r F_r) + \frac{\partial}{\partial \theta} (F_\theta) + \frac{\partial}{\partial \theta} (r F_z) \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (F_\theta) + \frac{\partial}{\partial \theta} (F_z). \end{aligned} \quad (6.191)$$

Derivation of the curl form.

We now turn to the curl operator. In this case, we need to evaluate

$$\nabla \times \mathbf{F} = \sum_{i=1}^{3} \nabla \times (F_i \hat{\mathbf{u}}_i).$$

Again we focus on one term, $\nabla \times (F_1 \hat{\mathbf{u}}_1)$. Using the vector identity 2e,

$$\nabla \times (f \mathbf{A}) = f \nabla \times \mathbf{A} - \mathbf{A} \times \nabla f,$$

we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla \times (F_1 h_1 \nabla u_1) \\ &= F_1 h_1 \nabla \times \nabla u_1 - \nabla (F_1 h_1) \times \nabla u_1. \end{aligned} \quad (6.192)$$

The curl of the gradient vanishes, leaving

$$\nabla \times (F_1 \hat{\mathbf{u}}_1) = \nabla (F_1 h_1) \times \nabla u_1.$$

Since $\nabla u_1 = \frac{\hat{\mathbf{u}}_1}{h_1}$, we have

$$\begin{aligned} \nabla \times (F_1 \hat{\mathbf{u}}_1) &= \nabla (F_1 h_1) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \left(\sum_{i=1}^{3} \frac{\hat{\mathbf{u}}_i}{h_i} \frac{\partial (F_1 h_1)}{\partial u_i} \right) \times \frac{\hat{\mathbf{u}}_1}{h_1} \\ &= \frac{\hat{\mathbf{u}}_2}{h_3 h_1} \frac{\partial (F_1 h_1)}{\partial u_3} - \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \frac{\partial (F_1 h_1)}{\partial u_2}. \end{aligned} \quad (6.193)$$

The other terms can be handled in a similar manner. The overall result is that

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{\hat{\mathbf{u}}_1}{h_2 h_3} \left(\frac{\partial (h_3 F_3)}{\partial u_2} - \frac{\partial (h_2 F_2)}{\partial u_3} \right) + \frac{\hat{\mathbf{u}}_2}{h_1 h_3} \left(\frac{\partial (h_1 F_1)}{\partial u_3} - \frac{\partial (h_3 F_3)}{\partial u_1} \right) \\ &\quad + \frac{\hat{\mathbf{u}}_3}{h_1 h_2} \left(\frac{\partial (h_2 F_2)}{\partial u_1} - \frac{\partial (h_1 F_1)}{\partial u_2} \right) \end{aligned} \quad (6.194)$$

This can be written more compactly as

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix} \tag{6.195}$$

Example 6.21. Write the curl operator in cylindrical coordinates.

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{1}{r} \begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix} \\ &= \left(\frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \hat{\mathbf{e}}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \hat{\mathbf{e}}_\theta \\ &\quad + \frac{1}{r} \left(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\mathbf{e}}_z. \end{aligned} \tag{6.196}$$

Finally, we turn to the Laplacian. In the next chapter we will solve higher dimensional problems in various geometric settings such as the wave equation, the heat equation, and Laplace's equation. These all involve knowing how to write the Laplacian in different coordinate systems. Since $\nabla^2 \phi = \nabla \cdot \nabla \phi$, we need only combine the results from Equations (6.188) and (6.190)for the gradient and the divergence in curvilinear coordinates. This is straight forward and gives

$$\begin{aligned} \nabla^2 \phi &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial u_2} \right) \right. \\ &\quad \left. + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial u_3} \right) \right). \end{aligned} \tag{6.197}$$

The results of rewriting the standard differential operators in cylindrical and spherical coordinates are shown in Problems **??** and **??**. In particular, the Laplacians are given as

Cylindrical Coordinates:

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{6.198}$$

Spherical Coordinates:

$$\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \tag{6.199}$$

Problems

1. A rectangular plate $0 \leq x \leq L$ $0 \leq y \leq H$ with heat diffusivity constant k is insulated on the edges $y = 0, H$ and is kept at constant zero temperature on the other two edges. Assuming an initial temperature of $u(x, y, 0) = f(x, y)$, use separation of variables t find the general solution.

2. Solve the following problem.

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad 0 < x < 2\pi, \quad 0 < y < \pi, \quad 0 < z < 1,$$
$$u(x, y, 0) = \sin x \sin y, \quad u(x, y, z) = 0 \text{on other faces.}$$

3. Consider Laplace's equation on the unit square, $u_{xx} + u_{yy} = 0, 0 \leq x, y \leq 1$. Let $u(0, y) = 0, u(1, y) = 0$ for $0 < y < 1$ and $u_y(x, 0) = 0$ for $0 < y < 1$. Carry out the needed separation of variables and write down the product solutions satisfying these boundary conditions.

4. Consider a cylinder of height H and radius a.

a. Write down Laplace's Equation for this cylinder in cylindrical coordinates.

b. Carry out the separation of variables and obtain the three ordinary differential equations that result from this problem.

c. What kind of boundary conditions could be satisfied in this problem in the independent variables?

5. Consider a square drum of side s and a circular drum of radius a.

a. Rank the modes corresponding to the first 6 frequencies for each.

b. Write each frequency (in Hz) in terms of the fundamental (i.e., the lowest frequency.)

c. What would the lengths of the sides of the square drum have to be to have the same fundamental frequency? (Assume that $c = 1.0$ for each one.)

6. We presented the full solution of the vibrating rectangular membrane in Equation 6.37. Finish the solution to the vibrating circular membrane by writing out a similar full solution.

7. A copper cube 10.0 cm on a side is heated to 100° C. The block is placed on a surface that is kept at 0° C. The sides of the block are insulated, so the normal derivatives on the sides are zero. Heat flows from the top of the block to the air governed by the gradient $u_z = -10°\text{C/m}$. Determine the temperature of the block at its center after 1.0 minutes. Note that the thermal diffusivity is given by $k = \frac{K}{\rho c_p}$, where K is the thermal conductivity, ρ is the density, and c_p is the specific heat capacity.

8. Consider a spherical balloon of radius a. Small deformations on the surface can produce waves on the balloon's surface.

a. Write the wave equation in spherical polar coordinates. (Note: ρ is constant!)

b. Carry out a separation of variables and find the product solutions for this problem.

c. Describe the nodal curves for the first six modes.

d. For each mode determine the frequency of oscillation in Hz assuming $c = 1.0$ m/s.

9. Consider a circular cylinder of radius $R = 4.00$ cm and height $H = 20.0$ cm which obeys the steady state heat equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz}.$$

Find the temperature distribution, $u(r,z)$, given that $u(r,0) = 0°\text{C}$, $u(r,20) = 20°\text{C}$, and heat is lost through the sides due to Newton's Law of Cooling

$$[u_r + hu]_{r=4} = 0,$$

for $h = 1.0 \text{ cm}^{-1}$.

10. The spherical surface of a homogeneous ball of radius one in maintained at zero temperature. It has an initial temperature distribution $u(\rho,0) = 100^o$ C. Assuming a heat diffusivity constant k, find the temperature throughout the sphere, $u(\rho,\theta,\phi,t)$.

11. Determine the steady state temperature of a spherical ball maintained at the temperature

$$u(x,y,z) = x^2 + 2y^2 + 3z^2, \quad \rho = 1.$$

[Hint - Rewrite the problem in spherical coordinates and use the properties of spherical harmonics.]

12.) A hot dog initially at temperature 50°C is put into boiling water at 100°C. Assume the hot dog is 12.0 cm long, has a radius of 2.00 cm, and the heat constant is 2.0×10^{-5} cm^2/s.

a. Find the general solution for the temperature. [Hint: Solve the heat equation for $u(r,z,t) = T(r,z,t) - 100$, where $T(r,z,t)$ is the temperature of the hot dog.]

b. indicate how one might proceed with the remaining information in order to determine when the hot dog is cooked; i.e., when the center temperature is 80°C.